Tripartite Entanglement Dynamics in the presence of Non-Markovian Environment

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Abstract

We study on the tripartite entanglement dynamics when each party is initially entangled with other parties, but they locally interact with their own non-Markovian environment. First, we consider three GHZ-type initial states, all of which have GHZ symmetry provided that the parameters are chosen appropriately. However, this symmetry is broken due to the effect of environment. The corresponding π-tangles, one of the tripartite entanglement measure, are analytically computed at arbitrary time. The revival phenomenon of entanglement occurs after complete disappearance of entanglement. We also consider two W-type initial states. The revival phenomenon also occurs in this case. On the analytical ground the robustness issue against the effect of environment is examined for both GHZ-type and W-type initial states.

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I. INTRODUCTION

Entanglement[1, 2] is a one of the important concepts from fundamental aspect of quantum mechanics and practical aspect of quantum information processing. As shown for last two decades it plays a crucial role in quantum teleportation[3], superdense coding[4], quantum cloning[5], and quantum cryptography[6, 7]. It is also quantum entanglement, which makes the quantum computer[1] outperform the classical one[9].

Quantum mechanics is a physics, which is valid for ideally closed system. However, real physical systems inevitably interact with their surroundings. Thus, it is important to study how the environment modifies the dynamics of the given physical system. There are two different tools for describing the evolution of open quantum system: quantum operation formalism[1] and master equation approach[10]. Both tools have their own merits.

It is known that the entanglement sudden death (ESD) occurs when the entangled multipartite quantum system is embedded in Markovian environments[11]. This means that the entanglement is completely disentangled at finite times. This ESD phenomenon has been revealed experimentally[12, 13].

It is also examined the dynamics of entanglement when the physical system is embedded in non-Markovian environment[10, 14]. It has been shown that there is a revival of entanglement after a finite period of time of its complete disappearance. This is mainly due to the memory effect of the non-Markovian environment. This phenomenon was shown in Ref.[14] by making use of the two qubit system and concurrence[15] as a bipartite entanglement measure. Subsequently, many works have been done to quantify the non-Markovianity[16–20].

In this paper we consider the tripartite entanglement dynamics when the qubit system interacts with the non-Markovian environment. For simplicity, we choose the same physical setting, i.e. there is no interaction between qubit and each qubit interacts with its own reservoir. We will compute the entanglement of three-types of initial Greenberger-Horne-Zeilinger(GHZ) state[21] and two types of initial W-state[22] in the presence of the non-Markovian environment.

Typical tripartite entanglement measures are residual entanglement[23] and π-tangle[24].

[1] The current status of quantum computer technology was reviewed in Ref.[8].

For three-qubit pure state $|\psi\rangle = \sum_{i,j,k=0}^{1} a_{ijk}|ijk\rangle$ the residual entanglement τ_{ABC} becomes

$$\tau_{ABC} = 4|d_1 - 2d_2 + 4d_3|, \tag{1.1}$$

where

$$\begin{aligned} d_1 &= a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2, \\ d_2 &= a_{000}a_{111}a_{011}a_{100} + a_{000}a_{111}a_{101}a_{010} + a_{000}a_{111}a_{110}a_{001} \\ &\quad + a_{011}a_{100}a_{101}a_{010} + a_{011}a_{100}a_{110}a_{001} + a_{101}a_{010}a_{110}a_{001}, \\ d_3 &= a_{000}a_{110}a_{101}a_{011} + a_{111}a_{001}a_{010}a_{100}. \end{aligned} \tag{1.2}$$

Thus, the residual entanglement of any three-qubit pure state can be computed by making use of Eq. (1.1). Although the residual entanglement can detect the GHZ-type entanglement, it can not detect the W-type entanglement:

$$\tau_{ABC}(GHZ) = 1 \qquad \tau_{ABC}(W) = 0, \tag{1.3}$$

where

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \qquad |W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{1.4}$$

For mixed states the residual entanglement is defined by a convex-roof method[25, 26] as follows:

$$\tau_{ABC}(\rho) = \min \sum_i p_i \tau_{ABC}(\rho_i), \tag{1.5}$$

where the minimum is taken over all possible ensembles of pure states. The pure state ensemble corresponding to the minimum τ_{ABC} is called the optimal decomposition. It is in general difficult to derive the optimal decomposition for arbitrary mixed states. Hence, the analytic computation of the residual entanglement can be done for rare cases[27]. Furthermore, recently, three-tangle[2] τ_3 of the whole GHZ-symmetric states[28] was explicitly computed[29].

The π-tangle defined in Ref.[24] is easier for analytic computation than the residual entanglement (or three tangle) because it does not rely on the convex-roof method. The π-tangle is defined in terms of the global negativities [30]. For a three-qubit state ρ they are given by

$$\mathcal{N}^A = ||\rho^{T_A}|| - 1, \qquad \mathcal{N}^B = ||\rho^{T_B}|| - 1, \qquad \mathcal{N}^C = ||\rho^{T_C}|| - 1, \tag{1.6}$$

[2] In this paper we will call $\tau_3 = \sqrt{\tau_{ABC}}$ three-tangle and $\tau_3^2 = \tau_{ABC}$ residual entanglement.

where $||R|| = \mathrm{Tr}\sqrt{RR^\dagger}$, and the superscripts T_A, T_B, and T_C represent the partial transposes of ρ with respect to the qubits A, B, and C respectively. Then, the π-tangle is defined as

$$\pi_{ABC} = \frac{1}{3}(\pi_A + \pi_B + \pi_C), \tag{1.7}$$

where

$$\pi_A = \mathcal{N}^2_{A(BC)} - (\mathcal{N}^2_{AB} + \mathcal{N}^2_{AC}) \quad \pi_B = \mathcal{N}^2_{B(AC)} - (\mathcal{N}^2_{AB} + \mathcal{N}^2_{BC}) \quad \pi_C = \mathcal{N}^2_{(AB)C} - (\mathcal{N}^2_{AC} + \mathcal{N}^2_{BC}). \tag{1.8}$$

The remarkable property of the π-tangle is that it can detect not only the GHZ-type entanglement but also the W-type entanglement:

$$\pi_{ABC}(GHZ) = 1 \qquad \pi_{ABC}(W) = \frac{4}{9}(\sqrt{5} - 1) \sim 0.55. \tag{1.9}$$

As commented earlier we will examine the tripartite entanglement dynamics of the three-qubit states in the presence of the non-Markovian environment. We will adopt the π-tangle as a entanglement measure for analytic computation as much as possible. In section II we consider how the three-qubit initial state is evolved when each qubit interacts with their own non-Markovian environment[14]. In section III we explore the entanglement dynamics of three GHZ-type initial states. The initial states are local unitary(LU) with each other. Thus, their entanglement are the same initially. Furthermore, if the parameters are appropriately chosen, they all have GHZ-symmetry, i.e. they are invariant under (i) qubit permutation (ii) simultaneous three-qubit flips (iii) qubit rotations about the z-axis. However, this symmetry is broken due to the non-Markovian effect. As a result, their entanglement dynamics are different with each other. In section IV we examine the entanglement dynamics of two W-type initial states. They are also LU with each other. However, the dynamics is also different with each other. In section V a brief conclusion is given.

II. GENERAL FEATURES

We consider three-qubit system, each of which interacts only and independently with its local environment. We assume that the dynamics of single qubit is governed by Hamiltonian

$$H = H_0 + H_I \tag{2.1}$$

where

$$H_0 = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k b_k^\dagger b_k \tag{2.2}$$
$$H_I = \sigma_+ \otimes B + \sigma_- \otimes B^\dagger \qquad with \quad B = \sum_k g_k b_k.$$

In Eq. (2.2) ω_0 is a transition frequency of the two-level system (qubit), and $\sigma_\pm$ are the raising and lowering operators. The index k labels the different field modes of the reservoir with frequencies ω_k, creation and annihilation operators $b_k^\dagger$, b_k, and coupling constants g_k. In the interaction picture the dynamics is governed by the Schrödinger equation

$$\frac{d}{dt}\psi(t) = -iH_I(t)\psi(t) \tag{2.3}$$

where

$$\begin{aligned} H_I(t) &\equiv e^{iH_0 t} H_I e^{-iH_0 t} = \sigma_+(t) \otimes B(t) + \sigma_-(t) \otimes B^\dagger(t) \\ \sigma_\pm(t) &\equiv e^{iH_0 t} \sigma_\pm e^{-iH_0 t} = \sigma_\pm e^{\pm i\omega_0 t} \\ B(t) &\equiv e^{iH_0 t} B e^{-iH_0 t} = \sum_k g_k b_k e^{-i\omega_k t}. \end{aligned} \tag{2.4}$$

The Hamiltonian (2.1) represents one of few exactly solvable model[31]. This means that the Schrödinger equation (2.3) can be formally solved if $\psi(0)$ is given. Then, the reduced state of the single qubit $\hat{\rho}^S(t) \equiv Tr_{env}|\psi(t)\rangle\langle\psi(t)|$ is given by[10, 32]

$$\hat{\rho}^S(t) = \begin{pmatrix} \rho_{00}^S(0) + \rho_{11}^S(0)\left(1 - |P_t|^2\right) & \rho_{01}^S(0) P_t \\ \rho_{10}^S(0) P_t^* & \rho_{11}^S(0)|P_t|^2 \end{pmatrix} \tag{2.5}$$

where $\hat{\rho}^S(0) = Tr_{env}|\psi(0)\rangle\langle\psi(0)|$ and Tr_{env} denotes the partial trace over the environment. The function P_t satisfies the differential equation

$$\frac{d}{dt}P_t = -\int_0^t dt_1 f(t - t_1) P_{t_1} \tag{2.6}$$

and the correlation function $f(t - t_1)$ is related to the spectral density $J(\omega)$ of the reservoir by

$$f(t - t_1) = \int J(\omega) exp[i(\omega_0 - \omega)(t - t_1)]. \tag{2.7}$$

We choose $J(\omega)$ as a effective spectral density of the damped Jaynes-Cummings model[10]

$$J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{(\omega_0 - \omega)^2 + \lambda^2} \tag{2.8}$$

where the parameter λ defines the spectral width of the coupling, which is connected to the reservoir correlation time τ_B by the relation $\tau_B = 1/\lambda$ and the relaxation time scale τ_R on which the state of the system changes is related to γ_0 by $\tau_R = 1/\gamma_0$.

By making use of the Residue theorem in complex plane the correlation function can be easily computed in a form

$$f(t - t_1) = \frac{\gamma_0 \lambda}{2} e^{-\lambda |t - t_1|}. \tag{2.9}$$

Inserting Eq. (2.9) into Eq. (2.6) and making use of Laplace transform one can compute P_t explicitly. While in a weak coupling (or Markovian) regime $\tau_R > 2\tau_B$ P_t becomes

$$P_t = e^{-\frac{\lambda}{2} t} \left[\cosh\left(\frac{\bar{d}}{2} t\right) + \frac{\lambda}{\bar{d}} \sinh\left(\frac{\bar{d}}{2} t\right) \right] \tag{2.10}$$

with $\bar{d} = \sqrt{\lambda^2 - 2\gamma_0 \lambda}$, in a strong coupling (or non-Markovian) regime $\tau_R < 2\tau_B$ P_t reduces to

$$P_t = e^{-\frac{\lambda}{2} t} \left[\cos\left(\frac{d}{2} t\right) + \frac{\lambda}{d} \sin\left(\frac{d}{2} t\right) \right] \tag{2.11}$$

with $d = \sqrt{2\gamma_0 \lambda - \lambda^2}$. Since, in the Markovian regime $\lambda > 2\gamma_0$, P_t in Eq. (2.10) exhibits an exponential decay in time, it seems to make a ESD phenomenon. However, in the non-Markovian regime $\lambda < 2\gamma_0$, P_t in Eq. (2.11) exhibits an oscillatory behavior in time with decreasing amplitude. It seems to be responsible for the revival phenomenon of entanglement[14], after a finite period of time of its complete disappearance.

The state $\hat{\rho}^T(t)$ at time t of whole three-qubit system, each of which interacts only and independently with its own environment, can be derived by the Kraus operators[33]. Introducing, for simplicity, $\{|0\rangle \equiv |000\rangle, |1\rangle \equiv |001\rangle, |2\rangle \equiv |010\rangle, |3\rangle \equiv |011\rangle, |4\rangle \equiv |100\rangle, |5\rangle \equiv |101\rangle, |6\rangle \equiv |110\rangle, |7\rangle \equiv |111\rangle\}$, the diagonal parts of $\hat{\rho}^T(t)$ are

$$\begin{aligned}
\hat{\rho}^T_{11}(t) &= P_t^2 \left[\hat{\rho}^T_{11}(0) + \left\{ \hat{\rho}^T_{33}(0) + \hat{\rho}^T_{55}(0) \right\} (1 - P_t^2) + \hat{\rho}^T_{77}(0)(1 - P_t^2)^2 \right] \\
\hat{\rho}^T_{22}(t) &= P_t^2 \left[\hat{\rho}^T_{22}(0) + \left\{ \hat{\rho}^T_{33}(0) + \hat{\rho}^T_{66}(0) \right\} (1 - P_t^2) + \hat{\rho}^T_{77}(0)(1 - P_t^2)^2 \right] \\
\hat{\rho}^T_{33}(t) &= P_t^4 \left[\hat{\rho}^T_{33}(0) + \hat{\rho}^T_{77}(0)(1 - P_t^2) \right] \\
\hat{\rho}^T_{44}(t) &= P_t^2 \left[\hat{\rho}^T_{44}(0) + \left\{ \hat{\rho}^T_{55}(0) + \hat{\rho}^T_{66}(0) \right\} (1 - P_t^2) + \hat{\rho}^T_{77}(0)(1 - P_t^2)^2 \right] \\
\hat{\rho}^T_{55}(t) &= P_t^4 \left[\hat{\rho}^T_{55}(0) + \hat{\rho}^T_{77}(0)(1 - P_t^2) \right] \\
\hat{\rho}^T_{66}(t) &= P_t^4 \left[\hat{\rho}^T_{66}(0) + \hat{\rho}^T_{77}(0)(1 - P_t^2) \right] \\
\hat{\rho}^T_{00}(t) &= 1 - \sum_{i=1}^{7} \hat{\rho}^T_{ii}(t)
\end{aligned} \tag{2.12}$$

and the non-diagonal parts are

$$
\begin{aligned}
&\hat{\rho}^T_{01}(t) = P_t \left[\hat{\rho}^T_{01}(0) + \left\{\hat{\rho}^T_{23}(0) + \hat{\rho}^T_{45}(0)\right\}(1-P_t^2) + \hat{\rho}^T_{67}(0)(1-P_t^2)^2\right] \\
&\hat{\rho}^T_{02}(t) = P_t \left[\hat{\rho}^T_{02}(0) + \left\{\hat{\rho}^T_{13}(0) + \hat{\rho}^T_{46}(0)\right\}(1-P_t^2) + \hat{\rho}^T_{57}(0)(1-P_t^2)^2\right] \\
&\hat{\rho}^T_{04}(t) = P_t \left[\hat{\rho}^T_{04}(0) + \left\{\hat{\rho}^T_{15}(0) + \hat{\rho}^T_{26}(0)\right\}(1-P_t^2) + \hat{\rho}^T_{37}(0)(1-P_t^2)^2\right] \\
&\hat{\rho}^T_{03}(t) = P_t^2 \left[\hat{\rho}^T_{03}(0) + \hat{\rho}^T_{47}(0)(1-P_t^2)\right] \quad \hat{\rho}^T_{05}(t) = P_t^2 \left[\hat{\rho}^T_{05}(0) + \hat{\rho}^T_{27}(0)(1-P_t^2)\right] \\
&\hat{\rho}^T_{06}(t) = P_t^2 \left[\hat{\rho}^T_{06}(0) + \hat{\rho}^T_{17}(0)(1-P_t^2)\right] \quad \hat{\rho}^T_{12}(t) = P_t^2 \left[\hat{\rho}^T_{12}(0) + \hat{\rho}^T_{56}(0)(1-P_t^2)\right] \\
&\hat{\rho}^T_{13}(t) = P_t^3 \left[\hat{\rho}^T_{13}(0) + \hat{\rho}^T_{57}(0)(1-P_t^2)\right] \quad \hat{\rho}^T_{14}(t) = P_t^2 \left[\hat{\rho}^T_{14}(0) + \hat{\rho}^T_{36}(0)(1-P_t^2)\right] \\
&\hat{\rho}^T_{15}(t) = P_t^3 \left[\hat{\rho}^T_{15}(0) + \hat{\rho}^T_{37}(0)(1-P_t^2)\right] \quad \hat{\rho}^T_{23}(t) = P_t^3 \left[\hat{\rho}^T_{23}(0) + \hat{\rho}^T_{67}(0)(1-P_t^2)\right] \\
&\hat{\rho}^T_{24}(t) = P_t^2 \left[\hat{\rho}^T_{24}(0) + \hat{\rho}^T_{35}(0)(1-P_t^2)\right] \quad \hat{\rho}^T_{26}(t) = P_t^3 \left[\hat{\rho}^T_{26}(0) + \hat{\rho}^T_{37}(0)(1-P_t^2)\right] \\
&\hat{\rho}^T_{45}(t) = P_t^3 \left[\hat{\rho}^T_{45}(0) + \hat{\rho}^T_{67}(0)(1-P_t^2)\right] \quad \hat{\rho}^T_{46}(t) = P_t^3 \left[\hat{\rho}^T_{46}(0) + \hat{\rho}^T_{57}(0)(1-P_t^2)\right] \\
&\hat{\rho}^T_{07}(t) = \hat{\rho}^T_{07}(0)P_t^3 \quad \hat{\rho}^T_{16}(t) = \hat{\rho}^T_{16}(0)P_t^3 \quad \hat{\rho}^T_{17}(t) = \hat{\rho}^T_{17}(0)P_t^4 \quad \hat{\rho}^T_{25}(t) = \hat{\rho}^T_{25}(0)P_t^3 \\
&\hat{\rho}^T_{27}(t) = \hat{\rho}^T_{27}(0)P_t^4 \quad \hat{\rho}^T_{34}(t) = \hat{\rho}^T_{34}(0)P_t^3 \quad \hat{\rho}^T_{35}(t) = \hat{\rho}^T_{35}(0)P_t^4 \quad \hat{\rho}^T_{36}(t) = \hat{\rho}^T_{36}(0)P_t^4 \\
&\hat{\rho}^T_{37}(t) = \hat{\rho}^T_{37}(0)P_t^5 \quad \hat{\rho}^T_{47}(t) = \hat{\rho}^T_{47}(0)P_t^4 \quad \hat{\rho}^T_{56}(t) = \hat{\rho}^T_{56}(0)P_t^4 \\
&\hat{\rho}^T_{57}(t) = \hat{\rho}^T_{57}(0)P_t^5 \quad \hat{\rho}^T_{67}(t) = \hat{\rho}^T_{67}(0)P_t^5
\end{aligned}
\tag{2.13}
$$

with $\hat{\rho}^T_{ij}(t) = \hat{\rho}^{T*}_{ji}(t)$. Now, we are ready to explore the tripartite entanglement dynamics in the presence of the non-Markovian environment.

III. ENTANGLEMENT DYNAMICS OF GHZ-TYPE INITIAL STATES

In this section we examine the tripartite entanglement dynamics when the initial states are GHZ-type states. All initial states have GHZ-symmetry[28] if the parameters are appropriately chosen. However, this symmetry is broken due to the effects of environment.

A. Type I

Let us choose the initial state in a form

$$\hat{\rho}^T_I(0) = |\psi_I\rangle\langle\psi_I| \tag{3.1}$$

where $|\psi_I\rangle = a|0\rangle + be^{i\delta}|7\rangle$ with $a^2+b^2=1$. As commented before $|\psi_I\rangle$ has a GHZ-symmetry when $a^2=b^2=1/2$ and $\delta=0$. Then the spectral decomposition of $\hat{\rho}^T_I(t)$ can be read directly from Eqs. (2.12) and (2.13) as a form:

$$\hat{\rho}_I^T(t) = \Lambda_+ |\psi_1\rangle\langle\psi_1| + \Lambda_- |\psi_2\rangle\langle\psi_2| + b^2 P_t^2 (1-P_t^2)^2 \left\{ |1\rangle\langle 1| + |2\rangle\langle 2| + |4\rangle\langle 4| \right\} \\ + b^2 P_t^4 (1-P_t^2) \left\{ |3\rangle\langle 3| + |5\rangle\langle 5| + |6\rangle\langle 6| \right\} \tag{3.2}$$

where

$$\Lambda_\pm = \frac{1}{2}\left[\left\{ 1 - 3b^2 P_t^2 (1-P_t^2) \right\} \pm \sqrt{\left[1 - 3b^2 P_t^2 (1-P_t^2) \right]^2 - 4b^4 P_t^6 (1-P_t^2)^2} \right] \tag{3.3}$$

and

$$|\psi_1\rangle = \frac{1}{N_I}\left(x|0\rangle + y e^{i\delta}|7\rangle \right) \qquad |\psi_2\rangle = \frac{1}{N_I}\left(y|0\rangle - x e^{i\delta}|7\rangle \right) \tag{3.4}$$

with

$$x = 1 - b^2 P_t^2 (3 - 3P_t^2 + 2P_t^4) + \sqrt{\left[1 - 3b^2 P_t^2 (1-P_t^2) \right]^2 - 4b^4 P_t^6 (1-P_t^2)^2} \\ y = 2ab P_t^2 \tag{3.5}$$

and $N_I = \sqrt{x^2 + y^2}$ is a normalization constant.

Since $\hat{\rho}_I^T(t)$ is a full rank, it seems to be highly difficult to compute the residual entanglement (or three-tangle) analytically. However, from Eq. (3.2) one can realize the upper bound of τ_{ABC} as

$$\tau_{ABC} \leq \left[1 - 3b^2 P_t^2 (1-P_t^2) \right] \frac{4x^2 y^2}{(x^2+y^2)^2}. \tag{3.6}$$

It is worthwhile noting that $\hat{\rho}_I^T(t)$ does not have the GHZ-symmetry even at $a^2 = b^2 = 1/2$ and $\delta = 0$. Thus, the symmetry which $\hat{\rho}_I^T(0)$ has is broken due to the effect of environment.

In order to explore the tripartite entanglement dynamics on the analytical ground, we compute the π-tangle of $\hat{\rho}_I^T(t)$. Using Eq. (1.6) it is straightforward to compute the induced bipartite entanglement quantities $\mathcal{N}_{A(BC)}$, $\mathcal{N}_{B(AC)}$, and $\mathcal{N}_{(AB)C}$. One can show that they are all the same with

$$\mathcal{N}_{A(BC)} = \mathcal{N}_{B(AC)} = \mathcal{N}_{(AB)C} \\ = \max\left[\sqrt{b^4 P_t^4 (1-P_t^2)^2 (1-2P_t^2)^2 + 4a^2 b^2 P_t^6} - b^2 P_t^2 (1-P_t^2), 0 \right]. \tag{3.7}$$

One can also show the two-tangles completely vanish, i.e. $\mathcal{N}_{AB} = \mathcal{N}_{AC} = \mathcal{N}_{BC} = 0$, easily. Thus the π-tangle of $\hat{\rho}_I^T(t)$ is

$$\pi_{GHZ}^I(t) = \mathcal{N}_{A(BC)}^2. \tag{3.8}$$

Since $P_{t=0} = 1$, the π-tangle is maximal initially. It reduces to zero with increasing t because $P_{t=\infty} = 0$. Furthermore, in the non-Markovian regime P_t exhibits an oscillatory behavior and becomes zero at $t_n = 2[n\pi - \tan^{-1}(d/\lambda)/d]$. Thus, complete disentanglement occurs at $t = t_n \ \ (n = 1, 2, \cdots)$.

B. Type II

Let us choose the initial state in a form

$$\hat{\rho}^T_{II}(0) = |\psi_{II}\rangle\langle\psi_{II}| \tag{3.9}$$

where $|\psi_{II}\rangle = a|1\rangle + be^{i\delta}|6\rangle$ with $a^2 + b^2 = 1$. Since $|\psi_I\rangle = \mathbb{1} \otimes \mathbb{1} \otimes \sigma_x |\psi_{II}\rangle$, $(\mathbb{1} \otimes \mathbb{1} \otimes \sigma_x)\hat{\rho}^T_{II}(0)(\mathbb{1} \otimes \mathbb{1} \otimes \sigma_x)^\dagger$ has a GHZ-symmetry provided that $a^2 = b^2 = 1/2$ and $\delta = 0$.

Using Eqs. (2.12) and (2.13) one can show that the spectral decomposition of $\hat{\rho}^T_{II}(t)$ becomes

$$\hat{\rho}^T_{II}(t) = \lambda_2|\phi_{II}\rangle\langle\phi_{II}| + (1 - P_t^2)\left[a^2 + b^2(1 - P_t^2)\right]|0\rangle\langle 0| + b^2 P_t^2 (1 - P_t^2)\left(|2\rangle\langle 2| + |4\rangle\langle 4|\right) \tag{3.10}$$

where

$$\lambda_2 = P_t^2 (a^2 + b^2 P_t^2) \tag{3.11}$$
$$|\phi_{II}\rangle = \frac{1}{\sqrt{a^2 + b^2 P_t^2}}\left(a|1\rangle + bP_t e^{i\delta}|6\rangle\right).$$

Unlike the case of type I $\hat{\rho}^T_{II}(t)$ is rank four tensor. From Eq. (3.10) one can derive the upper bound of τ_{ABC} for $\hat{\rho}^T_{II}(t)$, which is

$$\tau_{ABC} \leq \frac{4a^2 b^2 P_t^4}{a^2 + b^2 P_t^2}. \tag{3.12}$$

The negativities $\mathcal{N}_{A(BC)}$, $\mathcal{N}_{B(AC)}$, and $\mathcal{N}_{(AB)C}$ of $\hat{\rho}^T_{II}(t)$ can be computed by making use of Eq. (1.6). The final expressions are

$$\mathcal{N}_{A(BC)} = \mathcal{N}_{B(AC)} = \sqrt{b^4 P_t^4 (1 - P_t^2)^2 + 4a^2 b^2 P_t^6} - b^2 P_t^2 (1 - P_t^2) \tag{3.13}$$
$$\mathcal{N}_{(AB)C} = \sqrt{(1 - P_t^2)^2\left[a^2 + b^2(1 - P_t^2)\right]^2 + 4a^2 b^2 P_t^6} - (1 - P_t^2)\left[a^2 + b^2(1 - P_t^2)\right].$$

It is also easy to show $\mathcal{N}_{AB} = \mathcal{N}_{AC} = \mathcal{N}_{BC} = 0$. Thus the π-tangle of $\hat{\rho}^T_{II}(t)$ is

$$\pi^{II}_{GHZ}(t) = \frac{1}{3}\left[2\mathcal{N}^2_{A(BC)} + \mathcal{N}^2_{(AB)C}\right]. \tag{3.14}$$

When $t = 0$, $\pi^{II}_{GHZ}(0)$ becomes $4a^2 b^2$ and it reduces to zero as $t \to \infty$. Of course, the entanglement of $\hat{\rho}^T_{II}(t)$ is completely disentangled at $t = t_n \ \ (n = 1, 2, \cdots)$.

C. Type III

Let us choose the initial state in a form

$$\hat{\rho}^T_{III}(0) = |\psi_{III}\rangle\langle\psi_{III}| \tag{3.15}$$

where $|\psi_{III}\rangle = a|3\rangle + be^{i\delta}|4\rangle$ with $a^2 + b^2 = 1$. Since $|\psi_I\rangle = \mathbb{1} \otimes \sigma_x \otimes \sigma_x |\psi_{III}\rangle$, $(\mathbb{1} \otimes \sigma_x \otimes \sigma_x)\hat{\rho}^T_{III}(0)(\mathbb{1} \otimes \sigma_x \otimes \sigma_x)^\dagger$ has a GHZ-symmetry provided that $a^2 = b^2 = 1/2$ and $\delta = 0$.

Using Eqs. (2.12) and (2.13) one can show that the spectral decomposition of $\hat{\rho}^T_{III}(t)$ becomes

$$\hat{\rho}^T_{III}(t) = \lambda_3|\phi_{III}\rangle\langle\phi_{III}| + (1 - P_t^2)\left[a^2(1 - P_t^2) + b^2\right]|0\rangle\langle 0| + a^2P_t^2(1 - P_t^2)\left(|1\rangle\langle 1| + |2\rangle\langle 2|\right) \tag{3.16}$$

where

$$\lambda_3 = P_t^2(a^2P_t^2 + b^2) \tag{3.17}$$
$$|\phi_{III}\rangle = \frac{1}{\sqrt{a^2P_t^2 + b^2}}\left(aP_t|3\rangle + be^{i\delta}|4\rangle\right).$$

Unlike the case of type I $\hat{\rho}^T_{III}(t)$ is rank four tensor. From Eq. (3.16) one can derive the upper bound of τ_{ABC} for $\hat{\rho}^T_{III}(t)$, which is

$$\tau_{ABC} \leq \frac{4a^2b^2P_t^4}{a^2P_t^2 + b^2}. \tag{3.18}$$

The negativities $\mathcal{N}_{A(BC)}$, $\mathcal{N}_{B(AC)}$, and $\mathcal{N}_{(AB)C}$ of $\hat{\rho}^T_{III}(t)$ can be computed by making use of Eq. (1.6), whose explicit expressions are

$$\mathcal{N}_{A(BC)} = \sqrt{(1 - P_t^2)^2\left[a^2(1 - P_t^2) + b^2\right]^2 + 4a^2b^2P_t^6} - (1 - P_t^2)\left[a^2(1 - P_t^2) + b^2\right]$$
$$\mathcal{N}_{B(AC)} = \mathcal{N}_{(AB)C} = \sqrt{a^4P_t^4(1 - P_t^2)^2 + 4a^2b^2P_t^6} - a^2P_t^2(1 - P_t^2). \tag{3.19}$$

It is of interest to note that $\mathcal{N}_{A(BC)}$ and $\mathcal{N}_{B(AC)}$ of type III is the same with $\mathcal{N}_{(AB)C}$ and $\mathcal{N}_{A(BC)}$ of type II with $a \leftrightarrow b$ respectively. It is easy to show $\mathcal{N}_{AB} = \mathcal{N}_{AC} = \mathcal{N}_{BC} = 0$. Thus the π-tangle of $\hat{\rho}^T_{III}(t)$ is

$$\pi^{III}_{GHZ}(t) = \frac{1}{3}\left[\mathcal{N}^2_{A(BC)} + 2\mathcal{N}^2_{B(AC)}\right]. \tag{3.20}$$

One can also consider different types of initial GHZ-type states. For example, one can consider $\hat{\rho}^T_{IV}(0) = |\psi_{IV}\rangle\langle\psi_{IV}|$, where $|\psi_{IV}\rangle = a|2\rangle + be^{i\delta}|5\rangle$. Although, in this case, $\hat{\rho}^T_{IV}(t)$ is

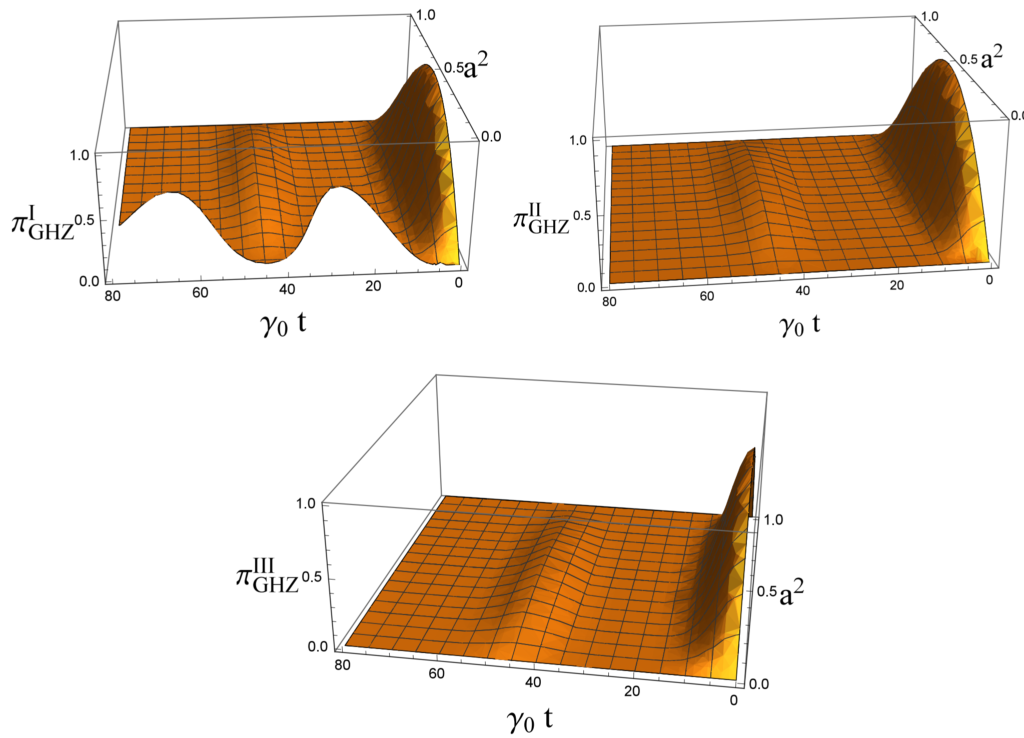

FIG. 1: (Color online) The π-tangle for the initial states (a) $a|000\rangle + be^{i\delta}|111\rangle$, (b) $a|001\rangle + be^{i\delta}|110\rangle$, and (c) $a|011\rangle + be^{i\delta}|100\rangle$ as a function of the parameters $\gamma_0 t$ and a^2. We choose λ as a $\lambda = 0.01\gamma_0$.

different from $\hat{\rho}^T_{II}(t)$, one can show that its π-tangle is exactly the same with that of type II. Thus, this case is not discussed in detail.

The π-tangle for each type is plotted in Fig. 1 as a function of dimensionless parameter $\gamma_0 t$ and a^2. We choose λ as a $\lambda = 0.01\gamma_0$. The white region in Fig. 1(a) corresponds to $\sqrt{b^4 P_t^4 (1 - P_t^2)^2 (1 - 2P_t^2)^2 + 4a^2 b^2 P_t^6} \leq b^2 P_t^2 (1 - P_t^2)$. Thus, Eq. (3.7) guarantees $\pi^I_{GHZ}(t) = 0$. As expected the tripartite entanglement reduces to zero with increasing time with oscillatory behavior.

The π-tangles $\pi^I_{GHZ}(t)$, $\pi^{II}_{GHZ}(t)$, and $\pi^{III}_{GHZ}(t)$ are compared in Fig. 2 when $\lambda/\gamma_0 = 0.001$. They are represented by red solid, black dashed, and blue dotted lines respectively. Fig. 2(a) and Fig. 2(b) correspond to $a^2 = 0.1$ and $a^2 = 0.9$. Both figures clearly show the revival of the tripartite entanglement, after a finite period of time of complete disappearance. The revival phenomenon seems to be mainly due to the memory effect of the non-Markovian environment. It is of interest to note that while $\pi^{III}_{GHZ}(t) \geq \pi^{II}_{GHZ}(t) \geq \pi^I_{GHZ}(t)$ when $a^2 = 0.1$, the order is changed as $\pi^I_{GHZ}(t) \geq \pi^{II}_{GHZ}(t) \geq \pi^{III}_{GHZ}(t)$ when $a^2 = 0.9$.

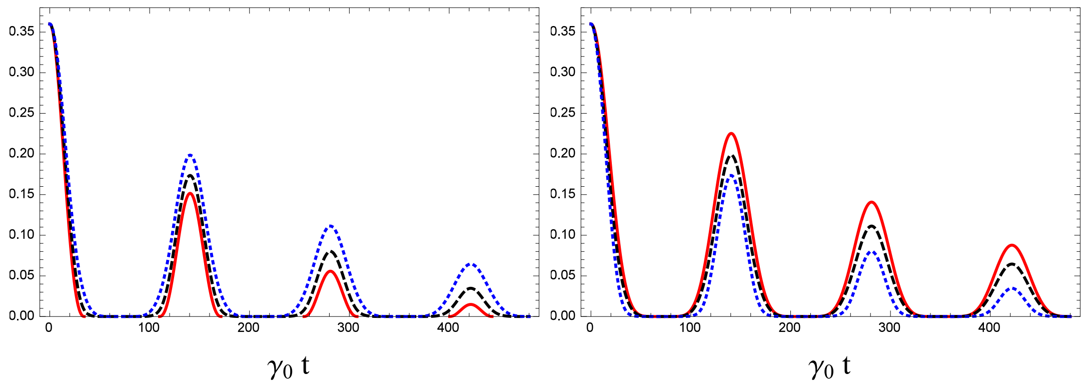

FIG. 2: (Color online) The $\gamma_0 t$ dependence of $\pi^I_{GHZ}(t)$ (red solid), $\pi^{II}_{GHZ}(t)$ (black dashed), and $\pi^{III}_{GHZ}(t)$ (blue dotted) when (a) $a^2 = 0.1$ and (b) $a^2 = 0.9$. We choose λ as a $\lambda = 0.001\gamma_0$.

IV. ENTANGLEMENT DYNAMICS OF W-TYPE INITIAL STATES

In this section we examine the tripartite entanglement dynamics when the initial states are two W-type states. Both initial states are LU to each other. However, their entanglement dynamics are different due to Eqs. (2.12) and (2.13).

A. Type I

In this subsection we choose the initial state as

$$\hat{\rho}^W_I(0) = |W_1\rangle\langle W_1| \tag{4.1}$$

where $|W_1\rangle = a|1\rangle + be^{i\delta_1}|2\rangle + ce^{i\delta_2}|4\rangle$ with $a^2 + b^2 + c^2 = 1$. Then, it is straightforward to show that the spectral decomposition of $\hat{\rho}^W_I(t)$ is

$$\hat{\rho}^W_I(t) = (1 - P_t^2)|0\rangle\langle 0| + P_t^2|W_1\rangle\langle W_1|. \tag{4.2}$$

Eq. (4.2) guarantees that the residual entanglement and three-tangle of $\hat{\rho}^W_I(t)$ are zero because the spectral decomposition exactly coincides with the optimal decomposition.

By making use of Eq. (1.6) one can compute the induced bipartite entanglement quantities $\mathcal{N}_{A(BC)}$, $\mathcal{N}_{B(AC)}$, and $\mathcal{N}_{(AB)C}$ of $\hat{\rho}_I^W(t)$ directly, whose expressions are

$$\begin{aligned}
\mathcal{N}_{A(BC)} &= \sqrt{(1-P_t^2)^2 + 4c^2(a^2+b^2)P_t^4} - (1-P_t^2) \\
\mathcal{N}_{B(AC)} &= \sqrt{(1-P_t^2)^2 + 4b^2(a^2+c^2)P_t^4} - (1-P_t^2) \\
\mathcal{N}_{(AB)C} &= \sqrt{(1-P_t^2)^2 + 4a^2(b^2+c^2)P_t^4} - (1-P_t^2).
\end{aligned} \tag{4.3}$$

Also, the two tangles $\mathcal{N}_{AB}$, $\mathcal{N}_{AC}$, and $\mathcal{N}_{BC}$ become

$$\begin{aligned}
\mathcal{N}_{AB} &= \sqrt{\left[(1-P_t^2) + a^2P_t^2\right]^2 + 4b^2c^2P_t^4} - \left[(1-P_t^2) + a^2P_t^2\right] \\
\mathcal{N}_{AC} &= \sqrt{\left[(1-P_t^2) + b^2P_t^2\right]^2 + 4a^2c^2P_t^4} - \left[(1-P_t^2) + b^2P_t^2\right] \\
\mathcal{N}_{BC} &= \sqrt{\left[(1-P_t^2) + c^2P_t^2\right]^2 + 4a^2b^2P_t^4} - \left[(1-P_t^2) + c^2P_t^2\right].
\end{aligned} \tag{4.4}$$

Thus, using Eqs. (1.7) and (1.8) one can compute the π-tangle of $\hat{\rho}_I^W(t)$, whose explicit expression is

$$\begin{aligned}
\pi_W^I(t) = \frac{2}{3}\Bigg[& 2\left[(1-P_t^2) + a^2P_t^2\right]\sqrt{\left[(1-P_t^2) + a^2P_t^2\right]^2 + 4b^2c^2P_t^4} \\
& +2\left[(1-P_t^2) + b^2P_t^2\right]\sqrt{\left[(1-P_t^2) + b^2P_t^2\right]^2 + 4a^2c^2P_t^4} \\
& +2\left[(1-P_t^2) + c^2P_t^2\right]\sqrt{\left[(1-P_t^2) + c^2P_t^2\right]^2 + 4a^2b^2P_t^4} \\
& -(1-P_t^2)\Bigg\{\sqrt{(1-P_t^2)^2 + 4a^2(b^2+c^2)P_t^4} \\
& +\sqrt{(1-P_t^2)^2 + 4b^2(a^2+c^2)P_t^4} + \sqrt{(1-P_t^2)^2 + 4c^2(a^2+b^2)P_t^4}\Bigg\} \\
& -2(a^4+b^4+c^4)P_t^4 - (1-P_t^2)(3+P_t^2)\Bigg].
\end{aligned} \tag{4.5}$$

When $t=0$, Eq. (4.5) reduces to

$$\pi_W^I(0) = \frac{4}{3}\left[a^2\sqrt{a^4+4b^2c^2} + b^2\sqrt{b^4+4a^2c^2} + c^2\sqrt{c^4+4a^2b^2} - (a^4+b^4+c^4)\right], \tag{4.6}$$

which exactly coincides with a result of Ref.[24]. Of course, when $t = t_n (n = 1, 2, \cdots)$ and $t = \infty$, the entanglement of $\hat{\rho}_I^W(t)$ is completely disentangled.

B. Type II

In this subsection we choose the initial state as

$$\hat{\rho}_{II}^W(0) = |W_2\rangle\langle W_2| \tag{4.7}$$

where $|W_2\rangle = a|6\rangle + be^{i\delta_1}|5\rangle + ce^{i\delta_2}|3\rangle$ with $a^2 + b^2 + c^2 = 1$. This initial state is LU to $|W_1\rangle$ because of $|W_2\rangle = (\sigma_x \otimes \sigma_x \otimes \sigma_x)|W_1\rangle$. Then, by making use of Eqs. (2.12) and (2.13) it is straightforward to show that $\hat{\rho}_{II}^W(t)$ is

$$\hat{\rho}_{II}^W(t) = (1 - P_t^2)^2|0\rangle\langle 0| + P_t^4|W_2\rangle\langle W_2| + 2P_t^2(1 - P_t^2)\sigma_{II}(t) \tag{4.8}$$

where

$$\begin{aligned}
\sigma_{II}(t) = \frac{1}{2}\Bigg[&(b^2 + c^2)|1\rangle\langle 1| + (a^2 + c^2)|2\rangle\langle 2| + (a^2 + b^2)|4\rangle\langle 4| \\
&+ab\left(e^{i\delta_1}|1\rangle\langle 2| + e^{-i\delta_1}|2\rangle\langle 1|\right) + ac\left(e^{i\delta_2}|1\rangle\langle 4| + e^{-i\delta_2}|4\rangle\langle 1|\right) \\
&+bc\left(e^{-i(\delta_1-\delta_2)}|2\rangle\langle 4| + e^{i(\delta_1-\delta_2)}|4\rangle\langle 2|\right)\Bigg].
\end{aligned} \tag{4.9}$$

The spectral decomposition of $\sigma_{II}(t)$ cannot be derived analytically. Also, analytic computation of π-tangle for $\hat{\rho}_{II}^W(t)$ is impossible. Thus, we have to reply on the numerical approach for computation of π-tangle.

However, some special cases allow the analytic computation. In this paper we consider a special case $a^2 = b^2 = c^2 = 1/3$. In this case the spectral decomposition of $\sigma_{II}(t)$ can be derived as

$$\sigma_{II}(t) = \frac{2}{3}|\alpha_1\rangle\langle\alpha_1| + \frac{1}{6}|\alpha_2\rangle\langle\alpha_2| + \frac{1}{6}|\alpha_3\rangle\langle\alpha_3| \tag{4.10}$$

where

$$\begin{aligned}
|\alpha_1\rangle &= \frac{1}{\sqrt{3}}\left(|1\rangle + e^{-i\delta_1}|2\rangle + e^{-i\delta_2}|4\rangle\right) \\
|\alpha_2\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle - e^{-i\delta_2}|4\rangle\right) \\
|\alpha_3\rangle &= \frac{1}{\sqrt{6}}\left(|1\rangle - 2e^{-i\delta_1}|2\rangle + e^{-i\delta_2}|4\rangle\right).
\end{aligned} \tag{4.11}$$

Thus, Eqs. (4.8) and (4.10) imply that $\hat{\rho}_{II}^W(t)$ with $a^2 = b^2 = c^2 = 1/3$ is rank-5 tensor, three of them are W-states and the remaining ones are fully-separable and bi-separable states. Thus, its residual entanglement and three-tangles are zero.

Using Eq. (1.6) one can show that $\mathcal{N}_{A(BC)}$, $\mathcal{N}_{B(AC)}$, and $\mathcal{N}_{(AB)C}$ are all identical as

$$\mathcal{N}_{A(BC)} = \mathcal{N}_{B(AC)} = \mathcal{N}_{(AB)C} = \frac{1}{3}P_t^2\left[\sqrt{9 - 18P_t^2 + 17P_t^4} - 3(1 - P_t^2)\right]. \tag{4.12}$$

Also $\mathcal{N}_{AB}$, $\mathcal{N}_{AC}$, and $\mathcal{N}_{BC}$ are all identical as

$$\mathcal{N}_{AB} = \mathcal{N}_{AC} = \mathcal{N}_{BC} = \begin{cases} \frac{\sqrt{9-24P_t^2+20P_t^4+2P_t^2(2-P_t^2)}}{3} - 1 & P_t^2 \geq 2 - \sqrt{2} \\ 0 & P_t^2 \leq 2 - \sqrt{2}. \end{cases} \tag{4.13}$$

Thus, the π-tangle for $\hat{\rho}_{II}^{W}(t)$ with $a^2 = b^2 = c^2 = 1/3$ is given by $\pi_W^{II} = \mathcal{N}_{A(BC)}^2 - 2\mathcal{N}_{AB}^2$.

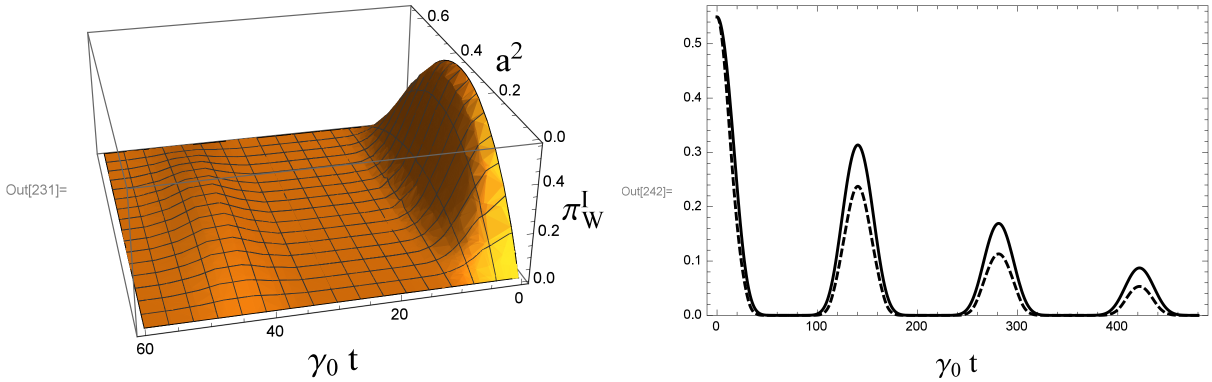

FIG. 3: (Color online) (a) The a^2 and $\gamma_0 t$ dependence of $\pi_I^W(t)$ when $c^2 = 1/3$. We choose $\lambda = 0.01\gamma_0$. (b) The $\gamma_0 t$ dependence of $\pi_I^W(t)$ (solid line) and $\pi_{II}^W(t)$ (dashed line) when $a^2 = b^2 = c^2 = 1/3$. We choose $\lambda = 0.001\gamma_0$. This figure implies that $\hat{\rho}_I^W(t)$ is more robust against the environment than $\hat{\rho}_{II}^W(t)$.

In fig. 3(a) we plot $\pi_I^W(t)$ as a function of a^2 and $\gamma_0 t$. We choose $c^2 = 1/3$ and $\lambda/\gamma_0 = 0.01$. As expected the π-tangle reduces to zero as $t \to \infty$. To compare $\pi_I^W(t)$ with $\pi_{II}^W(t)$ we plot both π-tangles as a function of $\gamma_0 t$ in Fig. 3(b). In this figure we choose $a^2 = b^2 = c^2 = 1/3$ and $\lambda/\gamma_0 = 0.001$. The π-tangles $\pi_I^W(t)$ and $\pi_{II}^W(t)$ are plotted as solid and dashed lines respectively. In this case as in the other cases the revival of entanglement occurs after complete disappearance. It is interesting to note that $\hat{\rho}_I^W(t)$ is more robust than $\hat{\rho}_{II}^W(t)$ against non-Markovian environment.

V. CONCLUSIONS

In this paper we examine the tripartite entanglement dynamics when each party is entangled with other parties initially, but they locally interact with their own non-Markovian environment. First, we consider three GHZ-type initial states $|\psi_I\rangle = a|000\rangle + be^{i\delta}|111\rangle$, $|\psi_{II}\rangle = a|001\rangle + be^{i\delta}|110\rangle$, and $|\psi_{III}\rangle = a|011\rangle + be^{i\delta}|100\rangle$. All states are LU to each other. They all have a GHZ-symmetry, i.e. they are invariant under (i) qubit permutation (ii) simultaneous three-qubit flips (iii) qubit-rotation about the z-axis, provided that $a^2 = b^2 = 1/2$, $\delta = 0$, and an appropriate LU is applied. It turns out that this symmetry is broken due to the effect of environment. We compute the corresponding π-tangles analytically at arbitrary time t in Eqs. (3.8), (3.14), and (3.20). The π-tangles completely vanish when $t_n = 2[n\pi - \tan^{-1}(d/\lambda)/d]$ $(n = 1, 2, \cdots)$ and $t \to \infty$. As shown in Fig. 2 the revival phenomenon of entanglement occurs after complete disappearance of entanglement. This is mainly due to the memory effect of non-Markovian environment. It is shown that while the robustness order against the effect of reservoir is $|\psi_I\rangle$, $|\psi_{II}\rangle$, $|\psi_{III}\rangle$ for large a^2 region, this order is reversed for small a^2 region.

We also examine the tripartite entanglement dynamics for two W-type initial states $|W_1\rangle = a|001\rangle + be^{i\delta_1}|010\rangle + ce^{i\delta_2}|100\rangle$ and $|W_2\rangle = a|110\rangle + be^{i\delta_1}|101\rangle + ce^{i\delta_2}|011\rangle$ with $a^2 + b^2 + c^2 = 1$. Like GHZ-type initial states they are LU to each other. For initial $|W_1\rangle$ state the π-tangle is analytically computed in Eq. (4.5). Since, however, $|W_2\rangle$ propagates to higher-rank states with the lapse of time, the analytic computation is impossible except few special cases. Thus, we compute the π-tangle analytically for special case $a^2 = b^2 = c^2 = 1/3$.

It is of interest to study the effect of non-Markovian environment when the initial state is a rank-2 mixture

$$\rho(p) = p|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)|\mathrm{W}\rangle\langle\mathrm{W}| \tag{5.1}$$

where $|\mathrm{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ and $|\mathrm{W}\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. The residual entanglement of $\rho(p)$ is known as

$$\tau(p) = \begin{cases} 0 & 0 \le p \le p_0 \\ g_I(p) & p_0 \le p \le p_1 \\ g_{II}(p) & p_1 \le p \le 1 \end{cases} \tag{5.2}$$

where

$$\begin{aligned} p_0 &= \frac{4\sqrt[3]{2}}{3 + 4\sqrt[3]{2}} = 0.626851\cdots \qquad & p_1 &= \frac{1}{2} + \frac{3\sqrt{465}}{310} = 0.70868\cdots \\ g_I(p) &= p^2 - \frac{8\sqrt{6}}{9}\sqrt{p(1-p)^3} \qquad & g_{II}(p) &= 1 - (1-p)\left(\frac{3}{2} + \frac{1}{18}\sqrt{465}\right). \end{aligned} \tag{5.3}$$

It is interesting, at least for us, how the non-Markovian environment modifies Coffman-Kundu-Wootters inequality $4\min[\det(\rho_A)] \ge \mathcal{C}(\rho_{AB})^2 + \mathcal{C}(\rho_{AC})^2$ in this model.

Acknowledgement: This research was supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of

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