POWER SERIES IN SEVERAL COMPLEX VARIABLES.

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Abstract. The purpose of this article is to provide an exposition of domains of convergence of power series of several complex variables without recourse to relatively advanced notions of convexity.

1. Notations, Preliminaries, Introduction.

A nice exposition of a multidimensional analogue of the Cauchy – Hadamard formula on the radius of convergence of power series, can be found in the book [6] by B. V. Shabat, which naturally leads one[1] to the conviction that domains of convergence of a power series in several complex variables constitute precisely, the class of logarithmically convex complete multi-circular domains. In the present expository essay, we provide an alternative route to this result which avoids relatively advanced notions of convexity, such as holomorphic convexity – this is natural in a systematic presentation of the subject of several complex variables, where a first goal lies in obtaining various characterizations of the collective of *all* domains of holomorphy, of which domains of convergence of power series, form a very small (and the simplest) sub-class. We emphasize that this is an expository essay that has been inspired by Shabat's treatment [6]. There have been other sources as well; instead of enlisting all the sources here, we shall cite them at appropriate places.

We show how one might guess the aforementioned result on the characterizing features of domains of convergence of power series in higher dimensions and help develop a feel for this simplest class of domains of holomorphy. Indeed, we shall show that on any given logarithmically convex complete multi-circular domain $D \subset \mathbb{C}^N$, all power series with its domain of convergence coinciding with D, can be seen to arise in one particular fashion. Namely, every power series with D as its domain of convergence, can be recast as a sum of monomials, indexed by sequences of rational points on the positive face of the standard simplex in $\mathbb{R}^N$, converging to prescribed points of a countable dense subset of the normalized effective domain of the support function of the logarithmic image of D! This then leads to a natural way of writing down explicit power series converging precisely on any such given D, without having to deal with the case of an unbounded D separately as done in the nice set of lecture notes by H. Boas, available at his web-page [2]. On the other hand given any power series, we shall see how to not only write down a defining function for the domain of convergence in $\mathbb{C}^N$ but also the support function of the convex domain in $\mathbb{R}^N$ formed by its logarithmic image, directly in terms of the coefficients of the given power series. All of this is perhaps folklore matter but our intent here is to provide a thoroughgoing treatment from an elementary standpoint.

Let us set up the stage for our discussion to begin in the next section. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $N \in \mathbb{N}$. For $J = (j_1, \ldots, j_N) \in \mathbb{N}_0^N$, define $|J| = |j_1| + \ldots + |j_N|$ and for $z \in \mathbb{C}^N$, let z^J

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[1]This is being written with a graduate student in mind or those with no prior knowledge of the matter here. The remaining footnotes may be ignored on a first reading.

stand for the monomial $z_1^{j_1} z_2^{j_2} \dots z_N^{j_N}$. Let $(\mathbb{R}^+)^N$ denote the N-fold Cartesian product of the multiplicative group of $\mathbb{R}^+$ of positive reals; its closure in $\mathbb{R}^N$ is the monoid $(\mathbb{R}_+)^N$ with $\mathbb{R}_+$ being the multiplicative monoid [2] of non-negative reals. . For $J \in \mathbb{N}_0^N$, define $J! = j_1! j_2! \dots j_N!$ with the understanding that $0! = 1$. We are interested here with the case $N > 1$. Unless explicitly specified, our indexing set in all countable summations is $\mathbb{N}_0^N$. A connected open subset of $\mathbb{C}^N$ is called a *domain* [3]. A viewpoint which has been decisive for the exposition here, is that the most tangible manner of describing a domain is by supplying sufficient data about its boundary, the simplest of which is specifying a defining function for the boundary of the domain and when the domain is convex, the supporting function for it. Two fundamental bounded domains which will appear often in the sequel are the unit ball with respect to the standard l^2-norm on $\mathbb{C}^N$ given by

$$\mathbb{B}^N = \{z \in \mathbb{C}^N \ : \ |z_1|^2 + \dots + |z_N|^2 < 1\}$$

and the unit ball with respect to the l^∞-norm on $\mathbb{C}^N$ given by the N-fold Cartesian product of Δ the unit disc in $\mathbb{C}$, namely

$$\mathbb{U}^N := \Delta^N := \{z \in \mathbb{C}^N \ : \ |z_j| < 1 \text{ for all } j = 1, 2, \dots, N\},$$

which is called the standard unit polydisc; while balls in the l^∞-norm will be called polydiscs, balls in the l^2-norm will simply be referred to as 'balls'. Further, N-fold Cartesian products of discs $\Delta(z_j^0, r_j)$ of varying radii r_j and varying centers z_j^0 for j varying through $\{1, 2, \dots, N\}$, called the polydisc with polyradius $r = (r_1, \dots, r_N)$ centered at the point z^0 in $\mathbb{C}^N$, will be denoted by $P(z^0, r)$. To indicate the practice of brevity in notation that will be adopted: the center of such sets will be dropped out of notation and denoted P or B, when it happens to be the origin or if they are not important for the discussion at hand; or for instance if the radius does need to be kept track of, discs in $\mathbb{C}$ about the origin with radius r will be denoted Δ_r. Finally, let us mention the one other norm to make an explicit appearance which is, the largest among all norms on $\mathbb{R}^N$ which assigns unit length to its standard basis vectors namely, the l^1-norm. Its unit ball is known by various names: co-cube/cross-polytope/orthoplex; the boundary of this orthoplex is the standard simplex S_N and its intersection with the non-negative orthant is called the probability simplex given by

$$PS_N = \{x \in \mathbb{R}^N \ : \ x_1 + \dots + x_N = 1, \text{ and } x_j \geq 0 \text{ for all } j\}$$

which may be noted to be the convex hull of the standard basis of $\mathbb{R}^N$.

We summarize several basic facts that will be used tacitly in the sequel. Let I be the unit interval $[0, 1] \subset \mathbb{R}$, which may be noted to be closed under a pair of basic algebraic binary operations: one, the arithmetic mean and the other, the geometric mean of any two numbers from $[0, 1]$; as is apparent, these come from the basic pair of algebraic/arithmetical operations on the field of reals. Infact, both these operations may be modified to give rise to a one-parameter family of operations of I on itself: for any pair of numbers a, b their weighted arithmetic mean, corresponding to any fixed $t \in I$, is given by $(1-t)a + tb$ while their weighted geometric mean is given by $a^{1-t}b^t$. For each fixed t, these operations make the set I into a monoid with the identity elements for these operations being situated at the opposite extremes of I. Furthermore, there is a relation between this pair of binary operations, given by the order relation, called the Hölder's inequality, namely,

$$a^{1-t}b^t \leq (1-t)a + tb.$$

[2]The definition of 'monoid' is obtained by removing precisely the condition on the existence of inverse in the definition of a 'group'.

[3]More generally, we shall refer to any connected open subset of any topological space X as a domain in X.

Just as the basic algebraic operations on $\mathbb{R}$ renders $\mathbb{R}^N$ the structure of a vector space, so does the monoid I with either of the above binary operations on certain subsets of $\mathbb{R}^N$. Indeed, let V be any finite dimensional vector space over the reals; there is for each $t \in [0,1]$ a pair of binary operations Φ_t, Ψ_t. While one of them, to be the one denoted Φ_t in the sequel – Φ_t corresponds to the action of forming the straight line joining [4] a pair points – requires only the affine-space structure of V, the other requires coordinatizing V. Indeed, $\Phi_t : V \times V \to V$ is given by

$$\Phi_t(v,w) = (1-t)v + tw.$$

For the other binary operation, we first make some identification of V with $\mathbb{R}^N$ for $N = \dim(V)$; it is best defined first in the connected component of the identity of the multiplicative Lie group $(\mathbb{R}^*)^N$, namely $(\mathbb{R}^+)^N$, as:

$$\Psi_t(v,w) = \Big(\psi_t(v_1,w_1), \ldots, \psi_t(v_N,w_N)\Big),$$

for $v,w \in (\mathbb{R}^+)^N$ with $\psi_t(a,b) = a^{1-t}b^t$. The former operation is facilitated by the scalar multiplication of $\mathbb{R}$ on V (henceforth identified with $\mathbb{R}^N$) and the latter [5] by its conjugate namely, the conjugate of scalar multiplication by the exponential/logarithm:

$$v \to \lambda^{-1}\big(t\lambda(v)\big)$$

where $\lambda(v) = (\log v_1, \ldots, \log v_N)$. This logarithmic mapping λ has an obvious extension: $v \to (\log|v_1|, \ldots, \log|v_N|)$ as a surjective group homomorphism $(\mathbb{C}^*)^N \to \mathbb{R}^N$ whose kernel is the torus $\mathbb{T}^N$. This map which we continue to denote by λ, may be further viewed to extend as a monoid morphism from the multiplicative monoid [6] $\mathbb{C}^N$ onto the additive monoid $[-\infty,\infty)^N$; this actually factors through the monoid morphism $\tau : (z_1, \ldots, z_N) \to (|z_1|, \ldots, |z_N|)$ mapping $\mathbb{C}^N$ onto the absolute space $(\mathbb{R}_+)^N$. The product[7] of $[-\infty,\infty)^N$ with $\mathbb{T}^N$ can be identified via the mapping $A : (x,\omega) \to (e^{x_1}\omega_1, \ldots, e^{x_N}\omega_N)$ (here ofcourse it is understood that $e^{-\infty} = 0$) with $\mathbb{C}^N$. Products of domains in $[-\infty,\infty)^N$ with $\mathbb{T}^N$ are pushed forward by this mapping A onto domains which are 'multi-circular' (invariant under the natural action of $\mathbb{T}^N$ on $\mathbb{C}^N$) and

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[4]Straight line formation and convex sets can be defined in any affine space; circular arcs, to be introduced later, in affine spaces (of dimension at least two) with an origin i.e., vector spaces and logarithmic convexity in normed vector spaces.

[5]This binary operation which consists of forming the coordinate-wise geometric mean of the given pair of points, may be extended to all other cosets of $(\mathbb{R}^+)^N$ in $(\mathbb{C}^*)^N$ by taking coordinate-wise product with the map which sends a complex number z to $z/|z|$, as:

$$\Psi_t(v,w) = \Big(v_1^{1-t}w_1^t\frac{v_1w_1}{|v_1w_1|}, \ldots, v_N^{1-t}w_N^t\frac{v_Nw_N}{|v_Nw_N|}\Big)$$

But we shall not pursue this here. We are more interested in sets closed under these binary operations – which admit alternative definitions – rather than the operations themselves.

[6]The multiplicative monoid structure on $\mathbb{C}^N$ is used in the operation Ψ_t which plays a central role in this article: $\Psi_t(v,w) = p\big(\lambda^{-1}\big(t\lambda(v)\big), \lambda^{-1}\big((1-t)\lambda(w)\big)\big)$ where $p(v,w) = (v_1w_1, \ldots, v_Nw_N)$ denotes the monoidal operation of coordinate-wise product. We remark in passing that the map λ whose components may be thought of as $\Re \circ \log$ (for a suitable local branch of the complex logarithm) applied to the respective coordinates, is continuous, infact smooth and (pluri-)harmonic, on all of $(\mathbb{C}^*)^N$ even though the complex logarithm fails to be continuous on $\mathbb{C}^*$; if we factor out τ from λ, it is a local diffeomorphism, in particular, an open mapping. These facts are convenient in assuring ourselves, while imaging Reinhardt domains in the logarithmic space as *domains*. Finally, let us mention that its extension to $\mathbb{C}^N$ is upper semi-continuous; indeed $z \to \log|z|$ furnishes the simplest upper semicontinuous subharmonic function whose polar set is non-empty.

[7]Direct product of the additive monoid $[-\infty,\infty)^N$ with the multiplicative group $\mathbb{T}^N$ can be identified – via the mapping $A : (x,\omega) \to (e^{x_1}\omega_1, \ldots, e^{x_N}\omega_N)$ – with $\mathbb{C}^N$ which is an additive group as well as a multiplicative monoid.

are termed Reinhardt domains [8]. Pull-backs of convex domains in $\mathbb{R}^N$ by λ are called logarithmically convex – formulated again precisely in definition (2.10) below. So, sets closed under Ψ_t are those whose logarithmic images are closed under the former binary operation Φ_t. As Φ_t requires no coordinatization, it is trivial that sets closed under this binary operation for all $t \in I$ namely the convex sets, remain convex under all affine transformations – convexity is an affine property. However, it is far more non-trivial that multi-circular logarithmically convex domains in $\mathbb{C}^N$ whose logarithmic images are complete/closed under translation by vectors from $(-\mathbb{R}^+)^N$, possess a property which remains invariant under all biholomorphic (not just affine!) transformations. This property known as pseudoconvexity will not be discussed much here (we refer the novice to Range's expository articles [4] and [5]). Pseudoconvexity is a subtle property; however, we hope that the present essay, among other extensive treatises such as [7], convinces the reader that it is possible to gain a 'hands-on' experience with the simplest examples of 'pseudoconvex' domains namely, domains of convergence of power series in several complex variables.

Among the most elementary functions of several complex variables are the monomial functions and their linear combinations.

Definition 1.1. A function of the form $p(z) = \sum_{|J| \leq m} c_J z^J$ is called a polynomial. Here, if at least one of the c_J's with $|J| = m$ is non-zero, the total degree of p is defined to be $\deg(p) = m$. For the zero polynomial, the degree is not defined. A polynomial is called homogeneous of degree m if the coefficients c_J for $|J| < m$ are all zero. Equivalently, a polynomial p of degree m is homogeneous if and only if $p(\lambda z) = \lambda^m p(z)$ for all $\lambda \in \mathbb{C}$.

Thus polynomials are for us by definition, functions on the coordinate space $\mathbb{C}^N$, defined by expressions from the (coordinate) ring $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_N]$. Such functions are annihilated by the operators $\partial/\partial \overline{z}_j$ for all $j = 1, \ldots, N$ and are sometimes referred to as 'holomorphic polynomials' to distinguish them from finite linear combinations of monomials in the $2N$ variables $z, \overline{z}$. Moving further, we may obtain more functions by taking limits of polynomials; but such limits will often not be well-defined on all of $\mathbb{C}^N$ and we need to identify the subset on which they exist. Before we investigate this, we must first be clear about issues of limits and convergence in several variables, which we review in the following sub-section.

1.1. **Series indexed by Lattices.** Suppose that for each $J \in \mathbb{N}_0^N$, a complex number c_J is given; we may form the series $\sum c_J$ and discuss the matter of its convergence. A trouble immediately arising is: there is no canonical order on $\mathbb{N}_0^N$. So to start with, we make the following

Definition 1.2. The series of complex numbers $\sum c_J$ indexed by $J \in \mathbb{N}_0^N$ is said to be convergent, if there exists at least one bijection $\phi : \mathbb{N} \to \mathbb{N}_0^N$ such that $\sum_{i=1}^{\infty} |c_{\phi(i)}| < \infty$. Then the number

$$\sum_{i=1}^{\infty} c_{\phi(i)}$$

is called the limit of the series. Now note that this notion of convergence is independent of the choice of the map ϕ and that it means absolute convergence, thus circumventing the ambiguities alluded to above; all possible rearranged-summing leads to the same sum.

Example 1.3 (The geometric series of several variables.). Let $r = (r_1, \ldots, r_N) \in \mathbb{R}_+^N$ with $r_i \in (0, 1)$ for all $i = 1, \ldots, N$. Then the number $r^J = r_1^{j_1} r_2^{j_2} \ldots r_N^{j_N}$ is again in $(0, 1)$. If I is a finite

[8] It is helpful to draw (for $N \leq 3$) images of Reinhardt domains in the absolute space as well as in the corresponding logarithmic space and we urge the reader to do so.

sub-lattice of $\mathbb{N}_0^N$, there is an integer L such that $I \subset \{0, 1, 2, \ldots, L\}^L$ so that we may write

$$|\sum_{J \in I} r^J| = \sum_{J \in I} r^J \leq \prod_{i=1}^{N} \sum_{k_i=0}^{L} r_i^{k_i} \leq \prod_{i=1}^{N} \Big(\frac{1}{1 - r_i}\Big) < \infty$$

and conclude that the series is convergent. Replacing r by $z \in \Delta^N$ shows likewise that the partial sums of the multi-variable geometric series $\sum_J z^J$ is also convergent on Δ^N – indeed, absolutely convergent – with sum being given by $\prod_{i=1}^{N} \big(1/(1 - z_i)\big)$.

1.2. **Convergence of functions.** Let M be an arbitrary subset of $\mathbb{C}^N$, $\{f_J \ : \ J \in \mathbb{N}_0^N\}$ a family of complex-valued functions on M. Denote by $|f_J|_M$ the supremum of $|f_J|$ on M.

Definition 1.4. The series $\sum_J f_J$ is said to *normally convergent* on M if the series of positive numbers $\sum |f_J|_M$ is convergent.

Proposition 1.5. *Suppose the series $\sum f_J$ is normally convergent on M. Then it is convergent for any $z \in M$ and for any bijective map $\phi : \mathbb{N}_0 \to \mathbb{N}_0^N$, the series $\sum_{i=1}^{\infty} f_{\phi(i)}$ is uniformly convergent on M.*

Set theoretic operations. A possibly not-so-often encountered operation shall arise naturally in the sequel, namely that of the limit infimum of a countable collection of sets, enumerated as say $\{C_n\}_{n=1}^{\infty}$; their limit infimum is given by

$$\liminf_{n \in \mathbb{N}} C_n = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} C_j.$$

Thus, $\omega \in \liminf_n C_n$ if and only if for some n, $\omega \in C_j$ for all $j \geq n$; in other words, $\omega \in \liminf_n C_n$ if and only if $\omega \in C_n$ eventually. A trivial fact that will be useful to keep in mind for the sequel is that the limit infimum of a countable collection of *convex* sets in $\mathbb{R}^N$ is convex. Rudiments of convex analysis is reviewed in the last section which may be useful as a reference for our notational practices as well. Indeed it will do well to keep the basics of convex calculus afresh in mind and the basics for the present essay are summarized in the last section.

1.3. **Recap of Convex Analysis and Geometry.** Refer to the last section.

Remark 1.6. A final remark about notations: an ambiguous notation to be used is the indexing of sequences of reals say, as $\{c^n\}$ rather than by a subscript, which may cause confusion with the notation of the n-th power of a number c. Such a notation will be employed only in connection with other objects; for instance, the first components of a vector sequence $v^n = (v_1^n, \ldots, v_N^n) \in \mathbb{R}^N$ is naturally denoted v_1^n. We hope such ambiguous notations will be clear from context.

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2. Power series in several variables.

Definition 2.1. Let c_J be a sequence of complex numbers indexed by $J \in \mathbb{N}_0^N$ and $z_0 \in \mathbb{C}^N$. Then the expression $\sum c_J(z - z_0)^J$ is called a formal power series about z_0. Without loss of generality, we shall assume henceforth that z_0 is the origin. If this series converges normally on a set M to a complex-valued function f then being a uniform limit of continuous functions, we first note that f is continuous on M.

Definition 2.2. Let $f(z) = \sum c_J z^J$ be a formal power series. Denote by B the set of all points of $\mathbb{C}^N$ at which the series S converges; it's interior B^0 is termed the 'domain' of convergence of the power series S.

Remark 2.3.

(i) There is a canonical way to sum a power series of several variables, even though the indexing set in the summation is $\mathbb{N}_0^N$. Namely, one first sums up all monomials of any given degree and then sums up the homogeneous polynomials of various degrees thus obtained:

$$\sum_{k=1}^{\infty} \sum_{|J|=k} c_J z^J$$

If we declare a power series to be convergent if the sum of its homogeneous constituents ordered by degree as above converges, instead of the (tacit) requirement made above that every rearrangement of the constituting monomials of a power series must lead to a convergent series with the same sum, then the domain of convergence gets enlarged. As a power series is thought of more as a sum of the monomials constituting occurring in it, this practice of summing by homogeneous components alone, is not adopted. It is even customary to write a power series as a sum of monomials arranged in non-decreasing order of their degree (i.e., with respect to the partial order on $\mathbb{N}_0^N$ by l^1-norm) though our requirement places no emphasis on such an ordering.

(ii) We shall refer to both the formal power series and the (holomorphic) function it defines, by the same symbol.

(iii) The quotes on the word 'domain' in the above definition, can and will be dropped as soon as we verify that the B^0 is connected. This requires the following lemma.

Lemma 2.4 (Abel's lemma). *Let $P' \Subset P$ be a pair of polydiscs about the origin. If the power series $\sum c_J z^J$ converges at some point of the distinguished boundary of P, then it converges normally on P'.*

Proof. Let $P = \Delta_1 \times \Delta_2 \times \ldots \times \Delta_N$ where Δ_j are discs of some radii about the origin and denote the distinguished boundary of P which is the thin subset $\partial\Delta_1 \times \partial\Delta_2 \times \ldots \times \partial\Delta_N$, of the boundary of P, by $\partial_0 P$. Let $w \in \partial_0 P$ be such that $\sum c_J w^J$ is convergent. Then firstly, there exists a constant $C > 0$ such that $|c_J w^J| \leq C$ for all $J \in \mathbb{N}_0^N$. Next, compare the modulii of the coordinates of points in P with that of w i.e., consider the ratios $r_j(z) = |z_j|/|w_j|$ for $j = 1, 2, \ldots, N$ – each of these ratios $r_j(z)$ is bounded above by a positive constant say q_j, strictly less than 1, owing to P' being compactly contained inside P. Note that the sup-norm of the monomial-function $c_J z^J$ on P' is bounded above by the constant Cq^J:

$$|c_J z^J| \leq |c_J q^J w^J| \leq C q^J$$

for every $z \in P'$. This comparison with the geometric series $\sum q^J$ – which converges because we know q_j are all strictly less than 1 – finishes the verification that $\sum |c_J z^J|_{P'}$ is convergent and subsequently that $\sum_J c_J z^J$ is normally convergent. Finally, since every compact subset of P is contained in some compact sub-polydisc P' of P, we see that our power series converges uniformly on each compact subset of P. □

Definition 2.5. We say that a power series $\sum_J c_J z^J$ converges compactly in a domain D, if it converges normally on every compact subset of D.

Lemma 2.6. *Let $P' \Subset P$ be a pair of polydiscs about the origin. Suppose $f(z) = \sum c_J z^J$ converges compactly on polydisc P and the multi-radius of P' is $r = (r_1, \ldots, r_N)$. Then the coefficients of the power series can be recovered from the knowledge of the values of f on the distinguished boundary of P' by the formula:*

$$c_K = \frac{1}{(2\pi)^N r^K} \int_{[0,2\pi]^N} f(z) e^{-i(k_1\theta_1 + \ldots + k_N\theta_N)} d\theta_1 \ldots d\theta_N$$

and consequently, we have the estimate

$$|c_K| \le \frac{1}{(2\pi)^N} \frac{|f|_{\mathbb{T}}}{r^K}$$

Proof. Set $z_j = r_j e^{i\theta_j}$ for each $j = 1, \ldots, N$ to write

$$f(z) = f(r_1 e^{i\theta_1}, \ldots, r_N e^{i\theta_N}) = \sum c_J r^J e^{i(j_1\theta_1 + \ldots + j_N\theta_N)}$$

and integrate with respect to each of the variables θ_j on $[0, 2\pi]$ to get

$$\int_{[0,2\pi]^N} f(z) e^{-i(k_1\theta_1 + \ldots + k_N\theta_N)} d\theta_1 \ldots d\theta_N$$

$$= \sum c_J r^J \int_{[0,2\pi]^N} e^{i(j_1-k_1)\theta_1 + \ldots + (j_N - k_N)\theta_N} d\theta_1 \ldots d\theta_N.$$

where the interchange of integral and summation on the right is justified by the uniform convergence of our power series on the boundary of P'. The integral appearing on the right in the last equation is zero except when $J = K$ in which case it is $[0, 2\pi]^N$. The formulae in assertion now follow. □

Definition 2.7. Let z^0 be any point of $\mathbb{C}^N$. The (open) polydisc centered at the origin with polyradius $(|z_1^0|, |z_2^0|, \ldots, |z_N^0|)$ is called the polydisc spanned by the point z^0.

We may rephrase Abel's lemma as follows. Let P be a polydisc and w a point of the distinguished boundary of P. If the power series $\sum c_J z^J$ converges (unconditionally) at w, then it converges compactly on P. Stated differently, if f converges at a point w, then it converges compactly on the polydisc spanned by w. This means that the interior of the set of convergence of the general power series f which we denoted B^0, can be expressed as the union of the (concentric) polydiscs spanned by points of B and subsequently that B^0 must be connected. This finishes the pending verification that B^0 is indeed a domain. In fact, we may note more here: B^0 is what is known as a Reinhardt domain, indeed a 'complete Reinhardt domain' as defined below and in particular therefore, a contractible domain.

Definition 2.8. A domain D in $\mathbb{C}^N$ is termed Reinhardt (about the origin) if $z \in D$ entails that $(e^{i\theta_1} z_1, \ldots, e^{i\theta_N} z_N) \in D$ for all possible choices of $(\theta_1, \ldots, \theta_N) \in \mathbb{R}^N$. Such a domain is also said to be *multi-circular*. A domain D in $\mathbb{C}^N$ is said to be circular if $z \in D$ entails (only) that $(e^{i\theta} z_1, \ldots, e^{i\theta} z_N) \in D$ for all $\theta \in \mathbb{R}$; it is said to be complete circular if it admits an action by the disc i.e., $z \in D$ entails that $(\lambda z_1, \ldots, \lambda z_N) \in D$ for all $\lambda \in \overline{\Delta}$; complete circular domains are sometimes also referred to as complex star-like domains and we note in passing that all complete circular domains are contractible domains. Likewise a Reinhardt domain is said to be *complete* if it is invariant under the action of the closed unit poydisc by coordinate-wise multiplication i.e., $z \in D$ entails that $(\lambda_1 z_1, \ldots, \lambda_N z_N) \in D$ for all choices of $(\lambda_1, \ldots, \lambda_N) \in \overline{\Delta}^N$.

Proposition 2.9. *The domain of convergence B^0 is a complete Reinhardt domain and $f(z)$ converges compactly in B^0.*

Now we may ask: is every complete Reinhardt domain, the domain of convergence of some power series? The answer is No. Domains of convergence have some additional properties.

Definition 2.10. Let $\lambda : (\mathbb{C}^*)^N \to \mathbb{R}^N$ be the map given by

$$\lambda(z) = (\log|z_1|, \log|z_2|, \ldots, \log|z_N|).$$

A set M in $\mathbb{C}^N$ is termed logarithmically convex if its logarithmic image $M^* = \lambda(M_0)$, where

$$M_0 = \{z \in M \ : \ z_1 z_2 \ldots z_N \neq 0\} = M \cap (\mathbb{C}^*)^N$$

is convex.

Remark 2.11. For a logarithmically convex complete Reinhardt domain D, we shall sometimes write $\lambda(D)$ for $\lambda(D^*)$. We may also consider the map $\lambda : D \to [-\infty, \infty)^N$ with the obvious extension of λ to points in $M \cap D$ i.e., with some of its coordinates zero. Then suppose $z^0 \in D \cap (\mathbb{C}^*)^N$. The Jacobian of λ at the point z^0 is given by

$$\frac{1}{|z_1^0|} \cdots \frac{1}{|z_N^0|}$$

which is evidently non-zero. Consequently by the inverse function theorem, λ is an open map when restricted to D^*; it is not difficult to check that λ (without restriction) is itself an open mapping. So if $p, q \in D^*$ then $p_\lambda = \lambda(p)$, $q_\lambda = \lambda(q)$ are interior points of the convex domain $G = \lambda(D^*)$. Observe that every point of the line segment joining the pair p_λ, q_λ is an interior point of G.

Suppose z, w are two different points in the domain of convergence of a given power series $\sum c_J z^J$. So, $\sum_J |c_J z^J|$ and $\sum_J |c_J w^J|$ converge to some finite positive numbers. Recall Hölder's inequality and write it, as applied to the pair of positive numbers $|z^J|, |w^J|$ and the conjugate exponents $1/t, 1/(1-t)$ where $t \in (0,1)$, as follows:

$$|z^J|^t |w^J|^{1-t} \leq \frac{\left(|z^J|^t\right)^{1/t}}{1/t} + \frac{\left(|w^J|^{1-t}\right)^{1/(1-t)}}{1/(1-t)}.$$

Hence,

$$|z^J|^t |w^J|^{1-t} \leq t|z^J| + (1-t)|w^J|$$

from which it is apparent that the given series converges at the point with real coordinates given by

$$(|z_1|^t |w_1|^{1-t}, |z_2|^t |w_2|^{1-t}, \ldots, |z_N|^t |w_N|^{1-t}).$$

Infact, this point lies in the interior of the set of points where the power series converges, namely $D = B^0$. Indeed to indicate the reasoning here, suppose $z, w \in D^* = D \cap (\mathbb{C}^*)^N$. Let B_z, B_w be balls centered at the points $\lambda(z), \lambda(w)$ of radius some $\epsilon > 0$, such that they are contained within $G = \lambda(D^*)$. Then, observe that the logarithmic image of the above point lies in the convex hull of B_z, B_w, which may be noted to be contained inside G. As λ pulls back open sets to open sets just by continuity of λ, the following basic result follows.

Proposition 2.12. *The domain of convergence B^0 is logarithmically convex.*

Well, how does one 'discover' this? How can one guess other properties, if any, that is possessed by all those domains which are precise domains of convergence of some power series? Is it possible to pin down all common features shared by domains of convergence of power series which characterize them completely? To answer all this, one needs to get to the roots of the theory of power series: first, the (precise/largest) domain of convergence of any given power series in a single variable is always a disc whose radius is read off from the coefficients of the given series, using the following

Theorem 2.13 (Cauchy – Hadamard formula)**.** *The radius of convergence of the power series $\sum c_j z^j$ is given by*

$$\frac{1}{\limsup_{j \to \infty} \sqrt[j]{|c_j|}}$$

It is natural to ask for a constructive method of describing the domain of convergence of a power series of several variables. Now, the uniformity in the shape of the domain of convergence of power series in several variables is not as trivial as in the case of one variable, for, as we shall see, the ball and the polydisc are each, the natural domain of convergence of some power series but they are not biholomorphically equivalent. All we know at this point, is that domains of convergence of power series in several variables are also completely determined by their absolute profile, so we focus on $\tau(D)$; but then $\tau(D)$ is not a domain and to avoid this annoyance, we pass to the logarithmic image $\lambda(D)$; more importantly, $\lambda(D)$ has a geometric property namely convexity, shared by all domains of convergence of power series. Further, they can be expressed as the union of concentric polydiscs.

Definition 2.14. A polydisc $U = U(z^0, r)$ is termed a polydisc of convergence of $\sum c_J z^j$ if $U \subset B$ but in any polydisc $U(z^0, R)$ where each $R_j \geq r_j$ for $j = 1, 2, \dots, N$ with at least one of the inequalities being strict, there are points in $U(z^0, R)$ where the series diverges.

Every such polyradii $(r_1, r_2, \dots, r_n)$ of $U(z^0, r)$ is called a conjugate polyradii i.e., the radii of each polydisc of convergence are called conjugate radii of convergence.

If we join the dots formed by the various conjugate radii in the absolute space, what do we get? The answer to this is facilitated by a higher dimensional analogue of the Cauchy – Hadamard formula:

Proposition 2.15. *The conjugate radii of convergence of the power series* $\sum_{k=1}^{\infty} \sum_{|J|=k} c_J z^J$ *satisfy the relation*

$$\limsup_{|J| \to \infty} \sqrt[|J|]{|c_J r^J|} = 1 \tag{2.1}$$

Proof. Let r be a conjugate radii of convergence of the given series

$$\sum_{k=1}^{\infty} \sum_{|J|=k} c_J z^J \tag{2.2}$$

Let $\zeta \in \Delta$. Then $z = \zeta \cdot r$ lies in the polydisc of convergence U, the series converges absolutely in U and after regrouping the terms, we obtain from (2.2), the following series in the variable ζ:

$$\sum_{|J|=1}^{\infty} |c_J| z^J = \sum_{|J|=1}^{\infty} |c_J| \zeta^{|J|} r^{|J|} = \sum_{k=1}^{\infty} \Big(\sum_{|J|=k} |c_J| r^J \Big) \zeta^{|J|}$$

So we obtain from (2.2) that the series

$$\sum_{k=1}^{\infty} \Big(\sum_{|J|=k} |c_J| r^J \Big) \zeta^{|J|}$$

which is a series in one complex variable ζ known to be convergent for $\zeta \in \Delta$.
If there exists ζ_0 outside the closed unit disc at which this series converges, then it must be convergent on the disc centered at the origin which will imply that the coefficients satisfy the following decay estimate:

$$|c_J| r^{|J|} \leq \frac{M}{|\zeta_0|^{|J|}}$$

that is,

$$|c_J| \leq \frac{M}{|(\zeta_0 r)^J|},$$

which means that the series at (2.2) must converge on the polydisc with polyradii $(\zeta_0 r_1, \zeta_0 r_2, \dots, \zeta_0 r_N)$ contradicting that U is a (maximal) polydisc of convergence. Thus, the series (2.2) diverges for every point ζ with $|\zeta| > 1$. By the Cauchy – Hadamard formula for one variable, we therefore have

$$\limsup_{k\to\infty} \sqrt[k]{\sum_{|J|=k} |c_J| r^J} = 1 \tag{2.3}$$

It only remains to show that this equation is equivalent to the one claimed in the statement of our proposition. For this, first choose among all the monomials $\{c_J z^J\}$ with $|J| = j_1 + \ldots + j_n = k$, the one for which the maximum in

$$\max_{|J|=k} |c_J| r^J$$

– the maximum of sup-norms of monomials on the polydisc of radius r – is attained. Let $M = (m_1, m_2, \ldots, m_N)$ be such that this maximum is attained i.e.,

$$|c_M| r^M = \max_{|J|=k} \{|c_J| r^J\}.$$

Then write down the obvious estimate

$$|c_M| r^M \leq \sum_{|J|=k} |c_J| r^J \leq (k+1)^N |c_M| r^M,$$

with the last inequality obtained by overestimating the number of terms appearing in the sum in the middle! Using this and the fact that $(k+1)^{N/K} \to 1$ as $K \to \infty$, we may rewrite (2.3) as the relation

$$\limsup_{k\to\infty} \sqrt[k]{|c_M| r^M} = 1,$$

from which the asserted relation of the proposition follows. $\square$

Now, note that the relation (2.1) in the proposition above, can be rewritten as the equation

$$\varphi(r_1, r_2, \ldots, r_N) = 0 \tag{2.4}$$

which 'ties together' a relation among the conjugate radii of convergence of the series (2.2). This equation determines the boundary of the domain $\tau(B^0)$ which depicts the domain of convergence B^0 in the absolute space. Next, substitute $r_j = e^{s_j}$ in (2.4). This leads to the last equation to be transformed as

$$\psi(s_1, s_2, \ldots, s_N) = 0$$

– the equation for the boundary of $\lambda(B^0)$, the logarithmic image of B^0, some convex domain in $\mathbb{R}^N$. Indeed, let us rewrite equation (2.1) after taking logarithms:

$$\limsup_{|J|\to\infty} \Big(\frac{j_1 \log r_1 + j_2 \log r_2 + \ldots + j_N \log r_N}{j_1 + j_2 + \ldots + j_N} + \log |c_J|/|J| \Big) = 0$$

So ultimately, in the variables $s_1, \ldots, s_N$, the relation (2.1) reads:

$$\limsup_{|J|\to\infty} \Big(\frac{j_1 s_1 + j_2 s_2 + \ldots + j_N s_N}{j_1 + j_2 + \ldots + j_N} + \log |c_J|/|J| \Big) = 0 \tag{2.5}$$

Indeed, the left hand side here is the function which we denoted by $\psi(s_1, \ldots, s_N)$ earlier; the above equation expresses ψ as the limsup of a family, infact a sequence, of affine functions. Thus, ψ must be convex. The domain of convergence D of our given power series corresponds to $\{s : \psi(s) \leq 0\}$. Let us rewrite this more precisely and record it for now: $D = \{z \in \mathbb{C}^N : \varphi(z) < 0\}$ where φ is given in terms of the coefficients of our power series $\sum c_J z^J$ by

$$\varphi(z_1, \ldots, z_N) = \limsup_{|J|\to\infty} \sqrt[|J|]{|c_J z^J|} - 1. \tag{2.6}$$

Thus, on the one hand, it *is* possible to read off the equation defining the boundary of its domain of convergence from its coefficients as in the one-variable case; on the other hand, as we shall see next, the possibilities for the boundary is going to be as varied as the whole range of convex functions.

Before moving on, a bit of notation: let p_K denote the point

$$(\frac{k_1}{k_1 + \ldots + k_N}, \ldots, \frac{k_N}{k_1 + \ldots + k_N})$$

for $K \in \mathbb{N}^N$. Let $PS\mathbb{Q}^N$ be the set of all such points p_K which forms a countable dense subset of P. We note that $PS\mathbb{Q}^N$ is precisely the set of all points on PS_N with rational coordinates.

We now proceed towards showing the existence for any given logarithmically convex complete multi-circular domain D in $\mathbb{C}^N$, a power series whose domain of convergence is precisely D; we shall actually describe a method for writing down one explicitly. As the key property of D is the convexity of the domain $G := \lambda(D)$, we first study the link between the domain G and the basic functions which constitute any power series namely, the monomial functions. Notice first that monomial functions on $\mathbb{C}^N$ correspond to linear functionals on its logarithmic image. More precisely, the monomial function $z^J = z^{j_1} \ldots z^{j_N}$ transforms into the linear functional $s \to j_1 s_1 + \ldots j_N s_N$ on $\mathbb{R}^N$ whose kernel is therefore $H_J = \{s \in \mathbb{R}^N \ : \ j_1 s_1 + \ldots j_N s_N = 0\}$. To spell out the result that we are after in brief, if an appropriate translate of H_J is a supporting hyperplane for G, then the (exponential of the) amount of translation required essentially renders the sought for coefficient of z^J in our candidate power series provided, the norm of the gradient vector $(j_1, \ldots, j_N)$ is one – we shall come to the appropriate choice of the norm in which we shall measure the amount of translation done, later. To ensure this condition on the norm of the gradient is easy: we just need to divide out the defining equation for H_J by $|J|$. But then notice that $z^J = z^{j_1} \ldots z^{j_N}$ with $J = (j_1, \ldots, j_N) = m(k_1, \ldots, k_N) = mK$ gives rise to the same $J/|J|$ as does $z^K = z^{k_1} \ldots z^{k_N}$. Our goal here, is to 'discover' the above-mentioned result.

Recall our observation around equation (2.5), that the logarithmic image G_g of the *domain* of convergence of a given power series $g = \sum c_J z^J$ is the convex domain given by

$$\Big\{ s \in \mathbb{R}^N \ : \ \limsup_{|J| \to \infty} \Big(\frac{j_1 s_1 + j_2 s_2 + \ldots + j_N s_N}{j_1 + j_2 + \ldots + j_N} + \frac{\log |c_J|}{|J|} \Big) < 0 \Big\}$$

Observe that this is essentially equivalent to the statement that the logarithmic image of the domain of convergence of every power series is the liminf of a sequence of half-spaces whose gradient vectors belong to $PS\mathbb{Q}^N$. Indeed,

$$G_g = \liminf_{J \in \mathbb{N}^N} \{H_J\} \tag{2.7}$$

with H_J denotes the half-space $\{s \ : \ \langle J/|J| \rangle + \log |c_J|^{1/|J|} < 0\}$.

The fact that the gradients of the bounding/supporting hyperplanes for G_g is 'positive', is contained within the conditions imposed on our D. Indeed, continuing our study of the logarithmic image G, notice by the convexity of G that any point $q \in \partial G$ has (possibly many) a supporting hyperplane for G in $\mathbb{R}^N$ passing through it; let H_q denote one such and be defined by say,

$$A_q(x) := \langle m, x \rangle + c$$

where $m \in \mathbb{R}^n \setminus \{0\}$. So $A_q(q) = 0$ and $A_q(x)$ is of the same sign throughout G. As usual, multiplying A_q by -1 if necessary, we may assume A_q is negative-valued throughout G. Just by the fact that D has a neighbourhood of the origin contained in it, G has a neighbourhood of $(-\infty, \ldots, -\infty)$ inside it; indeed, note that there is a positive number M such that all points with its coordinates all less than $-M$ must be contained in G giving an infinite box-neighbourhood of

$(-\infty, \dots, -\infty)$ which is contained inside G in its 'left-bottom'. Further, the *complete* circularity of D translates into the following condition about G: if $s^0 \in \overline{G}$ then all points s with $s_j \leq s_j^0$ for all j, must also be contained in $\overline{G}$ – this again gives an infinite box in the form of an orthant bounded by hyperplanes with gradients parallel to the axes, all passing through the point s^0. These features of G force all the components m_j of the gradient vector of A_q to be negative; for if m_j were negative for some j, then pick any $s^0 \in G$ and consider points of the form

$$p(s) = (s_1^0, \dots, s_{j-1}^0, s, s_{j+1}^0, \dots, s_n^0)$$

with s, a negative number to be chosen soon. Then, on the one hand $p(s) \in G$ will imply

$$A_q(p(s)) = m_1 s_1^0 + \dots + m_{j-1} s_{j-1}^0 + m_j s + m_{j+1} s_{j+1}^0 + \dots + m_n s_n^0 < 0,$$

which we rewrite as

$$m_j s < -\big(m_1 s_1^0 + \dots + m_{j-1} s_{j-1}^0 + m_{j+1} s_{j+1}^0 + \dots + m_n s_n^0\big).$$

On the other hand, we can use the freedom to take $p(s)$ to be points in G – indeed, within the aforementioned infinite box-neighbourhood of $(-\infty, \dots, -\infty)$ – with s negative and of modulus as large as we please; in particular, to contradict the above inequality whose right side is a constant. This shows that every component m_j of the normal vector m of every supporting hyperplane for G must be non-negative. Hence, every supporting hyperplane for G, is given by an equation of the form $\{s \in \mathbb{R}^N \ : \ A_q(s) = 0\}$ where

$$A_q(s) := m_1 s_1 + \dots + m_N s_n + d^q$$

for some positive real numbers m_j, which needless to say, depend on q. Actually, we may divide out the defining equation of this hyperplane by $|m_1| + \dots + |m_N|$ to assume that m_j's are all numbers in $[0, 1]$ with $|m_1| + \dots + |m_N| = 1$ and we shall suppose so, in the sequel; this also results in a change in the constant d^q but we shall continue to denote it by d^q. In other words m lies in the non-negative face, denoted earlier by PS_N, of the standard simplex. This will be important in the sequel; so, let us spell this out explicitly here: the defining function for every supporting hyperplane for G can be (and shall always be) written in a form such that its gradient vector belongs to PS_N. With this normalization made, d^q in modulus, gives the distance of the hyperplane H_q from the origin, as measured in the l^∞-metric. Indeed, note first that

$$|m_1 s_1 + \dots + m_N s_N| \leq |m|_{l^1} |s|_{l^\infty} = |s|_{l^\infty}.$$

But then for $s \in H_q$, the left hand side is equal to $|d^q|$, which means that $|d^q| \leq |s|_{l^\infty}$ for all $s \in H_q$; noting that the point $s^q := (-d^q, \dots, -d^q)$ satisfies $A_q(s^q) = 0$ i.e., lies on H_q and has $|s^q|_{l^\infty} = |d^q|$, we get that the foregoing lower bound for the l^∞-distance of points on H_q from the origin is actually attained at the point s^q and that this minimum distance is $|d^q|$. Let us keep these observations on record.

To discern the relationship between the coefficients defining a power series and the domain of convergence in more tangible terms, we now rephrase such relationships, (2.7) being one such for instance, in terms of the support function rather than the defining function; while the defining function is general tool to describe domains, the support function is a more convenient function specially adapted for convex domains. Let g be some general power series $\sum c_J z^J$ with the logarithmic image of its domain of convergence G_g. Then, as we know $G_g = \{s \in \mathbb{R}^N \ : \ \psi(s) < 0\}$ with the defining function ψ being given by

$$\psi(s) = \limsup_{|J| \to \infty} \Big\{ \langle \frac{J}{|J|}, s \rangle + \frac{1}{|J|} \log |c_J| \Big\}. \tag{2.8}$$

Given any $\alpha \in PS_N$ pick any sequence $R^n = J^n/|J^n| \in PS\mathbb{Q}^N$ for some sequence $\{J^n\} \subset \mathbb{N}^N$, such that $R^n \to \alpha$ as $n \to \infty$. Then

$$\psi(s) \geq \langle \alpha, s\rangle + \limsup_{n\to\infty} \frac{\log|c_{J^n}|}{|J^n|}.$$

Now, for all $s \in G_g$, $\psi(s) \leq 0$, so we must have

$$\sup_{s\in G_g} \langle \alpha, s\rangle \leq -\limsup_{n\to\infty} \{\frac{\log|c_{J^n}|}{|J^n|}\} = \liminf_{n\to\infty}\{-\log|c_{J^n}|^{1/|J^n|}\}.$$

This leads to the upper estimate for the support function $h := h_{G_g}$ of the convex domain G_g, given by

$$h(\alpha) \leq -\limsup_{n\to\infty} \frac{\log|c_{J^n}|}{|J^n|} \tag{2.9}$$

with this being valid for all $\alpha \in PS_N$ and any sequence $\{J^n\} \subset \mathbb{N}^N$ with $J^n/|J^n|$ converging to α as $n \to \infty$. Stated differently, for every sequence $\{J^n\} \subset \mathbb{N}^N$ with $J^n/|J^n|$ being convergent to say $\alpha \in PS_N$, we have:

$$-h(\alpha) \geq \limsup_{n\to\infty} \frac{\log|c_{J^n}|}{|J^n|}. \tag{2.10}$$

After passing to a subsequence to replace the limsup on the right by a limit, we may write

$$-h(\alpha) \geq \lim_{n\to\infty} \frac{\log|c_{J^n}|}{|J^n|}.$$

This means that every value assumed by $-h$ dominates some subsequential limit of $\log|c_J|/|J|$ leading us to the conclusion

$$\inf_{\alpha\in PS_N} \{-h(\alpha)\} \geq \liminf_{|J|\to\infty} \log|c_J|/|J|. \tag{2.11}$$

Next suppose $\{K^n\} \subset \mathbb{N}^N$ is a sequence which achieves the limit supremum for the sequence $\log|c_J|/|J|$ i.e., $\log|c_{K^n}|/|K^n|$ is a convergent sequence with limit $\limsup(\log|c_J|/|J|)$. Then after passing to a subsequence of $\{K^n\}$ to assume $K^n/|K^n| \to \gamma$ for some $\gamma \in PS_N$ and subsequently using (2.10), we get

$$\limsup_{|J|\to\infty} \log|c_J|/|J| \leq -h(\gamma) \leq \sup_{\alpha\in PS_N} \{-h(\alpha)\}. \tag{2.12}$$

On the other hand a lower bound may be obtained as follows. Pick *any* point $s^0 \in \overline{G}_g$, recall (2.8) and write

$$\limsup_{|J|\to\infty} \langle \frac{J}{|J|}, s^0\rangle + \limsup_{|J|\to\infty} \frac{\log|c_J|}{|J|} \geq \psi(s^0).$$

As every subsequential limit of the countable collection of numbers $\{\langle \frac{J}{|J|}, s^0\rangle \ : \ J \in \mathbb{N}^N\}$ is of the form $\langle \alpha, s^0\rangle$ for some $\alpha \in PS_N$, it follows that the left most term in the above, must be of the form $\langle \beta, s^0\rangle$ as well, for some $\beta \in PS_N$, so that we may write

$$h(\beta) \geq \psi(s^0) - \limsup_{|J|\to\infty} \frac{\log|c_J|}{|J|}.$$

As $s^0 \in \overline{G}_g$ was arbitrarily chosen, we may as well we might as well take s^0 to be on the boundary ∂G_g, to get the lower bound

$$h(\beta) \geq -\limsup_{|J|\to\infty} \frac{\log|c_J|}{|J|}.$$

Now, rewrite this as:

$$\limsup_{|J|\to\infty} \frac{\log|c_J|}{|J|} \geq -h(\beta) \tag{2.13}$$

to subsequently derive from this, the lower bound:

$$\limsup_{|J|\to\infty} \frac{\log|c_J|}{|J|} \geq \inf_{\alpha\in PS_N} \{-h(\alpha)\}. \tag{2.14}$$

Now, (2.10) and (2.13) together indicate the possibility that every value in the range of $-h$ can be realized as a subsequential limit of the sequence $\log|c_J|^{1/|J|}$. Indeed this is true: to this end, begin with the following rephrased version of (2.7):

$$G_g = \bigcap_{\alpha\in PS_N} \big\{\langle\alpha, s\rangle + \limsup_{\{J^n\}\in S_\alpha} \frac{\log|c_{J^n}|}{|J^n|} < 0\big\}$$

where S_α is the set of all sequences $\{J^n\}$ in $\mathbb{N}^N$ with $J^n/|J^n| \to \alpha$. On the other hand, if h is the support function of the convex domain G_g, we may write

$$G_g = \bigcap_{\alpha\in PS_N} \big\{\langle\alpha, s\rangle - h(\alpha)\big\}.$$

Comparing the foregoing pair of representations of G_g, using the basic fact that for any convex domain, there can be at most one supporting hyperplane with a given gradient, we conclude that: for every $\alpha \in PS_N$,

$$h(\alpha) = -\limsup_{\{J^n\}\in S_\alpha} \frac{\log|c_{J^n}|}{|J^n|}. \tag{2.15}$$

Thus, just as we have a formula connecting the coefficients of a power series g and the defining function of the logarithmic image G_g of its domain of convergence, we have a similar one linking it to the support function of G_g, as well. By picking a suitable sequence $\{J^n\} \subset \mathbb{N}^N$ then, we may write

$$h(\alpha) = \lim_{n\to\infty} \big\{ -\log|c_{J^n}|^{1/|J^n|}\big\} \tag{2.16}$$

where we are interested mainly in those α which lie in $PS_h = \{\alpha \;:\; h(\alpha) \text{ is finite}\}$. In short, $h(\alpha)$ is a subsequential limit of $-\log|c_J|^{1/|J|}$, allowing us to finally conclude that the range of h in $\mathbb{R}$ is contained in the set of all finite subsequential limits of the countable set of numbers:

$$\big\{ -\log|c_J|^{1/|J|} \;:\; J \in \mathbb{N}^N\big\}.$$

As every convex domain is characterized completely by its support function, it follows from (2.15) that: for any given convex domain G with support function h, the coefficients of every power series $\sum c_J z^J$ which converges precisely on the domain $\lambda^{-1}(G)$, must satisfy (2.15) or equivalently the following analogue of the Cauchy – Hadamard formula for the radius of (the polydiscs of) convergence:

$$e^{h(\alpha)} = \frac{1}{\limsup_{\{J^n\}\in S_\alpha} \big\{|c_{J^n}|^{1/|J^n|}\big\}}, \tag{2.17}$$

for each $\alpha \in PS_h$ (in fact, for all $\alpha \in PS_N$). Indeed, this formula gives the radius of convergence for any of the α-constituents of our power series, where by an α-constituent (or α-strand or α-section) of our generic power series $\sum c_J z^J$ we mean any of its sub-series given by

$$\sum_{n=1}^{\infty} c_{J^n} z_1^{J_1^n} \dots z_N^{J_N^n}$$

with $\{J^n\} \subset \mathbb{N}^N$ satisfying $J^n/|J^n| \to \alpha$.

As logarithm is an increasing (=order-preserving) function, (2.17) now leads to the result that the support function of the logarithmic image of the domain of convergence of any power series, can at least in principle be completely determined from the coefficients through the formula:

$$-h(\alpha) = \limsup_{\{J^n\} \in S_\alpha} \{\frac{\log |c_{J^n}|}{|J^n|}\}.$$

As this holds for all $\alpha \in PS_N$,

$$\limsup_{|J| \to \infty} \frac{\log |c_J|}{|J|} \leq \sup_{\alpha \in PS_N} \{-h(\alpha)\}$$

Getting back now to (2.14), we see that we have

$$\inf_{\alpha \in PS_N} \{-h(\alpha)\} \leq \limsup_{|J| \to \infty} \frac{\log |c_J|}{|J|} \leq \sup_{\alpha \in PS_N} \{-h(\alpha)\}$$

Combining this with (2.11), we may therefore write

$$\liminf_{|J| \to \infty} \frac{\log |c_J|}{|J|} \leq \inf_{\alpha \in PS_N} \{-h(\alpha)\} \leq \limsup_{|J| \to \infty} \frac{\log |c_J|}{|J|} \leq \sup_{\alpha \in PS_N} \{-h(\alpha)\}. \tag{2.18}$$

Now, recall our observation at (2.16) that, every member in the range of $-h$ is actually a subsequential limit of $\log |c_J|/|J|$; this gives

$$\sup_{\alpha \in PS_N} \{-h(\alpha)\} \leq \limsup_{|J| \to \infty} \frac{\log |c_J|}{|J|} \text{ and}$$
$$\inf_{\alpha \in PS_N} \{-h(\alpha)\} \geq \liminf_{|J| \to \infty} \frac{\log |c_J|}{|J|}$$

While the second inequality here is one that we already know, the former when combined with (2.18), gives in conclusion:

$$\limsup_{|J| \to \infty} \log |c_J|^{1/|J|} = \sup_{\alpha \in PS_N} \{-h(\alpha)\}.$$

As h is a convex function, its range is an interval of the extended real line $\overline{\mathbb{R}}$. In conclusion, we therefore have that the subsequential limits $-\log |c_J|/|J|$ do not shoot above the range of the support function: $[\inf_{\alpha \in PS_N}\{h(\alpha)\}, \sup_{\alpha \in PS_N}\{h(\alpha)\}]$ while the set of all such limits contains this interval. We remark in passing that since the range of h may well be an infinite interval despite G not being the whole space, it is (2.17) which will be more useful in practice.

Before proceeding to construct a power series which converges precisely on any given logarithmically convex multicircular domain D, let us take a look at two special cases: one when D is the unit polydisc and another when D is the pull-back of a half space under the map λ. For the former, the geometric series $\sum z^J$ which involves every monomial, converges precisely on the open unit polydisc (even though it *can* be analytically continued to a larger domain) whose support function is finite throughout PS_N. For the latter on the other hand, we may consider the power series $\sum c_k z^{kJ}$ for some fixed $J \in \mathbb{N}^N$ whose domain of convergence has its logarithmic image G_J, determined as the limit infimum of half-spaces given by:

$$(2.19)\quad \liminf_{k\to\infty} \Big\{ s \ :\ \frac{kj_1s_1+\ldots+kj_Ns_N}{k(j_1+\ldots+j_N)} + \frac{\log|c_k|}{k(j_1+\ldots+j_N)} < 0 \Big\}$$
$$= \liminf_{k\to\infty} \Big\{ s \ :\ \frac{j_1s_1+\ldots+j_Ns_N}{|J|} + \frac{1}{|J|}\log|c_k| < 0 \Big\}$$
$$= \Big\{ s\in\mathbb{R}^N \ :\ \langle J,s\rangle + \limsup_{k\to\infty}\log|c_k| < 0 \Big\}$$

which is a single half-space obtained by translating the ortho-complement of J, by a distance $\limsup_{k\to\infty}\log|c_k|$ in the direction opposite to J, unless the limsup in the above is infinite, in which case it is the whole space $\mathbb{R}^N$. The support function of a half-space is finite precisely at a single point of PS_N and for the above one, at $J/|J|$.

Now, we may wish to write any general power series $\sum c_Jz^J$ as a sum of series of the type just mentioned:

$$g(z) = \sum_{J\in\mathcal{P}} \Big(\sum_{k=0}^{\infty} c_{kJ}z^{kJ} \Big)$$

where $\mathcal{P}$ is the set of all N-tuples J of positive integers whose greatest common divisor is one. This representation is supported by the absolute convergence of the power series on its domain of convergence D. Let $G_g=\lambda(D)$ denote the logarithmic image of D. As noted above, for each $J\in\mathcal{P}$ fixed, the logarithmic image H_J^g, of the domain of convergence of $\sum_{k=0}^{\infty} c_{kJ}z^{kJ}$, is a half-space *or the whole space*. In fact, it may very well happen that every H_J^g is the whole space $\mathbb{R}^N$, while G_g is far from being so; this can be reconciled with the possibility that the set of points where the support function is finite avoids all of the rational points of PS_N. To address the question: how is the domain of convergence of g related to these half-spaces H_J^g? briefly, suppose $\lambda(z)\in G_g$; then for all $J\in\mathcal{P}$ we have that $\lambda(z)$ lies in the logarithmic image of the domain of convergence of of f_J, where

$$f_J = \sum_{k=0}^{\infty} c_{kJ}z^{kJ}.$$

Thus, $\lambda(z)$ belongs to $\cap H_J^g$ or in other words,

$$G_g \subset \bigcap_{R\in PS\mathbb{Q}^N} H_R^g.$$

with $H_R^g=\{s\in\mathbb{R}^N \ :\ \langle R,s\rangle + \limsup\log|c_J|^{1/|J|}<0\}$ where $R\in PS\mathbb{Q}^N$ is expressed as $J/|J|$. We wish ofcourse to know whether this inclusion can be improved to a better estimate, first of all an equality. The foregoing set-theoretic upper bound on G_g, may be totally useless because this inclusion may be far from equality, for instance when $H_J^g=\mathbb{R}^N$ for all J, as mentioned above – it is not difficult to conjure up examples when this takes place and in the forthcoming, we will see methods to do so; consider for instance the possibility of the domain of convergence of a power series g of two complex variables, being such that its logarithmic image in $\mathbb{R}^2$ is a half-space whose boundary is line of 'irrational slope' i. e., with gradient vector the $(1,\alpha)$ with α an irrational real number. On the one hand, we have the foregone equality

$$G_g = \bigcap_{\alpha\in PS_N} H_\alpha^g$$

where $H_\alpha=\{s\in\mathbb{R}^N \ :\ \langle\alpha,s\rangle - h(\alpha)<0\}$ with h being the *support function* of G_g. On the other hand, this equality does not immediately serve our purpose, as the intersection here is not countable; we shall redress this problem next – what we are seeking here, is a procedure to cast any given power series g as a sum of sub-series each with its logarithmic image of its domain of convergence being a half-space and such that the intersection of these half-spaces yields G_g. Actually, it is enough if we can recover G_g from the knowledge of these half-spaces by some tangible set-theoretic operation, not necessarily an intersection; infact, the operation of limit infimum for sets, is the one which comes up in this context. The key point here is that while the indexing set for our half-spaces must be a countable (dense) collection of vectors from PS_N, it need not be $PS\mathbb{Q}^N$. Subsequently therefore, we shall shift our considerations a bit, to starting with arbitrary countable dense subsets of PS_N.

Let G be any convex domain in $\mathbb{R}^N$ with support function $h = h_G$. The effective domain of h is the subset of those points of the domain of h where the support function is finite. We shall refer to the subset of the effective domain h, given by

$$PS_h = \{\alpha \in S_N \ : \ h(\alpha) \text{ is finite}\},$$

as the *normalized domain* (or normalized effective domain) of h, which is actually contained in PS_N, owing to the completeness of the given multicircular domain D, as noted earlier. Let $\mathcal{C} = \{\alpha^n\}$ be an arbitrary countable dense subset of PS_N. Pick a sequence $\{J^{1k}\} \subset \mathbb{N}^N$ with

$$\Big(\frac{J_1^{1k}}{|J^{1k}|}, \ldots, \frac{J_N^{1k}}{|J^{1k}|}\Big) \to (\alpha_1^1, \ldots, \alpha_N^1),$$

as $k \to \infty$ – it is trivial to see that such a sequence exists. Next, pick a sequence $\{J^{2k}\}$ this time in $\mathbb{N}^N \setminus \{J^{1k}\}$ such that

$$\Big(\frac{J_1^{2k}}{|J^{2k}|}, \ldots, \frac{J_N^{2k}}{|J^{2k}|}\Big) \to (\alpha_1^2, \ldots, \alpha_N^2).$$

Such a sequence exits, as $\pi\big(\mathbb{N}^N \setminus \{J^{1k}\}\big)$ is dense in $PS_N \setminus \{\alpha^1\}$, where $\pi(z) = z/|z|_{l^1}$. After l steps, we would have sequences $\{J^{lk} \ : \ k \in \mathbb{N}^N\}$ such that for any $m \leq l$, we have

$$\{J^{mk}\} \subset \mathbb{N}^N \setminus \bigcup_{i=1}^{m-1} \{J^{ik} : k \in \mathbb{N}\}$$

and $J^{mk}/|J^{mk}| \to \alpha^m$ as $k \to \infty$. Set $R_k^n = J^{nk}/|J^{nk}|$.

Keeping the notations as in the foregoing para, let $g(z) = \sum c_J z^J$ be a power series with its domain of convergence D and $\lambda(D) = G$ with support function h. We shall re-express the series g as a sum indexed essentially by any chosen countable dense subset $\mathcal{C}$ drawn out of the normalized effective domain PS_h of the support function. On the one hand, PS_h may fail to have any rational points i.e., the support function may fail to be finite on integral points; on the other hand, the standard indexing of power series is through the standard positive integral lattice. In order to pass to the desired rearranged sum, we first set up approximating sequences for our chosen $\mathcal{C}$ drawn from $PS\mathbb{Q}^N$ as in the foregoing para. We may pick out a strand (=sub-series) of terms interspersed in g, corresponding to each such subsequence. Thereafter, look upon the series g, as an interlaced sum of such strands. More simply put, re-express g in the following form

$$\sum_n \sum_k c_{kn} z_1^{l_{kn} R_{k1}^n} z_2^{l_{kn} R_{k2}^n} \ldots z_N^{l_{kn} R_{kN}^n} + \text{ the remaining terms of } g \tag{2.20}$$

with $l_{kn} = |J^{nk}|$ and $c_{kn} := c_{J^{nk}}$; the ordering of the 'remaining terms' in the above, can be ignored by the absolute convergence of g on D. Infact, the 'remaining terms' may be ignored altogether, because the values of the support function on the subset PS_h (of PS_N) where it is finite, gets determined as follows: firstly, on the chosen countable dense subset $\mathcal{C}$ by the asymptotic behaviour of the coefficients of g via the formula (2.17):

$$h(\alpha^n) = -\limsup_{k\to\infty} \log |c_{kn}|^{1/|J^{nk}|},$$

which subsequently, determines by continuity, the values of h on all points of the relative interior of PS_h. As these values suffice to determine the convex domain G, this explains why we may ignore the 'remaining terms', mentioned above. We have recast the power series g as in (2.20) to peel-off information from various strands [9] of coefficients of g about the support function h of its domain of convergence: (2.20) regroups g a sum of its α^n-constituents/sections and it is this organization of its terms, which splits up neatly to make apparent the links between the coefficients occurring in the various sections of the series g and the geometry of its domain of convergence. In conclusion, we thus observe here, how all power series arise 'essentially' in the same manner: the 'essential' limits being determined by a convex domain in $\mathbb{R}^N$ through its support function and a countable dense subset of the normalized domain of the support function.

A simple choice for getting a concrete/explicit power series converging precisely on a given log-convex Reinhardt D, now presents itself: take c_{kn} such that $|c_{kn}|^{1/l_{kn}} = e^{-h(\alpha^n)}$. To substantiate a bit more explicitly why this surmise may work, we first observe that the problem of constructing a power series which converges precisely on the prescribed domain D, is equivalent to the geometric problem of expressing its logarithmic image $G = \lambda(D)$ as the limit infimum of a sequence of half-spaces whose bounding hyperplanes have their gradient vectors from $PS\mathbb{Q}^N$ and converge to a 'dense' collection of supporting hyperplanes for the convex domain G. The gradient vectors of the supporting hyperplanes need not belong to $PS\mathbb{Q}^N$ at all; the foregoing prelude-para was to address this issue. So now, we choose a countable 'dense' collection of supporting hyperplanes for the logarithmic image G of our given domain, with the property that their (affine) defining functions all have gradient vectors whose components are all rational (and in PS_N); indeed, to be more carefully and correct, make the choice such that the gradient vectors of the aforementioned half-spaces, are in the above notation, of the form J^{nk}/l_{kn} where $l_{kn} = |J^{nk}|$ – in particular therefore vectors from $PS\mathbb{Q}^N$. In view of the experience gathered beginning from (2.7), we may surmise that: the constant terms in the defining functions of the above collection of supporting hyperplanes to $G = \lambda(D)$, would conceivably – a rigorous presentation is forthcoming – yield the coefficients of a power series convergent on D. As these constant terms ought to be the values of the support function h for G on a countable dense subset of PS_N, we may move higher in the ladder of precision. Keeping choices simple, the upshot is that we are led to consider the coefficients determined by the aforementioned prescription: take the coefficient of the monomial $z^{J^{nk}}$ to be $c_{kn} = e^{-l_{kn}h(\alpha^n)}$ with α^n being as in foregoing para. The resulting power series ought to work by the following geometric reasoning: as $h(\alpha^n)$ is the distance in the l^∞-metric from the origin to the supporting hyperplane for G with gradient α^n (this was recorded elaborately much earlier as well), it ought to follow that the half-spaces defined by affine functions with gradients J^{nk}/l_{kn} and with constant terms c_{kn}, being close to the supporting half-spaces, must yield the domain G upon passing to a (suitable) limit; that this indeed does follow is what is demonstrated next.

To work out the aforementioned strategy rigorously, pick a countable dense subset out of the set of all supporting hyperplanes for G. Indeed, this may be done by considering hyperplanes defined by affine functions of the form

$$A_n(s) := \langle \alpha^n, s\rangle - h(\alpha^n)$$

[9] A strand here means an infinite subset of the collection of coefficients; more precisely herein, one out of the infinitely many disjoint infinite subsets of the coefficients, each indexed by one of the sequences $\{J^{nk} : k \in \mathbb{N}\}$.

where h is the support function of the convex set G and $\{\alpha^n\}$ is any countable dense subset of PS_h. Let us mention in passing that it may well happen that PS_h is just a singleton; indeed, it will be instructive to keep the following example in mind: any complete multicircular domain in $\mathbb{C}^2$ the boundary of whose logarithmic image is a line. Next, the convexity of G and hence of the support function h (and subsequently the continuity of its restriction to PS_h), forces G to equal the countable intersection of the half spaces $\{s \in \mathbb{R}^N \ : \ A_n(s) < 0\}$. Next, for each α^n, choose a sequence R_j^n from $PS\mathbb{Q}^N$ which, as $j \to \infty$, converges to $(\alpha_1^n, \ldots, \alpha_N^n)$. Then, consider the power series

$$f(z) = \sum_{j,n \in \mathbb{N}} c_{jn} z_1^{k_{jn} R_{j1}^n} z_2^{k_{jn} R_{j2}^n} \ldots z_N^{k_{jn} R_{jN}^n} \tag{2.21}$$

where $c_{jn} = e^{-k_{jn} h(\alpha^n)}$ with k_{jn} being the least common multiple of the (+ve) denominators occurring in the reduced representation of the rational numbers $\{R_{j1}^n, \ldots, R_{jN}^n\}$. Now, the logarithmic image of the domain of convergence of the power series f, which we will denote by G_f, can be written using (2.7) as:

$$\begin{aligned} &\Big\{ s \in \mathbb{R}^N \ : \ \limsup_{j,n \in \mathbb{N}} \Big(\frac{k_{jn} \langle R_j^n, s \rangle + \log |e^{-k_{jn} h(\alpha^n)}|}{k_{jn}} \Big) < 0 \Big\} \\ &= \liminf_{j,n \in \mathbb{N}} \{ s \in \mathbb{R}^N \ : \ \langle R_j^n, s \rangle - h(\alpha^n) < 0 \} \end{aligned}$$

Thus G_f is the limit infimum of half spaces H_j^n defined by $B_j^n(s) = \langle R_j^n, s \rangle - h(\alpha^n)$. We wish to compare this representation of G_f with the representation of G as the intersection of half-spaces given by

$$G = \bigcap_{n \in \mathbb{N}} \{ s \in \mathbb{R}^N \ : \ A_n(s) < 0 \} \tag{2.22}$$

Indeed, to establish the claim that the domain of convergence of f is precisely G or in other words, to show the equality of domains: $G_f = G$, we proceed as follows. Pick any $s^0 \in G$. So s^0 belongs to every of the half-spaces appearing on the right of (2.22); so $\langle \alpha^n, s^0 \rangle - h(\alpha^n)$ is negative. We need to look at

$$B_j^n(s^0) = \langle R_j^n, s^0 \rangle - h(\alpha^n) = \langle R_j^n - \alpha^n, s^0 \rangle + \big(\langle \alpha^n, s^0 \rangle - h(\alpha^n) \big)$$

Depending on s^0 and n, choose $j(n, s^0) \in \mathbb{N}$ large enough for R_j^n to be so close to α^n that the second term at the right-most, is bigger in magnitude than its preceding term; more precisely, the 'close'-ness and the choice of $j(n, s^0)$ may be made by the following estimation:

$$|\langle R_j^n - \alpha^n, s^0 \rangle| \leq |R_j^n - \alpha^n| |s^0| < |\langle \alpha^n, s^0 \rangle - h(\alpha^n)|$$

which holds for all $j > j(n, s^0)$. This ensures that $\langle R_j^n, s^0 \rangle - h(\alpha^n)$ is negative whenever $j > j(n, s^0)$. However, we cannot immediately claim that s^0 lies in all but finitely many of the half-spaces H_j^n so as to conclude that s^0 belongs to their limit infimum, G_f. This will follow if we can remove the dependence of $j(n, s^0)$ on n. Indeed, it suffices to verify that $|\langle \alpha^n, s^0 \rangle - h(\alpha^n)|$ can be bounded below by a positive constant independent of n, for we may always choose the rate of convergence of $R_j^n \to \alpha^n$, to be independent of n – for instance, we may choose R_j^n so that $|R_j^n - \alpha^n| < 1/j$. To achieve the desired lower bound, notice first that $\langle \alpha^n, s^0 \rangle - h(\alpha^n)$ has a geometric meaning: it is the distance from s^0 to the supporting hyperplane H_α for G of gradient α, upto a factor of the length of α. To be precise and to proceed further, let s^1 denote the point where the perpendicular from s^0 on the supporting hyperplane H_α cuts the boundary ∂G – both the existence and uniqueness of such a point s^1 follows from the convexity of G; for instance, H_α is contained in the complement of G while both s^0 and ∂G are contained in the same one of the (closed) half spaces determined by H_α. An illustrative figure convinces us that the distance between H_α and the hyperplane of gradient α passing through s^0, satisfies the following lower bound:

$$\langle \alpha^n, s^0 \rangle - h(\alpha^n) \geq \frac{1}{|\alpha^n|_{l^2}} \text{dist}(s^0, s^1).$$

Recalling that $\alpha^n \in PS_N$ and that $|\cdot|_{l^2} \leq |\cdot|_{l^1}$, renders the desired independence of n in the lower bound:

$$\langle \alpha^n, s^0 \rangle - h(\alpha^n) \geq \text{dist}(s^0, \partial G).$$

As noted before this enables us to drop the dependence of $j(n, s^0)$ in the above, which we shall now write as $j(s^0)$. This ensures that for all n, $\langle R_j^n, s^0 \rangle - h(\alpha^n)$ is negative except possibly when $j \leq j(s^0)$. Thus, s^0 lies in all but finitely many of the halfspaces H_j^n whose limit infimum is G_f. This is exactly the requirement for s^0 to belong to this limit infimum. Thus, $G_f \subset G$.

To obtain the reverse inclusion start again with a point s^0, this time in G_f. Then, for all but finitely many values of the indices (j, n), we must have

$$\langle R_j^n, s^0 \rangle - h(\alpha^n) < 0.$$

Let

$$\mathcal{R} = \mathcal{R}_G = \{R_j^n \ : \ \langle R_j^n, s^0 \rangle - h(\alpha^n) \text{ is negative}\},$$

which differs from the set of all R_j^n's, only by a finite set. Then $\{\alpha^n \ : \ n \in \mathbb{N}\}$ is contained in the closure of $\mathcal{R}$. For each n, the continuous function

$$\langle \cdot, s^0 \rangle - h(\alpha^n)$$

is (finite and) negative on $\mathcal{R}$ and therefore non-positive on $\overline{\mathcal{R}}$. Therefore for every $n \in \mathbb{N}$, $\langle \alpha^n, s^0 \rangle - h(\alpha^n)$ must be non-positive. This means that $s^0 \in \overline{G}$ and subsequently that $G_f \subset \overline{G}$. As G_f contains G and is open in $\mathbb{C}^n$, it follows that $G_f = G$.

Logarithmic convexity may not be a property as intuitive as standard geometric convexity; nevertheless, let us not be amiss to note certain easy consequential visible properties common to all domains of convergence of power series; for instance: all of them are topologically trivial i.e., are contractible domains. However, this does not mean that they are holomorphically equivalent, even if we restrict ourselves to bounded domains. Indeed, two of the simplest logarithmically convex complete Reinhardt domains namely, the polydisc $\mathbb{U}^N$ and the ball $\mathbb{B}^N$ are holomorphically inequivalent. Or take the unbounded domain $\{z \in \mathbb{C}^N \ : \ |z_1 \ldots z_N| < 1\}$ obtained as the inverse image of a half-space under the logarithmic map λ; this is not biholomorphic to $\mathbb{B}^N$ or $\mathbb{U}^N$. One way to see this inequivalence is via a theorem due to H. Cartan about biholomorphic mappings between circular domains, in conjunction with the fact that the automorphism group of $\mathbb{B}^N$ or $\mathbb{U}^N$ act transitively on their respective domains. We mention in passing, as a matter of (a non-trivial!) fact that any pair of such domains (domain of convergence of some power series) will generically fail to be biholomorphically equivalent. Now, while what we have shown in the foregoing paras, means for instance, that there is a power series convergent precisely on $\mathbb{B}^N$, we have not shown that every holomorphic function on $\mathbb{B}^N$ can be represented by a single convergent power series, as in dimension one. In fact, we have thus far, not really dealt with 'holomorphicity'.

Definition 2.16. Let $D \subset \mathbb{C}^N$ be a domain. A function $f : D \to \mathbb{C}$ is said to be holomorphic if it admits a local representation by convergent power series i.e., every point $p \in D$ has corresponding to it a countable set of complex numbers $\{c_J(p) : J \in \mathbb{N}_0^N\}$ and a neighbourhood U_p such that the power series about p, $\sum c_J(p)(z-p)^J$ converges for all $z \in U_p$ to $f(z)$.

We shall not digress into complex analysis of several variables here; in particular not even pause to discuss the uniqueness of the numbers $c_J(p)$ in the possibility of multiple local representation by power series in the definition above. We refer the reader to standard references (such as [9] or [6]) or wherein familiar basic properties such as the (local) Cauchy integral formula, maximum modulus principle, open mapping theorem, theorems of Weierstrass and Montel etc., are established for holomorphic functions of several variables; alternative definitions for holomorphic functions are provided and the equivalences established therein as well. Concerning the representation of holomorphic functions by a single power series on discs in dimension 1, we must remark here that: it should not be concluded from the foregoing considerations it is only on logarithmically convex complete Reinhardt domains that every holomorphic function has a representation by a single power series. Infact, such a representation is valid on any complete Reinhardt domain – logarithmic convexity is inessential here. Infact, we may expand any holomorphic function on any complete circular domain, into a series of homogeneous polynomials compactly convergent on such a domain. All this and much more can be found in the text [7].

Among the first fundamental and strikingly new phenomenon in complex dimensions N greater than one, is the Hartogs phenomenon: every holomorphic function on the punctured ball $\mathbb{B}^N \setminus \{0\}$ extends to the origin to be holomorphic on $\mathbb{B}^N$ (so that in particular, there are no isolated singularities for holomorphic functions in dimensions $N > 1$). It is then natural to single out domains maximal with respect to this phenomenon of simultaneous extension of holomorphic functions i.e., domains D for which there exists at least one holomorphic function which does not extend holomorphically across the boundary near any point in ∂D; it turns out that this property is equivalent to the seemingly weaker property that for each boundary point $p \in \partial D$, there is a function f_p holomorphic on D resisting holomorphic continuation to any neighbourhood of p. A domains possessing this property is called a domain of holomorphy. A celebrated problem going by the name of the Levi problem and taking several decades for its complete resolution, was to obtain a geometric characterization of domains of holomorphy. This is best left for another essay; suffice it to say here that the answer lies in a subtle convexity property and we refer the reader again to [4], [5] and other texts of the subject. Our next goal here will be to show that domains of convergence of power series are indeed domains of holomorphy.

The question to be dealt with now is: given a domain D which is the domain of convergence of some power series (equivalently, a logarithmically convex complete Reinhardt domain D) in $\mathbb{C}^N$ and an arbitrary point p of its boundary ∂D, is it possible to construct (another) power series $f_p(z)$ which converges on D and whose limit supremum as $z \to p$ is ∞? Note that this question does not get trivially settled with the knowledge of the existence of a power series converging precisely on D, owing to the possibility of the existence some (tiny) piece of ∂D across which all such power series can somehow be continued holomorphically. As already seen at (2.21), while constructing power series with certain desired properties, it is best to use the freedom in expressing them as a sum of monomials in any order that we wish – in a manner that is telling about the desired properties. With this flexibility, let us demonstrate that domains of convergence of power series are (what are known as 'weak'-) domains of holomorphy by constructing the function f_p in question. We cannot help but narrate here the concise but clear treatment in Ohsawa's little text [11]. First, note that given any point p in the exterior of D (i.e., $p \in \mathbb{C}^N \setminus \overline{D}$), there exists a monomial $m_p(z)$ such that

$$\sup_{z \in D} |m_p(z)| < m_p(p) = 1. \tag{2.23}$$

Indeed, this follows by passing to the logarithmic image $G = \lambda(D)$, applying to it a standard separation theorem to the convex domain G and then exponentiating back. Among other things, what (2.23) means is that we may arrange for the supremum on D appearing therein to be arbitrarily small, by taking powers of the monomial m_p, while maintaining the value at p to be at unity; in symbols, $m_p(z)^{n_k}$ for a suitable $n_k \in \mathbb{N}$, will satisfy

$$\sup_{z \in D} |\big(m_p(z)\big)^{n_k}| < 1/2^k.$$

The sum of such monomials gives a power series uniformly convergent on D; actually, we only need compact convergence – it suffices if the supremums on compact subdomains satisfy a bound as above. Moreover, we need to modify this series to make it take arbitrarily large values along some sequence approaching p. Thus on the one hand, we need the supremums on compact subdomains of the monomials constituting our power series to decrease exponentially and on the other hand we need its values along some sequence approaching the boundary to blow up. In order to have these requirements met, it is natural to exhaust the given domain D by a sequence of relatively compact subdomains expanding out to the boundary and then apply (2.23) to each member of this sequence. Before proceeding to work this out rigorously, note that we may further multiply the monomial m_p as above, by a constant C independent of k to get a monomial, denoted again by m_p, which assumes the value C at p and satisfies an exponential decay rate in k:

$$\sup_{z \in D} |\big(m_p(z)\big)^{n_k}| < C/2^k.$$

Now, let $D_j = \lambda^{-1}(G_j)$ where

$$G_j = \{s \in \mathbb{R}^N \ : \ \text{dist}(s, \partial G) > 1/j\}.$$

Recall that as D is a *complete* Reinhardt domain, the infinite box-neighbourhood of $(-\infty, \ldots, -\infty)$ (at the 'left-bottom') arising as the logarithmic image of the polydisc spanned by any point is contained in G and consequently in all the G_j's as well owing to the concavity of the function $\text{dist}(\cdot, \partial G)$ on G; this ensures that all the D_j's are complete Reinhardt domains as well. If z, w are a pair of points in G whose distance from ∂G are at least δ, then concavity of the function $\text{dist}(\cdot, \partial G)$ on G, ensures that the minimum distance of every point of the line segment joining z, w in G lies at a distance at least δ from ∂G. This fact ensures that all the domains G_j's are convex and thereby the logarithmic convexity of the D_j's. Thus, the D_j's form an (increasing) exhaustion of D by logarithmically convex complete Reinhardt domains. By intersecting them with balls centered at the origin of radii increasing to infinity, we may further suppose that these D_j's are bounded as well. To construct an f_p with $\limsup_{z \to p} |f_p(z)| = \infty$, what could be more simple than to arrange for a function whose values at some sequence p_j of points in D approaching p, is at least as big as j? In trying to arrange for such a function f_p, we must not loose sight of the requirement that f_p is to be given by a power series which *converges on all* of D. Recall the availability of a characterizing test to determine whether or not a point belongs to the domain of convergence of any given power series:

Proposition 2.17. *A point p belongs to the domain of convergence of a power series $\sum c_J z^J$ if and only if there exits a neighbourhood U of p and positive constants M and $r < 1$ such that*

$$|c_J z_1^{j_1} z_2^{j_2} \ldots z_N^{j_N}| \le M r^{j_1 + \ldots + j_N}$$

for all $J = (j_1, \ldots, j_N) \in \mathbb{N}^N$ and $z \in U$.

This is essentially contained in our discussion of Abel's lemma in Section 1. Put in words, according to this proposition, a point is within the domain of convergence of a power series if the sequence of complex numbers obtained by evaluating the monomials constituting the power series (in the standard partial ordering by degree) at that point, decays to zero at at least an exponential rate; stated differently, faster than a geometric progression (of ratio < 1). The last statement holds with the word 'point' replaced by 'any point from the set of all points whose distance to the boundary of the domain of convergence is bounded below by a positive constant'. We choose the standard geometric progression namely $\{1/2^k\}$ for measuring/controlling the rate in what follows. First, let p^j be sequence in D which converges to p; indeed, choose the sequence so that (it escapes out of the D_j's linearly as:) $p^j \in D_{j+1} \backslash D_j$ and converges to p. Corresponding to each such p^j, by (2.23) choose a monomial m_{p^j} whose value at p^j exceeds the supremum of its values on D_j. We wish to arrange our series f_p in such a way that the value of the n-th term of the series, at p^n, exceeds n – the amount by which it exceeds, is arranged to cancel out the possible negative contributions of the remaining terms, so as to ensure ($f_p(n) > n-1$) ultimately that $f_p(p^n) \to \infty$. For instance, we may take the n-th term to be $c_n m_{p^n}(z)$ with $c_n > n$, whose value at p^n is c_n. The major part of the 'negative contributions' to possibly pull down the value of f_p at p^n, will conceivably due to the terms preceding the n-th term, as the remaining tail of the series f_p (assuming convergence) will be small. Put together with the aforementioned convergence criterion, we are then led to seek for sequences $n_k \in \mathbb{N}$ and real numbers c_k such that

$$c_k = k + \Big| \sum_{j=1}^{k-1} c_j \big(m_{p^j}(p_k)\big)^{n_j} \Big|$$

together with the requirement

$$\sup_{z \in D_k} \big| c_k (m_{p^k}(z))^{n_k} \big| < 1/2^k.$$

It is easy to construct the sequences c_k and n_k inductively, satisfying the above conditions at each stage. Then the series

$$\sum_{j=1}^{\infty} c_j \big(m_{p^j}(z)\big)^{n_j}$$

is compactly convergent (recall D_j's are relatively compact) on D and thus defines a holomorphic function $f_p(z)$ on D. As $f_p(p^n) > n-1$, we must have $\limsup_{z \to p} f_p(z) = \infty$, with which we have attained our goal of checking out that domains of convergence of power series are indeed domains of holomorphy.

Remark 2.18. The series just constructed may converge on a domain larger than D; so, there is no guarantee that it is also a power series which converges 'precisely' on the given logarithmically convex complete Reinhardt domain D.

It is almost a foregone conclusion now, that all considerations of this essay on power series, can be modified to yield analogous results for Laurent series. We leave the details to the reader who may also want to check out for a concrete power series for the ball:

Example 2.19. Show that the domain of convergence in $\mathbb{C}^2$ of the power series of two complex variables z, w given by

$$\sum_{j,k \in \mathbb{N}} \frac{f(j)f(k)}{f(j+k)} z^j w^k$$

where $f(t) = \sqrt{t^t}$, is the unit ball $\mathbb{B}^2$.

Remark 2.20. The fact that the subject of power series is fundamental and elementary does not mean that all basic questions about them have more or less been settled. Among many recent works concerning power series, we call attention to the semi-expository article [3] concerning the Bohr phenomenon arising out of functions defined by power series on logarithmically convex complete Reinhardt domains; associated to such domains are certain curious numbers called the

‘Bohr radius’. For an exposition of this as well as for open problems, the ambitious reader may consult [3].

Now, we know that given any power series, we may read off the equation defining the boundary of its domain of convergence from its coefficients; it is given precisely by (2.6). Conversely, we have been discussing methods to explicitly write down power series which converge on any given logarithmically convex multicircular domain. Now, when we say, we are ‘given a domain’, what could this mean in practice? The most tangible meaning would be that we are given (the knowledge of all connected components of) the boundary of the domain as the zero set of a defining function. How does one plot points of the boundary, given the defining function ϱ, say? Well, the immediate answer would be write down solutions to the equation $\varrho = 0$. But then, such an equation is in general is never going to be linear and very likely, difficult to solve. One way out of this problem, while dealing with convex domains and thereby for our problem of constructing power series, is to express everything in terms of the support function (as we have already done) and then seek a link between the support function and the defining function, which is the matter that we take up next.

Suppose $G \subset \mathbb{R}^N$ is a convex domain with support function h. Then G can be written as the intersection of *open* half-spaces

$$G = \bigcap_{\alpha \in \mathbb{R}^N} \{x \in \mathbb{R}^N \ : \ \langle \alpha, x \rangle - h(\alpha) < 0\}$$

However, we cannot claim from this that G equals $\{x \ : \ \sup_{\alpha \in \mathbb{R}^N} \{\langle \alpha, x \rangle - h(\alpha)\} < 0\}$ nor that it equals $\{x \ : \ \sup_{\alpha \in \mathbb{R}^N} \{\langle \alpha, x \rangle - h(\alpha)\} \leq 0\}$. On the other hand, we may restrict the parameter α to vary over the *compact* set S_N, the standard simplex, and still write

$$G = \bigcap_{\alpha \in S_N} \{x \in \mathbb{R}^N \ : \ \langle \alpha, x \rangle - h(\alpha)\}$$

That is, G equals the set of all those points x which satisfy $\langle \alpha, x \rangle - h(\alpha) < 0$ for all $\alpha \in S_N$. So, for each fixed $p \in G$, the function $\langle \alpha, p \rangle - h(\alpha)$ is an upper-semicontinuous concave function which is strictly negative on S_N and therefore attains its supremum on S_N at some point therein and consequently this supremum must be strictly negative. This proves that

$$G = \{p \ : \ \sup_{\alpha \in S_N} \{\langle \alpha, p \rangle - h(\alpha)\} < 0\},$$

a claim that cannot be made if α were allowed to vary over all of $\mathbb{R}^N$ in the above. In other words, this is saying that G is precisely the domain defined by the Legendre transform (also called Fenchel – Legendre transform or convex conjugate) of the restriction of the support function of G to S_N. On the other hand, given a defining function ψ for a convex domain G in $\mathbb{R}^N$, it is straightforward to write down the value of the support function for the normal vector at boundary points $p \in \partial G$, as:

$$h(\nabla \psi(p)) = \langle p, \nabla \psi(p) \rangle$$

which agrees with the Legendre transform of ψ for normal vectors at all points of the boundary. If we normalize the normal vectors at all points of ∂G, so as to be unit vectors in the l^1-norm, we obtain a convex subset of S_N, by virtue of the convexity of G. We may then extend h by the general property of positive homogeneity of the support function to obtain its values on a convex cone and subsequently thereafter, take the lower semicontinuous regularization, to completely obtain the support function $h : \mathbb{R}^N \to (-\infty, +\infty]$ from a given defining function ψ for G. The Legendre transform, among other notions of duality, is of fundamental importance in the subject of convex analysis which we shall only briefly review in the next and last section, and end.

The reader is assumed to have some familiarity with convexity. So instead of saying that a convex set is a subset some of $\mathbb{R}^N$ of closed under the geometric operation of formation of straight line segments joining any pair of its points, we are going to say: that a convex set is a subset C of some $\mathbb{R}^N$, which is closed under the one-parameter family of algebraic operations given by the weighted arithmetic mean $(p, q) \to (1-t)p + tq$ for $t \in I$ and $p, q \in C$. Our purpose here is to gather together results in convex analysis to serve as a convenient reference for the main text. Proofs therefore, are omitted. They can be found in the systematic treatment in [8] or in many good expository texts such as [12]. Henceforth V shall denote a real vector space of finite dimension. Given an arbitrary subset E of V, the intersection $\mathrm{ah}(E)$ of all affine subspaces containing E is an affine subspace called the affine hull of E, which has the following analytic expression

$$\mathrm{ah}(E) = \{\sum_{j=1}^{n} \lambda_j x_j \ : \ \sum_{j=1}^{n} \lambda_j,\ x_j \in E,\ n = 1, 2 \ldots\},$$

If the λ_j's in the above are further required to be positive, we obtain what is called the convex hull of E, denoted $\mathrm{ch}(E)$. Let $C \subset \mathbb{R}^N$ be convex. A point x is said to be in the relative interior of C if x has a neighbourhood U open in $\mathbb{R}^N$ such that $U \cap \mathrm{aff}(C) \subset C$. Note that the relative interior of a convex set is always a (non-empty) convex set and the closure of the relative interior of C is the closure of C.

Trivially, every affine subspace of V is convex. An affine subspace of codimension 1 is termed a hyperplane, which divides V into two connected components; each of these connected components of the complement of a hyperplane is an open *half-space*. Each half-space is convex and is denoted generally by H overloaded by some subscript or superscript when it is desirable to specific about its gradient or a point through which it passes. The closure of a half-space – often denoted by H again – is convex, as is more generally the closure of any convex set. Another fundamental example of convex set is provided by the class of convex cones: a cone is any set set A which is invariant under homotheties i.e., $x \in A \Rightarrow \alpha x \in A$ for all $\alpha \geq 0$; therefore, convex cones are those cones which are convex sets. Among basic examples of bounded convex sets are balls with respect to any norm. Of course all norms on the finite dimensional V are equivalent; but they are far from being affinely equivalent – note that convexity is preserved by invertible affine maps of V. We next pass onto the notion of convex functions.

Definition 2.21. Let X be a convex set. A function $f : X \to (-\infty, +\infty]$ is termed convex if

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

for all pairs of positive numbers λ_1, λ_2 with $\lambda_1 + \lambda_2 = 1$ and $x_1, x_2 \in X$. Equivalently, a function is convex iff its epigraph

$$\{(x, t) \in V \oplus \mathbb{R} \ : \ x \in X, t \geq f(x)\}$$

is a convex set.

Theorem 2.22. *If f is a convex function on V, then*

$$X = \{x \in V \ : \ f(x) < \infty\}$$

is a convex set and f is continuous in the relative interior of X i.e, in the interior of X in $\mathrm{ah}(X)$.

Remark 2.23. It is not always possible to redefine f at boundary points of X in $\mathrm{ah}(X)$, so as to have f become continuous with values in $(-\infty, +\infty]$.

This problem is redressed by taking the lower-semicontinuous regularization.

Proposition 2.24. *Let f be a convex function on V. Define for all $x \in V$:*

$$f_1(x) = \liminf_{y \to x} f(y)$$

Then f_1 is convex and $f_1(x) \le f(x)$ for all x, with equality if x lies in the interior of $X = \{x \in V : f(x) < \infty\}$ in $\mathrm{ah}(X)$ or interior in $V \setminus X$. The function f_1 is lower semi-continuous and is termed the lower semi-continuous regularization of f.

If f is not given to be defined on all of V but given on a convex set X, we first extend by setting its values equal to $+\infty$ at all points where it is not apriori given i.e., on $V \setminus X$; the above proposition then applies to furnish its lower semicontinuous regularization. The role of lower semi-continuity here is explained as follows. While the epigraph of a function f is convex iff its epigraph is convex, the epigraph is a closed iff f is lower semi-continuous. This will be important in the subsection on the Legendre transform.

Definition 2.25. Let $E \subset V$. The indicator function I_E is the function whose value at points of E is set equal to 0 and equal to $+\infty$ at all points outside E. Such a function is convex precisely when E is convex.

Separation theorems. The following four results go by the name of Hahn – Banach theorems.

Theorem 2.26. *Let D be a convex domain in V. If $x_0 \notin D$, there is an affine hyperplane H such that $x_0 \in H$ but $H \cap D = \emptyset$. Thus there is an affine function f on V with $f(x_0) = 0 > f(x)$ for all $x \in D$.*

Corollary 2.27. *Let X be a closed convex subset of V. If $x_0 \notin X$, there is an affine hyperplane containing x_0 which does not intersect X i.e., there is an affine function f with $f(x) \le 0 < f(x_0)$ for all $x \in X$.*

Corollary 2.28. *If X is a closed convex subset of V and if y is on the boundary of X, then one can find a non-constant affine function f such that $f(x) \le 0 = f(y)$ for all $x \in X$. The affine hyperplane $\{x \in V \ : \ f(x) = 0\}$ is called a supporting hyperplane of X.*

Corollary 2.29. *An open (closed) convex set K in a finite dimensional vector space is the intersection of the (open) closed half-spaces containing it.*

As a closed convex set is the intersection of its supporting half-spaces, such a set can alternatively be described by specifying the position of its supporting hyperplanes, given their gradient vectors.

Definition 2.30. Let $C \subset \mathbb{R}^N$ be a closed convex set. The *support function* $h = h_C : \mathbb{R}^N \to (-\infty, +\infty]$ of C is defined by

$$h(u) = \sup\{\langle x, u\rangle \ : \ x \in C\}.$$

The set of all $u \in \mathbb{R}^N$, for which $h(u)$ is finite is called the *effective* domain of h and we call its subset consisting unit vectors thereof, as the *normalized effective* domain of h.

The geometric meaning of the support function is: for a unit vector u with $h(u)$ finite, the number $h(u)$ is the signed distance of the supporting hyperplane to C with normal vector u, from the origin; the distance is negative if and only if u points into the open half-space containing the origin. From the definition, it is straight-forward to check that $h_C(\cdot) = \langle z, \cdot\rangle$ is a linear functional iff C is a singleton. More importantly, h is *positively homogeneous*: $h(\lambda u) = \lambda h(u)$ for all $\lambda \ge 0$ and is sub-additive:

$$h(u + v) \le h(u) + h(v).$$

These conditions constitute what is sometimes referred to as sub-linearity, from which it follows in particular that h is a convex function. If $x \in \mathbb{R}^N \setminus C$, a separation theorem yields the existence of a vector u_0 with $\langle x, u_0\rangle > h(u_0)$. The support function of a convex set C may also be defined as the Legendre transform of its indicator function I_C; the Legendre transform being defined in the following subsection.

The Legendre transform.

Definition 2.31. Let $f : \mathbb{R}^N \to \mathbb{R}$ any function. The Legendre transform (= Fenchel – Legendre transform), also called the convex conjugate, of f, is defined by

$$f^*(y) = \sup_x \{\langle x, y\rangle - f(x)\}.$$

We restrict attention to taking convex conjugates only of convex functions as in this case we have the the following key result: The convex conjugate of the convex conjugate of any given convex function is the given function itself. This is only recorded differently, in theorem 2.32 below. We have been silent about the domain of f^*; we shall allow $+\infty$ to be in the range of f^*. Actually it is convenient here to have functions defined on all of our vector space and in order to do this, we extend them by setting them equal to $+\infty$ outside the convex hull of the set of all points where it's value is specified. Let f be a convex function such that the set X of all points where it is finite, has non-empty interior which we denote by X^0. Then it is possible to argue that X^0 must be a convex domain (the basic idea can be found in lemma 2.2 of [1]) and f must be continuous herein. Next and further, by taking a liminf of f at points on the boundary of X^0 we may redefine f at these points, so that it becomes a lower semi-continuous function on the whole. This will pave the way for using the above definition of the Legendre transform for functions not apriori given to be defined on all of $\mathbb{R}^N$ and more importantly take the domain of f^* to be all of $\mathbb{R}^N$.

Theorem 2.32. *The Legendre transform is an involution on the space of all lower semi-continuous convex functions on* $\mathbb{R}^N$.

Thus,

$$f(x) = \sup_m \{A_m(x)\} \tag{2.24}$$

where $A_m(x) = \langle m, x\rangle - f^*(m)$.

As a corollary to the foregoing theorem, one may derive another fundamental fact: every 'sub-linear' function on a finite dimensional real vector space V arises essentially as the support function of a closed convex set.

Theorem 2.33. *If $C \subset \mathbb{R}^N$ is a closed convex set, then its support function is lower semicontinuous, convex and positively homogeneous.*
Conversely, every lower semicontinuous function h on $\mathbb{R}^N$, which is positively homogeneous and convex (equivalently, positively homogeneous and subadditive) is the supporting function of one and only one closed convex set C, given by

$$C = \big\{x \in \mathbb{R}^N \ : \ h(v) \geq \langle v, x\rangle \ \textit{for all } v \in \mathbb{R}^N\big\}.$$

We remark in passing to the next sub-section that, if ϱ is a defining function for a convex domain G, the support function of G is given by the Legendre transform of $I_{\mathbb{R}_-} \circ \varrho$, where $I_{\mathbb{R}_-}$ is the indicator function of $\mathbb{R}_-$, the ray of non-positive reals; while this remark may not be useful, the concept of defining function surely is, which we review next.

Defining functions for convex domains.

Theorem 2.34. *Let $D \subset \mathbb{R}^N$ be a convex domain. There exists a convex function which is negative on D, vanishes precisely on ∂D and is positive on the complement of $\overline{D}$.*

Proof. Let p be an arbitrary point of ∂D. The convexity of D guarantees the existence of a supporting hyperplane for D at p i.e., an affine subspace L of $\mathbb{R}^N$ of codimension 1 through p with D contained entirely in one, out of the 2 connected components of $\mathbb{R}^N \setminus L$. Now, if we let

$a_p(x)$ denote the affine function which defines L, then after a change of sign if necessary we may – and will! – assume that a_p is negative throughout D. Needless to say, $a_p(p) = 0$. Now denote by $\mathcal{F}$ the family of all such affine functions a_p as p varies through ∂D. Let

$$A(x) = \sup_{\mathcal{F}}\{a_p(x)\}.$$

Clearly, $A(x)$ is a convex function which is non-negative on $\overline{D}$ which vanishes precisely on ∂D. Further, by invoking a suitable separation theorem, we may assure ourselves that A is actually positive on all of $\mathbb{R}^N \setminus \overline{D}$. □

We shall refer to the function guaranteed by the above theorem as a defining function. With some regularity assumptions about the boundary of the domain, it is natural to impose further conditions on the defining function so that it encodes the additional regularity features. A customary definition for defining functions for smoothly bounded domains – not necessarily convex – is as follows:

Definition 2.35. Let D be a domain in $\mathbb{R}^N$. Then D is said to have smooth boundary, if there there exists a smooth function $\varrho : \mathbb{R}^N \to \mathbb{R}$ such that ϱ is positive on the complement of $\overline{\mathbb{D}}$,

$$D = \{x \in \mathbb{R}^N \ : \ \varrho(x) < 0\},$$

ϱ vanishes precisely on ∂D and its gradient vector is non-zero at all points of ∂D. The function ϱ is said to be a (global) smooth defining function.

It is not necessary to have ϱ defined on all of $\mathbb{R}^N$, a tubular neighbourhood of ∂D will suffice; there is also the notion of a local defining function and how one may obtain a global defining function by patching together local defining functions via standard partition-of-unity techniques and other results about the relationships between any two defining functions. These matters can be found in standard texts; a reference relevant for the present subsection is [10]. We shall only state the condition of convexity for smoothly bounded domains formulated via the defining function as:

Theorem 2.36. *Let D be a domain in $\mathbb{R}^N$ with C^2-smooth boundary. Let ϱ be a C^2-defining function for D near $p \in \partial D$. Then there is an open ball U centered at p such that $U \cap D$ is convex if and only if the Hessian of ϱ satisfies the condition:*

$$\sum_{j,k=1}^{N} \frac{\partial^2 \varrho}{\partial x_j \partial x_k} v_j v_k \geq 0$$

for all $p \in \partial D$ and $v \in T_p(\partial D)$.

Thus at least for domains whose boundaries are C^2-smooth, there is a local characterization of convexity and their convexity is determined by their boundaries.

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