Large Block Properties of the Entanglement Entropy of Free Disordered Fermions

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Abstract

We consider the macroscopic disordered system of free lattice fermions with the one-body Hamiltonian, which is the Schrödinger operator with ergodic potential. We assume that the expectation $\mathbf{E}\{|P(x,y)|\}$ of the entries of the Fermi projection $P = \{P(x,y)\}_{x,y\in\mathbb{Z}^d}$ of the Hamiltonian decays exponentially: $\mathbf{E}\{|P(x,y)|\} \le Ce^{\gamma|x-y|}$, $C < \infty$, $\gamma > 0$ (a typical behavior in the localization regime). We prove that if S_Λ is the entanglement entropy of the cubic block Λ of side length L of the system, then for any $d \ge 1$ $\mathbf{E}\{L^{-(d-1)}S_\Lambda\}$ has a finite limit as $L \to \infty$ and we identify the limit. We then prove that for $d = 1$ and under the same assumption on the potential the entanglement entropy admits a well defined asymptotic form with probability 1, which is not selfaveraging as $L \to \infty$, i.e., its fluctuations do not vanish as $L \to \infty$ even if the ergodic potential consists of i.i.d. random variables according to recent numerical results by Pastur L., Slavin V.: Phys. Rev. Lett. **113** (2014) 150404. On the other hand, if $d \ge 2$ and the ergodic potential consists of i.i.d. random variables with sufficiently smooth probability distribution, then the variance of $L^{-(d-1)}S_\Lambda$ vanishes as $L \to \infty$, i.e., in this case the entanglement entropy is selfaveraging.

1 Introduction

Entanglement is a fundamental feature of quantum mechanics manifesting non-local intrinsically quantum correlations between separated quantum systems. Having been first used by Einstein, Rosen and Podolsky in 1935 to demonstrate the incompleteness of quantum description and explicitly introduced by Schrödinger in the same year, entanglement is nowadays an object of extensive studies ranging from general relativity, cosmology and foundation of quantum mechanics through quantum optics and quantum statistical mechanics to quantum information and computation. Among the wide variety of ideas, problems and results there are those dealing with many-body (macroscopic) systems, common in statistical mechanics and condensed matter physics. Here one often uses the so called bipartite setting, in which a macroscopic quantum systems $\mathcal{S}$ being in its pure state is divided in two parts $\mathcal{E}$ and $\mathcal{B}$ of characteristic sizes $\mathcal{L}$ and L in space and one asks how the parts are correlated in the asymptotic regime

$$1 \ll L \ll \mathcal{L}. \tag{1.1}$$

A widely used measure (a quantifier) of the corresponding correlations is the von Neumann entropy

$$S_{\mathcal{B}} = -\mathrm{tr}_{\mathcal{B}}\, \rho_{\mathcal{B}} \log_2 \rho_{\mathcal{B}} \tag{1.2}$$

of the reduced density matrix $\rho_{\mathcal{B}}$ of the block $\mathcal{B}$, i.e., the density matrix of the (pure) state of $\mathcal{S}$ traced with respect to the degree of freedom of $\mathcal{E}$. One of important problems of the field is the asymptotic behavior of the entanglement entropy in the regime (1.1). Since the r.h.s. inequality of (1.1) is usually implemented via the macroscopic limit $\mathcal{L} \to \infty$ for $\mathcal{S}$, in which the entanglement entropy is usually well defined, the problem is to find the asymptotics as $L \gg 1$ of the entanglement entropy of a block of size L of an infinite many body system.

It has been found in the last decades that these asymptotics may be unusual if the whole system is in its ground state, more generally, in a pure state. Namely, it was shown in several physics works that the entanglement entropy is proportional to the surface area L^{d-1} of the block but not to its volume L^d. The latter (extensive) length scaling is standard in quantum statistical mechanics for non-zero temperature, while the former was found first in cosmology and quantum field theory and then in other fields and is known as the *area law*. Moreover, the area law is not always valid, e.g., at quantum critical points of several one-dimensional translation invariant quantum spin chains, where the entropy is proportional to $\log L,\ L \gg 1$. (recall that the area law in the one-dimensional case is just the boundedness of the entanglement entropy in L).

More generally, the area law const$\cdot L^{d-1}$ is to be valid for quantum systems with finite range interaction and a spectrum gap, while for gapless systems other asymptotics are possible, const$\cdot L^{d-1}\log L$ in particular, which again has to be closely related to the existence of a quantum phase transition in the corresponding system [5]. This is, however, not simple to deal with, since the spectrum of many-body interacting quantum systems is rather complex and is known mostly for certain one dimensional exactly solvable models. On the other hand, there is a simpler model having the both types of spectrum (i.e, gapless and gaped) and the both type of asymptotics. These are the quasi-free fermions described by Hamiltonians quadratic in the creation and annihilation operators. The Hamiltonians arise in condensed matter theory and statistical physics (e.g., electrons in metals, including superconductivity, and other mean field type approximations, exactly solvable spin chains, etc.). For these Hamiltonians with finite range and translation invariant coefficients the large-L behavior of the entanglement entropy for any $d \geq 1$ and a gapless spectrum was obtained first via the upper and lower bounds both of order $O(L^{d-1}\log L)$ and certain conjectures on the subleading term in the Szegö theorem for Töplitz determinants [13, 15] and then rigorously [20], by using a rather sophisticated techniques of modern operator theory [29, 30].

All the above concerns the translation invariant systems. Following a widely accepted paradigm of condensed matter physics, it is natural to consider a disordered version of the free fermion model replacing the translation invariant coefficients of the fermionic Hamiltonian by random coefficients, which are translation invariant in the mean and have sufficiently fast decaying spatial correlations, i.e., ergodic.

The analysis of many body quadratic fermionic Hamiltonians reduces to that of certain one body operators determined by the coefficients of the form. Thus, in the case of random coefficients we obtain a problem of the theory of one body disordered systems, the Anderson localization in particular.

It was rigorously shown in [24] (see also related works [22, 32]) that if $\mathcal{S}$ is in the ground (or even a pure) state of the fermionic quadratic form determined by the Anderson model (discrete Schrödinger operator with an i.i.d. random potential) and the Fermi energy μ, then the expectation $\mathbf{E}\{S_\Lambda\}$ of the entanglement entropy S_Λ of the $d \geq 1$ dimensional cube Λ of side length L admits the two sided bounds

$$C_- L^{d-1} \leq \mathbf{E}\{S_\Lambda\} \leq C_+ L^{d-1}, \; 0 < C_- \leq C_+ < \infty \tag{1.3}$$

The result is valid in the both cases, i.e., if the Fermi energy μ lies in a spectral gap of the Anderson model and if μ lies in the localized (pure point) spectrum of the model. The first fact (gaped case) is fairly simple and follows from the general principle of spectral theory while the second (gapless case) is essentially based on the exponential decay of the Fermi projection of the Anderson model, one of fundamental results of the localization theory. For $d = 1$ and $L \gg 1$ two sided bounds for the entanglement entropy of the almost all realizations of disorder were also obtained in [24] and then used to show numerically that the entanglement entropy of one-dimensional disordered fermions is not selfaveraging, i.e., has non vanishing random fluctuations even if $L \gg 1$

In this paper we first prove that for any $d \geq 1$ there exists a "surface macroscopic" limit of the entanglement entropy per unit of a cubic block

$$\lim_{L\to\infty} L^{-(d-1)} \mathbf{E}\{S_\Lambda\}, \tag{1.4}$$

which is non-zero and finite in view of (1.3). In other words, the entanglement entropy of disordered fermions satisfies the *area law in the mean*.

We then show that for $d \geq 2$ the variance of S_Λ vanishes as $L \to \infty$, i.e., that for $d \geq 2$ the entanglement entropy of disordered fermions is selfaveraging.

As for $d = 1$, we show that S_Λ has the limit as $L \to \infty$ with probability 1 (non-zero and finite with probability 1 in view of (1.3)). According to the numerical results of [24] the limit is random, i.e., the entanglement entropy of disordered fermions is not selfaveraging in the one-dimensional case.

Note that the selfaveraging property, i.e., the vanishing of fluctuations of extensive observables in the macroscopic limit is widely known in condensed matter theory and statistical mechanics of disordered systems. In the entanglement studies an essentially analogous property is known as the entanglement typicality (see e.g. the recent review [10]) and is used to simplify the problem in hand by considering entanglement characteristics of a pure state, which is "typical", i.e., random with respect to a certain multivariate probability distribution provided that the distribution is strongly picked in the number of variables. In the most of known cases the distribution is related to the Haar measure on the multidimensional unitary group $U(N)$, which is not always easy to interpret physically in the corresponding context, in particular, to determine the system physical dimension entering explicitly in large block asymptotics of the entanglement entropy (see (1.3) and (1.4) and many formulas below). On the other hand, the random ground state (more generally, pure states) of N free disordered fermions are just the Slater determinants of N eigenfunctions of the Schrödinger equation with random (more generally, ergodic) potential in the d-dimensional space.

This allows one to study in a simple setting of free fermions various depending on d entanglement properties, in particular, to conclude that the corresponding quantum states are typical for $d \geq 2$ and are not typical for $d = 1$.

Throughout the paper we will use the symbols C, C_1, c, c_1, etc. for quantities, which can be different in different expressions and which value is not essential for the validity of the expressions.

2 Main Results

In this section we formulate and prove our main results (see Theorems 2.2, 2.4, 2.6 and 2.8). Various technical results are given in the next section

2.1 Generalities

We consider the system of spinless lattice fermions confined to a finite domain $\mathcal{D} \subset \mathbb{Z}^d$ and described by the quadratic Hamiltonian

$$\mathcal{H}_{\mathcal{D}} = \sum_{x,y \in \mathcal{D}} A(x,y) c_x^+ c_y, \tag{2.1}$$

where $c_x^+, c_x,\ x \in \mathcal{D}$ are the Fermi creation and annihilation operators, $A_{\mathcal{D}} = \{A(x,y)\}_{x,y \in \mathcal{D}}$ is a $N \times N$ hermitian matrix, the restriction to $\mathcal{D} \subset \mathbb{Z}^d$ of a bounded hermitian operator $A = \{A(x,y)\}_{x,y \in \mathbb{Z}^d}$ acting in $l^2(\mathbb{Z}^d)$. We will choose

$$A_{\mathcal{D}} = (H_{\mathcal{D}} - \mu), \tag{2.2}$$

where μ is a parameter (Fermi energy) to be chosen below and $H_{\mathcal{D}}$ is the restriction to $\mathcal{D}$ of the discrete Schrödinger operator

$$H = -\Delta + V \tag{2.3}$$

in $\mathbb{Z}^d$, where Δ is the d-dimensional discrete Laplacian

$$(\Delta \psi)(x) = \sum_{|x-y|=1} \psi(y),\ x \in \mathbb{Z}^d \tag{2.4}$$

and

$$(V\psi)(x) = V(x)\psi(x),\ x \in \mathbb{Z}^d. \tag{2.5}$$

is the potential.

It follows from a quite standard second quantization calculation (see e.g. [9, 25, 32]) that the entanglement entropy of a cube $\Lambda \in \mathcal{D}$ is

$$S_\Lambda^{\mathcal{D}} = \mathrm{Tr}\ h(P_\Lambda^{\mathcal{D}}), \tag{2.6}$$

where

$$h(x) = -x \log_2 x - (1-x) \log_2 (1-x), \tag{2.7}$$

is a binary Shannon entropy, $P_\Lambda^{\mathcal{D}}$ is the restriction to Λ of the spectral projection measure $\mathcal{E}_{H_{\mathcal{D}}}$ of $H_{\mathcal{D}}$ (2.3), corresponding to the interval $(-\infty, \mu]$ ($[\mu_0, \mu]$, where $\mu_0 > -\infty$ is the finite lower edge of the spectrum of H, if we assume that V is bounded):

$$P_\Lambda^{\mathcal{D}} = P^{\mathcal{D}}\big|_\Lambda,\ P^{\mathcal{D}} := \mathcal{E}_{H_{\mathcal{D}}}((-\infty, \mu]). \tag{2.8}$$

It is easy to show that $H_{\mathcal{D}}$ converges strongly to H as $\mathcal{D} \nearrow \mathbb{Z}^d$, say in the van Hove sense [27], hence $P^{\mathcal{D}}$ converges strongly to

$$P = \mathcal{E}_H((-\infty, \mu)) \tag{2.9}$$

and since Λ is finite, $S_\Lambda^{\mathcal{D}}$ of (2.6) converges to

$$S_\Lambda = \operatorname{Tr} h(P_\Lambda) \tag{2.10}$$

We will assume in this paper that the potential (2.5) is an ergodic field in $\mathbb{Z}^d$. Recall that the field is defined by a measurable function v on a probability space $(\Omega, \mathcal{F}, P)$ endowed with a measure preserving and ergodic group of transformations $\{T_a\}_{a\in\mathbb{Z}^d}$:

$$V(x,\omega) = v(T_x\omega), \tag{2.11}$$

or

$$V(x, T_a\omega) = V(x+a, w),\ v(\omega) = V(0,\omega). \tag{2.12}$$

As a result, the whole operator $H = \{H(x,y;\omega)\}_{x,y\in Z^d}$ is an ergodic operator (see [23]), i.e., satisfies with probability 1 the relation

$$H(x,y;T_a\omega) = H(x+a, y+a;\omega),\ \forall a,x,y \in \mathbb{Z}^d. \tag{2.13}$$

More generally, an operator $A = \{A(x,y)\}_{x,y\in\mathbb{Z}^d}$ in $l^2(\mathbb{Z}^d)$ is called ergodic if it satisfies (2.13).

It follows then (see [23], Theorem 2.7) that P of (2.9) is an ergodic orthogonal projection, i.e., P is selfadjoint, $P^2 = P$ and

$$P = \{P(x,y,\omega)\}_{x,y\in\mathbb{Z}^d},\ P(x,y,T_a\omega) = P(x+a,y+a,\omega). \tag{2.14}$$

In particular, for any collection $\{(x_i,y_i)\}_{j=1}^k$ of pairs of points of $\mathbb{Z}^d$ the expectations

$$\mathbf{E}\Big\{\prod_{i=1}^k P(x_i,y_i)\Big\} = \mathbf{E}\Big\{\prod_{i=1}^k P(x_i+a,y_i+a)\Big\}, \quad \forall a \in \mathbb{Z}^d. \tag{2.15}$$

are translation invariant.

One of the main results of spectral theory of the Schrödinger operator with ergodic potential, which we will use extensively below, is the bound

$$\mathbf{E}\{|P(x,y)|\} \le C_0 e^{-\gamma|x-y|}, \quad C_0 < \infty,\ \gamma > 0,\ |x-y| = \sum_{j=1}^d |x_j - y_j|. \tag{2.16}$$

The bound is a manifestation of the exponential localization for the Schrödinger operator (pure point spectrum and the exponential decay of eigenfunctions). It is generically valid for random i.i.d. potential with sufficiently regular probability distribution (see (3.35)) and for certain classes of quasiperiodic potentials for any μ in the spectrum if the amplitude of the potential is large enough and for μ belonging to a certain neighborhood of spectrum edges if the amplitude of the potential is fixed. In the one dimensional case and i.i.d. potential with regular probability distribution the bound is valid for any μ in the specrum and any amplitude of potential. We refer the reader to the works [1, 3, 12, 17, 21, 31] and references therein, where the validity of the bound is proved and discussed.

However, it is worth mentioning that the proofs of a considerable amount of our results, in particular, those of Section 2 do not use that P is the spectral projection of an ergodic Schroedinger operator (see (2.8)). In other words, the results are valid for any ergodic orthogonal projection (2.14) – (2.16) satisfying (2.16) and even having a sufficiently fast power law decay.

Given the domains $\mathcal{C}_1 \subset \mathbb{Z}^d$ and $\mathcal{C}_2 \subset \mathbb{Z}^d$, consider the selfadjoint operator acting in $l^2(\mathcal{C}_1)$ (see Lemma 3.4 for its properties)

$$\Pi_{\mathcal{C}_1,\mathcal{C}_2} = \{\Pi_{\mathcal{C}_1,\mathcal{C}_2}(x,y)\}_{x,y\in\mathcal{C}_1},\ \Pi_{\mathcal{C}_1,\mathcal{C}_2}(x,y) = \sum_{z\in\mathcal{C}_2} P(x,z)P(z,y). \tag{2.17}$$

The operator can also be viewed as acting in the whole $l^2(\mathbb{Z}^d)$, if we continue by zero its matrix for x, y outside $\mathcal{C}_1$, i.e.,

$$\Pi_{\mathcal{C}_1,\mathcal{C}_2}(x,y) = \sum_{z\in\mathcal{C}_2} P(x,z)P(z,y)\mathbf{1}_{x\in\mathcal{C}_1}\mathbf{1}_{y\in\mathcal{C}_1}, \tag{2.18}$$

or, alternatively, as the restriction $\Pi_{\mathcal{C}_1',\mathcal{C}_2}|_{\mathcal{C}_1}$of the operator $\Pi_{\mathcal{C}_1',\mathcal{C}_2}$ corresponding to any $\mathcal{C}_1' \subset \mathbb{Z}^d$.

We will confine ourselves to the case where Λ in (2.6) – (2.10)) is

$$\Lambda = [-M, M]^d \subset \mathbb{Z}^d,\ |\Lambda| = L^d,\ L = 2M + 1. \tag{2.19}$$

We will also use the change of variables for the function h of (2.7):

$$h(x)_0 = h(x(1-x)),\ x \in [0,1]. \tag{2.20}$$

It is easy to check that h_0 is monotone increasing, convex and $h_0(0) = 0$ (see Lemma 3.1).

It follows then from (2.10), (2.7), (2.17) with $\mathcal{C}_1 = \Lambda$ and $\mathcal{C}_2 = \mathbb{Z}^d \setminus \Lambda$ that

$$P_\Lambda(1_\Lambda - P_\Lambda) = \Pi_{\Lambda,\mathbb{Z}^d\setminus\Lambda} \tag{2.21}$$

and

$$S_\Lambda = \operatorname{Tr} h_0(\Pi_{\Lambda,\mathbb{Z}^d\setminus\Lambda}). \tag{2.22}$$

We will prove now a simple general result, manifesting already that the large block behavior of the entanglement entropy is different from that of the thermodynamic entropy, which is extensive, i.e., asymptotically proportional to the volume L^d of the box Λ. The theorem also shows the advantage to use formula (2.22) rather than (2.10), since (2.22) takes explicitly into account the fact that the main contribution into S_Λ is from a neighborhood of the surface of Λ. This is because the matrix of the operator $\Pi_{\mathcal{C}_1,\mathcal{C}_2}$ of (2.17) is essentially concentrated on the boundary between $\mathcal{C}_1$ and $\mathcal{C}_2$, the fact, which is systematically used below, especially in view of the exponential bound (2.16). Here we will use only the general and quite weak decay of $P(x,y)$ given by its property

$$\sum_{y\in\mathbb{Z}^d} |P(x,y)|^2 = P(x,x) \le 1. \tag{2.23}$$

Theorem 2.1 *Let P be an ergodic orthogonal projection (2.14) – (2.15) and S_Λ be defined by (2.10), (2.20) and (2.22). Then*

$$\lim_{L\to\infty} L^{-d}\mathbf{E}\{S_\Lambda\} = 0. \tag{2.24}$$

Proof. We have according to (2.22):

$$\begin{aligned} L^{-d}\mathbf{E}\{S_\Lambda\} &= L^{-d}\mathbf{E}\{\mathrm{Tr}\ h_0(\Pi_{\Lambda,\mathbb{Z}^d\setminus\Lambda})\} \\ &= L^{-d}\sum_{x\in\Lambda}\mathbf{E}\{(h_0(\Pi_{\Lambda,\mathbb{Z}^d\setminus\Lambda}))(x,x)\}, \end{aligned}$$

where $\{(h_0(\Pi_{\Lambda,\mathbb{Z}\setminus\Lambda})(x,y)\}_{x,y\in\Lambda}$ is the matrix of $h_0(\Pi_{\Lambda,\mathbb{Z}\setminus\Lambda})$. We will use twice the convexity of h_0 of (2.20), see Lemma 3.1. First, we use (3.22) for $f = h_0$ yielding

$$(h_0(\Pi_{\Lambda,\mathbb{Z}\setminus\Lambda}))(x,x) \le h_0(\Pi_{\Lambda,\mathbb{Z}\setminus\Lambda}(x,x)).$$

Then, treating the operation $L^{-d}\sum_{x\in\Lambda}\mathbf{E}\{...\}$ as a generalized "averaging" and using the Jensen inequality, we obtain

$$L^{-d}\mathbf{E}\{S_\Lambda\} \le h_0\Big(L^{-d}\sum_{x\in\Lambda}\mathbf{E}\{\Pi_{\Lambda,\mathbb{Z}\setminus\Lambda}(x,x)\}\Big).$$

Now, using (2.17) and denoting

$$Q(x-y) := \mathbf{E}\{|P(x,y)|^2\}, \tag{2.25}$$

we get

$$L^{-d}\mathbf{E}\{S_\Lambda\} \le h_0(Q_\Lambda), \tag{2.26}$$

where

$$Q_\Lambda = L^{-d}\sum_{x\in\Lambda}\sum_{y\in\mathbb{Z}^d\setminus\Lambda} Q(x-y).$$

Let us show that $Q_\Lambda = o(1),\ L\to\infty$, since then (2.26) implies (2.24) in view of the equality $h_0(0) = 0$.

We write

$$Q_\Lambda = L^{-d}\sum_{x\in\Lambda}\sum_{y\in\mathbb{Z}^d} Q(x-y) - L^{-d}\sum_{x\in\Lambda}\sum_{y\in\Lambda} Q(x-y).$$

Since P is an orthogonal projection, it follows from (2.23) and (2.25) that $\{Q(x\}_{x\in\mathbb{Z}} \in l^1(\mathbb{Z})$. Hence, the first term on the right is

$$\widehat{Q} := \sum_{x\in\mathbb{Z}^d} Q(x) < \infty.$$

On the other hand, the second term on the right can be written as

$$\sum_{x=(x_1,\ldots,x_d)=-2M}^{2M} Q(x)\prod_{j=1}^{d}(1-|x_j|/L)\,,$$

thus its limit as $L \to \infty$ is also $\widehat{Q}$. ■

We are ready now to present our main results and their proofs modulo certain auxiliary facts given in the next section and a short discussion of a link with the Szegö theorem just now.

Recall that the Szegö theorem treats the large box asymptotic behavior of $\operatorname{Tr} \varphi(A_\Lambda)$, where $A = \{A(x-y)\}_{x,y\in\mathbb{Z}^d}$ is a selfadjoint convolution operator in $l^2(\mathbb{Z}^d)$, $A_\Lambda := A|_\Lambda$ is its restriction to the box Λ of (2.19) and $\varphi : \mathbb{R} \to \mathbb{C}$ is a function. According to the theorem [6, 28]

$$\operatorname{Tr} \varphi(A_\Lambda) = L^d \int_{\mathbb{T}^d} \varphi(a(p))dp + L^{d-1}T_d + o(L^{d-1}),\ L \to \infty, \tag{2.27}$$

where a is the Fourier transform of $\{A(x)\}_{x\in\mathbb{Z}^d}$ and T_d is a functional of a and φ. Functions a and φ are known as the symbol of A and the test function. It is important that (2.27) is valid only if a and φ are smooth enough. If, however, a is piece-wise constant, then the second (subleading) term of the formula is $\widetilde{T}_d\ L^{d-1}\log L$. These asymptotic formulas play an important role in the description of the entanglement entropy of translation invariant free fermions [13, 15, 20, 29].

Let us confine ourselves to the case $d = 1$ and view the operator of multiplication by p in $L^2(\mathbb{T})$ as the symbol of the selfadjoint operator $\widehat{p}$ in $l^2(\mathbb{Z})$. Then we can write the r.h.s. of (2.27) as $\operatorname{Tr} \varphi((a(\widehat{p}))_\Lambda)$. This make natural to consider a more general setting, where one chooses a sufficiently standard convolution operator B and studies asymptotic behavior of $\operatorname{Tr}\ \varphi((a(B))_\Lambda)$ depending on the choice of the pair (a, φ).

Note now that convolution operators can be viewed as a particular case of ergodic operators defined by (2.13). Indeed, it suffices to take in (2.13) the trivial case of the one-point space of events $\Omega = \{0\}$. Thus, we can extend the above general setting for the Szegö theorem to ergodic operators just choosing as B a certain "standard" ergodic operator, say the discrete Schrödinger operator H with ergodic potential (2.3) and studying the asymptotic behavior of the random variable $\operatorname{Tr} \varphi((a(H))_\Lambda)$, e.g., its expectation, its behavior for typical realizations of potential, i.e., with probability 1, etc. A particular case of this setting, where a and φ are smooth enough, was considered in [18]. It was found, that in this case and for i.i.d. random potential the subleading term in the analog of (2.27) is not T_1 but has the order $L^{1/2}$ and is a Gaussian random variable. Likewise, the case, where $\varphi = h$ of (2.7) and $a = \chi_{(-\infty,\mu]}$, the indicator of the interval $(-\infty, \mu]$ (see (2.8)) corresponds to the entanglement entropy (2.22). We prove below that in this apparently non-smooth case the subleading term does not depend on L and is a random variable, i.e., again the corresponding result differs from that for convolution operators.

2.2 One-dimensional Case

2.2.1 Limit of Expectation

The proofs of (1.4) for the cases $d = 1$ and $d \geq 2$ are similar. We will consider, however, the case $d = 1$ separately, since it is simpler and makes more transparent several important steps of the proof strategy for $d \geq 2$.

Theorem 2.2 *Let P be a non-trivial ($P \neq 0, \mathbf{1}_{\mathbb{Z}^d}$) ergodic projection of (2.14) – (2.15) satisfying (2.16), h be defined by (2.7) and $\Lambda = [-M, M], |\Lambda| =: L = 2M + 1$ (cf. (2.19)).*

Then there exists

$$\lim_{L\to\infty} \mathbf{E}\{S_\Lambda\} = \mathbf{E}\{\operatorname{Tr} h(P_{\mathbb{Z}_-})\} + \mathbf{E}\{\operatorname{Tr} h(P_{\mathbb{Z}_+})\}, \tag{2.28}$$

where

$$\mathbb{Z}_\pm = [0, \pm\infty) \tag{2.29}$$

and $P_{\mathbb{Z}_\pm}$ *are the restrictions of* P *to* $\mathbb{Z}_\pm$.

Remark 2.3 Assume that in addition to the ergodic group $\{T_x\}_{x\in\mathbb{Z}}$ of (2.11) – (2.14) the probability space is endowed with the measure preserving transformation R such that $V(x, R\omega) = V(-x, \omega)$, hence

$$P(x, y, R\omega) = P(-x, -y, \omega). \tag{2.30}$$

This is, for instance, the case for any random i.i.d. potential as well as for the quasiperiodic potential $V(x, \omega) = v(\alpha x + \omega)$, where $v : \mathbb{T}^1 \to \mathbb{R}^1$ is an even 1-periodic function, α is an irrational number and ω is uniformly distributed over the one-dimensional torus $\mathbb{T}^1$. Under assumption (2.30) we can write (2.28) in a simpler form

$$\lim_{L\to\infty} \mathbf{E}\{S_\Lambda\} = 2\mathbf{E}\{\operatorname{Tr} h(P_{\mathbb{Z}_-})\} = 2\sum_{x\in\mathbb{Z}_-} \mathbf{E}\{(h(P_{\mathbb{Z}_-}))(x, x)\}, \tag{2.31}$$

where $\{(h(P_{\mathbb{Z}_-}))(x, y)\}_{x,y\in\mathbb{Z}_-}$ is the matrix of $h(P_{\mathbb{Z}_-})$.

Proof. Given $M \in \mathbb{N}$, we can write in the one dimensional case, i.e., for $\Lambda = [-M, M]$:

$$\mathbb{Z} \setminus \Lambda = \Lambda_{+M} \cup \Lambda_{-M},\ \Lambda_{+M} = (+M, \infty),\ \Lambda_{-M} = (-\infty, -M)$$

and in view of (2.21) and Lemma 3.4 (iv)

$$\Pi_{\Lambda,\mathbb{Z}\setminus\Lambda} = \Pi_{\Lambda,\Lambda_{+M}} + \Pi_{\Lambda,\Lambda_{-M}}, \tag{2.32}$$

hence

$$\operatorname{Tr} h_0(\Pi_{\Lambda,\mathbb{Z}\setminus\Lambda}) = \operatorname{Tr} h_0(\Pi_{\Lambda,\Lambda_{+M}} + \Pi_{\Lambda,\Lambda_{-M}}). \tag{2.33}$$

This, (2.22) and (2.28) show that we have to prove the limiting relation

$$\lim_{L\to\infty} \mathbf{E}\{\operatorname{Tr} h_0(\Pi_{\Lambda,\mathbb{Z}\setminus\Lambda})\} = \mathbf{E}\{\operatorname{Tr} h_0(\Pi_{\mathbb{Z}_-,\mathbb{N}_+})\} + \mathbf{E}\{\operatorname{Tr} h_0(\Pi_{\mathbb{Z}_+,\mathbb{N}_-})\}, \tag{2.34}$$

where

$$\mathbb{N}_\pm = (0, \pm\infty). \tag{2.35}$$

To this end we will use Lemma 3.2 (i) for $d = 1$, $p = 2$, $A_1 = \Pi_{\Lambda,\Lambda_{+M}}$ and $A_2 = \Pi_{\Lambda,\Lambda_{-M}}$, hence $A_1 + A_2 = \Pi_{\Lambda,\mathbb{Z}\setminus\Lambda}$ (see (2.32). Let us check the first condition of (3.1), i.e., the uniform in L bound

$$\mathbf{E}\{(\operatorname{Tr} \Pi^{1/4}_{\Lambda,\mathbb{Z}\setminus\Lambda})^2\} < \infty.$$

It follows from Lemma 3.5 (i) that

$$\begin{aligned}\mathbf{E}\{(\operatorname{Tr} \Pi^{1/4}_{\Lambda,\mathbb{Z}\setminus\Lambda})^2\} &\le C\Big(\sum_{x\in\Lambda}\sum_{z\in\mathbb{Z}\setminus\Lambda} e^{-\gamma|z-x|/4}\Big)^2 \\ &\le \Big(\sum_{x\in\Lambda}\sum_{z\in\Lambda_{+M}} e^{-\gamma(z-x)/4} + \sum_{x\in\Lambda}\sum_{z\in\Lambda_{-M}} e^{-\gamma(z-x)/4}\Big)^2 \le C_1 < \infty.\end{aligned} \tag{2.36}$$

Hence we have verified the first inequality in (3.1).

Likewise, it follows from the Hölder inequality for expectations and (2.16) that for any collection $\{(x_j,y_j)\}_{j=1}^8$:

$$\Big|\mathbf{E}\Big\{\prod_{j=1}^8 P(x_j,y_j)\Big\}\Big| \le \prod_{j=1}^8 \mathbf{E}^{1/8}\{|P(x_j,y_j)|\} \le C_0 \prod_{j=1}^8 e^{-\gamma|x_j-y_j|/8}.$$

This implies the inequality

$$\mathbf{E}\{(\mathrm{Tr}\,\Pi_{\Lambda,\Lambda_{+M}}\Pi_{\Lambda,\Lambda_{-M}})^2\} \le C\ (\mathrm{Tr}\,\overline{\Pi}_{\Lambda,\Lambda_{+M}}\overline{\Pi}_{\Lambda,\Lambda_{-M}})^2, \tag{2.37}$$

where $\overline{\Pi}_{\mathcal{C}_1,\mathcal{C}_2}$ is obtained from $\Pi_{\mathcal{C}_1,\mathcal{C}_2}$ of (2.17) by replacing $P(x,y)$ by $C_0^{1/8}e^{-\gamma|x-y|/8}$. Hence, it suffices to estimate $\mathrm{Tr}\,\overline{\Pi}_{\Lambda,\Lambda_{+M}}\overline{\Pi}_{\Lambda,\Lambda_{-M}}$. We have

$$\mathrm{Tr}\,\overline{\Pi}_{\Lambda,\Lambda_{+M}}\overline{\Pi}_{\Lambda,\Lambda_{-M}} = C_0^{1/2}\sum_{x,y\in\Lambda}\sum_{z_\pm\in\Lambda_{\pm M}} e^{-\gamma(|x-z_+|+|y-z_+|+|x-z_-|+|y-z_-|)/8}.$$

Taking into account that $\Lambda=[-M,M]$ and $\Lambda_{\pm M}=(\pm M,\pm\infty)$, hence $z_-<x<z_+$, the exponent on the right of the above expression is $-\gamma(z_+-z_-)/4$, hence the expression is bounded by $CL^2e^{-\gamma L/4}$ and

$$\mathbf{E}\{(\mathrm{Tr}\,\Pi_{\Lambda,\Lambda_{+M}}\Pi_{\Lambda,\Lambda_{-M}})^2\} \le CL^4e^{-\gamma L/2}. \tag{2.38}$$

Thus, the second inequality in (3.1) is also verified and then Lemma 3.2 (i) yields

$$\begin{aligned}\mathbf{E}\{S_\Lambda\} &= \mathbf{E}\{\mathrm{Tr}\,h_0(\Pi_{\Lambda,\Lambda_{+M}}+\Pi_{\Lambda,\Lambda_{-M}})\} \\ &= \mathbf{E}\{\mathrm{Tr}\,h_0(\Pi_{\Lambda,\Lambda_{+M}})\}+\mathbf{E}\{\mathrm{Tr}\,h_0(\Pi_{\Lambda,\Lambda_{-M}})\}+O(e^{-cL}),\ \ c>0.\end{aligned} \tag{2.39}$$

We are left to prove that (see (2.34))

$$\lim_{L\to\infty}\mathbf{E}\{\mathrm{Tr}\,h_0(\Pi_{\Lambda,\Lambda_{\pm M}})\} = \mathbf{E}\{\mathrm{Tr}\,h_0(\Pi_{\mathbb{Z}_\mp,\mathbb{N}_\mp})\}. \tag{2.40}$$

Since the proofs of both relations are identical, we consider the case of $\Pi_{\mathbb{Z}_-,\mathbb{N}_+}$ and $\Pi_{\Lambda,\Lambda_{+M}}$. It follows from (2.14) – (2.15) that

$$\mathbf{E}\{\mathrm{Tr}\,h_0(\Pi_{\Lambda,\Lambda_{+M}})\} = \mathbf{E}\{\mathrm{Tr}\,h_0(\Pi_{(-L,0],\mathbb{N}_+})\},$$

hence we have to prove that

$$\lim_{L\to\infty}\mathbf{E}\{\mathrm{Tr}\,h_0(\Pi_{(-L,0],\mathbb{N}_+})\} = \mathbf{E}\{\mathrm{Tr}\,h_0(\Pi_{\mathbb{Z}_-,\mathbb{N}_+})\}. \tag{2.41}$$

We will use Lemma 3.2 (ii) with $A_1=\Pi_{\mathbb{Z}_-,\mathbb{N}_+}$ and $A_2=\Pi_{[-2L,0],\mathbb{N}_+}$ and we will view $\Pi_{[-2L,0],\mathbb{N}_+}$ as the restriction $\Pi_{\mathbb{Z}_-,\mathbb{N}_+}|_{(-L,0]}$ to have the both operators A_1 and A_2 acting in the same space $l^2(\mathbb{Z}_-)$ (see Lemma 3.4 (iii)). To check the condition (3.3), we will use Lemma 3.5 (iii) with $\mathcal{C}_1'=(-L,0]$, $\mathcal{C}_1''=\mathbb{Z}_-$ and $\mathcal{C}_2=\mathbb{N}_+$. Thus, we have from (3.28)

$$\begin{aligned}\mathbf{E}\big\{\mathrm{Tr}\,|\Pi_{\mathbb{Z}_-,\mathbb{N}_+}-\Pi_{(-L,0],\mathbb{N}_+}|\big\} &\le C\sum_{x\in(-L,0]}\ \sum_{y\in(-\infty,-L]}\ \sum_{z\in\mathbb{N}_+} e^{-\gamma(2z-x-y)/2} \\ &\quad + C\sum_{x,y\in(-\infty,-L]}\ \sum_{z\in\mathbb{N}_+} e^{-\gamma(2z-x-y)/2} = O(e^{-\gamma L}).\end{aligned} \tag{2.42}$$

This implies (2.41), hence (2.40) and, in view of (2.10) and (2.22), the theorem. ■

2.2.2 Asymptotics with probability 1

We will now take into account that all the bounds for expectations above are exponential in L (see e.g. (2.39) and (2.41)) and find the asymptotic form of the entanglement entropy valid with probability 1.

Theorem 2.4 *Let $S_\Lambda,\ \Lambda = [-M, M] \subset \mathbb{Z}$ be the entanglement entropy (2.22) of the one dimensional system of disordered fermions having the discrete Schrödinger operator with ergodic potential as the one body operator. Then we have with probability 1*

$$S_\Lambda = S_+(T_{+M}\omega) + S_-(T_{-M}\omega) + o(1),\ L := (2M+1) \to \infty, \tag{2.43}$$

where

$$S_\pm = \text{Tr } h(P_{\mathbb{Z}_\mp}), \tag{2.44}$$

$\Pi_{\mathbb{Z}_\pm,\mathbb{N}_\mp}$ *are defined in (2.17) and (2.29) and* $T_{\pm M}$ *are the shift transformations, see (2.11) – (2.14).*

The random variables (2.44) are finite and not identically zero with probability 1.

Proof. It follows from the exponential bound (2.39) and the Borel-Cantelli lemma that we have with probability 1

$$S_\Lambda = \text{Tr}\, h_0(\Pi_{\Lambda,\Lambda_{+M}}) + \text{Tr}\, h_0(\Pi_{\Lambda,\Lambda_{-M}}) + o(1),\ L := 2M+1 \to \infty. \tag{2.45}$$

Next, it follows from the exponential bound (2.41) and analogous bound for the pair

$$\text{Tr}\, h_0(\Pi_{[-M,M],(-\infty,-M)}) := \text{Tr}\, h_0(\Pi_{\Lambda,\Lambda_{-M}})$$

and

$$\text{Tr}\, h_0(\Pi_{[-M,\infty),(-\infty,-M)}) := \text{Tr } h_0(P_{\mathbb{Z}_+,\mathbb{N}_-})$$

that we have with probability 1

$$\text{Tr}\, h_0(\Pi_{\Lambda,\Lambda_{+M}}) = \text{Tr}\, h_0(\Pi_{(-\infty,M),(M,\infty)}) + o(1),\ L := 2M+1 \to \infty \tag{2.46}$$

and

$$\text{Tr}\, h_0(\Pi_{\Lambda,\Lambda_{-M}}) = \text{Tr}\, h_0(\Pi_{[-M,\infty),(-\infty,-M)}) + o(1),\ L := 2M+1 \to \infty. \tag{2.47}$$

Now, combining (2.45), (2.46) and (2.47), we obtain with probability 1

$$S_\Lambda = \text{Tr}\, h_0(\Pi_{(-\infty,M),(M,\infty)}) + \text{Tr}\, h_0(\Pi_{[-M,\infty),(-\infty,-M)}) + o(1),\ L := 2M+1 \to \infty. \tag{2.48}$$

Denoting

$$S_\pm(\omega) = \text{Tr}\, h_0(\Pi_{\mathbb{Z}_\mp,\mathbb{N}_\pm}),$$

we find that the first term on the right of (2.48) is $S_+(T_{+M}\omega)$ and the second term on the right of (2.48) is $S_-(T_{+M}\omega)$. This and the equalities

$$\text{Tr}\, h_0(\Pi_{\mathbb{Z}_\mp,\mathbb{N}_\pm}) = \text{Tr}\, h(P_{\mathbb{Z}_\mp}),$$

following from (2.20) and (2.17), cf. (2.10) and (2.22), prove (2.43).

To prove that (2.44) are finite and not identically zero with probability 1 it suffices to prove that $\mathbf{E}\{S_\pm\}$ is finite and positive:

$$0 < \mathbf{E}\{S_\pm\} < \infty. \tag{2.49}$$

We have by (2.44), (2.20) – (2.22) and Lemma 3.1 (iii):

$$4\mathbf{E}\{\operatorname{Tr}\Pi_{\mathbb{Z}_\mp,\mathbb{N}_\pm}\} \le \mathbf{E}\{\operatorname{Tr} h_0(\Pi_{\mathbb{Z}_\mp,\mathbb{N}_\pm})\} = \mathbf{E}\{\operatorname{Tr} h(P_{\mathbb{Z}_\mp})\} \le 2\mathbf{E}\{\operatorname{Tr}\Pi^{1/2}_{\mathbb{Z}_\mp,\mathbb{N}_\pm}\}. \tag{2.50}$$

By using the Peierls inequality (3.23), (2.17) and (2.16) in the r.h.s., we obtain

$$2\mathbf{E}\{\operatorname{Tr}\Pi^{1/2}_{\mathbb{Z}_\mp,\mathbb{N}_\pm}\} \le 2\sum_{x\in\mathbb{Z}_\mp}\Big(\sum_{y\in\mathbb{N}_\pm}\mathbf{E}\{|P(x,y)|^2\}\Big)^{1/2} \le C < \infty.$$

This proves a finite upper bound for r.h.s. (2.50), hence, for the r.h.s. of (2.49).

Furthermore, we have for the l.h.s. of (2.50):

$$4\mathbf{E}\{\operatorname{Tr}\Pi_{\mathbb{Z}_\mp,\mathbb{N}_\pm}\} = 4\sum_{x\in\mathbb{Z}_\mp}\sum_{y\in\mathbb{N}_\pm}\mathbf{E}\{|P(x,y)|^2\} = 4\sum_{l=1}^{\infty} lQ(l),$$

with Q defined in (2.25). Thus, if the l.h.s. of (2.50) is zero, then $P(x,y)=0,\ x\neq y$ with probability 1 by (2.25), i.e., the projection P is diagonal, $P(x,y) = P(x,x)\delta_{xy}$. Since P commutes with the Schrödinger operator (2.3), we have $P(x,x) = P(x+1,x+1),\ \forall x\in\mathbb{Z}$, i.e., $P = p\mathbf{1}_{\mathbb{Z}},\ p\in\{0,1\}$ with probability 1. This contradicts the hypothesis of the theorem. ■

Remark 2.5 The lower bound for the entanglement entropy (see (2.50)), which follows from Lemma 3.1, can also be used in the translation invariant case, where the operator A of (2.1) is a convolution operator in $l^2(\mathbb{Z}^d)$, e.g. the Schrödiger operator with a constant potential:

$$A = \{A(x-y)\}_{x,y\in\mathbb{Z}^d},\ \sum_{x\in\mathbb{Z}^d}|A(x)| < \infty. \tag{2.51}$$

In this case

$$P(x,y) = \frac{\sin\kappa\ (x-y)}{\pi(x-y)}, \tag{2.52}$$

where $\kappa\in[0,\pi)$ depends on the spectral interval in question and the Fourier transform (symbol) of $\{A(x)\}_{x\in\mathbb{Z}^d}$. In this case we have from (2.10) and Lemma 3.1 for $d=1$:

$$\begin{aligned} S_\Lambda &\ge 4\sum_{|x|\le M,|y|>M}\frac{\sin^2\kappa(x-y)}{\pi^2(x-y)^2} \\ &= 8\pi^2\sum_{l=1}^{\infty}\min\{L,l\}\sin^2\kappa l/l^2 = 4\pi^{-2}\log L + O(1),\ L\to\infty. \end{aligned} \tag{2.53}$$

Analogous argument yields for $d\ge 1$

$$S_\Lambda \ge C_d L^{d-1}\log L + O(L^{L-1}),\ L\to\infty, \tag{2.54}$$

where C_d is universal, i.e., does not depend on the operator A.

The bounds (2.53) – (2.54) provide a simple manifestation of the logarithmic corrections to the area law in the translation invariant macroscopic systems, see [8, 11, 13, 15, 20, 30].

2.3 Multidimensional case

We will deal now with the entanglement entropy S_Λ of (2.10) for the $d \geq 2$ dimensional cube Λ of (2.19). We will prove first that the limit of expectation of $L^{d-1}S_\Lambda$ exists and give a formula for the limit generalizing formula (2.28) of the one dimensional case. We then prove a power law decaying bound for the variance of $L^{d-1}S_\Lambda$Thus, for $d \geq 2$ the entanglement entropy per unit surface is selfaveraging, i.e., converges in probability to the nonrandom limit equals to the limit of its expectation.

2.3.1 Limit of Expectation

We will use the same strategy as in the one-dimensional case, i.e., the replacement of the operator $\Pi_{\Lambda,\mathbb{Z}^d\setminus\Lambda}$ in (2.22) by an appropriate "limiting" operator.

To present our results in a compact form we will assume certain symmetry properties of the ergodic potential (general case is described in a remark after the theorem).

Assume, just as in the one-dimensional case (see Remark 2.3), that our probability space possesses the measure preserving transformation R (reflection) such that with probability 1 (cf. (2.30))

$$V(x, R\omega) = V(-x, \omega),\ x \in \mathbb{Z}^d. \tag{2.55}$$

Assume also that there exists a collection of measure preserving transformations $\{\Sigma_\sigma\}$ (permutations) of the probability space that form a representation of the d-dimensional symmetric group S_d and such that with probability 1

$$V(x, \Sigma\omega) = V(\sigma x, \omega),\ x \in \mathbb{Z}^d,\ \sigma \in S_d. \tag{2.56}$$

Note that an important case of an i.i.d. potential has the both properties. Since the d-dimensional discrete Laplacian commutes with the reflection $x \to -x$ and permutations of the components $x = (x_1, ..., x_d) \to \sigma x = (x_{\sigma(1)}, ..., x_{\sigma(d)})$ of vectors of $\mathbb{Z}^d$, the Schrödinger operator H, hence its spectral projection (see Theorem 2.7 of [23]), also possesses these properties:

$$P(x, y, R\omega) = P(-x, -y, \omega),\ x, y \in \mathbb{Z}^d \tag{2.57}$$

and

$$P(x, y, \Sigma_\sigma\omega) = P(\sigma x, \sigma y, \omega),\ x, y \in \mathbb{Z}^d \tag{2.58}$$

Theorem 2.6 *Let P be an ergodic orthogonal projection (2.14) – (2.15) satisfying condition (2.16) as well as (2.57) and (2.58), h be defined by (2.7) and $\Lambda = [-M, M]^d,\ L = 2M + 1$. Then there exists*

$$\lim_{L\to\infty} L^{-(d-1)}\mathbf{E}\{S_\Lambda\} = 2d \sum_{x_1 \leq 0} \mathbf{E}\{(h(P_{\mathbb{Z}^d_-}))(x_1, 0; x_1, 0)\}, \tag{2.59}$$

where

$$\left\{\left(h(P_{\mathbb{Z}^d_-})\right)(x_1, \xi; y_1, \eta)\right\}_{x_1,y_1\in\mathbb{Z}_-;\xi,\eta\in\mathbb{Z}^{d-1}}$$

is the matrix of the operator $h(P_{\mathbb{Z}^d_-})$ and we write $\mathbb{Z}^d_- = \mathbb{Z}_- \times \mathbb{Z}^{d-1}$, $\mathbb{Z}_- = \{0, -1, ..., -\infty\}$, i.e., the explicit splitting of $x \in \mathbb{Z}^d$ into the "longitudinal" $x_1 \in \mathbb{Z}_-$ and the "transversal" $\xi \in \mathbb{Z}^{d-1}$ components.

Remark 2.7 We will give here the general form of Theorem 2.6 where we do not assume the symmetry properties (2.55) and (2.56) of ergodic projection. Denote

$$\mathbb{Z}^d_{-s}(j) = \{x = (x_1, ..., x_d) \in \mathbb{Z}^d : sx_j \le 0\},\ s = \pm.$$

Then have instead of (2.59)

$$\lim_{L\to\infty} L^{-(d-1)}\mathbf{E}\{S_\Lambda\} = \sum_{j=1}^{d}\sum_{s=\pm}\sum_{sx_j\le 0} \times \mathbf{E}\{(h(P_{\mathbb{Z}^d_{-s}(j)}))(0, ..., 0, x_j, 0, ..., 0; 0, ..., 0, x_j, 0, ..., 0\}, \tag{2.60}$$

where

$$\left\{\left(h(P_{\mathbb{Z}^d_{-s}})\right)(x;y)\right\}_{x,y\in\mathbb{Z}^d_{-s}(j)}$$

is the matrix of the operator $h(P_{\mathbb{Z}^d_{-s}})$.

Proof. Denote by $\Lambda_1^{(0)}, \dots, \Lambda_{2d}^{(0)}$ the faces of Λ, by $\Lambda_1, \dots, \Lambda_{2d}$ the semi-infinite cylindric domains adjacent to each of $\Lambda_1^{(0)}, \dots, \Lambda_{2d}^{(0)}$ from the exterior of Λ and by

$$\widetilde{\Lambda} = \mathbb{Z}^d \setminus \big(\cup_{j=0}^{2d} \Lambda_j\big). \tag{2.61}$$

This and additivity of the operator $\Pi_{\mathcal{C}_1,\mathcal{C}_2}$ of (2.17) with respect to $\mathcal{C}_2$ allow us to write (cf. (2.32))

$$\Pi_{\Lambda,\mathbb{Z}^d\setminus\Lambda} = \sum_{j=1}^{2d}\Pi_{\Lambda,\Lambda_j} + \Pi_{\Lambda,\widetilde{\Lambda}}. \tag{2.62}$$

Let us prove first that

$$\lim_{L\to\infty}\Big(L^{-(d-1)}\mathbf{E}\{S_\Lambda\} - \sum_{j=1}^{2d}\lim_{L\to\infty}\mathbf{E}\{L^{-(d-1)}\mathrm{Tr}\, h_0(\Pi_{\Lambda,\Lambda_j})\}\Big) = 0,$$

i.e., that in view of (2.22) and (2.62)

$$\begin{aligned} L^{-(d-1)}\mathbf{E}\{S_\Lambda\} &= \mathbf{E}\Big\{L^{-(d-1)}\mathrm{Tr}\, h_0\Big(\sum_{j=1}^{2d}\Pi_{\Lambda,\Lambda_j} + \Pi_{\Lambda,\widetilde{\Lambda}}\Big)\Big\} \\ &= \mathbf{E}\Big\{L^{-(d-1)}\mathrm{Tr}\, h_0\Big(\sum_{j=1}^{2d}\Pi_{\Lambda,\Lambda_j}\Big)\Big\} + o(1) \\ &= \sum_{j=1}^{2d}\mathbf{E}\Big\{L^{-(d-1)}\mathrm{Tr}\, h_0\Big(\Pi_{\Lambda,\Lambda_j}\Big)\Big\} + o(1),\ L\to\infty. \end{aligned} \tag{2.63}$$

We will prove this in two steps both based on Lemmas 3.2 and 3.5. The first step is the justification of the omission of $\Pi_{\Lambda,\widetilde{\Lambda}}$ in the second line of (2.63) and the second is the proof of the approximate additivity of h_0 given in the third line of (2.63).

For the first step we will use Lemma 3.2 (ii) with $A_1 = \Pi_{\Lambda,\mathbb{Z}^d\setminus\Lambda}$ and $A_2 = \sum_{j=1}^{2d}\Pi_{\Lambda,\Lambda_j}$, hence $A_1 - A_2 = \Pi_{\Lambda,\widetilde{\Lambda}}$ and (3.29) . According to the lemma we have to check conditions (3.3). The first of conditions in our case is the validity of the uniform in L bounds

$$\mathbf{E}\{(L^{-(d-1)}\mathrm{Tr}\,\Pi^{1/4}_{\Lambda,\mathbb{Z}^d\setminus\Lambda})^2\} < \infty, \;\; \mathbf{E}\{(L^{-(d-1)}\mathrm{Tr}\,\Pi^{1/4}_{\Lambda,\cup_{j=1}^{2d}\Lambda_j})^2\} < \infty \tag{2.64}$$

To prove the first bound above we will use Lemma 3.5 (i) according to which the bound is valid if

$$L^{-(d-1)}\sum_{x\in\Lambda, z\in\mathbb{Z}^d\setminus\Lambda} e^{-\gamma|x-z|/4} < \infty.$$

The sum in $z \in \mathbb{Z}^d \setminus \Lambda$ is not less than the sum over the all half-spaces, where one of coordinates z_j of $z = (z_1, ..., z_d)$ is either $z_j > M$ or $z_j < -M$. For instance, the term with the sum over the half-space $\{z = (z_1, \zeta) \in \mathbb{Z}^d : z_1 > M, \; \zeta \in \mathbb{Z}^{d-1}\}$ is

$$\begin{aligned} &L^{-(d-1)}\sum_{|x_1|\le M, z_1>M}\;\sum_{\xi\in[-M,M]^{d-1},\zeta\in\mathbb{Z}^{d-1}} e^{-\gamma(z_1-x_1)/4-\gamma|\xi-\zeta|/4} \\ &\quad\le CL^{-(d-1)}\sum_{\xi\in[-M,M]^{d-1},\zeta\in\mathbb{Z}^{d-1}} e^{-\gamma|\xi-\zeta|/4} \le C_1 < \infty \end{aligned}$$

since the double sum over x_1 and z_1 is uniformly bounded in $L \to \infty$, the sum of the exponential $e^{-\gamma|\xi-\zeta|/4}$ over $\zeta \in \mathbb{Z}^d$ is independent of ξ and the sum over ξ is $L^{(d-1)}$. The proof of the second bound in (2.64) is analogous. Thus, the first of conditions of (3.3) holds.

Let us consider the second condition of (3.3), i.e., determine the order of magnitude of ε in $L \to \infty$ from the asymptotic relation

$$\mathbf{E}\Big\{L^{-(d-1)}\mathrm{Tr}\,\Pi_{\Lambda,\widetilde{\Lambda}}\Big\} = O(\varepsilon), \tag{2.65}$$

where we took into account that $\Pi_{\Lambda,\widetilde{\Lambda}}$ is positive definite, hence $|\Pi_{\Lambda,\widetilde{\Lambda}}| = \Pi_{\Lambda,\widetilde{\Lambda}}$. We will use Lemma 3.5 (ii) with $A = \Pi_{\Lambda,\widetilde{\Lambda}}$, according to which it suffices to determine the order of magnitude of ε in L guarantying the bound

$$L^{-(d-1)}\sum_{x\in\Lambda}\sum_{z\in\widetilde{\Lambda}} e^{-\gamma|x-z|} \le CL^{-1} := \varepsilon. \tag{2.66}$$

The bound follows from the first inequality of (3.29), if we split $\widetilde{\Lambda}$ in parallelepipeds and apply twice the first inequality of (3.29) to each of the corresponding sums, the first in the initial dimension d and the second in the dimension $d-1$. Then Lemma 3.2 (ii) implies

$$\mathbf{E}\Big\{\Big(L^{-(d-1)}\mathrm{Tr}\,(h_0(\Pi_{\Lambda,\cup_{j=0}^{2d}\Lambda_j\cup\widetilde{\Lambda}}) - h_0(\Pi_{\Lambda,\cup_{j=0}^{2d}\Lambda_j}))\Big)^2\Big\} = O(L^{1/5}\log^2 L), \tag{2.67}$$

hence the second line of (2.63).

Let us prove the passage from the second line to the third line, i.e., the approximate additivity of h_0. Here we will use Lemma 3.2 (i) with $A_j = \Pi_{\Lambda,\Lambda_j}$ and $p = 2d$, i.e., we are to check conditions (3.1) for these operators.

Since in our case (see (2.17) and (2.62))

$$\sum_{j=1}^{q}\Pi_{\Lambda,\Lambda_j} = \Pi_{\Lambda,\cup_{j=1}^{q}\Lambda_j}, \; 1 \le q \le 2d,$$

the verification of the first of conditions (3.1) is quite similar to that of (2.36) and (2.64). Thus, consider the second of conditions (3.1), i.e., the determination of ε from the bounds (cf. (2.38))

$$\mathbf{E}\{(L^{-(d-1)}\mathrm{Tr}\,\Pi_{\Lambda,\Lambda_j}\Pi_{\Lambda,\Lambda_k})^2\} \le \varepsilon^2,\ 1 \le j \ne k \le 2d. \tag{2.68}$$

The corresponding argument is quite similar to that proving (2.38) for $d = 1$. The only difference is that for $d = 1$ the distance between the "neighboring" sets Λ_{+M} and Λ_{-M} is L (hence the exponential bound (2.38)), while for $d \ge 2$ every Λ_j has nearest neighbors with a non-empty boundary with Λ_j. However, the codimension of the boundary is at least 2 (cf. the proof of (2.67)) and we obtain that $\varepsilon = O(L^{-1})$, cf. the proof of (2.65), hence $o(1)$ in the third line of (2.63) is $O(L^{-1/5}\log^2 L)$, since according to Lemma 3.2 (i)

$$\mathbf{E}\Big\{\Big(L^{-(d-1)}\mathrm{Tr}\,\Big(h_0\Big(\sum_{j=1}^{2d}\Pi_{\Lambda,\Lambda_1}\Big) - \sum_{j=1}^{2d} h_0(\Pi_{\Lambda,\Lambda_j})\Big)\Big)^2\Big\} = O(L^{-1/5}\log^2 L), \tag{2.69}$$

hence we get the third line of (2.63)) or, in the case, where the ergodic potential possesses the symmetries (2.55) and (2.56), that

$$L^{-(d-1)}\mathbf{E}\{S_\Lambda\} = 2d\ \mathbf{E}\{L^{-(d-1)}\mathrm{Tr}\,h_0(\Pi_{\Lambda,\Lambda_1})\} + O(L^{-1/11}),\ L \to \infty. \tag{2.70}$$

Moreover, even in the case, where the above symmetries are absent, it suffices to prove the existence of the limit of the expectation on the right of (2.70), since the proof for any $j > 1$ is identical to that for $j = 1$ modulo the change of notation. Thus, we will prove the existence of the limit

$$\lim_{L\to\infty} L^{-(d-1)}\mathbf{E}\{\mathrm{Tr}\,h_0(\Pi_{\Lambda,\Lambda_1})\} \tag{2.71}$$

assuming for convenience that

$$\Lambda_1 = [M+1,\infty) \times \Lambda^{(0)} \subset \mathbb{Z}^d, \quad \Lambda^{(0)} = [-M,M]^{d-1} \in \mathbb{Z}^{d-1}. \tag{2.72}$$

It is also convenient to shift Λ and Λ_1 to the left by the vector $(-L,0) \in \mathbb{Z}^d$ in order to deal with the cube (cf. (2.19)) and the semi-infinite parallelepiped

$$\Lambda^* = [-2m,0] \times \Lambda^{(0)}, \quad \Lambda_1^* = \mathbb{N}_+ \times \Lambda^{(0)}, \tag{2.73}$$

where $\mathbb{N}_+$ is defined in (2.35), since this shift does not change (2.71), i.e.,

$$L^{-(d-1)}\mathbf{E}\{S_\Lambda\} = 2d\ \mathbf{E}\{L^{-(d-1)}\mathrm{Tr}\,h_0(\Pi_{\Lambda^*,\Lambda_1^*})\} + O(L^{-1/11}),\ L \to \infty, \tag{2.74}$$

or, if $\{(h_0(\Pi_{\Lambda^*,\Lambda_1^*})(x_1,\xi;y_1,\eta)\}_{x_1,y_1\in(-L.0],\xi,\eta\in\Lambda_1^{(0)}}$ is the matrix of $h_0(\Pi_{\Lambda^*,\Lambda_1^*})$,

$$\begin{aligned} &L^{-(d-1)}\mathbf{E}\{S_\Lambda\} \\ &= 2dL^{-(d-1)} \sum_{x_1\in(-L,0]}\sum_{\xi\in\Lambda^{(0)}} \mathbf{E}\{h_0(\Pi_{\Lambda^*,\Lambda_1^*})(x_1,\xi;y_1,\eta)\} + O(L^{-1/11}),\ L \to \infty. \end{aligned} \tag{2.75}$$

Let us show first that it suffices to justify the replacement of $h_0(\Pi_{\Lambda^*,\Lambda_1^*})$ in the first term on the right of (2.75) by the restriction $h_0(\Pi_{\mathbb{Z}_-^d,\mathbb{N}_+^d})\Big|_{\Lambda^*}$ of $h_0(\Pi_{\mathbb{Z}_-^d,\mathbb{N}_+^d})$ to Λ^* where

$$\mathbb{N}_\pm^d = \mathbb{N}_\pm \times \mathbb{N}^{d-1}. \tag{2.76}$$

Indeed, let

$$\{(h_0(\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+})(x_1,\xi;y_1,\eta)\}_{x_1,y_1\in\ \mathbb{Z}_-,\,\xi,\eta\in\mathbb{Z}^{d-1}}, \tag{2.77}$$

be the matrix of $h_0(\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+})$. Then the first term on the right of (2.74) becomes after the replacement

$$\begin{aligned} &L^{-(d-1)}\mathbf{E}\{\mathrm{Tr}\, h_0(\Pi_{\Lambda^*,\Lambda_1^*})\} \\ &= L^{-(d-1)}\sum_{x_1\in(-L,0]}\sum_{\xi\in\Lambda^{(0)}}\mathbf{E}\{(h_0(\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+}))(x_1,\xi;x_1,\xi)\}+o(1),\ L\to\infty. \end{aligned} \tag{2.78}$$

Let $T_{(0,\alpha)}$ be the measure preserving shift transformation (see, e.g. (2.11) – (2.14)) by a vector $a=(0,\alpha)\in\mathbb{Z}^d,\ \alpha\in\mathbb{Z}^{d-1}$, so that we have with probability 1

$$\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+}(x_1,\xi;y_1,\eta;T_{(0,\alpha)}\omega)=\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+}(x_1,\xi+\alpha;y_1,\eta+\alpha;\omega), \tag{2.79}$$

i.e., the equality is valid for every $(x_1,y_1)\in\mathbb{Z}_+\times\ \mathbb{Z}_+$ and all $\omega\in\Omega_{x_1,y_1},\ \mathbf{P}(\Omega_{x_1,y_1})=1$. Since the set of values of $(x_1,y_1)\in\mathbb{Z}_+\times\ \mathbb{Z}_+$ is countable, the equality is valid for all $(x_1,y_1)\in\mathbb{Z}_+\times\ \mathbb{Z}_+$ on the set of event $\Omega_0=\cap_{(x_1,y_1)\in\mathbb{Z}_+\times\ \mathbb{Z}_+}\Omega_{x_1,y_1},\ \mathbf{P}(\Omega_0)=1$, i.e., the matrix

$$\{(\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+}(x_1,\xi;y_1,\eta)\}_{x_1,y_1\mathbb{Z}_-,\xi,\eta\in\mathbb{Z}^{d-1}}$$

determines a random operator in $l^2(\mathbb{Z}^d_+)$, which is ergodic with respect to the "transversal" coordinates $(\xi,\eta)\in\mathbb{Z}^{d-1}\times\ \mathbb{Z}^{d-1}$. It follows then from an extended version of Theorem 2.7 of [23] that the matrix (2.77) determines the operator $h_0(\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+})$, which has the same property. In particular, since T_a is a measure preserving transformation of the event space, $\mathbf{E}\{(h_0(\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+}))(x_1,\xi;x_1,\xi)\}$ does not depend on $\xi\in\mathbb{Z}^{d-1}$ in view of (2.79), hence the first term on the right of (2.78) is

$$\sum_{x_1\in(-L,0]}\mathbf{E}\{(h_0(\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+}))(x_1,0;x_1,0)\}. \tag{2.80}$$

We are left then to find the limit of the above expression as $L\to\infty$. Since the terms of the sum are non-negative, it suffices to show that the sum is bounded uniformly in $L\to\infty$.We will use the bounds $0\le h_0(x)\le 2x^{1/2}$ (see Lemma 3.1 (iii)) and the Peierls inequality for $f(x)=2x^{1/2}$ (see Lemma 3.3 (i)) to obtain

$$\begin{aligned} (h_0(\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+}))(x_1,0;x_1,0) &\le 2\left(\Pi^{1/2}_{\mathbb{Z}^d_-,\mathbb{N}^d_+}\right)(x_1,0;x_1,0) \\ &\le 2\left(\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+}(x_1,0;x_1,0)\right)^{1/2} \end{aligned}$$

and then the Schwarz inequality for expectations, (2.17), the bound,

$$|P(x,y)|\le 1 \tag{2.81}$$

valid for any projection in $l^2(\mathcal{C})$, and (2.16) yield

$$\begin{aligned} &\mathbf{E}\{(h_0(\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+}))(x_1,0;x_1,0)\}\ \le 2\mathbf{E}^{1/2}\{\Pi_{\mathbb{Z}^d_-,\mathbb{N}^d_+}(x_1,0;x_1,0)\} \\ &= 2\mathbf{E}^{1/2}\Big\{\sum_{z\in\mathbb{N}^d_+}|P(x_1,0;z)|^2\Big\}\le 2\Big(C_0\sum_{z_1\le 1,\zeta\in\mathbb{Z}^{d-1}}e^{-\gamma(z_1-x_1)-\gamma|\zeta|}\Big)^{1/2}\le Ce^{\gamma x_{1/2}}, \end{aligned}$$

guarantying evidently the uniform in $L = 2M + 1 \to \infty$ boundedness of (2.80).

In view of the above it suffices to justify (2.78), i.e., the replacement of $h_0(\Pi_{\Lambda^*,\Lambda_1^*})$ in (2.74) by the restriction $\pi_{\Lambda^*} h_0(\Pi_{\mathbb{Z}_-^d,\mathbb{N}_+^d})\pi_{\Lambda^*}$ of $h_0(\Pi_{\mathbb{Z}_-^d,\mathbb{N}_+^d})$ to Λ^* of (2.73) and we write here and below $\pi_{\mathcal{C}}$ for the orthogonal projection onto the subspace $l^2(\mathcal{C}) \subset l^2(\mathbb{Z}_-^d)$.

This will be carried out in two steps: (i) $\Pi_{\Lambda^*,\Lambda_1^*} \to \Pi_{\Lambda^*,\mathbb{N}_+^d}$ and (ii) $\Pi_{\Lambda^*,\mathbb{N}_+^d} \to \Pi_{\mathbb{Z}_-^d,\mathbb{N}_+^d}$.

(i) The replacement

$$\Pi_{\Lambda^*,\Lambda_1^*} \to \Pi_{\Lambda^*,\mathbb{N}_+^d}.$$

We will use Lemma 3.2 (ii) with $A_1 = \Pi_{\Lambda^*,\mathbb{N}_+^d}$ and $A_2 = \Pi_{\Lambda^*,\Lambda_1^*}$. To check the first of conditions (3.3), we note first that $(\mathrm{Tr}\,(A_1^{1/4} + A_2^{1/4}))^2 \le 2(\mathrm{Tr}\,A_1^{1/4})^2 + 2(\mathrm{Tr}\,(A_2^{1/4})^2$, hence we have to prove that uniformly in $L \to \infty$

$$\mathbf{E}\{(L^{-(d-1)}\mathrm{Tr}\,(\Pi_{\Lambda^*,\Lambda_1^*})^{1/4})^2\} < \infty, \quad \mathbf{E}\{(L^{-(d-1)}\mathrm{Tr}\,(\Pi_{\Lambda^*,\mathbb{N}_+^d})^{1/4})^2\} < \infty.$$

The both bounds follow from Lemma 3.5 (ii) (cf. (2.36) and (2.64)). Thus, let us check the second bound in (3.3). Since $\Lambda_1^* \subset \mathbb{N}_+^d$, have from Lemma 3.4 (v)

$$\Pi_{\Lambda^*,\mathbb{N}_+^d} - \Pi_{\Lambda^*,\Lambda_1^*} = \Pi_{\Lambda^*,\mathbb{N}_+^d \setminus \Lambda_1^*} \ge 0$$

and in view of Lemma 3.2 (ii) we have to determine the order of magnitude of ε in $L \to \infty$ from the expression

$$L^{-(d-1)} \sum_{x \in \Lambda^*} \mathbf{E}\{\ \mathrm{Tr}\,\Pi_{\Lambda^*,\mathbb{N}_+^d \setminus \Lambda_1^*}\} = O(\varepsilon).$$

We have from Lemma 3.5 (ii) and (3.29) the following bound for the l.h.s. of the expression:

$$C_0 L^{-(d-1)} \sum_{x \in \Lambda^*} \sum_{z \in \mathbb{N}_+^d \setminus \Lambda_1^*} e^{-\gamma|x-z|} \le C L^{-1}.$$

Indded, the bound follows from the first inequality of (3.29), if we apply it twice: first in the dimension d and then in the dimension $d-1$. Thus, $\varepsilon = O(L^{-1})$ and Lemma 3.2 (ii) yields the asymptotic relation

$$\begin{aligned} &L^{-(d-1)}\mathbf{E}\{\mathrm{Tr}\,h_0(\Pi_{\Lambda^*,\Lambda_1^*})\} \\ &= L^{-(d-1)}\mathbf{E}\{\mathrm{Tr}\,h_0(\Pi_{\Lambda^*,\mathbb{N}_+^d})\} + O(L^{-1/11}),\ L \to \infty \end{aligned} \tag{2.82}$$

justifying the first step of replacement $\Pi_{\Lambda^*,\Lambda_1^*} \to \Pi_{\Lambda^*,\mathbb{N}_+^d}$ in (2.74).

(ii) The replacement $\Pi_{\Lambda^*,\mathbb{N}_+^d} \to \Pi_{\mathbb{Z}_-^d,\mathbb{N}_+^d}$. We have to prove that

$$L^{-(d-1)}\mathbf{E}\{\mathrm{Tr}\,(\pi_{\Lambda^*} h_0(\Pi_{\Lambda^*,\mathbb{N}_+})\pi_{\Lambda^*} - \pi_{\Lambda^*} h_0(\Pi_{\mathbb{Z}_-^d,\mathbb{N}_+})\pi_{\Lambda^*})\} = o(1),\ L \to \infty, \tag{2.83}$$

where we inserted the projection π_{Λ^*} in the first term on the left to make the subsequent argument more transparent. It is convenient to introduce the operator

$$\Pi_{\overline{\Lambda^*},\mathbb{N}_+},\ \ \overline{\Lambda^*} = \mathbb{Z}_-^d \setminus \Lambda^* \tag{2.84}$$

acting in $l^2(\overline{\Lambda^*})$ and the block operator $\Pi_{\Lambda^*,\mathbb{N}_+^d} \oplus \Pi_{\overline{\Lambda^*},\mathbb{N}_+^d}$ acting in $l^2(\mathbb{Z}_-^d)$. Since

$$\mathrm{Tr}\,\pi_{\Lambda^*} h_0(\Pi_{\Lambda^*,\mathbb{N}_+^d})\pi_{\Lambda^*} = \mathrm{Tr}\,\pi_{\Lambda^*} h_0(\Pi_{\Lambda^*,\mathbb{N}_+^d} \oplus \Pi_{\overline{\Lambda^*},\mathbb{N}_+^d})\pi_{\Lambda^*}$$

we can write (2.83) as

$$L^{-(d-1)}\mathbf{E}\{\mathrm{Tr}\,(\pi_{\Lambda^*}h_0(\Pi_{\Lambda^*,\mathbb{N}_+}\oplus\Pi_{\overline{\Lambda^*},\mathbb{N}_+^d})\pi_{\Lambda^*}-\pi_{\Lambda^*}h_0(\Pi_{\mathbb{Z}_-^d,\mathbb{N}_+})\pi_{\Lambda^*})\}=o(1),\ L\to\infty,\quad (2.85)$$

Relations analogous to (2.83) but without the projection π_{Λ^*} have been proved above by using Lemma 3.2 (ii), see e.g. (2.41). Here the lemma, as it was formulated, is not applicable since the involved operators are infinite dimensional. However it is easy to check that the l.h.s. of (2.85) can be estimated by using an analog of the lemma valid under the conditions (cf. (3.3))

$$L^{-2(d-1)}\mathbf{E}\{(\mathrm{Tr}\,\Pi_{\Lambda^*,\mathbb{N}_+}^{1/4})^2\}=L^{-2(d-1)}\mathbf{E}\{(\mathrm{Tr}\,\pi_{\Lambda^*}\Pi_{\Lambda^*,\mathbb{N}_+}^{1/4}\oplus\Pi_{\overline{\Lambda^*},\mathbb{N}_+^d}^{1/4}\pi_{\Lambda^*})^2\}<\infty,\quad (2.86)$$
$$L^{-2(d-1)}\mathbf{E}\{(\mathrm{Tr}\,\pi_{\Lambda^*}\Pi_{\mathbb{Z}_-^d,\mathbb{N}_+^d}^{1/4}\pi_{\Lambda^*})^2\}<\infty,$$

and

$$\mathbf{E}\Big\{L^{-(d-1)}\mathrm{Tr}\,\big|R\big|\Big\}=O(\varepsilon(L)),\ R=\Pi_{\mathbb{Z}_-^d,\mathbb{N}_+^d}-\Pi_{\Lambda^*,\mathbb{N}_+^d}\oplus\Pi_{\overline{\Lambda^*},\mathbb{N}_+^d}),\ L\to\infty.\quad (2.87)$$

Conditions (2.86) are analogous to those verified several time above (see (2.36) and (2.64)). Let us find $\varepsilon(L)$ from (2.87). By (3.26) and (3.31) with $l=1$, $L_1=L$

$$L^{-(d-1)}\mathbf{E}\{\mathrm{Tr}\,|R|\}\le AL^{-(d-1)}\sum_{x\in\Lambda^*}\sum_{y\in\overline{\Lambda^*}}\sum_{z\in\mathbb{N}_+^d}e^{-\gamma(|x-z|+|y-z|)/2}\le CL^{-(d-1)}L^{d-2}=CL^{-1}.$$

We conclude that ε in (2.85) is $\varepsilon=O(L^{-1})$ and according to Lemma 3.2 (ii)

$$L^{-(d-1)}\mathbf{E}\Big\{\Big|\mathrm{Tr}\,\pi_{\Lambda^*}\Big(h_0(\Pi_{\Lambda^*,\mathbb{N}_+})-h_0(\Pi_{\mathbb{Z}_-^d,\mathbb{N}_+})\Big)\pi_{\Lambda^*}\Big|\Big\}=O(L^{-1/11}).\quad (2.88)$$

Thus, we have justified the replacement of $\Pi_{\Lambda^*,\mathbb{N}_+^d}$ by $\Pi_{\mathbb{Z}_-^d,\mathbb{N}_+^d}$ in (2.74). This, (2.10) and (2.22) prove the theorem. ■

2.3.2 Variance

The above result on the limit of the entanglement entropy per unit area, hence for the validity of the area law in the mean, is valid for any ergodic orthogonal projection P of (2.22) satisfying the exponential bound (2.16) and even a weaker power law bound, guarantying the validity of the above proofs. Confining ourselves to the spectral projections (2.8) of the Schrödinger operator with ergodic potentials, we note that the most studied class of the operators for which (2.16) holds consists of operators with i.i.d. potential having a regular common probability distribution, more generally, potentials with sufficiently fast decay of correlations [1, 3, 12, 31]. However, the bound (2.16) holds also for the one dimensional Schrödinger operators with quasiperiodic potential (see, e.g., [17] for the corresponding proofs and references), which have, so to say, a minimum amount of randomness. Thus, the sufficiently fast decay of correlations of ergodic potential is not necessary for the sufficiently fast decay of entries of ergodic projections, hence for the validity of Theorems 2.2, 2.4 and (2.6).

On the other hand, the next theorem on the decay, although rather weak, of the variance of the entanglement entropy per unit area as $L\to\infty$ is proved for i.i.d. random potentials. We believe that our bound on the decay of variance is not optimal and that this and more strong bounds are also valid for random potentials with sufficiently fast decaying correlations.

Theorem 2.8 *Let P be the spectral projection (2.8) corresponding to a spectral interval I of the Schrödinger operator (2.3) in $l^2(\mathbb{Z}^d)$ with i.i.d. random potential such that its common probability distribution F is the Hölder continuous: $F((a-\varepsilon, a+\varepsilon)) \le A|\varepsilon|^s$, $a \in \mathrm{supp}F$, $\varepsilon > 0$, $s \in (0,1]$. Let S_Λ be the corresponding entanglement entropy (2.10). Assume that the exponential bound (2.16) is valid on I. Then there exists L-independent $C < \infty$ providing the bound*

$$\mathbf{Var}\{L^{-(d-1)}S_\Lambda\} := \mathbf{E}\{(L^{-(d-1)}S_\Lambda)^2\} - (\mathbf{E}\{L^{-(d-1)}S_\Lambda\})^2 \le CL^{-1/11}. \tag{2.89}$$

Proof. The scheme of the proof is as follows. We present the entanglement entropy as a sum of sufficiently large number of independent random variables modulo error terms vanishing as $L \to \infty$ at least as fast as $L^{-1/11}$ and use then the additivity of the variance of a collection of independent random variables choosing their number to guaranty the order $O(L^{-1/11})$ for the sum. Since, however, the variance is not simple to estimate, we will often use its rather rough bounds via the expectations of squares of the corresponding random variables, in particular those obtained above. This is why our final bound (2.89) is $O(L^{-1/11})$, cf. (2.70) and (2.82).

Here is the simple inequality, which we will often use below:

$$\mathbf{Var}\{\xi_1\} \le 2\mathbf{Var}\{\xi_2\} + 2\mathbf{E}\{|\xi_1 - \xi_2|^2\}. \tag{2.90}$$

Using this inequality with

$$\xi_1 = L^{-(d-1)}S_\Lambda, \quad \xi_2 = \sum_{j=1}^{2d} L^{-(d-1)} \mathrm{Tr}\, h_0\left(\Pi_{\Lambda,\Lambda_j}\right).$$

and taking into account (2.67) and (2.69), we get

$$\mathbf{Var}\{L^{-(d-1)}S_\Lambda\} \le 2\mathbf{Var}\{\sum_{j=1}^{2d} L^{-(d-1)}\mathrm{Tr}\ h_0(\Pi_{\Lambda,\Lambda_j})\} + O(L^{-1/5}\log^2 1/L^2). \tag{2.91}$$

We will use now the general inequality

$$\mathbf{Var}\Big\{\sum_{j=1}^{p}\xi_j\Big\} \le \left(\sum_{j=1}^{p}\mathbf{Var}^{1/2}\{\xi_j\}\right)^2$$

with $\xi_j = L^{-(d-1)}\mathrm{Tr}\ h_0(\Pi_{\Lambda,\Lambda_j})$ and $p = 2d$ to obtain instead of (2.91)

$$\mathbf{Var}\Big\{\sum_{j=1}^{2d} L^{-(d-1)}\mathrm{Tr}\, h_0(\Pi_{\Lambda,\Lambda_j})\Big\} \le \left(\sum_{j=1}^{p}\mathbf{Var}^{1/2}\{L^{-(d-1)}\mathrm{Tr}\, h_0(\Pi_{\Lambda,\Lambda_j})\}\right)^2.$$

Recall now that we are dealing in this section with a i.i.d. random potential, hence satisfying (2.57) and (2.58). This, (2.17) and the above inequality allow us to write instead of (2.91)

$$\mathbf{Var}\left\{L^{-(d-1)}S_\Lambda\right\} \le 8d^2\ \mathbf{Var}\left\{L^{-(d-1)}\mathrm{Tr}\, h_0\left(\Pi_{\Lambda,\Lambda_1}\right)\right\} + O(L^{-1/5}\log L). \tag{2.92}$$

We conclude that the problem on the variance of the entanglement entropy reduces to that on $\mathbf{Var}\{L^{-1}\mathrm{Tr}\, h_0\left(\Pi_{\Lambda,\Lambda_1}\right)\}$.

To proceed it is convenient, again as in the case of expectations, to pass from the cube Λ of (2.19) and the semi-infinite parallelepiped Λ_1 of (2.72) to the shifted cube Λ^* and parallelepiped Λ_1^* of (2.73). Choose $l = [\log^2 L]$ and split the cubic face $\Lambda^{(0)}$ as

$$\Lambda^{(0)} = \Lambda_1^{(0)} \cup \Lambda_2^{(0)}, \quad \Lambda_1^{(0)} \cap \Lambda_2^{(0)} = \emptyset,$$

where

$$\Lambda_1^{(0)} = \bigcup_{k=1}^{m} \Lambda_{1k}^{(0)}, \quad m = O\left(\left(L/(L_1 + l)\right)^{(d-1)}\right),$$

with $\Lambda_{1k}^{(0)},\ k = 1, ..., m$ being $(d-1)$-dimensional cubes of side length L_1 to be chosen later, separated by corridors of width l, and $\Lambda_2^{(0)} = \Lambda_1^{(0)} \setminus \Lambda_{1,0}^{(0)}$ is the set of all corridors between the cubes $\Lambda_{1k}^{(0)}$'s. Denote Λ_1' and Λ_2' the cylindrical sets adjacent to $\Lambda_1^{(0)}$ and $\Lambda_2^{(0)}$ from the exterior of Λ^*. Then evidently

$$\Pi_{\Lambda^*,\Lambda_1^*} = \Pi_{\Lambda^*,\Lambda_1'} + \Pi_{\Lambda^*,\Lambda_2'}.$$

It follows from (3.27) and (3.29) with $A_1 = \Pi_{\Lambda^*,\Lambda_1^*}$ and $A_2 = \Pi_{\Lambda^*,\Lambda_1'}$ that

$$\begin{aligned} &L^{-(d-1)}\mathbf{E}\{\mathrm{Tr}\,|\Pi_{\Lambda^*,\Lambda_1^*} - \Pi_{\Lambda^*,\Lambda_1'}|\} = L^{-(d-1)}\mathbf{E}\{\mathrm{Tr}\,\Pi_{\Lambda^*,\Lambda_2'}\} \\ &\le C \sum_{x\in\Lambda^*, y\in\Lambda_2'} e^{-\gamma|x-y|} \le CL^{-(d-1)}|\Lambda_2| \le Cl/L_1. \end{aligned} \tag{2.93}$$

Similarly, denote Λ_1'' and Λ_2'' the cylindrical sets adjacent to $\Lambda_1^{(0)}$ and $\Lambda_2^{(0)}$ from the interior of Λ^* and use Lemma 3.5 (iii) and (3.30) to obtain

$$\begin{aligned} &L^{-(d-1)}\mathbf{E}\{\mathrm{Tr}\,|\Pi_{\Lambda^*,\Lambda_1'} - \Pi_{\Lambda_1'',\Lambda_1'}|\} \le C \sum_{z\in\Lambda_1'} \sum_{x\in\Lambda^*, y\in\Lambda_2''} e^{-\gamma|x-z|-\gamma|y-z|} \\ &\le CL^{-(d-1)}|\Lambda_2| \le Cl/L_1. \end{aligned} \tag{2.94}$$

Let $\Lambda_{1k}'',\ k = 1, ..., m$ be the cylindrical sets adjacent to $\Lambda_{1k}^{(0)}$ from the interior of Λ^*. Since $\Lambda_1'' = \cup_{k=1}^m \Lambda_{1k}''$, one can consider the matrix $\Pi_{\Lambda_1'',\Lambda_1'}$ as a block matrix with the diagonal part D consisting of m blocks $D_k = \{\Pi_{\Lambda_1'',\Lambda_1'}(x, y)\}_{x,y\in\Lambda_{1k}''},\ k = 1, ..., m$ and the off diagonal part $R = \Pi_{\Lambda_1'',\Lambda_1'} - D$. Then Lemma 3.5 (iii) and (3.31) yield

$$\begin{aligned} &L^{-(d-1)}E\{\mathrm{Tr}\,|\Pi_{\Lambda_1'',\Lambda_1'} - D|\} \le C \sum_{z\in\Lambda_1'} \sum_{j\neq k} \sum_{x\in\Lambda_{1j}'', y\in\Lambda_{1k}''} e^{-\gamma|x-z|-\gamma|y-z|} \\ &\le Cm^2 e^{-\gamma l/2} \le Ce^{-\gamma[l/d]/4}. \end{aligned} \tag{2.95}$$

Combining the bound (2.93) – (2.95) with Lemma 3.2 (ii) and (2.90), we get

$$\begin{aligned} &\mathbf{Var}\{L^{-(d-1)}\mathrm{Tr}\,h_0\left(\Pi_{\Lambda,\Lambda_1}\right)\} = \mathbf{Var}\{L^{-(d-1)}\mathrm{Tr}\,h_0\left(\Pi_{\Lambda^*,\Lambda_1^*}\right)\} \\ &\le 2\mathbf{Var}\{L^{-(d-1)}\mathrm{Tr}\,h_0\left(D\right)\} + O((l/L_1)^{1/5}) + O(e^{-\gamma l/3}). \end{aligned} \tag{2.96}$$

Since D is block diagonal, we can write

$$\mathrm{Tr}\,h_0\left(D\right) = \sum_{k=1}^{m} \mathrm{Tr}\,h_0\left(D_k\right)$$

and if $\Lambda'_{1k},\ k=1,...,m$ are the cylindrical sets adjacent to $\Lambda^{(0)}_{1k}$ from the exterior of Λ^*, then

$$\mathbf{E}\{\mathrm{Tr}\,|D_k-\Pi_{\Lambda''_{1k},\Lambda'_{1k}}|\}=\mathbf{E}\{\mathrm{Tr}\,|\Pi_{\Lambda''_{1k},\Lambda'_{1}}-\Pi_{\Lambda''_{1k},\Lambda'_{1k}}|\}\le\sum_{x\in\Lambda''_{1k}}\sum_{y\in\Lambda'_1\setminus\Lambda'_{1k}}e^{-\gamma|x-y|}\le Ce^{-\gamma l/3}.$$

The last two bounds and (2.96) yield finally

$$\mathbf{Var}\{L^{-(d-1)}\mathrm{Tr}\,h_0\,(\Pi_{\Lambda,\Lambda_1})\}\le 2\mathbf{Var}\left\{L^{-(d-1)}\sum_{k=1}^{m}\mathrm{Tr}\,h_0\left(\Pi_{\Lambda''_{1k},\Lambda'_{1k}}\right)\right\}+O(l/L_1). \tag{2.97}$$

Let $\widetilde{\Lambda}_{1k},\ k=1,...,m$ be the cylinders such that each of them contains the cylinder $\Lambda'_{1k}\cup\Lambda''_{1k}$ and the distance between them is $l/3$. Let $H_{\widetilde{\Lambda_k}}$ be the restriction to $\widetilde{\Lambda_k}$ of the discrete Schrödinger operator H of (2.3), i.e., the "Dirichlet" Schrödinger operator in $l^2(\widetilde{\Lambda_k})$. We will replace in $\Pi_{\Lambda''_{1k},\Lambda'_{1k}}$ of above formula the spectral projection P (2.8) of H by the spectral projection $P_{\widetilde{\Lambda}_{1k}}$ of $H_{\widetilde{\Lambda}_{1k}}$ corresponding the same spectral interval I. It is important that since the random potential (2.5) in H consists of i.i.d. random variables and the distance between the different $\widetilde{\Lambda_{1k}}$ is $l/3$, the operators $H_{\widetilde{\Lambda_k}},\ k=1,..,m$ and hence their spectral projections $P_{\widetilde{\Lambda}_{1k}},\ k=1,...,m$ are independent. Introduce the operators (cf. (2.17))

$$\widetilde{\Pi_k}=\{\widetilde{\Pi_k}(x,y)\}_{x,y\in\Lambda_{1k}},\ k=1,...,m,\ \ \widetilde{\Pi_k}(x,y)=\sum_{z\in\Lambda'_{1k_1}}P_{\widetilde{\Lambda}_{1k}}(x,z)P_{\widetilde{\Lambda}_{1k}}(z,y), \tag{2.98}$$

then by Lemmas 3.6 and (3.4)

$$\mathbf{E}\Big\{L^{-2(d-1)}\big(\mathrm{Tr}\,h_0\left(\Pi_{\Lambda''_{1k},\Lambda'_{1k}}\right)-\mathrm{Tr}\,h_0(\widetilde{\Pi_k})\big)^2\Big\}\le Ce^{-cl}. \tag{2.99}$$

Bounds (2.97) – (2.99) combined with (2.90) yield

$$\mathbf{Var}\{L^{-(d-1)}\mathrm{Tr}\,h_0\,(\Pi_{\Lambda,\Lambda_1})\}\le 2\mathbf{Var}\{L^{-(d-1)}\sum_{k=1}^{m}\mathrm{Tr}\,h_0(\widetilde{\Pi_k})\}+O(l/L_1). \tag{2.100}$$

We will now take into account that by construction (see (2.98)) $\{\mathrm{Tr}\,h_0(\widetilde{\Pi}_k)\}_{k=1}^m$ are independent random variables, thus

$$\mathbf{Var}\Big\{L^{-(d-1)}\sum_{k=1}^{m}\mathrm{Tr}\,h_0(\widetilde{\Pi_k})\Big\}=L^{-(d-1)}\sum_{k=1}^{m}\mathbf{Var}\Big\{\mathrm{Tr}\,h_0(\widetilde{\Pi_k})\Big\}\le C(L_1/L)^{d-1}. \tag{2.101}$$

Choosing here $L_1=L^{5/(5d-4)}$ and using (2.92), (2.100) and (2.101), we obtain (2.89). ■

3 Auxiliary Results

In this section we present technical results, which we use to prove the above theorems. First is an elementary facts on the function h_0 of (2.20).

Lemma 3.1 *Let the functions $h : [0,1] \to [0,1]$ and $h_0 : [0,1/4] \to [0,1]$ be defined by (2.7). Then h_0:*

(i) is nonnegative, monotone and convex:

(ii) admits the representation: $h_0(x) = \sqrt{x}\varphi(x)$, where $0 \le \varphi(x \le Cx^{1/4}$;

(iii) satisfies the bound: $4x \le h_0(x) \le 2\sqrt{x}$.

We pass now to the main technical result of the paper.

Lemma 3.2 *Let Λ be a parallelepiped in $\mathbb{Z}^d$ with sides of lengths $L_1, ..., L_d$ and parallel to the coordinate axes and $A_1, ..., A_p$ be positive definite random operators acting in the $l_2(\Lambda)$ such that $0 \le A_1 + ... + A_q \le 1/4$, $q = 1, ..., p$. We have*

(i) if $L^{(d-1)} := L_2...L_d$, uniformly in $L \to \infty$,

$$\mathbf{E}\big\{(L^{-(d-1)}\mathrm{Tr}\,(\sum_{j=1}^{p} A_j)^{1/4})^2\big\} \le C_1 < \infty, \tag{3.1}$$

$$\mathbf{E}\{(L^{-(d-1)}\mathrm{Tr}\,A_jA_k)^2\} \le \varepsilon^2(L),\ 1 \le j < k \le p$$

and h_0 is defined by (2.20), then there exists $C \in (0,\infty)$, depending only on C_1 and p and such that

$$\mathbf{E}\Big\{\Big(L^{-(d-1)}\mathrm{Tr}\Big(h_0\Big(\sum_{j=1}^{p} A_j\Big) - \sum_{j=1}^{p} h_0(A_j)\Big)\Big)^2\Big\} \le C_1\varepsilon^{1/5}\log^2 1/\varepsilon. \tag{3.2}$$

(ii) If $0 \le A_1, A_2 \le \frac{1}{4}$ and uniformly in $L \to \infty$

$$\mathbf{E}\Big\{(L^{-(d-1)}\mathrm{Tr}\,(A_1^{1/4} + A_2^{1/4}))^2\Big\} \le C_1 < \infty, \quad L^{-(d-1)}\mathbf{E}\{\mathrm{Tr}\,|A_1 - A_2|\} \le \varepsilon(L), \tag{3.3}$$

where $|A| := (A^2)^{1/2}$, then there exists $C \in (0,\infty)$, depending only on C_1 and such that

$$\mathbf{E}\big\{(L^{-(d-1)}\mathrm{Tr}\,\big(h_0(A_1) - h_0(A_2)\big))^2\big\} \le C\varepsilon^{1/5}\log^2 1/\varepsilon. \tag{3.4}$$

Proof. (i). Let us prove first the case $p = 2$ of the assertion. Write

$$h_0(\lambda) = r(\lambda)\varphi(\lambda), \tag{3.5}$$

where

$$r(\lambda) = \sqrt{\lambda}\,\mathbf{1}_{[0,1]}(\lambda) \tag{3.6}$$

and

$$\varphi(\lambda) = (h_0(\lambda)/\sqrt{\lambda})\,1_{[0,1/4]}(\lambda). \tag{3.7}$$

Let us prove first that

$$L^{-2(d-1)}\mathbf{E}\big\{\mathrm{Tr}^{\,2}\big(r(A_1 + A_2) - r((\sqrt{A_1} + \sqrt{A_2})^2)\big)\varphi(A_1 + A_2)\big\} = O(\varepsilon^{-1/5}). \tag{3.8}$$

Set $\eta = \varepsilon^{-1/5}$ and consider the convolution of r

$$r_\eta = r * \mathcal{P}_\eta, \tag{3.9}$$

and the Poisson kernel.

$$\mathcal{P}_\eta = \pi^{-1}\frac{\eta}{\lambda^2 + \eta^2}.$$

Then

$$\sup_{\lambda\in[0,1/2]}|r_\eta(\lambda)-r(\lambda)|\le C\eta^{1/2}$$

and

$$\|r_\eta(A_1+A_2)-r(A_1+A_2)\|\le C\eta^{1/2}.$$

This and the inequality

$$|\mathrm{Tr}\, M_1M_2|\ \le ||M_1||\mathrm{Tr}\, M_2, \tag{3.10}$$

valid for a hermitian M_1 and a positive definite M_2, yield

$$\begin{aligned} L^{-2(d-1)}\mathbf{E}\Big\{(\mathrm{Tr}\,(r_\eta(A_1+A_2)-r(A_1+A_2))\varphi(A_1+A_2))^2\Big\} \\ \le C\eta L^{-2(d-1)}\mathbf{E}\big\{(\mathrm{Tr}\,\varphi(A_1+A_2))^2\big\}\le C_1\eta. \end{aligned} \tag{3.11}$$

Here in the last inequality we use the first condition of (3.1) combined with the bounds

$$\varphi(\lambda)\le C|\lambda|^{1/4}. \tag{3.12}$$

Similarly

$$L^{-2(d-1)}\mathbf{E}\Big\{\Big(\mathrm{Tr}\,\Big(\Big|r_\eta\big((\sqrt{A_1}+\sqrt{A_2})^2\big)-r\big((\sqrt{A_1}+\sqrt{A_2})^2\big)\Big|\varphi(A_1+A_2)\Big)\Big)^2\Big\}\le C\eta.$$

We will use now the resolvent identity

$$G_1(z)-G_2(z)=G_1(z)(M_2-M_1)G_2(z), \tag{3.13}$$

where

$$G_{1,2}(z)=(M_{1,2}-z)^{-1} \tag{3.14}$$

are the resolvents of hermitian matrices $M_{1,2}$. The identity and the spectral theorem for hermitian matrices yield

$$\begin{aligned} r_\eta(M_1)-r_\eta(M_2)&=\int_0^{1/4}d\lambda r(\lambda)\Im(G_1(\lambda+i\eta)-G_2(\lambda+i\eta)) \\ &=\int_0^{1/4}d\lambda r(\lambda)\Im(G_1(\lambda+i\eta)(M_2-M_1)G_2(\lambda+i\eta)). \end{aligned} \tag{3.15}$$

Hence, taking $M_1=A_1+A_2$ and $M_2=\left(\sqrt{A_1}+\sqrt{A_2}\right)^2$, we get

$$\begin{aligned} &L^{-(d-1)}|\mathrm{Tr}\,(r_\eta(A_1+A_2)-r_\eta((\sqrt{A_1}+\sqrt{A_2})))\varphi(A_1+A_2)| \\ &\le C\eta^{-2}L^{-(d-1)}\mathrm{Tr}^{\,1/2}(A_1A_2)\,\mathrm{Tr}^{\,1/2}\varphi^2(A_1+A_2), \end{aligned} \tag{3.16}$$

where we used the Schwarz inequality for traces, (3.10) and the bound

$$\|G(z)\|\le|\Im z|^{-1} \tag{3.17}$$

valid for the resolvent of a hermitian matrix.

The bound (3.16) combined with the Schwarz inequality for expectation, (3.11) and $\eta = \varepsilon^{1/5}$ yields (3.8).

Now let us prove that

$$L^{-2(d-1)}\mathbf{E}\{\mathrm{Tr}^2\sqrt{A_1}\big(\varphi(A_1+A_2)-\varphi(A_1)\big)\} \le C\varepsilon^{1/5}\log^2\varepsilon^{-1}, \qquad (3.18)$$
$$L^{-2(d-1)}\mathbf{E}\{\mathrm{Tr}^2\sqrt{A_2}\big(\varphi(A_1+A_2)-\varphi(A_2)\big)\} \le C\varepsilon^{1/5}\log^2\varepsilon^{-1}.$$

Similarly to (3.9) we introduce the convolution

$$\varphi_\eta = \varphi * \mathcal{P}_\eta, \quad \eta = \varepsilon^{1/5},$$

satisfying (see (2.7) and (3.6))

$$\sup_{0\le\lambda\le1} |\varphi_\eta(\lambda)-\varphi(\lambda)| \le C\eta^{1/2}\log\eta^{-1}.$$

Then, an argument similar to that proving (3.11) yields

$$\begin{aligned} &L^{-2(d-1)}\mathbf{E}\{\mathrm{Tr}^2\sqrt{A_1}\big(\varphi(A_1+A_2)-\varphi_\eta(A_1+A_2)\big)\} \\ &\qquad \le C\eta\log^2\eta^{-1}L^{-2(d-1)}\mathbf{E}\{\mathrm{Tr}^2\sqrt{A_1}\} \le C\eta\log^2\eta^{-1}. \end{aligned} \qquad (3.19)$$

We can also write an analog of (3.15):

$$\begin{aligned} &L^{-(d-1)}\mathrm{Tr}\sqrt{A}\big(\varphi_\eta(A_1+A_2)-\varphi_\eta(A_1)\big) \\ &= L^{-(d-1)}\int_0^{1/4} d\lambda\varphi(\lambda)\mathrm{Tr}\sqrt{A_1}G_1(\lambda+i\eta)A_2G_2(\lambda+i\eta), \end{aligned}$$

where now

$$G_1(z) = (A_1-z)^{-1}, \quad G_2(z) = (A_1+A_2-z)^{-1}.$$

Since $G_1(z)$ and $A_1^{1/2}$ commute, we have by (3.10), (3.17) and with $\eta = \varepsilon^{1/5}$

$$\begin{aligned} &L^{-2(d-1)}\mathbf{E}\{(\mathrm{Tr}\,A_1^{1/2}G_1A_2G_2)^2\} \\ &= L^{-2(d-1)}\mathbf{E}\Big\{\big|(\mathrm{Tr}\,A_1^{1/2}A_2^{1/2}A_2^{1/2}G_2G_1)^2\big|\Big\} \le C\eta^{-4}L^{-2(d-1)}\mathbf{E}\{\mathrm{Tr}\,A_1A_2\mathrm{Tr}\,A_2\} \\ &\le C\eta^{-4}L^{-2(d-1)}\mathbf{E}^{1/2}\{(\mathrm{Tr}\,A_1A_2)^2\}\mathbf{E}^{1/2}\{(\mathrm{Tr}\,A_2)^2\} \le C\eta^{-4}\varepsilon = C\varepsilon^{1/5}. \end{aligned}$$

The above inequality combined with (3.19) implies the first inequality of (3.18). The second one can be obtained similarly. Combining (3.18) with (3.8) we get (3.2).

We have proved the case $p=2$ of assertion (i) of the lemma. The case of an arbitrary $p \ge 2$ follows from the above by induction in p.

(ii). The proof is similar to that of assertion (i). First we prove the inequality

$$L^{-2(d-1)}\mathbf{E}\{(\mathrm{Tr}\,\big(r(A_1)-r(A_2)\big)\varphi(A))^2\} \le C\varepsilon^{1/5}\log^2\varepsilon^{-1}, \qquad (3.20)$$

and then

$$L^{-2(d-1)}\mathbf{E}\{(\mathrm{Tr}\,r(A_2)\big(\varphi(A_1)-\varphi(A_2)\big))^2\} \le C\varepsilon^{1/5}\log^2\varepsilon^{-1}. \qquad (3.21)$$

■

The next three lemmas are useful to check the conditions of the previous lemma.

Lemma 3.3 *Let M be an $n \times n$ hermitian matrix and $f : \mathbb{R} \to \mathbb{R}$ be a convex function. We have:*

(i) for any vector $e \in \mathbb{C}^n$ of norm 1

$$(f(M)e, e) \leq f((Me, e)); \tag{3.22}$$

(ii) for any orthonormal basis $\{e_j\}_{j=1}^n$ in $\mathbb{C}^n$

$$\mathrm{Tr} f(M) \leq \sum_{j=1}^{n} f(M_{jj}),\ M_{jj} = (Me_j, e_j) \tag{3.23}$$

Proof. According to the spectral theorem for hermitian matrices

$$(f(M)e, e) = \int_{-\infty}^{\infty} f(\lambda)\mu_e(d\lambda),$$

where μ_e is non-negative and of the total mass 1. Hence, by the Jensen inequality and again by spectral theorem the r.h.s. is bounded from above by

$$f\left(\int_{-\infty}^{\infty} \lambda\mu_e(d\lambda)\right) = f\left((Me, e)\right).$$

This proves (3.22). As for (3.23), known as the Peierls inequality (see e.g. [27]), it follows from (3.22) with $e = e_j$ by summation over j. ■

Lemma 3.4 *Let $\Pi_{\mathcal{C}_1,\mathcal{C}_2}$ be defined by (2.17) where P is an orthogonal projection in $l^2(\mathbb{Z}^d)$. We have*

(i) $\Pi_{\mathcal{C}_1,\mathcal{C}_2}$ is positive definite;

(ii) $||\Pi_{\mathcal{C}_1,\mathcal{C}_2}|| \leq 1$ and if $\mathcal{C}_1 \subset \mathbb{Z}^d \setminus \mathcal{C}_2$, then $||\Pi_{\mathcal{C}_1,\mathcal{C}_2}|| \leq 1/4$

(iii) if $\mathcal{C}_1^{'} \subset \mathcal{C}_1''$, then $\Pi_{\mathcal{C}_1',\mathcal{C}_2}$ is the restriction of $\Pi_{\mathcal{C}_1'',\mathcal{C}_2}$ to $l^2(\mathcal{C}_1') : \Pi_{\mathcal{C}_1',\mathcal{C}_2} = \Pi_{\mathcal{C}_1'',\mathcal{C}_2}\big|_{\mathcal{C}_1'}$;

(iv) if $\mathcal{C}_2 = \cup_{j=1}^p \mathcal{C}_{2j}$ and $\mathcal{C}_{2j} \cap \mathcal{C}_{2k} = \varnothing,\ j \neq k$, then $\Pi_{\mathcal{C}_1,\mathcal{C}_2} = \sum_{j=1}^p \Pi_{\mathcal{C}_1,\mathcal{C}_2 j}$;

(v) if $\mathcal{C}_2^{'} \subset \mathcal{C}_2''$, then $\Pi_{\mathcal{C}_1,\mathcal{C}_2''} - \Pi_{\mathcal{C}_1,\mathcal{C}_2'} = \Pi_{\mathcal{C}_1,\mathcal{C}_2''\setminus\mathcal{C}_2'}$.

Lemma 3.5 *Let $\mathcal{C}$ be a domain in $\mathbb{Z}^d$ and $A = \{A(x, y)\}_{x,y\in\mathcal{C}}$ be a random selfadjoint operator acting in $l_2(\mathcal{C})$. We have the bounds:*

(i) for $\alpha = 1/2, 1/4$ and a positive definite A

$$\mathbf{E}\big\{(\mathrm{Tr}\, A^\alpha)^2\big\} \leq \Big(\sum_{x\in\mathcal{C}} \mathbf{E}^\alpha\big\{A(x, x)\big\}\Big)^2 \tag{3.24}$$

and for $A = \Pi_{\mathcal{C}_1,\ \mathcal{C}_2}$ of (2.17), (2.81) and (2.16)

$$\mathbf{E}\big\{(\mathrm{Tr}\, \Pi^\alpha_{\mathcal{C}_1,\mathcal{C}_2})^2\big\} \leq C\Big(\sum_{x\in\mathcal{C}_1, z\in\mathcal{C}_2} e^{-\alpha\gamma|x-z|}\Big)^2; \tag{3.25}$$

(ii) for a bounded A and $|A| = (A^2)^{1/2}$

$$\mathbf{E}\{\mathrm{Tr}\,|A|\} \leq \sum_{x,y\in\mathcal{C}} \mathbf{E}\{|A(x,y)|\}, \tag{3.26}$$

and

$$\mathbf{E}\{\mathrm{Tr}\,|\Pi_{\mathcal{C}_1,\mathcal{C}_2}|\} = \mathbf{E}\{\mathrm{Tr}\,\Pi_{\mathcal{C}_1,\mathcal{C}_2}\} \leq C \sum_{x\in\mathcal{C}_1, z\in\mathcal{C}_2} e^{-\gamma|x-z|}; \tag{3.27}$$

(iii) for $\mathcal{C}_1' \subset \mathcal{C}_1$

$$\mathbf{E}\{\mathrm{Tr}\,|\Pi_{\mathcal{C}_1,\mathcal{C}_2} - \Pi_{\mathcal{C}_1',\mathcal{C}_2}|\} \leq 2C_0 \sum_{x\in\mathcal{C}_1} \sum_{y\in\mathcal{C}_1\setminus\mathcal{C}_1'} \sum_{z\in\mathcal{C}_2} e^{-\gamma|x-z|/2-\gamma|y-z|/2}; \tag{3.28}$$

(iv) for $\mathcal{C}_1 = \mathbb{Z}_- \times \Lambda_1$, $\mathcal{C}_2 = \mathbb{N}_+ \times \Lambda_2$, *and* $\mathcal{C}_1' = \mathbb{Z}_- \times \Lambda_1'$ *with* $\Lambda_1\Lambda_1', \Lambda_2 \subset \mathbb{Z}^{d-1}$, $\Lambda_1 \subset \Lambda_1'$

$$\Sigma_1 := \sum_{x\in\mathcal{C}_1} \sum_{y\in\mathcal{C}_2} e^{-\gamma|x-y|} \leq C \sum_{x'\in\Lambda_1} \sum_{y'\in\Lambda_2} e^{-\gamma|x-y|} \leq C \min\{|\Lambda_1|, |\Lambda_2|\} \tag{3.29}$$

$$\Sigma_2 := \sum_{x\in\mathcal{C}_1} \sum_{y\in\mathcal{C}_1\setminus\mathcal{C}_1'} \sum_{z\in\mathcal{C}_2} e^{-\gamma|x-z|/2-\gamma|y-z|/2} \leq C|\Lambda_1 \setminus \Lambda_1'|, \tag{3.30}$$

where $x = (x_1, \xi)$, (y_1, η), $x_1 \in \mathbb{Z}_-$, $y_1 \in \mathbb{N}_+$, $\xi \in \Lambda_1$, $\eta \in \Lambda_2$ *and if* Λ_1 *is a* $(d-1)$-*dimensional cube with an edge length* L_1, Λ_1' *is a* $(d-1)$-*dimensional parallelepiped (may be infinite), and* $\mathrm{dist}\{\Lambda_1, \Lambda_1'\} > l \geq 1$, *then*

$$\Sigma_3 := \sum_{x\in\mathcal{C}_1} \sum_{y\in\mathcal{C}_1'} \sum_{z\in\mathcal{C}_2} e^{-\gamma|x-z|/2-\gamma|y-z|/2} \leq Ce^{-\gamma l/4d} L_1^{d-2}. \tag{3.31}$$

Proof. (i). To prove (3.24), we use the Peierls inequality (3.23) for $f(x) = x^\alpha$, $x \in [0, \infty)$ yielding

$$\mathbf{E}\{(\mathrm{Tr}\,A^\alpha)^2\} \leq \sum_{x,y\in\mathcal{C}} \mathbf{E}\{(A(x,x))^\alpha (A(y,y))^\alpha\}.$$

We have then by the Hölder inequality for expectations with $\alpha = 1/2, 1/4$:

$$\mathbf{E}\{(A(x,x))^\alpha (A(y,y))^\alpha\} \leq \mathbf{E}^\alpha\{A(x,x)\}\mathbf{E}^\alpha\{(A(y,y)\},$$

thus, (3.24).

If $A = \Pi_{\mathcal{C}_1,\,\mathcal{C}_2}$ of (2.17), then

$$A(x,x) = \sum_{z\in\mathcal{C}_2} |P(x,z)|^2$$

and we have by (2.81) and (2.16)

$$\mathbf{E}\{A(x,x)\} \leq \sum_{z\in\mathcal{C}_2} \mathbf{E}\{|P(x,y)|\} \leq C \sum_{z\in\mathcal{C}_2} e^{-\gamma|x-z|}.$$

Plugging this into (3.24) and using (3.32), we obtain (3.25).

(ii). We have by (3.23) with $f(x)=x^{1/2},\ x\in[0,\infty)$:

$$\begin{aligned}\mathbf{E}\Big\{\operatorname{Tr}|A|\Big\}\le&\mathbf{E}\Big\{\sum_{x\in\mathcal{C}}((A^2)(x,x))^{1/2}\Big\}\\ =&\mathbf{E}\Big\{\sum_{x\in\mathcal{C}}\Big(\sum_{y\in\mathcal{C}}|A(x,y)|^2\Big)^{1/2}\Big\}\le E\Big\{\sum_{x\in\mathcal{C}}\sum_{y\in\mathcal{C}}|A(x,y)|\Big\}.\end{aligned}$$

Here we have used a simple inequality

$$\Big(\sum|a_j|\Big)^s\le\sum|a_j|^s,\quad s\in[0,1]. \tag{3.32}$$

If $A=\Pi_{\mathcal{C}_1,\mathcal{C}_2}$ of (2.17), then we have by (2.81) and (2.16):

$$\mathbf{E}\big\{\operatorname{Tr}\Pi_{\mathcal{C}_1,\mathcal{C}_2}\big\}=\sum_{x\in\mathcal{C}_1,z\in\mathcal{C}_2}\mathbf{E}\big\{|P(x,z)|^2\big\}\le C_0\sum_{x\in\mathcal{C}_1,z\in\mathcal{C}_2}e^{-\gamma|x-z|}.$$

(iii). If in the previous proof $A=\Pi_{\mathcal{C}_1,\mathcal{C}_2}-\Pi_{\mathcal{C}_1',\mathcal{C}_2}$, then by Lemma 3.4 (iii) $A=\Pi_{\mathcal{C}_1,\mathcal{C}_2}-\Pi_{\mathcal{C}_1,\mathcal{C}_2}|_{\mathcal{C}_1'}$ and we have for the r.h.s. of (3.26) by (2.17), the Schwarz inequality, (2.81) and (2.16)

$$\sum_{x,y\in\mathcal{C}}\mathbf{E}\big\{|A(x,y)|\big\}\le2\sum_{x\in\mathcal{C}_1}\sum_{y\in\mathcal{C}_1\setminus\mathcal{C}_1'}\mathbf{E}\big\{|\Pi_{\mathcal{C}_1'',\mathcal{C}_2}(x,y)|\big\}\le2C_0\sum_{x\in\mathcal{C}_1}\sum_{y\in\mathcal{C}\setminus\mathcal{C}_1'}\sum_{z\in\mathcal{C}_2}e^{-\gamma(|x-z|+|y-z|)/2}.$$

(iv) To prove (3.29), we write $x=(x_1,\xi),\ (y_1,\eta),\ x_1\in\mathbb{Z}_-,\ y_1\in\mathbb{N}_+,\ \xi\in\Lambda_1,\ \eta\in\Lambda_2$ and take into account that

$$|x-y|=|x_1-y_1|+|\xi-\eta| \tag{3.33}$$

and that $x_1\le0,\ y_1>0$. Then the sums with respect to x_1 and y_1 give a factor C in the r.h.s. of (3.29) and we obtain the first inequality of (3.29). Next, summing in the r.h.s. first with respect to ξ and then with respect to η, we obtain $\Sigma_1\le C|\Lambda_1|$ and summing in the opposite order we obtain $\Sigma_1\le C|\Lambda_1|$. This proves the second inequality of (3.29)

To prove (3.30), we write again $x=(x_1,\xi)$, $y=(y_1,\eta$, $z=(z_1,\zeta)$, and take into account the analogs of (3.33) for $|x-z|$ and $|y-z|$. Since $x_1,y_1\le0$, $z_1>0$ we can take a sum with respect to x_1,y_1 and z_1 which gives us a multiplier C Then, since for any ξ,η

$$\sum_{z'}e^{-\gamma|\xi-\zeta|/2-\gamma|\eta-\zeta|/2}\le Ce^{-\gamma|\xi-\eta|/4}$$

we obtain the sum, similar to that we have in the proof of (3.29).

To prove (3.31), we repeat first the argument used in the proof of (3.30) and obtain

$$\Sigma_3\le C\sum_{x'\in\Lambda_1}\sum_{y'\in\Lambda_1'}e^{-\gamma|x'-y'|/4}=\Sigma_3'.$$

We then note that it follows from the condition $\operatorname{dist}\{\Lambda_1,\Lambda_1'\}>l$ that there is $\alpha\in[2,d]$ such that we have for $\xi=\{\xi_\alpha'\}_{\alpha'=1}^{d-1},\ \eta=\{\eta_{\alpha'}\}_{\alpha'=1}^{d-1}$

$$\inf_{\xi\in\Lambda_1,\eta\in\Lambda_1}|\xi_\alpha-\eta_\alpha|>l/d$$

and that without loss of generality we can assume $\alpha = 1$ and $\xi_1 > \eta_1$ for any $\xi \in \Lambda_1$, $\eta \in \Lambda_1'$. Then, denoting e_1 the first basic vector in $\mathbb{Z}^{d-1}$, we obtain

$$-|\xi_1 - \eta_1| \le -[l/d] - |\xi_1 - \eta_1 - [l/d]|,$$

hence

$$\Sigma_3' \le Ce^{-\gamma l/4d} \sum_{\xi \in \Lambda_1} \sum_{\eta \in \Lambda_1' + [l/d]e_1} e^{-\gamma|\xi - \eta|/4}.$$

Since in the last sum $\xi_1 \le L_0 := \sup_{\xi \in \Lambda_1}\{\xi_1\} \le \eta_1$, we can sum with respect to ξ_1, η_1 like we did in the proof of (3.29) and we get (3.31). ■

We will prove now an important bound, which we use in proof of Theorem 2.8. Let H be a discrete Schrödinger operator in $l^2(\mathbb{Z}^d)$ with i.i.d. potential

$$V = gQ,\ Q = \{Q(x)\}_{x \in \mathbb{Z}^d} \tag{3.34}$$

such that the common probability distribution F of every $Q(x)$ satisfies the condition

$$|F(q + \delta) - F(q - \delta)| \le C|\delta|^s,\ C < \infty,\ s \in (0, 1] \tag{3.35}$$

for all $q \in \operatorname{supp} F$ and sufficiently small δ. Assume that on an interval I of the spectrum of H the bound

$$\mathbf{E}\{|(H - \lambda - i\varepsilon)^{-1}(x, y)|^s\} \le Ce^{-\widetilde{\gamma}|x-y|},\ x, y \in \mathbb{Z}^d \tag{3.36}$$

holds uniformly in $\varepsilon > 0$ and $\lambda \in I$ for some $s \in (0, 1)$ and $C < \infty$. According to [3] the bound is valid for any I in the spectrum of H if g in (3.34) – (3.35) is large enough or for any g and I lying in a certain depending on g neighborhood of spectrum edges. The bound (3.36) is a manifestation of the Anderson localization, i.e., the pure point spectrum and exponential decay of eigenfunctions. In fact, (3.36) implies various other manifestations, which are commonly associated with the Anderson localization [1, 2, 3, 12, 31].

We have

Lemma 3.6 *Let H be the discrete Schrödinger operator in $l^2(\mathbb{Z}^d)$, H_Λ be its restriction to a domain $\Lambda \in \mathbb{Z}^d$ and P and $P^{(\Lambda)}$ be the spectral projections of H and H_Λ corresponding to a spectral interval I. Then we have for any I on which (3.36) holds*

$$\mathbf{E}\{|P(x, y) - P^{(\Lambda)}(x, y)| \le C|\partial\Lambda|e^{-\widetilde{\gamma}R/2},\ x, y \in \Lambda,\ \operatorname{dist}(\{x, y\}, \partial\Lambda) \ge R,$$

where $\partial\Lambda$ denotes the boundary of Λ, $\widetilde{\gamma} > 0$, $C < \infty$.

Proof. It follows from the spectral theorem that if $G(z) := (H - z)^{-1} = \{G(z; x, y)\}_{x,y \in \mathbb{Z}^d}$ and $G_\Lambda(z) := (H_\Lambda - z)^{-1} = \{G_\Lambda(z; x, y)\}_{x,y \in \Lambda}$ are the resolvents of H and H_Λ, then we have with probability 1

$$P(x, y) = \frac{1}{2\pi i} \oint_K G(\zeta; x, y) d\zeta \tag{3.37}$$

and

$$P^{(\Lambda)}(x, y) = \frac{1}{2\pi i} \oint_K G_\Lambda(\zeta; x, y) d\zeta, \tag{3.38}$$

where K is a rectangular contour, which encircles I and crosses transversally the real axis at the endpoints of I (recall that the probability that a given point of real axis is an eigenvalue of H is zero, see [23], see Theorems 2.10, 2.12 and 4.21).

We will now use the resolvent identity (3.13) for $M_1 = H$ and $M_2 = H_\Lambda \oplus H_{\mathbb{Z}^d\setminus\Lambda}$, taking into account that the non-diagonal parts of H and H_Λ are $-\Delta$ (see (2.3)) and $-\Delta_\Lambda \oplus \Delta_{\mathbb{Z}^d\setminus\Lambda}$:

$$G(\zeta; x, y) - G_\Lambda(\zeta; x, y) = -\sum_{(u,v)\in\mathcal{L}} G(\zeta; x, u)G_\Lambda(\zeta; v, y), \quad x, y \in \Lambda,$$

where $\mathcal{L}$ is a collection of bonds between the points $v \in \partial\Lambda$ and their nearest neighbors in $\mathbb{Z}^d \setminus \partial\Lambda$. It follows from (3.32)

$$|G(\zeta; x, y) - G_\Lambda(\zeta; x, y)|^{s/2} \le \sum_{(u,v)\in\mathcal{L}} |(G(\zeta; x, u)G_\Lambda(\zeta; v, y)|^{s/2}, \quad x, y \in \Lambda,$$

and then the Hölder inequality for expectation yields

$$\begin{aligned}
&\mathbf{E}\{|G(\zeta; x, y) - G_\Lambda(\zeta; x, y)|^{s/2}\} \\
&\le \sum_{(u,v)\in\mathcal{L}} \mathbf{E}^{1/2}\{|G(\zeta; x, u)|^s\}\mathbf{E}^{1/2}\{G_\Lambda(\zeta; v, y)|^s\}, \quad x, y \in \Lambda.
\end{aligned}$$

We have also the bound

$$\mathbf{E}\{|G_\Lambda(\zeta; x, y)|^s\} \le C < \infty, \; x, y \in \Lambda$$

valid uniformly in $\Im\zeta$ under the conditions of the lemma (see e.g. [3]).

This, (3.37), (3.38), (3.17), and (3.36) imply

$$\begin{aligned}
&\mathbf{E}\{|P(x, y) - P^{(\Lambda)}(x, y)| \\
&\le \frac{1}{2\pi}\oint_K |G(\zeta; x, y) - G_\Lambda(\zeta.x, y)|\}^{s/2}|\Im\zeta|^{s/2-1}|d\zeta| \\
&\le C_0|\partial\Lambda|e^{-\widetilde{\gamma}R/2}\oint_K |\Im\zeta|^{s/2-1}||d\zeta| \le C|\partial\Lambda|e^{-\widetilde{\gamma}R/2}.
\end{aligned}$$

■

4 Conclusion

Here we discuss obtained results and their possible meaning.

We have proved in Section 2 that the entanglement entropy of the system of free disordered fermions with the Schrödinger operator with ergodic potential as one body Hamiltonian satisfies the area law in the mean for any dimension $d \ge 1$ and area law with probability 1 for $d = 1$. The condition of validity of these results is the exponential decay of the entries of the matrix of the spectral projection of Schrödinger operator (see (2.16) and the text below). Note also that the exponential decay (2.16) of spectrum projection does not require the complete localization for all the energies below the Fermi level μ (see (2.2) and (2.8)), moreover a single (connected) spectral interval of localized states. It applies also to the case in which the Fermi level is in a localized segment of spectrum above the bands of extended states.

Our results may also be compared with those of work [7], where the area law for many body quantum system is derived from the exponential decay of all multipoint correlations (exponential clustering property). The property, in turn, is closely related to the existence of the gap between the energy of the pure state in question and the rest of the many body spectrum [14]. Note, however, that in the disordered case (for the disordered fermions at least) the exponential decay results not from the spectrum gap but from the exponential localizations of the one body states in the gapless spectrum.

In this paper we use the version of the setting in which one first carry out the macroscopic limit $\mathcal{L} \to \infty$ and then the large block limit $L \to \infty$ (cf. (1.1)). Another possible version is where these limits are carried out simultaneously: $\mathcal{L} \to \infty,\ L \to \infty,\ L/\mathcal{L} \to c \in (0,1)$, see e.g. [19, 26]. Our basic results are also valid in this setting since there is an analog of bound (2.16) for large but finite systems [3, 12].

The above has to be contrasted with the case of constant (zero) potential of the Schrödinger operator, more generally, for convolution operators in $l^2(\mathbb{Z}^d)$, where the entanglement entropy is asymptotically proportional to $L^{d-1} \log L$, see (2.54) and [15, 20, 30]. Thus, our results can be viewed as a manifestation of the instability of the $L^{d-1} \log L$ asymptotics with respect to the replacement of convolution operators by ergodic operators having the pure point component of the spectrum. This is especially well pronounced in the one dimensional case, where the exponential bound (2.16) is valid for i.i.d. potential of any non zero amplitude of random potential (although with $\gamma \to 0$ as the amplitude tends to zero) [21]. This instability seems reminiscent to the instability of the conducting state or the rounding effect of disorder on phase transitions, see e.g. [4].

References

[1] Aizenman, M.: Localization at weak disorder: some elementary bounds Rev. Mod. Phys. **6**, 1163 – 1182 (1994)

[2] Aizenman, M., Graf, G. M.: Localization bounds for an electron gas. J. Phys. A **31**, 6783 – 6806 (1998)

[3] Aizenman, M., Schenker, J.M., Friedrich, R. M., Hundertmark, D.: Finite-volume fractional-moment criteria for Anderson localization. Comm. Math. Phys. **224**, 219 – 253 (2001)

[4] Aizenman, M., Greenblatt, R. L., Lebowitz J.L.: Proof of rounding by quenched disorder of first order transitions in low-dimensional quantum systems J. Math. Phys. **53**, 023301 (2012)

[5] Amico, L., Fazio, R., Osterloh, A., Vedral, V.: Entanglement in many body systems, Rev. Mod. Phys. **80**, 517 – 578 (2008)

[6] Böttcher, A., Silbermann, B.: *Analysis of Toeplitz Operators*. Berlin: Springer-Verlag, 1990.

[7] Brandao, F., Horodecki M.: Exponential decay of correlations implies area law. Comm. Math. Phys. 333, 761 – 798. (2015).

[8] Calabrese, P., Cardy, J., Doyon, B.: Entanglement entropy in extended systems J. Phys. A: Math. Theor. 42 500301 (2009).

[9] Cheong, S.-A., Henley C. L.: Many-body density matrices for free fermions, Phys . Rev. B. **69**, 075111 (2004).

[10] Dahlsten, O.C.O., Lupo, C., Mancini, S., Serafini, A.: Entanglement typicality, J. Phys. A: Math. Theor. **47**, 363001 (2014).

[11] Eisert, J., Cramer, M., Plenio, M. B.: Area laws for the entanglement entropy, Rev. Mod. Phys. **82**, 277 (2010).

[12] Germinet, F., Taarabt, A.: Spectral properties of dynamical localization for Schrödinger operators. Rev. Math. Phys. **25**, 1350016 (2013).

[13] Gioev, D., Klich, I.: Entanglement entropy of fermions in any dimension and the Widom conjecture, Phys. Rev. Lett. **96**, 100503 (2006).

[14] Hastings,M.: An area law for one dimensional quantum systems. J. Stat. Mech. Theory Exp. P08024 (2007).

[15] Helling, R. C., Leschke, H., Spitzer, W. L.: A special case of a conjecture by Widom with implications to fermionic entanglement entropy, Int. Math. Res. Notices, 1451 – 1482 (2011).

[16] Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. **81** 865 – 942 (2009)

[17] Jitomirskaya, S., Kruger, H.: Exponential dynamical localization for the almost Mathieu operator. Commun. Math. Phys. **322** 877 – 882 (2013).

[18] Kirsch, W., Pastur, L.: On the analogues of Szegö's theorem for ergodic operators, Sbornik: Mathematics. **206:1** 93 – 119 (2015).

[19] Lai, H.-H., Young, K., Bonesteel N. E.: Violation of the Entanglement Area Law in Bosonic Systems with Bose Surfaces: Possible Application to Bose Metals. Phys. Rev. Lett. 111, 210402 (2013).

[20] Leschke, H., Sobolev, A., Spitzer, W.: Scaling of Rényi entanglement entropies of the free Fermi-gas ground state: a rigorous proof, Phys. Rev. Lett. **112**, 160403 (2014).

[21] Minami, N.: Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. Commun. Math. Phys. **177**, 709 – 725 (1996).

[22] Abdul-Rahman, H., Nachtergaele, B., Sims, R., Stolz, G.: Entanglement dynamics of disordered quantum XY chains. arXiv:1510.00262 (2015).

[23] Pastur, L., Figotin, A.: *Spectra of Random and Almost Periodic Operators*. Berlin: Springer-Verlag, 1992.

[24] Pastur, L., Slavin, V.: Area law scaling for the entropy of disordered quasifree fermions, Phys. Rev. Lett. **113**, 150404 (2014).

[25] Peschel, I., Eisler, V.: Reduced density matrices and entanglement entropy in free lattice models. J. Phys. A **42** 504003 (2009).

[26] Pouranvari, M., Zhang, Y., Yang, K.: Entanglement area law in disordered free fermion Anderson model in one, two, and three dimensions. Advances in Condensed Matter Physics, 397630 (2015).

[27] Ruelle, D: *Statistical Mechanics: Rigorous Results*. Reading: Benjamin, 1977.

[28] Simon, B.: *Szëgo's Theorem and its Descendants. Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*. Princeton NJ: Princeton University Press, 2011.

[29] Sobolev, A.V.: Quasi-classical asymptotics for pseudodifferential operators with discontinuous symbols: Widom's conjecture. Functional Analysis and Its Applications, **44** 313 – 317 (2010).

[30] Sobolev, A.V.: *Pseudodifferential Operators with Discontinuous Symbols: Widoms Conjecture*. Memoirs of the AMS, Vol. 222 Providence, RI, AMS, 2013.

[31] Stolz, G.: An introduction to the mathematics of Anderson localization. In: *Entropy and the Quantum II*. Contemp. Math., **552**, 71 – 108. Providence, RI, AMS, 2011.

[32] Stolz, G., Abdul-Rahman, H.: A uniform area law for the entanglement of the states in the disordered XY chain, arXiv:1504.231.