Uncountable locally free groups and their group rings

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Abstract. In this note, we show that an uncountable locally free group, and therefore every locally free group, has a free subgroup whose cardinality is the same as that of G. This result directly improve the main result in [4] and establish the primitivity of group rings of locally free groups.

1. INTRODUCTION

A group G is called locally free if all of its finitely generated subgroups are free. As a consequence of Nielsen-Schreier theorem, a free group is always locally free. If the cardinality $|G|$ of G is countable, then G is locally free if and only if G is an ascending union of free groups. In particular, G is a locally free group which is not free provided that it is a properly ascending union of non-abelian free groups of bounded finite rank. In fact, in this case, G is infinitely generated and Hopfian and so it is not free (also see [5] and [1]). If $|G|$ is uncountable, that is $|G| > \aleph_0$, then it was studied in the context of almost free groups, and it is also known that there exists an uncountable locally free group which is not free ([2]).

Now, clearly, if G is a locally free group with $|G| = \aleph_0$, then G has a free subgroup whose cardinality is the same as that of G. In the present note, we shall prove that it is true for locally free groups of any cardinality. In fact, we shall prove the following theorem:

Theorem 1.1. *If G is a locally free group with $|G| > \aleph_0$, then for each finitely generated subgroup A of G, there exists a subgroup H of G with $|H| = |G|$ such that $AH \simeq A * H$, the free product of A and H.*

In particular, G has a free subgroup of the same cardinality as that of G.

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Theorem 1.1 means that there is no need to assume the existence of free subgroups in [4, Theorem 1]. That is, we can improve the theorem and establish the primitivity of group rings of locally free groups, where a ring R is (right) primitive provided it has a faithful irreducible (right) R-module.

2. Proof of the theorem

In order to prove Theorem 1.1, we prepare necessary notations and some lemmas which include a result due to Mal'cev [3]. Some of them might be trivial for experts but we include their proofs for completeness.

For a finitely generated subgroup H of a locally free group G, $\mu_G(H)$ is defined to be the least positive integer m such that $H \subseteq F_m$ for some free subgroup F_m of rank $m > 0$ in G. The rank $r(G)$ of G is defined to be the maximum element in

$$\{\mu_G(H) \mid H \text{ is a finitely generated subgroup in } G\}$$

or $r(G) = \infty$. We should note that $r(G)$ is finite if and only if for each finitely generated subgroup H of G, there exists a free subgroup N of rank $r(G)$ such that $H \subseteq N$. We should also note that for subgroups H, M and N of G with $H \subseteq M \subseteq N$, $\mu_M(H) \geq \mu_N(H)$ holds.

Lemma 2.1. *Let G be a locally free group, D a finitely generated subgroup of G with $\mu_G(D) = m$ and M a subgroup of G such that $D \subseteq M$ and $r(M) = m$. For $g \in G \setminus M$, set $M_g = M\langle g\rangle$; the subgroup of G generated by g and the elements in M, and let $r(M_g) = n$.*

Then, if $n > m$, then $n = m + 1$ and there exists a free subgroup F_m of rank m in M such that $D \subseteq F_m$ and $F_m\langle g\rangle$ is isomorphic to the free product $F_m * \langle g\rangle$.

Proof. Since $r(M_g) = n$, there exists a finitely generated subgroup C of M_g such that $\mu_{M_g}(C) = n$. Since C is finitely generated, it can be easily seen that there exists finite number of elements $a_1, \ldots, a_l$ in M such that $C \subseteq \langle a_1, \ldots, a_l, g\rangle$. We have then that there exists a free subgroup F_m of rank m in M such that $\langle a_1, \ldots, a_l\rangle D \subseteq F_m$ because of $r(M) = m$. Since $C \subseteq F_m\langle g\rangle$ and $\mu_{M_g}(C) = n$, we see that $r(F_m\langle g\rangle) \geq n$. On the other hand, $r(F_m\langle g\rangle) \leq m + 1$ because of $g \notin F_m$. Combining these with the assumption $n > m$, we get that $n = m + 1 = r(F_m\langle g\rangle)$, which implies $F_m\langle g\rangle \simeq F_m * \langle g\rangle$. □

For a locally free group of finite rank, the following result due to Mal'cev is well-known.

Lemma 2.2. (See [3]) *If G is a locally free group of finite rank, then the cardinality of G is countable; namely $|G| = \aleph_0$.*

On the other hand, if G is not of finite rank, then we have the following property:

Lemma 2.3. *If G is a locally free group whose rank is not finite, then for each finitely generated subgroup A of G, there exists an element $x \in G$ with $x \notin A$, such that $A\langle x\rangle \simeq A * \langle x\rangle$, the free product of A and $\langle x\rangle$.*

Proof. Let A be a finitely generated subgroup of G. We have then that A is a free group of finite rank, because G is locally free. Since the rank of G is not finite, there exists a free subgroup F of G such that $A \subsetneq F$ and $r(F) > r(A)$. Let $A = \langle y_1, \ldots, y_l\rangle$ and $F = \langle x_1, \ldots, x_m\rangle$, where $l = r(A)$ and $m = r(F)$. If for $A_1 = \langle y_1, \ldots, y_l, x_1\rangle$, $r(A_1) = l + 1$, then $A_1 \simeq A * \langle x_1\rangle$. If $r(A_1) \leq l$ then there exists $i \in \{2, \ldots, m\}$ such that $r(A_i) = r(A_{i-1}) + 1$, where $A_i = \langle y_1, \ldots, y_l, x_1, \ldots, x_i\rangle$. We have then that $A_i \simeq A_{i-1} * \langle x_i\rangle$. Since $A \subseteq A_{i-1}$, we have thus seen that $A\langle x\rangle \simeq A * \langle x\rangle$ for some $x \in G \setminus A$. □

Let $G = A * B$ be the free product of $A \neq 1$ and $B \neq 1$. Clearly, if $|G| = \aleph_0$, then G has a free subgroup whose cardinality is the same as that of G. If $|G| > \aleph_0$, then either $|A| = |G|$ or $|B| = |G|$, say $|A| = |G|$. Let I be a set with $|I| = |A|$, and for each $i \in I$, let a_i be in A such that $a_i \neq a_j$ for $i \neq j$. We have then that for $1 \neq b \in B$, the elements $(a_i b)^2$ over $i \in I$ freely generate the subgroup of G whose cardinality is the same as that of G. Hence we have

Lemma 2.4. *If $G = A * B$ is the free product of A and B, then G has a free subgroup whose cardinality is the same as that of G.*

We are now read to prove Theorem 1.1.

Proof of Theorem 1.1. Let A be a finitely generated subgroup of G. We set

$$\mathcal{B} = \{B \mid B \text{ is a non-trivial subgroup of } G \text{ such that } AB \simeq A * B\}.$$

Since G is locally free, A is a free group of finite rank. By assumption, $|G| > \aleph_0$, and so the rank of G is not finite by Lemma 2.2. Hence, by Lemma 2.3, there exists an element $g \in G \setminus A$ such that $A\langle g\rangle \simeq A * \langle g\rangle$, whence $\langle g\rangle \in \mathcal{B}$; thus $\mathcal{B} \neq \emptyset$. Let $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_i \subseteq \cdots$ be a chain of B_i's in $\mathcal{B}$, and let $B^* = \bigcup_{i=1}^{\infty} B_i$. We can see that B^* belongs to $\mathcal{B}$. In fact, if not so, then $AB^* \not\simeq A * B^*$, and so there exists a finitely generated subgroup C of B^* such that $AC \not\simeq A * C$. However, because C is finitely generated in B^*, we have $C \subseteq B_i$ for some i, which implies $AB_i \not\simeq A * B_i$, a contradiction. We have thus shown that $(\mathcal{B}, \subseteq)$ is an inductively ordered set. By Zorn's lemma, there exists a maximal element H in $(\mathcal{B}, \subseteq)$. We shall show $|H| = |G|$, which completes the proof of the theorem. In fact, $AH \simeq A * H$ and it has also a free subgroup whose cardinality is the same as that of G by Lemma 2.4.

Suppose, to the contrary, that $|H| < |G|$. Set $N = AH(\simeq A*H)$, and for a finitely generated subgroup D of N with $\mu_G(D) = m$, let $\mathcal{M}(D)$ be the set of subgroups M of G such that $D \subseteq M$ and $r(M) = m$. We can see that $(\mathcal{M}(D), \subseteq)$ is an inductively ordered set as follows: Since $m = \mu_G(D)$, there exists a free subgroup F_m of rank m in G such that $D \subseteq F_m$. Hence $F_m \in \mathcal{M}(D)$, whence $\mathcal{M}(D) \neq \emptyset$. Let $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq \cdots$ be a chain of M_i's in $\mathcal{M}$, and let $M^* = \bigcup_{i=1}^{\infty} M_i$. Clearly, $D \subseteq M^*$. By the definition of the rank, $r(M^*) \geq \mu_{M^*}(D)$. Since $M^* \subseteq G$, we have that $\mu_{M^*}(D) \geq \mu_G(D) = m$; thus $r(M^*) \geq m$. On the other hand, by the definition of $r(M^*)$, there exists a finitely generated subgroup C of M^* such that $\mu_{M^*}(C) = r(M^*)$. Since C is finitely generated, there exists $i > 0$ such that $C \subseteq M_i$, and then $\mu_{M_i}(C) \leq r(M_i)$, which implies that $r(M^*) = \mu_{M^*}(C) \leq \mu_{M_i}(C) \leq r(M_i) = m$; thus $r(M^*) \leq m$. Hence we have $r(M^*) = m$. We have thus proved that $M^* \in \mathcal{M}(D)$ and that $(\mathcal{M}(D), \subseteq)$ is an inductively ordered set.

Again by Zorn's lemma, there exists a maximal element $M(D)$ in $(\mathcal{M}(D), \subseteq)$. Let $L = \bigcup_{D \in \mathcal{D}} M(D)$, where $\mathcal{D}$ is the set consisting of all finitely generated subgroups of N. Since $r(M(D))$ is finite for each $D \in \mathcal{D}$, it follows from Lemma 2.2 that $|M(D)| = \aleph_0$ for each $D \in \mathcal{D}$. Hence we have $|L| < |G|$ because $|\mathcal{D}| = |N| < |G|$. In particular, there exists $g \in G$ such that $g \notin L$. Note that $g \notin N$ because of $N \subset L$, and so $g \notin H$. We shall show that $N\langle g\rangle \simeq N * \langle g\rangle$. In order to do this, it suffices to show that for each $D \in \mathcal{D}$, $D\langle g\rangle \simeq D * \langle g\rangle$ holds.

Let $r(M(D)) = m$ for $D \in \mathcal{D}$ and let $M_g = M(D)\langle g\rangle$. Since $D \subseteq M_g$ and $\mu_{M_g}(D) \geq \mu_G(D) = m$, it follows that $r(M_g) \geq m$. Moreover, since $g \notin M(D)$, we have $M(D) \subsetneq M_g$. Hence the maximality of $M(D)$ implies $r(M_g) > m$. It follows from Lemma 2.1 that there exists a free subgroup F_m of rank m in $M(D)$ such that $D \subseteq F_m$ and $F_m\langle g\rangle \simeq F_m * \langle g\rangle$. Hence we have $D\langle g\rangle \simeq D * \langle g\rangle$.

We have thus shown that $N\langle g\rangle \simeq N * \langle g\rangle$, which contradicts the maximality of H because $N * \langle g\rangle = A * (H * \langle g\rangle)$. This completes the proof of the theorem. □

Theorem 1.1 shows that the assumption on existence of free subgroups in [4, Theorem 1] can be dropped. That is, we have the following theorem:

Theorem 2.5. *Let G be a non-abelian locally free group. If R is a domain with $|R| \leq |G|$, then RG is primitive.*

In particular, KG is primitive for any field K.

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