Sums of quadratic endomorphisms of an infinite-dimensional vector space

Clément de Seguins Pazzis*†

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Abstract

We prove that every endomorphism of an infinite-dimensional vector space splits as the sum of four idempotents and as the sum of four square-zero endomorphisms, a result that is optimal for general fields.

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1 Introduction

In trying to decompose an endomorphism of a vector space into a sum/linear combination/product of endomorphisms of special type, two situations are traditionally studied:

- The one of finite-dimensional vector spaces, i.e. the matrix case, see e.g. [1, 2, 3, 4, 7, 12];
- In the infinite-dimensional setting one considers a real or complex Hilbert space (or Banach space) and the endomorphisms and the summands (or factors) are required to be bounded operators (see e.g. [5, 8, 9, 12, 13]).

*Université de Versailles Saint-Quentin-en-Yvelines, Laboratoire de Mathématiques de Versailles, 45 avenue des Etats-Unis, 78035 Versailles cedex, France

†e-mail address: dsp.prof@gmail.com

In this article, we shall explore a somewhat neglected territory, in which the vector space is assumed to be infinite-dimensional and the ground field is totally arbitrary. Hence, there is no structure from analysis involved here and the problem is a purely algebraic one. Here are basic questions: Which endomorphisms can be written as a (finite) sum of idempotents? of involutions? of square-zero endomorphisms? If so, what is the minimal number of summands required in such a decomposition? In those questions, the special endomorphisms are all **quadratic**, a quadratic endomorphism u being one that satisfies $u^2 \in \operatorname{span}(\mathrm{id}, u)$. Hence, we will consider more generally the question of decomposing an arbitrary endomorphism into the sum of quadratic endomorphisms with prescribed split annihilating polynomials of degree 2.

Before we go on, we introduce some notation. Throughout the text, $\mathbb{F}$ is an arbitrary field, and t is an indeterminate which we use to write polynomials over $\mathbb{F}$. We use the French convention for the set of all non-negative integers, which we denote by $\mathbb{N}$. All the vector spaces that we consider have $\mathbb{F}$ as ground field. An endomorphism u of an $\mathbb{F}$-vector space V endows V with a structure of $\mathbb{F}[t]$-module so that $t.x = u(x)$ for all $x \in V$: We use the notation V^u when we speak of V as an $\mathbb{F}[t]$-module. The endomorphism u is called **elementary** when the module V^u is free. Our basic method will be to start from an endomorphism u of V and, by subtracting well-chosen "special" endomorphisms, to obtain an elementary one. In this prospect, the key notion is the one of a *stratification* of an $\mathbb{F}[t]$-module, which we will define and study in Section 4.

2 Main results

We start with the main results of our article.

Definition 1. Let $p_1, \dots, p_n$ be polynomials with coefficients in $\mathbb{F}$. An endomorphism u of a vector space is called a **$(p_1, \dots, p_n)$-sum** when there exists an n-tuple $(u_1, \dots, u_n)$ of endomorphisms of V such that

$$u = \sum_{k=1}^{n} u_k \quad \text{and} \quad \forall k \in [\![1, n]\!],\ p_k(u_k) = 0.$$

Theorem 1. *Let p_1 and p_2 be split polynomials of degree 2 with coefficients in $\mathbb{F}$. Then, every elementary endomorphism of a vector space is a (p_1, p_2)-sum.*

Theorem 2. *Let p_1, p_2, p_3, p_4 be split polynomials of degree 2 with coefficients in $\mathbb{F}$. Then, every endomorphism of an infinite-dimensional vector space is a (p_1, p_2, p_3, p_4)-sum.*

Corollary 3. *Let V be an infinite-dimensional vector space. Then, every endomorphism of V is the sum of four square-zero endomorphisms and the sum of four idempotent endomorphisms.*

R. Slowik recently showed [11] that over a field with characteristic different from 2, any endomorphism of a vector space with (infinite) countable dimension is the sum of ten square zero endomorphisms. On the other hand, in Hilbert spaces it is known that every bounded operator is the sum of five square-zero ones, and the result is optimal [9].

By checking the details of the proof of Theorem 2, the reader will easily convince herself that Corollary 3 can be extended to left vector spaces over an arbitrary division ring.

Before we go on, let us discuss the optimality of Corollary 3:

Proposition 4. *Let u be a finite-rank endomorphism of a vector space V with trace different from 0. Then, u is not the sum of three square-zero endomorphisms of V.*

In particular, a rank 1 idempotent of V is not the sum of three square-zero endomorphisms, which proves that Corollary 3 is optimal as far as square-zero endomorphisms are concerned.

Proof of Proposition 4. Assume on the contrary that $u = a+b+c$ for square-zero endomorphisms a, b, c of V. We claim that the finite-dimensional subspace

$$W := \operatorname{Im} u + \operatorname{Im}(au) + \operatorname{Im}(bu) + \operatorname{Im}(cu) + \operatorname{Im}(abu) + \operatorname{Im}(acu) + \operatorname{Im}(bcu) + \operatorname{Im}(abcu)$$

is stable under a, b and c. It is obvious that W is stable under a since $a^2 = 0$. Next, we claim that W includes

$$W' := \operatorname{Im} u + \operatorname{Im}(au) + \operatorname{Im}(bu) + \operatorname{Im}(cu) + \operatorname{Im}(bau) + \operatorname{Im}(acu) + \operatorname{Im}(bcu) + \operatorname{Im}(bacu).$$

Indeed, we note first that $ab + ba = (a + b)^2 = (u - c)^2 = u(u - c) + cu$, and hence $\operatorname{Im}(bau) \subset \operatorname{Im}(abu) + \operatorname{Im}(u) + \operatorname{Im}(cu) \subset W$. Likewise, we find $\operatorname{Im}(bacu) \subset \operatorname{Im}(abcu) + \operatorname{Im}(u) + \operatorname{Im}(cu) \subset W$, which proves the claimed inclusion. Symmetrically, one obtains that $W' \subset W$, and hence $W' = W$. As W'

is stable under b – again, because $b^2 = b$ – we conclude that W is stable under a. Finally, W includes $\operatorname{Im} u$ and hence it is stable under u, whence W is stable under $c = u - a - b$.

From there, we denote by u', a', b', c' the endomorphisms of W induced by u, a, b, c respectively, so that $u' = a' + b' + c'$. Then, $\operatorname{tr}(u') = \operatorname{tr}(u)$ since W includes $\operatorname{Im} u$, and a', b' and c' are square-zero endomorphisms of W. It would follow that $\operatorname{tr}(a') = \operatorname{tr}(b') = \operatorname{tr}(c') = 0$, whence $\operatorname{tr}(u') = 0$, a contradiction. □

Now, let us turn to idempotents:

Proposition 5. *Assume that $\mathbb{F}$ has characteristic not 2 and let $\alpha \in \mathbb{F} \smallsetminus \{0, 1, 2, 3\}$ and V be a vector space over $\mathbb{F}$. Then, $\alpha \operatorname{id}_V$ is not the sum of three idempotent endomorphisms of V.*

Proof. Assume on the contrary that $\alpha . \operatorname{id}_V = p + q + r$ for some idempotents p, q, r. Note first that q and r both commute with $(q+r)(2 \operatorname{id}_V - q - r) = q + r - qr - rq$ (for instance, $q(q + r - qr - rq) = q - qrq = (q + r - qr - rq)q$). However, $(q + r)(2 \operatorname{id}_V - q - r) = (\alpha \operatorname{id}_V - p)((2 - \alpha) \operatorname{id}_V + p) = \alpha(2 - \alpha) \operatorname{id}_V + 2(\alpha - 1)p$. Since $2(\alpha - 1) \neq 0$, we deduce that both q and r commute with p. Symmetrically, q commutes with r. Then, p, q, r are simultaneously diagonalizable, which leads to α being the sum of three elements of $\{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$, contradicting the assumption that $\alpha \notin \{0, 1, 2, 3\}$. □

If $\mathbb{F}$ has characteristic 2 and more than 2 elements, it can be shown that any finite rank endomorphism of V with trace outside of the prime subfield of $\mathbb{F}$ is not the sum of three idempotents: the proof is quite similar to the one of Proposition 4 and we shall therefore leave the details to the reader. We suspect however that every endomorphism of V is the sum of three idempotents if $\mathbb{F} = \mathbb{F}_2$: indeed, the result is known to hold over finite-dimensional spaces, see [10].

Remark 1. Let u be an endomorphism of a vector space V. Let $p_1, \dots, p_r$ be polynomials with coefficients in $\mathbb{F}$. Assume that V splits as $V = \bigoplus_{i \in I} V_i$ where each V_i is stable under u and we denote by u_i the induced endomorphism. Assume finally that for all $i \in I$, the endomorphism u_i splits as $u_i = \sum_{k=1}^{r} u_{i,k}$ where $u_{i,k} \in \operatorname{End}(V)$ and $p_k(u_{i,k}) = 0$ for all $k \in [\![1, r]\!]$. Then, by setting $u^{(k)} := \bigoplus_{i \in I} u_{i,k}$, we see that $u = \sum_{k=1}^{r} u^{(k)}$ and $p_k(u^{(k)}) = 0$ for all $k \in [\![1, r]\!]$.

Remark 2 (The canonical situation). In both Theorems 1 and 2, we can reduce the situation to the one where each polynomial under consideration has the form $t^2 - at$ for some $a \in \mathbb{F}$. Indeed, let $p_1, \dots, p_r$ be split polynomials with degree 2 over $\mathbb{K}$. For each k, denote by x_k, y_k the roots of p_k and note that an endomorphism v is annihilated by p_k if and only if $v - x_k \,\mathrm{id}$ is annihilated by $t^2 - (y_k - x_k)t$. Given $u \in \mathrm{End}(V)$, we deduce that u is a $(p_1, \dots, p_k)$-sum if and only $u - \Big(\sum_{k=1}^{r} x_k\Big).\mathrm{id}$ is a $\big(t^2 - (y_1 - x_1)t, \dots, t^2 - (y_r - x_r)t\big)$-sum. In both Theorems 1 and 2, we note that the assumption on the endomorphism u is left invariant by subtracting a scalar multiple of the identity from u (for Theorem 1, note that V^u is free if and only if $V^{u-\lambda\,\mathrm{id}}$ if free, owing to the fact that $p(t) \mapsto p(t+\lambda)$ is an automorphism of the $\mathbb{F}$-algebra $\mathbb{F}[t]$). Hence, in both theorems, it will suffice to consider the case when each polynomial p_k has the form $t^2 - at$ for some $a \in \mathbb{F}$ (depending on k).

Theorem 1 will be proved in Section 3. The proof of Theorem 2 is spread over Sections 4 and 5.

3 Decomposing an elementary operator

Here, we give a quick proof of Theorem 1. We will need the following basic lemma:

Lemma 6. *Let u be an endomorphism of a vector space V with countable dimension, and let $(e_n)_{n\in\mathbb{N}}$ be a basis of V. Assume that $u(e_n) = e_{n+1}$ mod $\mathrm{span}(e_0, \dots, e_n)$ for all $n \in \mathbb{N}$. Then, $(u^n(e_0))_{n\in\mathbb{N}}$ is a basis of V.*

Proof. We prove by induction that $\mathrm{span}(e_0, \dots, e_n) \subset F_n := \mathrm{span}(u^k(e_0))_{0\leq k\leq n}$ for all $n \in \mathbb{N}$. This is obvious if $n = 0$. Assume that it holds for some $n \geq 0$. In order to prove that $\mathrm{span}(e_0, \dots, e_{n+1}) \subset F_{n+1}$, it suffices to see that $e_{n+1} \in F_{n+1}$. Yet, $e_{n+1} = u(e_n) + x$ for some $x \in \mathrm{span}(e_0, \dots, e_n)$. By induction x and e_n belong to F_n, and as u obviously maps F_n into F_{n+1} we deduce that $e_{n+1} \in F_{n+1}$, which completes the inductive step.

From there, we immediately deduce that $(u^n(e_0))_{n\in\mathbb{N}}$ spans V. Moreover, for all $n \in \mathbb{N}$, we see that $e_0, u(e_0), \dots, u^n(e_0)$ are linearly independent because $n + 1 = \dim \mathrm{span}(e_0, \dots, e_n) \leq \dim F_n$. We conclude that $(u^n(e_0))_{n\in\mathbb{N}}$ is a linearly independent sequence, which completes the proof. ☐

In light of Remark 2, we can limit the discussion to the following situation: Let u be an elementary endomorphism of a vector space V, and a and b be scalars. We have to prove that there exist endomorphisms v and w of V such that $u = v + w$, $v^2 = a\,v$ and $w^2 = b\,w$. In this prospect, we know from Remark 1 that it suffices to prove this when V^u has a single generator, say x. Since all the non-zero free $\mathbb{F}[t]$-modules with one generator are isomorphic, we can turn the question on its head: It suffices to *construct* a non-zero vector space U over $\mathbb{F}$ and a pair (v, w) of endomorphisms of U such that $v^2 = a\,v$, $w^2 = b\,w$, and the $\mathbb{F}[t]$-module U^{v+w} has a generator.

To do so, we take an arbitrary vector space U over $\mathbb{F}$ with a countable basis $(e_n)_{n\in\mathbb{N}}$ on which we define two endomorphisms v and w as follows:

- $v(e_k) = a\,e_k + e_{k+1}$ for every even $k \in \mathbb{N}$, otherwise $v(e_k) = 0$;
- $w(e_k) = b\,e_k + e_{k+1}$ for every odd $k \in \mathbb{N}$, otherwise $w(e_k) = 0$.

For every $k \in \mathbb{N}$, we have $v^2(e_k) = 0 = a\,v(e_k)$ if k is odd, otherwise $v^2(e_k) = v(a\,e_k + e_{k+1}) = a\,v(e_k) + 0 = a\,v(e_k)$ since $k + 1$ is even. Hence, $v^2 = a\,v$. Likewise, one proves that $w^2 = b\,w$.

Setting $u := v + w$, we see that u satisfies the condition of Lemma 6, whence $(u^n(e_0))_{n\in\mathbb{N}}$ is a basis of U. It follows that U^u is a free $\mathbb{F}[t]$-module with one generator, which completes the proof of Theorem 1.

4 Stratifications

4.1 Stratifications and associated objects

Definition 2. Let V be an $\mathbb{F}[t]$-module. A **stratification** of V is a sequence $(V_\alpha)_{\alpha\in D}$, indexed over a well-ordered set D, of submodules of V in which:

- For all $\alpha \in D$, the quotient module $V_\alpha / \left(\sum_{\beta<\alpha} V_\beta\right)$ is non-zero and has a generator;
- $V = \sum_{\alpha\in D} V_\alpha$.

To any such stratification, we assign the **dimension sequence** $(n_\alpha)_{\alpha\in D}$ defined by

$$n_\alpha := \dim_{\mathbb{F}} \left(V_\alpha / \sum_{\beta<\alpha} V_\beta \right)$$

(in the infinite-dimensional case, we consider the dimension to be $+\infty$, not the first infinite ordinal $\aleph_0$).

Let $(V_\alpha)_{\alpha\in D}$ be a stratification of V. For every $\alpha \in D$, we can choose a vector $x_\alpha \in V_\alpha$ such that $V_\alpha = \mathbb{F}[t]x + \sum_{\beta<\alpha} V_\beta$, and we note that if n_α is finite then $V_\alpha = \mathbb{F}_{n_\alpha-1}[t]\, x_\alpha \oplus \sum_{\beta<\alpha} V_\beta$ and $(t^k x_\alpha)_{0\leq k<n_\alpha}$ is linearly independent, otherwise $(t^k x_\alpha)_{0\leq k<+\infty}$ is linearly independent and $V_\alpha = \mathbb{F}[t]\, x_\alpha \oplus \sum_{\beta<\alpha} V_\beta$. We shall say that the **vector sequence** $(x_\alpha)_{\alpha\in D}$ is attached to $(V_\alpha)_{\alpha\in D}$. In this case, an obvious transfinite induction shows that, for all α and β in D with $\beta<\alpha$, the family $(t^k.x_\delta)_{\beta\leq\delta\leq\alpha,\, 0\leq k<n_\delta}$ is linearly independent and

$$\begin{cases} \left(\sum_{\gamma<\beta} V_\gamma\right) \oplus \operatorname{span}\bigl((t^k\, x_\delta)_{\beta\leq\delta\leq\alpha,\, 0\leq k<n_\delta}\bigr) & = V_\alpha \\ \left(\sum_{\gamma<\beta} V_\gamma\right) \oplus \operatorname{span}\bigl((t^k\, x_\delta)_{\beta\leq\delta<\alpha,\, 0\leq k<n_\delta}\bigr) & = \sum_{\gamma<\alpha} V_\gamma. \end{cases}$$

In particular, $(t^k\, x_\alpha)_{\alpha\in D,\, 0\leq k<n_\alpha}$ is a basis of V. As a special case, we get the obvious consequence:

Lemma 7. *Let V be an $\mathbb{F}[t]$-module with a stratification $(V_\alpha)_{\alpha\in D}$. Assume that the corresponding dimension sequence $(n_\alpha)_{\alpha\in D}$ is constant with sole value $+\infty$. Then, V is free.*

Conversely, consider a sequence $(x_\alpha)_{\alpha\in D}$, indexed over a well-ordered set D, of vectors of V such that $x_\alpha \notin \sum_{\beta<\alpha} \mathbb{F}[t]\, x_\beta$ for all $\alpha \in D$, and $V = \sum_{\alpha\in D} \mathbb{F}[t]\, x_\alpha$. Then, one sees that $\left(\sum_{\beta\leq\alpha} \mathbb{F}[t]\, x_\beta\right)_{\alpha\in D}$ is a stratification of V, with corresponding vector sequence $(x_\alpha)_{\alpha\in D}$.

Of course, any stratification can be re-indexed over an ordinal, and in general we shall often assume that the stratifications we are dealing with are indexed over ordinals.

4.2 Connectors for a stratification

Definition 3. Let u be an endomorphism of a vector space V. Let $(V_\alpha)_{\alpha\in D}$ be a stratification of V^u, with attached dimension sequence $(n_\alpha)_{\alpha\in D}$ and an associated vector sequence $(x_\alpha)_{\alpha\in D}$.

An endomorphism v of V is called a **connector** for u with respect to the vector sequence $(x_\alpha)_{\alpha\in D}$ whenever it acts as follows on the basis $(t^k\, x_\alpha)_{\alpha\in D, 0\leq k<n_\alpha}$:

for all $\alpha \in D$ such that $n_\alpha < +\infty$, we have $v(t^{n_\alpha - 1} x_\alpha) = x_{\alpha+1} \mod V_\alpha$, and all the other vectors are mapped to 0.

The definition is motivated by the following result:

Proposition 8. *Let u be an endomorphism of a vector space V. Let $(V_\alpha)_{\alpha\in\kappa}$ be a stratification of V^u, with attached dimension sequence $(n_\alpha)_{\alpha\in\kappa}$ and an associated vector sequence $(x_\alpha)_{\alpha\in\kappa}$.*

Assume that if κ has a maximum M then $n_M = +\infty$. Then, for any connector v for u with respect to $(x_\alpha)_{\alpha\in\kappa}$, then endomorphism $u+v$ is elementary.

Proof. Without loss of generality, we can assume that κ is an ordinal.

We define D as the set of all $\alpha \in \kappa$ such that either α has no predecessor or α has a predecessor and $n_{\alpha-1} = +\infty$. Note that D is a non-empty well-ordered set.

Set $w := u + v$.

Fix $\alpha \in D$. Either $n_{\alpha+k} < +\infty$ for all $k \in \mathbb{N}$, in which case we set $m_\alpha := +\infty$, otherwise we denote by m_α the smallest positive integer k such that $\alpha+k-1 \in \kappa$ and $n_{\alpha+k-1} = +\infty$ (it exists because if κ has a maximum M then $n_M = +\infty$). In any case, we set

$$W_\alpha := \sum_{k<m_\alpha} V_{\alpha+k} = \bigcup_{k<m_\alpha} V_{\alpha+k}.$$

We shall prove that $(W_\alpha)_{\alpha\in D}$ is a stratification of V^w and that the corresponding dimension sequence takes no finite value.

To help us, we need additional notation. Let $\beta \in \kappa$. If $\beta \notin D$ then β has a predecessor $\beta - 1$. As there is no infinite decreasing sequence in κ, it follows that there is a uniquely-defined element $g(\beta) \in D$ together with an non-negative integer m such that $g(\beta) + m = \beta$ and $g(\beta) + k \notin D$ for all $k \in [\![1, m]\!]$. It follows from the above definition that

$$V_\beta \subset W_{g(\beta)}.$$

Now, for all $\beta \in D$, the endomorphism v maps V_β into $V_{\beta+1}$ unless $n_\beta = +\infty$ in which case V_β is stable under D. Hence, we deduce from the definition of m_α that W_α is stable under v. Moreover, we readily see that it is stable under u, and we conclude that W_α is a submodule of V^w.

Next, we see that $(W_\alpha)_{\alpha\in D}$ is increasing. Let indeed $(\alpha, \beta) \in D^2$ be such that $\alpha < \beta$. From the definition of m_α, it follows that $\alpha + k < \beta$ for every integer k such that $0 \leq k < m_\alpha$, and hence $W_\alpha \subset \sum_{\gamma\in\kappa, \gamma<\beta} V_\gamma \subsetneq V_\beta \subset W_\beta$.

Finally, let us fix $\alpha \in D$. Denote by y the class of x_α in the quotient module $E := W_\alpha / \sum_{\beta\in D,\, \beta<\alpha} W_\beta$. Let us prove that E is non-zero and is the free module generated by y. We start by proving that

$$\sum_{\beta\in D,\, \beta<\alpha} W_\beta = \sum_{\beta\in\kappa,\, \beta<\alpha} V_\beta. \tag{1}$$

Let $\beta \in \kappa$ be such that $\beta < \alpha$. Then, $V_\beta \subset W_{g(\beta)}$ with $g(\beta) < \alpha$ and $g(\beta) \in D$. Conversely, let $\beta \in D$ be such that $\beta < \alpha$. Then, $\beta + k < \alpha$ for all k such that $0 \le k < m_\alpha$, and hence it follows from the definition of W_β that $W_\beta \subset \sum_{\gamma\in\kappa,\, \gamma<\alpha} V_\gamma$.

Using equality (1), we deduce that

$$E = \Big(\sum_{0\le k<m_\alpha} V_{\alpha+k}\Big) \Big/ \Big(\sum_{\beta\in\kappa,\, \beta<\alpha} V_\beta\Big).$$

Assume first that $m_\alpha = +\infty$. Then, $n_{\alpha+k}$ is finite for each integer k and we set

$$(e_n)_{n\in\mathbb{N}} = \big(x_\alpha, u(x_\alpha), \dots, u^{n_\alpha-1}(x_\alpha), x_{\alpha+1}, u(x_{\alpha+1}), \dots, u^{n_{\alpha+1}-1}(x_{\alpha+1}), \dots, \\ x_{\alpha+k}, u(x_{\alpha+k}), \dots, u^{n_{\alpha+k}-1}(x_{\alpha+k}), \dots\big)$$

If m_α is finite, then we take

$$(e_n)_{n\in\mathbb{N}} = \big(x_\alpha, u(x_\alpha), \dots, u^{n_\alpha-1}(x_\alpha), x_{\alpha+1}, u(x_{\alpha+1}), \dots, u^{n_{\alpha+1}-1}(x_{\alpha+1}), \dots, \\ x_{\alpha+m_\alpha-1}, u(x_{\alpha+m_\alpha-1}), \dots, u^{l}(x_{\alpha+m_\alpha-1}), \dots\big)$$

In any case we see by induction that $\big(x_\alpha, \dots, u^{n_{\alpha+k}-1}(x_{\alpha+k})\big)$ is linearly independent and

$$\operatorname{span}\big(x_\alpha, \dots, u^{n_{\alpha+k}-1}(x_{\alpha+k})\big) \oplus \Big(\sum_{\beta\in\kappa,\, \beta<\alpha} V_\beta\Big) = V_{\alpha+k}$$

for all $0 \le k < m_\alpha$.

Next, we prove that $w(e_n) = e_{n+1} \mod \sum_{\beta\in\kappa,\, \beta<\alpha} V_\beta + \operatorname{span}(e_0, \dots, e_n)$, for all $n \in \mathbb{N}$. Let indeed $n \in \mathbb{N}$. Then $e_n = t^l x_{\alpha+k}$ for some $0 \le k < m_\alpha$ and some $0 \le l < n_{\alpha+k}$. If $l < n_{\alpha+k} - 1$ then we know that $e_{n+1} = u(e_n)$ and $v(e_n) = 0$, whence $w(e_n) = e_{n+1}$. Assume that $l = n_{\alpha+k} - 1$. Then,

$u(e_n) \in V_{\alpha+k}$ whereas $v(e_n) - e_{n+1} \in V_{\alpha+k}$, and hence $w(e_n) - e_{n+1} \in V_{\alpha+k} = \sum_{\beta\in\kappa,\, \beta<\alpha} V_\beta + \operatorname{span}(e_0, \dots, e_n)$.

Hence, Lemma 6 applies to the endomorphism $\overline{w}$ of E induced by w, and it yields that the resulting module $E^{\overline{w}}$ is non-zero and free with generator y. Hence, $(W_\alpha)_{\alpha\in D}$ is a stratification of V^w and all the terms of its dimension sequence equal $+\infty$. By Lemma 7, we conclude that $E^{\overline{w}}$ is elementary. ☐

5 Decompositions into four quadratic operators

Here, we shall prove Theorem 2. Combining Remark 2 with Theorem 1, one sees that we only need to prove the following result.

Proposition 9. *Let u be an endomorphism of an infinite-dimensional space V, and let a and b be scalars. Then, there exist endomorphisms u_1 and u_2 of V such that $u - u_1 - u_2$ is elementary, $u_1^2 = a\,u_1$ and $u_2^2 = b\,u_2$.*

We will prove Proposition 9 by constructing a "well-behaved" stratification of V^u. In the first section, we perform such a construction, and in the subsequent one we use it to construct the claimed endomorphisms u_1 and u_2 by use of a connector.

5.1 On the existence of a well-behaved stratification

Proposition 10. *Let V be an $\mathbb{F}[t]$-module with infinite dimension as a vector space over $\mathbb{F}$. Then, there is a stratification $(V_\alpha)_{\alpha\in\kappa}$ of V such that:*

(i) κ is a cardinal;

(ii) If κ is finite then $\dim V_0 = +\infty$.

Proof. We construct such a stratification by transfinite induction. First we denote by ν the dimension of the $\mathbb{F}$-vector space V, seen as a cardinal. We choose a basis $(e_k)_{k\in\nu}$ of V. If V is no torsion module, we can further assume that e_0 does not belong to the torsion submodule of V, i.e. $\mathbb{F}[t]\, e_0$ is a free submodule of V. Then, we construct an ordinal $\kappa \leq \nu$ and an increasing sequence $(V_\alpha)_{\alpha\in\kappa}$ of submodules of V as follows.

We put $V_0 := \mathbb{F}[t]\, e_0$.

Let $\alpha \in \nu$ and assume that we have constructed an increasing sequence $(V_\beta)_{\beta<\alpha}$ of submodules of V such that, for all $\beta < \alpha$:

(i) the $\mathbb{F}[t]$-module $V_\beta / \sum_{\gamma<\beta} V_\gamma$ is non-zero and has a generator;

(ii) the vector e_β belongs to V_β.

Put $W := \sum_{\beta<\alpha} V_\beta$. If $V = W$, then the process terminates at α. Otherwise, we take the least $k \in \nu$ such that $e_k \notin W$ (note that $\alpha \leq k$), and we put $V_\alpha := W + \mathbb{F}[t]e_k$. The module V_α/W is non-zero and has a generator (namely, the class of e_k). Finally, we note that $e_\alpha \in V_\alpha$. Hence, the inductive step is climbed.

Assume first that the process never terminates. Then, by condition (ii), we have $V = \sum_{\alpha\in\nu} V_\alpha$. It follows that $(V_\alpha)_{\alpha\in\nu}$ is a stratification of V, and ν is a cardinal.

In the rest of the proof, we assume that the process terminates at some ordinal α, so that $(V_\beta)_{\beta<\kappa}$ is a stratification of V. First of all, we prove that ν is countable. Let us take an associated vector sequence $(x_\beta)_{\beta<\alpha}$. Then, we know that $V = \sum_{\beta<\alpha} \mathbb{F}[t]x_\beta$, and hence, denoting by μ the cardinality of α, we find

$$\dim_{\mathbb{F}} V \leq \mu \times \aleph_0.$$

Note that $\mu < \nu$. Yet, if $\aleph_0 < \nu$, then $\mu \times \aleph_0 < \nu$ and we have a contradiction. If follows that $\nu = \aleph_0$ and hence α is a finite ordinal. Next, as V is infinite-dimensional at least one of the $\mathbb{F}$-vector spaces $\mathbb{F}[t]x_\beta$, for $\beta < \alpha$, must be infinite-dimensional. Hence, V is not a torsion module and by our assumptions $\dim V_0 = \dim \mathbb{F}[t]e_0 = +\infty$. Hence, the claimed conclusion follows. ☐

5.2 The reduction to an elementary endomorphism

We are now ready to prove Theorem 2. Here is the key step.

Proposition 11. *Let u be an endomorphism of a vector space V. Let $(V_\alpha)_{\alpha\in\kappa}$ be a stratification of V^u that is indexed over an ordinal κ. Assume that κ is a limit ordinal or V_0 is infinite-dimensional.*

Let a and b be scalars. Then, there exist endomorphisms u_1 and u_2 of V such that $u_1^2 = a\,u_1$, $u_2^2 = b\,u_2$ and $u - (u_1 + u_2)$ is elementary.

The combination of this result with Proposition 10 yields Proposition 9, which in turn yields Theorem 1.

Proof. We take the dimension sequence $(n_\alpha)_{\alpha\in\kappa}$ of V^u and a vector sequence $(x_\alpha)_{\alpha\in\kappa}$. Given an ordinal α, there is a largest non-negative integer k for which there exists an ordinal γ satisfying $\alpha=\gamma+k$: we say that α is **even** when k is even, and **odd** otherwise.

Case 1: κ is a limit ordinal.
We define two endomorphisms v_1 and v_2 of V as follows on the basis $(u^k(x_\alpha))_{\alpha\in\kappa, 0\le k<n_\alpha}$:

For each ordinal $\alpha\in\kappa$ such that $n_\alpha<+\infty$, we set

$$v_1\bigl(u^{n_\alpha-1}(x_\alpha)\bigr) := \begin{cases} a\,u^{n_\alpha-1}(x_\alpha)-x_{\alpha+1} & \text{if } \alpha \text{ is even} \\ 0 & \text{if } \alpha \text{ is odd}\end{cases}$$

$$v_2\bigl(u^{n_\alpha-1}(x_\alpha)\bigr) := \begin{cases} 0 & \text{if } \alpha \text{ is even} \\ b\,u^{n_\alpha-1}(x_\alpha)-x_{\alpha+1} & \text{if } \alpha \text{ is odd}\end{cases}$$

and v_1 and v_2 are required to map all the other basis vectors to 0. We check on the above basis that $v_1^2=a\,v_1$ and $v_2^2=b\,v_2$. Indeed, let α be an even ordinal such that $n_\alpha<+\infty$. Then, $v_1(u^{n_\alpha-1}(x_\alpha))=a\,u^{n_\alpha-1}(x_\alpha)-x_{\alpha+1}$ and $\alpha+1$ is odd whence $v_1(x_{\alpha+1})=0$, which leads to $v_1^2(u^{n_\alpha-1}(x_\alpha))=a\,v_1(u^{n_\alpha-1}(x_\alpha))$. For any other basis vector y, we have $v_1(y)=0$ and hence $v_1^2(y)=0=a\,v_1(y)$. The proof is similar for v_2.

Next, it is obvious that $v:=-v_1-v_2$ is a connector for u with respect to the vector sequence $(x_\alpha)_{\alpha\in\kappa}$. We deduce from Proposition 8 that $u-v_1-v_2$ is elementary.

Case 2: κ has a maximum M.
Then, we know that V_0 is infinite-dimensional. If $\kappa=0$, then u is already elementary and we just take $u_1=u_2=0$. In the rest of the proof, we assume that $M>0$. Set $W:=\operatorname{span}(u^k(x_\alpha))_{0<\alpha\le M,\, 0\le k<n_\alpha}$ and note that $V=V_0\oplus W$. Denote by π the projection of V onto V_0 along W, and set $w:=\pi\circ u\circ(\mathrm{id}-\pi)$. Then, $u':=u-w$ stabilizes both V_0 and W. For $\alpha\in\kappa\smallsetminus\{0\}$, set $W_\alpha:=\operatorname{span}(u^k(x_\beta))_{0<\beta\le\alpha,\, 0\le k<n_\beta}$ and note that u' stabilizes W_α. Set finally $W_0:=V$. Now, define $D:=\kappa$ with the same order on $\kappa\smallsetminus\{0\}$, but with $x<0$ for all $x\in\kappa\smallsetminus\{0\}$. One sees that D is still a well-ordered set and now $(W_k)_{k\in D}$ is a stratification for $V^{u'}$ for which the last dimension equals $+\infty$. An associated vector sequence is $(x_k)_{k\in D}$.

Like in Case 1, we define a connector v of u' for this stratification and this vector sequence. Hence, $u-(w-v)$ is elementary. In order to conclude, it only remains to split $(w-v)$ into v_1+v_2 where $v_1^2=a\,v_1$ and $v_2^2=b\,v_2$.

First of all, we see that $(w-v)(u^k(x_0))=0$ for all $k\in\mathbb{N}$.

For each $\alpha\in\kappa\setminus\{0\}$ and each integer k such that $0\leq k<n_\alpha$, either $k\neq n_\alpha-1$ and hence $(w-v)(u^k(x_\alpha))=0$, or $k=n_\alpha-1$ and hence there is a vector y_α of V_0 such that

$$(w-v)(u^k(x_\alpha))=\begin{cases} a\,u^k(x_\alpha)-x_{\alpha+1}+y_\alpha & \text{if } \alpha \text{ is odd} \\ b\,u^k(x_\alpha)-x_{\alpha+1}+y_\alpha & \text{if } \alpha \text{ is even.} \end{cases}$$

Now, we define endomorphisms v_1 and v_2 as follows on the basis $(u^k(x_\alpha))_{\alpha\in\kappa,0\leq k<n_\alpha}$: for all $\alpha\in\kappa$ such that $n_\alpha<+\infty$,

$$v_1\bigl(u^{n_\alpha-1}(x_\alpha)\bigr)=\begin{cases} a\,u^{n_\alpha-1}(x_\alpha)-x_{\alpha+1}+y_\alpha & \text{if } \alpha \text{ is odd} \\ 0 & \text{if } \alpha \text{ is even,} \end{cases}$$

and

$$v_2\bigl(u^{n_\alpha-1}(x_\alpha)\bigr)=\begin{cases} 0 & \text{if } \alpha \text{ is odd} \\ b\,u^{n_\alpha-1}(x_\alpha)-x_{\alpha+1}+y_\alpha & \text{if } \alpha \text{ is even,} \end{cases}$$

and v_1 and v_2 are required to map all the basis vectors to 0. Obviously $v_1+v_2=w-v$. On the other hand, as in Case 1 it is easily checked that $v_1^2=a\,v_1$ and $v_2^2=b\,v_2$ by using the fact that v_1 and v_2 vanish everywhere on V_0 (and in particular they map any y_α to 0). As $u-(v_1+v_2)$ is elementary, the proof is complete. □

Therefore, Theorem 2 is now fully established.

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