HIGHER THEORIES OF ALGEBRAIC STRUCTURES

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Abstract. The notion of (symmetric) coloured operad or "multicategory" can be obtained from the notion of commutative algebra through a certain general process which we call "theorization" (where our term comes from an analogy with William Lawvere's notion of algebraic theory). By exploiting the inductivity in the structure of higher associativity, we obtain the notion of "n-theory" for every integer $n \geq 0$, which inductively *theorizes* n times, the notion of commutative algebra. As a result, (coloured) morphism between n-theories is a "graded" and "enriched" generalization of $(n-1)$-theory. Theorization is moreover, a generalization of the process of categorification in the sense of Louis Crane, and the inductive hierarchy of those *higher theories* extends in particular, the hierarchy of higher categories. In a part of low "theoretic" order of this hierarchy, graded and enriched 1- and 0-theories vastly generalize symmetric, braided, and many other kinds of enriched multicategories and their algebras in various places.

We make various constructions of/with higher theories, and obtain some fundamental notions and facts. We also find iterated theorizations of more general kinds of algebraic structure including (coloured) properad of Bruno Vallette and various kinds of topological field theory (TFT). We show that a "TFT" in the extended context can reflect a very different type of data from a TFT in the conventional sense, despite close formal similarity of the notions.

This work is intended to illustrate use of simple understanding of the coherence for associativity.

Contents

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0. Introduction

0.0. **Higher theories.**

0.0.0. Among variations of the notion of operad, the symmetric, the planar and the braided (see Fiedorowicz [9]) versions are particularly simple to describe, and are very commonly worked with. Over such an operad, an *algebra* can be considered respectively in a symmetric, associative and braided (see Joyal and Street [12]) monoidal category.

In general, each operad governs algebras over it, and this role is important. In other words, the notion of operad is important since it gives a way to do *universal algebra* in (e.g., symmetric) monoidal categories. For this role, we consider an operad as analogous to Lawvere's *algebraic theory* [14].

By visiting the conceptual origin of the notion of operad, one finds that the notion of symmetric operad arises naturally through a certain process, which we call "theorization" (inspired by Lawvere's notion), from the notion of commutative algebra. Moreover, the same process can be started from the notion of $\mathcal{U}$-algebra for any operad $\mathcal{U}$ in sets or groupoids, instead of the commutative operad $\mathrm{Com} = E_\infty$, to produce a new kind of algebraic structure which we call "$\mathcal{U}$-graded" operad (Section 0.2.0). It turns out that planar and braided operads are $\mathcal{U}$-graded operads where $\mathcal{U} = E_1, E_2$ respectively.

The notion of $\mathcal{U}$-graded operad in groupoids has a deceptively simple description, namely, a $\mathcal{U}$-graded operad in groupoids can be described as an operad $\mathcal{X}$ in groupoids equipped with a morphism $P\colon \mathcal{X} \to \mathcal{U}$. ($\mathcal{X}$ is a $\mathcal{U}$-graded operad in *sets* if the maps induced by P on the groupoids of operations have everywhere, homotopy fibre with a discrete homotopy type. The details will be discussed in Remark 0.8.) This might hide the notion of theorization from a non-obsessed mind. However, we have come to think that theorization is an important notion.

One purpose of this long introductory section is to introduce the notion of theorization, which will be a generalization with its own mathematical content, of the notion of *categorification* in the sense of Crane [7, 6] (about which the most influential pioneer may have been Grothendieck), but will be at least as informal a notion as categorification. Even though a very precise understanding of the notion of theorization is not technically necessary for the body of the article, at least a rough understanding will be essential for understanding the ideas of our work. In the body, we shall use the language of theorization to navigate the reader through ideas.

Let us, however, start with a sketch of what we actually do in this work. After that, the main purpose of this introduction will be to introduce the idea of theorization, see its role in our work, and describe more exactly what we do in the body.

0.0.1. In this work, we introduce and study 'higher order' generalizations of (coloured) operads, which we call "higher theories" of algebras. Higher theories will be obtained by iterating the process of theorization starting from the notion of coloured operad. Introduction of these objects leads to (among other things) a framework for a natural explanation and a vast generalization of the fact which we formulate below as Proposition 0.0. Let us describe it now.

The proposition will be about places where the notion of $\mathcal{U}$-graded operad can be enriched, but the reader may assume that $\mathcal{U} = E_1$ or E_2 (see above). As another notice, colours in operads will not play an essential role for a while, so we shall consider just uncoloured operads everywhere till we start taking colours explicitly into consideration.

For a symmetric monoidal category $\mathcal{A}$, let us denote by $\mathrm{Op}_{\mathcal{U}}(\mathcal{A})$ the category of $\mathcal{U}$-graded operads in $\mathcal{A}$. $\mathrm{Op}_{\mathcal{U}}$ is a category-valued functor on the 2-category $\mathrm{Alg}_{\mathrm{Com}}(\mathrm{Cat})$ of symmetric monoidal categories, where Cat denotes the Cartesian symmetric monoidal 2-category of categories (with a fixed limit for size).

For an operad $\mathcal{U}$, let us mean by an **$(\mathcal{U} \otimes E_1)$-monoidal** category, an associative monoidal object in the 2-category of $\mathcal{U}$-monoidal categories, or equivalently, a $\mathcal{U}$-monoidal object in the 2-category of associative monoidal categories. Note that there is a forgetful functor $\mathrm{Alg}_{\mathrm{Com}}(\mathrm{Cat}) \to \mathrm{Alg}_{\mathcal{U}}\bigl(\mathrm{Alg}_{\mathrm{Com}}(\mathrm{Cat})\bigr) \to \mathrm{Alg}_{\mathcal{U}}\bigl(\mathrm{Alg}_{E_1}(\mathrm{Cat})\bigr)$ from the symmetric monoidal categories to $(\mathcal{U} \otimes E_1)$-monoidal categories, where we have used the canonical functor $\mathcal{C} \to \mathrm{Alg}_{\mathcal{U}}(\mathcal{C})$ exsisting for every *coCartesian* symmetric monoidal 2-category $\mathcal{C}$ (namely, a 2-category $\mathcal{C}$ closed under the finite coproducts, made symmetric monoidal by the finite coproduct operations) obtained by letting every operation on a given object of $\mathcal{C}$ be the codiagonal map of the object.

The formulation of the proposition is as follows. (See Remark 0.1 for a technical point.)

Proposition 0.0. *For every symmetric operad* $\mathcal{U}$*, the functor* $\mathrm{Op}_{\mathcal{U}}$ *on* $\mathrm{Alg}_{\mathrm{Com}}(\mathrm{Cat})$ *has an extention to* $\mathrm{Alg}_{\mathcal{U}}\bigl(\mathrm{Alg}_{E_1}(\mathrm{Cat})\bigr)$ *(in a manner which is functorial in* $\mathcal{U}$*).*

The meaning of Proposition is that there is a natural notion of $\mathcal{U}$-graded operad in every $(\mathcal{U} \otimes E_1)$-monoidal category, such that the notion of $\mathcal{U}$-graded operad in a symmetric monoidal category $\mathcal{A}$ coincides naturally with the notion of $\mathcal{U}$-graded operad in the $(\mathcal{U} \otimes E_1)$-monoidal category underlying $\mathcal{A}$. Our definition will generalize the familiar notions of

- associative algebra in a *associative* monoidal category,
- planar operad in a *braided* monoidal category,
- braided operad in a E_3-*monoidal* infinity 1-category

(in addition to vacuously, the notion of symmetric operad in a symmetric monoidal category).

Remark 0.1. In order to actually have these examples, Proposition needs to be interpreted in the framework of sufficiently high dimensional category theory. (Infinity 1-category theory is sufficient.) However, let us not emphasize this technical point in this introduction, even though our work will eventually be about higher category theory.

Remark 0.2. The author has unfortunately failed to find a reference for Proposition as stated, or any generalization of it in the literature. A positive is that we have found that natural ideas lead to vast generalization of Proposition, even though this does not relieve our failure of attribution.

In fact, we introduce in this work much more general notion of "grading", generally for higher theories, and find quite general but natural places where the notions of graded higher theory can be enriched. An explanation of Proposition 0.0 from the general perspective to be so acquired, will be given in Section 0.6.1. In fact, all of these will result from extremely simple ideas, which we would like to describe with their main consequences.

0.0.2. Our starting point is the idea that an operad and more generally, a *coloured* operad or "*multicategory*" (see Lambek [13] or Section 1.1.2) is analogous to an algebraic theory for its role of governing algebras over it. We extend this by defining, for every integer $n \geq 0$, the notion of n-theory, where a 0-theory is an algebra (commutative etc.), a 1-theory will be a multicategory (symmetric etc.), and, for $n \geq 2$, each n-theory comes with a natural notion of algebra over it, in such a way that an $(n-1)$-theory coincides precisely with an algebra over the terminal n-theory, generalizing from the case $n = 1$, the fact that, e.g., a commutative algebra is an algebra over the terminal symmeric multicategory Com.

This will be realized by defining an n-theory as a *theorized* form of an $(n-1)$-theory, where the theorization we consider is a "coloured" version of it. Theorization in the coloured sense, of commutative algebra, will be symmetric multicategory, and in general, coloured theorization generalizes *categorification* in the same way as how symmetric multicategory generalizes symmetric monoidal category, i.e., the standard categorification of commutative algebra. (Details will be seen in Sections 0.3, 0.4.)

In particular, the hierarchy of n-theories as n varies, contains the hierarchy of n-categories or iterated categorifications of category, as in fact a very small part of it. Quite a variety of other hierarchies of iterated categorifications, such as operads in n-categories and so on, also form very small parts of the same hierarchy.

Importance of these "higher theories" lies in the significance of the notion of n-theory, to be revealed in this work, to our understanding of $(n-1)$-theories, and hence of 0-theories or algebras in fact, by induction. Indeed, the notion of algebra over an n-theory naturally generalizes the notion of $(n-1)$-theory, and one important role of an n-theory is to govern algebras over it. In another important role, an n-theory provides a place in which one can consider $(n-1)$-theories. Namely, an n-theory allows one to enrich the notion of $(n-1)$-theory (including the "graded" one) along it.

These two points can actually be combined into one expression that a graded and enriched $(n-1)$-theory is a generalized (i.e., "coloured") functors between n-theories. We shall see these points later, but in summary, higher theories are important since an $(n-1)$-theory can be understood as closely similar to a functor of n-theories.

In this work, we solve difficulties for iteratively theorizing the notion of algebra (see Sections 0.3, 0.6 for more on this) in a fruitful manner as described, and then study fundamental notions and basic facts about higher theories. In particular, we investigate relationship among various mathematical structures related to these objects, as well as do and investigate various fundamental constructions. See Sections 0.6 and 0.7 in particular.

Remark 0.3. While algebraic theory in Lawvere's sense is about a kind of algebraic structure which makes sense in any *Cartesian* symmetric monoidal category, the notion of algebra over an n-theory makes sense (in particular) in any, e.g., symmetric, monoidal category. In this work, we only consider kinds of algebraic structure which are definable at least in every symmetric monoidal category.

It has not been clear whether there is an analogous hierarchy starting from Lawvere theory. Possible iterated *categorifications* of Lawvere's notion are algebraic theories enriched in n-categories, but we are looking for a larger hierarchy than iterated categorifications.

An interesting consequence of the existence of the hierarchy of higher theories is that, by considering an algebra over a *non*-terminal higher theory, an algebra over such a thing, and so on, we obtain various exotic new kinds of structure, all of whom can nevertheless be treated in a unified manner. These structures include the hierarchy of n-theories for every kind of "grading", specified by a choice of a (trivially graded) higher theory $\mathcal{U}$ (and can in fact be exhausted essentially by all of these). The adjectives "commutative" or "symmetric" above refer to this choice (the choice being the "trivial" grading in these cases, to be specific). Hierarchies in different gradings are related to each other in some specific way which will be clarified in Section 3. (See Sections 0.6, 0.7 for a sketch.)

We expect that higher theories would lead to new methods for studying algebra, generalizing use of operads and multicategories, which are just the second bit, coming next of the algebras, in the hierarchy of higher theories. Moreover, the existence of a hierarchy of iterated theorizations can be asked more generally, starting from much more general kinds of "algebraic" structure than we have talked about so far. Our construction of the hierarchy of n-theories (already of various kinds) may be showing the meaningfulness of such a question, and this may be our deepest contribution at the conceptual level.

The author indeed expects a similar hierarchy to exist starting from a more general kind of algebraic structure which can be expressed as defined by an "associative" operation. Indeed, at the heart of our method is a technology of producing from a given kind of associative operation, a new kind of associative operation, which is based on fundamental understanding of the higher structure of associativity.

Indeed, even though we shall discuss in this introduction only higher theoretic structures related to algebras over multicategories, we shall also consider in Section 5, a modest generalization involving higher theoretic structures related to some algebraic structures in which the operations may have multiple inputs *and* multiple outputs, such as various versions of topological field theories.

Remark 0.4. A similar method also leads to a new model [17] for higher category theory, including a model of "the infinity infinity category of infinity infinity categories" (given a limit for the size). This will not excessively be surprising since higher theory will be a more general kind of structure than higher category; we have already mentioned that the hierarchy of n-theories contains the hierarchy of n-categories.

This new model moreover has a certain convenient feature which has not been realized on any other known model (even of infinity 1-*categories*). Even thought this is unfortunately not a convenient place for describing the mentioned feature, *other* features of the model include that

- it is "algebraic" in the sense that the composition etc. are given by actual operations, and
- its construction employs only a tiny amount of combinatorics, and no model category theory, topology or geometry.

0.0.3. While the main focus of our work will be on *higher* theories, Proposition 0.0 can be considered as an instance in a low "theoretic" level of algebra, of a consequence of this work.

0.1. **The conceptual origin of the notion of multicategory.**

0.1.0. Theorization will be a process which produces a new kind of algebraic structure from a given kind. In order to start a discussion of the idea of theorization, we would like to be able to talk about *kinds* of algebraic structure.

An example of a kind of structure definable in a symmetric monoidal category, is a symmetric monoidal functor from a fixed symmetric monoidal category, say $\mathcal{B}$. Thus, let us mean by a **$\mathcal{B}$-algebra** in a symmetric monoidal category $\mathcal{A}$, simply a symmetric monoidal functor $F\colon \mathcal{B} \to \mathcal{A}$. It is a 'representation' of $\mathcal{B}$ in $\mathcal{A}$ (or an $\mathcal{A}$-valued point of the 'affine scheme' $\operatorname{Spec}\mathcal{B}$).

Concretely, by describing the structure of $\mathcal{B}$ using a collection of generating objects and generating maps (as well as decomposition of each of the source and the target of every generating map into a monoidal product of generating objects), one may obtain a presentation of the form of the structure of a $\mathcal{B}$-algebra in terms of structure maps and equations satisfied by the structure maps. For example, the coCartesian symmetric monoidal category $\mathcal{B} = \mathrm{Fin}$ of finite sets, is generated under the symmetric monoidal multiplication operations $\otimes = \amalg$, by the terminal object $*$, and one map $\nabla_S\colon *^{\otimes S} \to *$ for each finite set S. It follows that the data of a symmetric monoidal functor $F\colon \mathrm{Fin} \to \mathcal{A}$ can be described as the object $A = F(*)$ of $\mathcal{A}$ equipped with one operation $F(\nabla_S)\colon A^{\otimes S} = F(*^{\otimes S}) \to A$ for each finite set S, satisfying suitable equations resulting from the relations one has in Fin. Thus, we have obtained a presentation of the form of the structure of a Fin-algebra, as the form of data for defining a commutative algebra.

For a general $\mathcal{B}$, if objects of $\mathcal{B}$ are generated under the monoidal multiplication by a family $b = (b_\lambda)_{\lambda\in\Lambda}$ of objects, then a $\mathcal{B}$-algebra defined by a symmetric monoidal functor $F\colon \mathcal{B} \to \mathcal{A}$, can be considered similarly as structured on the family Fb of objects of $\mathcal{A}$, by structure maps satisfying equations imposed by the structure of $\mathcal{B}$. For example, the universal $\mathcal{B}$-algebra defined by $\mathrm{id}\colon \mathcal{B} \to \mathcal{B}$, is structured on the family b of objects of $\mathcal{B}$, where the structure maps will be the chosen generating maps of $\mathcal{B}$.

Algebra over a symmetric monoidal category in our sense, is simply the most obvious formalization of kind of structure which can be presented as defined by structure maps satisfying some specific equations. Lawvere's theory is based on this idea. Indeed, a *multi-sorted*, i.e., "coloured", Lawvere theory is essentially a Cartesian symmetric monoidal category $\mathcal{B}$ which is given a nice collection of generating objects. A *PROP* [16] or "*category of operators*" [3], with colours [4], is similar.

An algebra over a multicategory is also covered. Indeed, given a multicategory $\mathcal{U}$, one can freely generate from it a symmetric monoidal category, say $L\mathcal{U}$, so a $\mathcal{U}$-algebra in a symmetric monoidal category $\mathcal{A}$ will be equivalent as data to an $L\mathcal{U}$-algebra in $\mathcal{A}$.

Even though we are not particularly interested in the notion of algebra over a general symmetric monoidal category, this can be the starting point for our purpose of finding kinds of algebraic structure which generalize nicely. For example, recall that the basic idea of categorification is that a *categorification* of a certain kind of algebraic structure, is a kind of structure on category obtained by replacing structure maps by functors, and structural equations by suitably coherent isomorphisms, forming a part of the structure. We have a canonical categorification of $\mathcal{B}$-algebra, which we shall call a **$\mathcal{B}$-monoidal category**, and it is simply a symmetric monoidal functor $\mathcal{B} \to \mathrm{Cat}$, where Cat denotes the Cartesian symmetric monoidal category enriched in groupoids, of categories (with a fix limit for size), where for $\mathcal{X}, \mathcal{Y} \in \mathrm{Cat}$, we let $\mathrm{Map}_{\mathrm{Cat}}(\mathcal{X}, \mathcal{Y})$ be the groupoid formed by functors $\mathcal{X} \to \mathcal{Y}$ and isomorphisms between them.

Remark 0.5. This is a technical remark.

Here and everywhere else in this introduction, a functor which we consider to a category enriched in groupoids (or in categories) is a functor in the usual "weakened" sense (which is sometimes called a *pseudo*-functor, see Grothendieck [11, Section 8], [10]). Even though we shall not need to look into the details of this till we enter the body, a symmetric monoidal structure on such a functor can also be defined in an appropriate manner.

In fact, it should be understood that every categorical term in this introduction is used in the similarly appropriate sense when there is enrichment of the relevant categorical structures in groupoids or categories, where erichment itself should be understood to be done in the standard "weakened" manner. See Bénabou [1]. (The reader who is comfortable with homotopy theory may instead replace all sets/groupoids with infinity groupoids, and understand everything as enriched in infinity groupoids, and this will trivialize the process of categorification since infinity 1-categories (of size up to a fixed limit) are already forming a (larger) infinity 1-category.)

However, we notify the reader of a circularity here. Namely, we have used one particular categorification of the notion of commutative algebra, i.e., the notion of symmetric monoidal category, to categorify kinds of structure which are similar to commutative algebra. Even though the result obtained is not bad, one may not be able to expect that the same framework would also be the most useful for categorifying very different kinds of structure.

For example, for categorification of the notion of multicategory, a method which takes account of the categorical dimensionality appears to lead to a cleaner and less redundant presentation of the result than the method of reformulating the notion of multicategory as algebra over a symmetric monoidal category, and then applying the previous definition. (It appears simpler to treat a multicategory as an algebra over a categorically 2-*dimensional* algebraic structure, e.g., the terminal "2-theory", than to treat it as an algebra over a multicategory or a symmetric monoidal category, which can naturally be seen as 1-dimensional structures.)

Therefore, the general idea which we have described of a kind of algebraic structure and its categorification, seems more important for practical purposes, than precise but limited formulations of the notions in particular contexts, such as $\mathcal{B}$-algebra and $\mathcal{B}$-monoidal category. Nevertheless, we hope that the examples above was clarifying on our view on algebraic structures.

Finally, we remark that, in order to consider a categorification of the notion of $\mathcal{B}$-algebra, the target of the symmetric monoidal functor on $\mathcal{B}$ to define a categorified structure, did not need to be Cat. Namely, the notion of symmetric monoidal functor $\mathcal{B} \to \mathcal{A}$, where $\mathcal{A}$ is *any* symmetric monoidal category enriched in groupiods, formalizes the idea of replacing structural equations for the structure of a $\mathcal{B}$-algebra (in some presentation of the notion) by coherent isomorphisms.

Therefore, it seems reasonable to expect in general, that a meaningful categorification of a kind of structure definable in a symmetric monoidal category, should be a kind of structure definable in any symmetric monoidal category $\mathcal{A}$ enriched in groupoids. In fact, the right way to consider categorification is perhaps as about enrichment in groupoids, and the resulting weakening in a coherent manner, of the structure.

While the expectation above is indeed fulfilled in the concrete cases which we consider in this work, we shall keep concentrating on the case $\mathcal{A} = \mathrm{Cat}$ of categorified structures for the time being, since this will keep things simpler. One relation between the mentioned general form of categorification and the idea of theorization, will be seen in Section 0.3. On the other hand, for the kinds of structure which we theorize in this work, the general form of categorification can be understood in any case, as a specific kind of structure residing in a suitably associated theorized structure, leaving us no need to consider more general situation than $\mathcal{A} = \mathrm{Cat}$ in this early stage. See e.g., Corollary 2.12 (or Theorem 2.10). For example, the case "$n = 0$" of this result applies to the categorified form of the notion of $\mathcal{B}$-algebra.

0.1.1. In order to get to the idea of theorization, we first recall that a basic feature expected of the categorified structure is that, if a category $\mathcal{C}$ is equipped with a categorified form of a certain kind of algebraic structure, then the original, uncategorified form of the same structure should naturally make sense in $\mathcal{C}$. For example, if $\mathcal{C}$ is given a monoidal structure over a symmetric monoidal category $\mathcal{B}$, then a "$\mathcal{B}$-algebra" in $\mathcal{C}$ means a lax $\mathcal{B}$-monoidal functor to $\mathcal{C}$ from the unit $\mathcal{B}$-monoidal category.

However, for some kinds of algebraic structure, we have more general instances of this phenomenon. In a symmetric monoidal category for example, the notion of algebra makes sense over any symmetric operad or multicategory, and the same moreover makes sense also *in* any symmetric multicategory. Indeed, an algebra over a symmetric multicategory $\mathcal{U}$ in a symmetric multicategory $\mathcal{V}$, is simply a morphism $\mathcal{U} \to \mathcal{V}$. Similarly, the notion of algebra over any planar multicategory makes sense in any associative monoidal category, and more generally, in any planar multicategory in the same manner.

While the notion of associative monoidal category categorifies the notion of associative algebra, the process of theorization, which will be more general than the process of categorification, will produce the notion of planar multicategory from the notion of associative algebra, and symmetric multicategory from commutative algebra. In general, theorization will produce from a given kind of algebraic structure, a new kind of algebraic structure generalizing its categorification, in such a manner that the original notion of algebra reduces to the notion of algebra over the terminal one among the theorized objects (meaning symmetric multicategories, for commutative algebras, so generalizing the simple fact that an commutative algebra is an algebra over the terminal symmetric multicategory).

Let us thus recall how one may naturally arrive at the notion of symmetric operad, starting from the notion of commutative algebra (and we are suggesting that the same procedure will produce the notion of planar operad from the notion of associative algebra, for example). Specifically, let us try to find the notion of symmetric operad (in sets) out of the desire of generalizing the notion of commutative algebra to the notions of certain other kinds of algebra which makes sense in a symmetric monoidal category. Indeed, one of the most important roles of a multicategory is definitely the role of governing algebras over it.

The way how we generalized the notion of commutative algebra is as follows. Recall, as we have already seen, that the structure of a commutative algebra on an object A of a symmetric monoidal category, was given by a single S-ary operation $A^{\otimes S} \to A$ for every finite set S which, collected over all S, had appropriate consistency. We get a generalization of this by allowing not just a single S-ary operation, but a *family* of S-ary operations parametrized by a set prescribed for S. This "set of S-ary operations" for each S, is the first bit of the data defining an operad in sets. Having this, we next would like to compose these operations just as we can compose multiplication operations of a commutative algebra, and the composition should have appropriate consistency. A symmetric multicategory is simply a more general version of this, with many objects, or "colours". (We shall take a look at colours in a theorized structure in Section 0.3.1.)

Remark 0.6. In this formulation of an operad, part of the composition structure makes the set of S-ary operations functorial with respect to bijections of "S". This gives the "action of the symmetric group" in another common formulation of an operad.

A similar procedure can be imagined once a kind of "algebraic" structure in a broad sense is specified as a specific kind of system of operations, in place of commutative or associative algebra. Inspired by Lawvere's notion of algebraic theory, we call a multicategory also a (**symmetric**) **1-theory**, and then generally call **theorization**, a process similar to the process above through which we have obtained 1-theories from the notion of commutative algebra. The result of such a process will also be called a **theorization**. Thus, the notion of 1-theory is a *theorization* of the notion of commutative algebra. We shall see in Sections 0.3 and 0.4, how the process of theorization indeed generalizes the process of categorification.

Remark 0.7. Recall that operad was a kind of structure which made sense in any symmetric monoidal category, the meaning of which for us was that the form of data to define an operad, could be presented in terms of structure maps and structural equations. In general, it seems reasonable to expect that a meaningful theorization of a given kind of algebraic structure should have a similar presentation, and in particular, should make sense in any symmetric monoidal category $\mathcal{A}$. This ability of presentation will be important when we would like to theorize a theorized kind of structure once again, even though we shall till that time, stick normally to the case $\mathcal{A} = \mathrm{Set}$ in order to keep our exposition simpler.

0.2. **Theorization of algebra.**

0.2.0. As the simplest example of a theorization process next to the one which we have seen in the previous section, let us consider theorization of the notion of $\mathcal{U}$-algebra for a symmetric operad $\mathcal{U}$ in sets (see Remark 0.8 below for the case of an operad in groupoids). By using the same method as in the previous section, we shall obtain a theorization of the notion of $\mathcal{U}$-algebra, which we call *$\mathcal{U}$-graded operad* (in the uncoloured version). Let us assume for simplicity, that $\mathcal{U}$ is an uncoloured operad.

Recall that the structure of a $\mathcal{U}$-algebra on an object A of a symmetric monoidal category, is defined by an associative action on A, of operations in $\mathcal{U}$. If u is an

S-ary operation in $\mathcal{U}$ for a finite set S, then it should act as an S-ary operation $A^{\otimes S} \to A$. Now, to theorize the notion of $\mathcal{U}$-algebra given by an action of the operators of $\mathcal{U}$, means to modify the definition of this structure by replacing an action of every operator u in $\mathcal{U}$, by a choice of the set "of operations of shape (so to speak) u". We call an element of this set an **operation of degree** u.

Thus the data of a $\mathcal{U}$-graded operad $\mathcal{X}$ in sets should include, for every operation u in $\mathcal{U}$, a set whose element we shall call an operation in $\mathcal{X}$ of degree u. If the operation u is S-ary in $\mathcal{U}$, then we shall say that any operation of degree u in $\mathcal{X}$ has **arity** S.

There should further be given a consistent way to compose the operations in $\mathcal{X}$ which moreover respects the degrees of the operations. These will be a complete set of data for a **$\mathcal{U}$-graded operad** $\mathcal{X}$ in sets.

There is also a coloured version of this, which we call $\mathcal{U}$-graded **1-theory**, **multicategory** or **coloured** operad, and this is a theorization of $\mathcal{U}$-algebra in a more general sense. From the general discussions of theorization in Sections 0.3 and 0.4, $\mathcal{U}$-graded multicategory will turn out to be also a generalization of $\mathcal{U}$-monoidal category, generalizing the way how symmetric multicategory generalizes symmetric monoidal category.

0.2.1. By reflecting on what we have done above, we immediately find that a $\mathcal{U}$-graded operad in sets is in fact exactly a symmetric operad $\mathcal{Y}$ in sets equipped with a morphism $P\colon \mathcal{Y} \to \mathcal{U}$. The relation between $\mathcal{X}$ above and $\mathcal{Y}$ here is that an S-ary operation in $\mathcal{Y}$ is an S-ary operation in $\mathcal{X}$ of *any* degree. The map P maps an operation in $\mathcal{Y}$ to the degree which the operation had when it was in $\mathcal{X}$. Conversely, given an S-ary operation u in $\mathcal{U}$, an operation in $\mathcal{X}$ of degree u is an S-ary operation in $\mathcal{Y}$ which lies over u. For example, $\mathcal{U}$, lying terminally over itself, indeed corresponds in this manner, to the terminal $\mathcal{U}$-graded operad, which has exacly one operation of each degree.

Remark 0.8. In the case where $\mathcal{U}$ is an operad in groupoids, the set of S-ary operations in $\mathcal{X}$ of degree u should be functorial in u on the groupoid of S-ary operations in $\mathcal{U}$. The S-ary operations in $\mathcal{Y}$ will then be the corresponding groupoid projecting to the groupoid of S-ary operations in $\mathcal{U}$ (obtained by the Grothendieck construction [11, Section 8] or the (homotopy) colimit in groupoids). For $\mathcal{U} = E_2$, the mentioned functoriality corresponds to the action of the pure braid groups in a braided operad. See Fiedorowicz [9].

In general, a symmetric operad $\mathcal{Y}$ in groupoids equipped with $P\colon \mathcal{Y} \to \mathcal{U}$, corresponds to a $\mathcal{U}$-graded operad $\mathcal{X}$ in *groupoids*. $\mathcal{X}$ is in sets if for every operation u in $\mathcal{U}$, the groupoid of operations in $\mathcal{X}$ of degree u, obtained as the (homotopy) fibre over u in $\mathcal{Y}$, is a *homotopy* 0-*type*, namely, a groupoid in which every pair of maps $f, g\colon x \xrightarrow{\sim} y$ between the same pair of objects are equal.

Remark 0.9. Inclusion of operads in groupoids in the discussion leads to a subtle situation. For example, if $\mathcal{U} = E_2$, then a $\mathcal{U}$-algebra in a symmetric monoidal category is simply a commutative algebra. Therefore, we are considering both E_2-graded multicategory and symmetric multicategory as theorizations of commutative algebra. The crucial difference between the two theorizations is between the categorifications being generalized, namely, braided monoidal category and symmetric monoidal category.

Similarly, a $\mathcal{U}$-graded multicategory enriched in sets is a symmetric multicategory in which multimaps are graded by multimaps of $\mathcal{U}$. In other words, it is just a symmetric multicategory (enriched in sets) equipped with a morphism to $\mathcal{U}$. Now, given a $\mathcal{U}$-graded multicategory $\mathcal{X}$, an $\mathcal{X}$-algebra in a $\mathcal{U}$-graded 1-theory $\mathcal{Y}$ will be just a functor $\mathcal{X} \to \mathcal{Y}$ of $\mathcal{U}$-graded 1-theories.

Example 0.10. A multicategory graded by the initial operad Init, is a multicategory with only unary multimaps, which is equivalent as data to a category.

A similar theorization of the notion of $\mathcal{U}$-algebra, can be defined for a *coloured* symmetric operad $\mathcal{U}$ enriched in sets, and a $\mathcal{U}$-graded 1-theory enriched in sets will be again a symmetric multicategory $\mathcal{X}$ (enriched in sets) equipped with a functor to $\mathcal{U}$. In other words, $\mathcal{X}$ will be such that not only multimaps in it are graded, but objects are also graded by objects of $\mathcal{U}$. For an object u of $\mathcal{U}$, an object of $\mathcal{X}$ of *degree* u, will be just an object of $\mathcal{X}$ lying over u.

Example 0.11. Recall, as noted in Example 0.10, that a category $\mathcal{C}$ can be considered as a symmetric multicategory having only unary maps. If we consider $\mathcal{C}$ as a multicategory in this way, then a $\mathcal{C}$-graded 1-theory enriched in sets is a category equipped with a functor to $\mathcal{C}$, and this theorizes $\mathcal{C}$-algebra, or functor on $\mathcal{C}$ (which one might also call a left $\mathcal{C}$-module). On the other hand, a categorification of $\mathcal{C}$-algebra is a category valued functor $\mathcal{C} \to \mathrm{Cat}$, and among the theorizations, categorifications correspond to op-fibrations over $\mathcal{C}$.

Suppose given a category $\mathcal{C}$ and two functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathrm{Cat}$, corresponding respectively to categories $\mathcal{X}, \mathcal{Y}$ lying over $\mathcal{C}$, mapping down to $\mathcal{C}$ by op-fibrations. Note that, by Example 0.11, $\mathcal{F}$ and $\mathcal{G}$ are categorified $\mathcal{C}$-modules, and $\mathcal{X}$ and $\mathcal{Y}$ as categories over $\mathcal{C}$, are the corresponding $\mathcal{C}$-graded 1-theories.

In this situation, the relation between maps $\mathcal{F} \to \mathcal{G}$ and maps $\mathcal{X} \to \mathcal{Y}$, is as follows. Namely, a functor $\phi \colon \mathcal{X} \to \mathcal{Y}$ of categories over $\mathcal{C}$ (see Remark 0.5, to be technical), corresponds to a map $\mathcal{F} \to \mathcal{G}$ if and only if ϕ preserves coCartesian maps, and an arbitrary functor ϕ over $\mathcal{C}$ only corresponds to a *lax* map $\mathcal{F} \to \mathcal{G}$ (defined with respect to the standard 2-*category* structure on Cat).

A similar pattern can be observed on theorization in general (Theorem 2.10).

0.3. **Theorization in general.**

0.3.0. For the idea for theorization of a general algebraic structure, the notion of *profunctor/distributor/bimodule* is useful. For categories $\mathcal{C}, \mathcal{D}$, a **$\mathcal{D}$–$\mathcal{C}$-bimodule** (in the category Set of sets) is a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$. The category of $\mathcal{D}$–$\mathcal{C}$-bimodules contains the opposite of the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as a full subcategory, where a functor $F \colon \mathcal{C} \to \mathcal{D}$ is identified with the bimodule $\mathrm{Map}_{\mathcal{D}}(F-, -)$. Let us say that this bimodule is **corepresented** by F. By symmetry, the category of bimodules also contains $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$. However, for the purpose of theorization, we treat $\mathcal{C}$ and $\mathcal{D}$ asymmetrically, and mostly consider only *co*representation of bimodules. Bimodules compose by tensor product, to make categories form a 2-category, extending the 2-category formed with (opposite) functors as 1-morphisms, by the identification of a functor with the bimodule corepresented by it.

0.3.1. We would like to consider the general idea of theorization, while having in mind as an example, the case of the notion of $\mathcal{B}$-algebra for a symmetric monoidal category $\mathcal{B}$.

We shall find that theorization is in fact more general than *lax* (or "op-lax"; see below) categorification, where, by a **lax categorification** of a kind of structure, we mean a *relaxation* of a specific categorification of the same kind of structure in the sense that it is a specific generalization of the specified categorification such that, in a lax categorified structure, a non-invertible map (going in a specified direction) is allowed in place of every one of some specified structure isomorphisms in some presentation of the categorified form of structure of the kind. Let us denote by **Cat**, the 2-*category* of categories and functors. Then the notion of $\mathcal{B}$-algebra has a canonical lax categorification, which we shall call *lax* $\mathcal{B}$-monoidal category, where a **lax $\mathcal{B}$-monoidal category** is by definition, a lax functor $\mathcal{B} \to \mathbf{Cat}$ which is given data of (not lax) commutation with the symmetric monoidal structures. In the case where $\mathcal{B}$ is Fin, this coincides with the standard notion of lax symmetric monoidal category.

There is another lax categorification of the notion of $\mathcal{B}$-algebra, which we shall call *op-lax* $\mathcal{B}$-monoidal category. An **op-lax** $\mathcal{B}$-monoidal category is similar to a lax $\mathcal{B}$-monoidal category except that the laxness of the functor $\mathcal{B} \to \mathbf{Cat}$ should be opposite, namely, the non-invertible structure maps should go in the opposite direction. In other words, it should be an symmetric monoidal op-lax functor in one of the common conventions, in which a *lax* functor $\mathbf{1} \to \mathbf{Cat}$ from the unit category $\mathbf{1}$, is a category equipped with a monad on it (see Benabou [1]).

The idea of **theorization** which we have described in Sections 0.1, 0.2, can now be expressed as that it is a *virtualization* of an op-lax categorification, where by a **virtualization** of op-lax $\mathcal{B}$-monoidal structure, or any kind of structure as far as the following makes sense, we mean a specific generalization of the structure such that, in a virtualized structure, non-corepresentable bimodules are allowed in place of some specified structure *functors*. (Note here also that the structure maps on bimodules should be understood to be in the opposite direction to the structure maps on functors, owing to the contravariance of the corepresentation of bimodules by functors.)

To be cautious, specification of a theorization of $\mathcal{B}$-algebra for example, includes specification of the notion of “structure functor” for $\mathcal{B}$-monoidal structure, which should at least include specification of a collection of generating objects of $\mathcal{B}$. Specifically, given a family $b = (b_\lambda)_{\lambda\in\Lambda}$ of objects of $\mathcal{B}$ which generates all objects under the monoidal multiplication, we consider for a family $\mathcal{X} = (\mathcal{X}_\lambda)_\lambda$ of categories, a symmetric monoidal functor $\mathcal{F}\colon \mathcal{B} \to \mathrm{Cat}$ with $\mathcal{F}b = \mathcal{X}$ as a structure on $\mathcal{X}$, and then see a theorization as a more general kind of structure which we can consider on $\mathcal{X}$ (although a slightly more precise understanding will be that the structure is on the collection of colours to be described shortly, as will be concluded from the discussions of Section 0.3.3 below). In this manner, a theorized structure is generalizing $\mathcal{B}$*-monoid* (i.e., $\mathcal{B}$-algebra in sets) considered as a kind of structure on similar family of *sets*.

For example, theorization of algebra over a multicategory $\mathcal{U}$, can be considered as the case where $\mathcal{B}$ is freely generated by $\mathcal{U}$, and the generating objects which we choose will be the indecomposable objects, i.e., the objects which come from $\mathcal{U}$. See Section 0.3.2 below.

Note in particular, that the idea above does not determine the theorization from a categorification uniquely, so there is a question on which theorization if any, we would like. In the case of algebra over a multicategory $\mathcal{U}$, we achieve the following through theorization. Firstly, our categorification is $\mathcal{U}$-monoidal category, and this already allows us to define a $\mathcal{U}$-algebra internal in a $\mathcal{U}$-monoidal category $\mathcal{A}$ as a lax $\mathcal{U}$-monoidal functor $\mathbf{1} \to \mathcal{A}$, where $\mathbf{1}$ denotes the unit $\mathcal{U}$-monoidal category. Now, through the process of theorization, this notion of $\mathcal{U}$-algebra becomes generalized to the notion of functor of $\mathcal{U}$-graded multicategories. See Example 0.17.

In general, given a specific kind of structure in a specific context, and some reasonable categorification of it, we would like a similar extension of the notion through theorization. It is a theorization which allows this that we would like, and existence of such a theorization appears to be usually a non-trivial question.

Another thing to note is that the notion of theorization which we have formulated is the “coloured” version which we did not discuss in detail in Section 0.1 or 0.2. A **colour** in the theorized structure is an object of a category in the underlying family of categories, e.g., $(\mathcal{X}_\lambda)_{\lambda\in\Lambda}$ above. This generalizes the colours in a multicategory. See Section 0.3.2 below.

To be more detailed, in the case discussed above, one can consider an object of $\mathcal{X}_\lambda$ as a colour having **degree** λ.

Remark 0.12. As mentioned in Remark 0.7, it is better to have a presentation of the form of data for a theorized structure, so in particular, we have a definition of a theorized structure *enriched* in a symmetric monoidal category. In practice, it is perhaps not difficult usually, to write down a presentation by looking at the process of the theorization carefully. Essentially, one simply needs to run the virtualization process using the enriched version of bimodules, even though, to be rigourous, there is a minor issue here that bimodules in a general symmetric monoidal category do not necessarily compose, so we need to work actually in a 2-*theory* formed by enriched categories and bimodules. However, we shall not worry about this in this introduction, and shall mainly consider only *un*enriched theorized structures.

In the concrete situations which we treat in the body, another, simpler method for theorization (which demands more concrete data as an input) will in fact give a simpler solution for enriching the theorized structure in a symmetric monoidal category, as will be seen in Section 0.3.4.

0.3.2. For a multicategory $\mathcal{U}$, let us try to interpret $\mathcal{U}$-graded multicategory as a "theorization" of $\mathcal{U}$-algebra in the defined sense.

Firstly, we consider the structure of a $\mathcal{U}$-algebra as a structure on family of objects (of a symmetric monoidal category) indexed by the objects of $\mathcal{U}$. Namely, we consider a $\mathcal{U}$-algebra A in a symmetric monoidal category $\mathcal{A}$, as consisting of

- for every object $u \in \mathrm{Ob}\,\mathcal{U}$, an object $A(u)$ of $\mathcal{A}$,
- for every finite set S and an S-ary operation $f\colon u \to u'$ in $\mathcal{U}$, where $u = (u_s)_{s\in S}$ is a family of objects of $\mathcal{U}$ indexed by S, a map $Af\colon A(u) \to A(u')$ in $\mathcal{A}$, where $A(u) := \bigotimes_{s\in S} A(u_s)$,

and then consider the latter as a structure on the family $\mathrm{Ob}\,A := \big(A(u)\big)_{u\in \mathrm{Ob}\,\mathcal{U}}$ of objects of $\mathcal{A}$. The structure is thus an action of every multimap f in $\mathcal{U}$ on the relevant members of the family $\mathrm{Ob}\,A$.

We would like next to obtain from a $\mathcal{U}$-graded multicategory $\mathcal{X}$, a family similar to $\mathrm{Ob}\,A$ above, *of categories*, to underlie $\mathcal{X}$. For this, we take the family $\mathrm{Ob}\,\mathcal{X} := (\mathcal{X}_u)_{u\in\mathrm{Ob}\,\mathcal{U}}$, where $\mathcal{X}_u$ denotes the category formed by the objects of $\mathcal{X}$ of degree u, and maps (i.e., unary multimaps) between them of degree id_u.

The rest of the structure of $\mathcal{X}$ can then be considered as a lax associative action of the rest of the multimaps f in $\mathcal{U}$, on these categories $\mathcal{X}_u$, each f acting as the bimodule formed by the multimaps in $\mathcal{X}$ of degree f, so $\mathcal{X}$ can be interpreted as obtained by putting a theorized $\mathcal{U}$-algebra structure on the family $\mathrm{Ob}\,\mathcal{X}$. (See Section 0.3.3 below to be more precise.)

Remark 0.13. Lax associativity of an action through bimodules generalizes *op*-lax associativity of an action through functors.

If a $\mathcal{U}$-graded multicategory $\mathcal{X}$ is seen as a theorized structure in this manner, then a colour in this theorized structure is an object of a category $\mathcal{X}_u$, where u is any object of $\mathcal{U}$. In other words, it is an object of the multicategory $\mathcal{X}$.

0.3.3. We would like to give a minor and technical remark.

For a multicategory $\mathcal{U}$, a natural theorization which our definition expects of $\mathcal{U}$-algebra would appear to be lax $\mathcal{U}$-algebra in the 2-category mentioned above formed by categories and bimodules between them. (Note Remark 0.13.) This does not coincide with our desired theorization, which is $\mathcal{U}$-graded multicategory.

Indeed, for an object u of $\mathcal{U}$, if a category, say $\mathcal{X}_u$, is associated to u, and the identity map of u acts on $\mathcal{X}_u$ in our 2-category of bimodules, then this acton gives another category, say $\mathcal{Y}_u$, with objects the objects of $\mathcal{X}_u$, and a map, say

$F\colon \mathcal{X}_u \to \mathcal{Y}_u$, of the structures of categories on the same collection of objects. However, the theorization in the idea described in the previous sections, is not where id_u acts on an already existing category $\mathcal{X}_u$, but where the structure of $\mathcal{X}_u$ itself as a category, is the action of id_u.

In other words, we usually do not just want to consider a relaxed structure in the 2-category of categories and bimodules, but we would further like to require that the resulting map corresponding to F in the example above, associated to each member of the underlying family of categories, to be an isomorphism.

Remark 0.14. In the example of Section 0.3.2, there is another category structure on the objects of $\mathcal{X}_u$, in which a map is a (unary) map in $\mathcal{X}$ of degree an *arbitrary* endomorphism of u in $\mathcal{U}$, rather than just the identity. This clearly does not interfere with the remark here.

0.3.4. For a final remark, for the kinds of structure which we know how to theorize, we actually have a more economical description of the theorizations than we have given above. This uses the construction of a 2-category by 'categorically delooping' an associative monoidal category $\mathcal{A}$. See Bénabou [1, 2.2] or the review in Section 1.1.2. Note that, if $\mathcal{A}$ is a symmetric monoidal category, then the resulting 2-category, which we shall denote by $B\mathcal{A}$, inherits a symmetric monoidal structure.

To turn to the description of the theorization, for the case of the notion of algebra over a multicategory $\mathcal{U}$ for example, a $\mathcal{U}$-graded multicategory enriched in a symmetric monoidal category $\mathcal{A}$, can be described as a coloured version of a lax $\mathcal{U}$-algebra in the symmetric monoidal 2-category $B\mathcal{A}$. We refer the reader to Section 1.1.2 for the case $\mathcal{U} = \mathrm{Com}$ of this. In general, it will be seen in Section 3.2.3 that, for an n-theory $\mathcal{U}$, our theorization of $\mathcal{U}$-algebra will, in the $\mathcal{A}$-enriched form, be an ("n-tuply") coloured lax $\mathcal{U}$-algebra enriched in $B\mathcal{A}$.

This idea of *coloured lax structure enriched in* $B\mathcal{A}$, is actually less redundant than the idea of theorization which we have expressed above for a general situation, and does not produce the issue discussed in Section 0.3.3, either. This idea is the one along which we actually theorize kinds of algebraic structure which we can theorize so far. See the definitions in the body. In particular, the simplest case will be observed explicitly in Proposition 2.9.

However, compared to our previous formulation of the idea of theorization, the formulation of the notion of "coloured lax structure" enriched in a symmetric monoidal 2-category, would rely more on the manner how we generate the relevant structure (e.g., the symmetric monoidal category $\mathcal{B}$ for the case of "$\mathcal{B}$-algebra"). Since the author does not know what exact data is needed for theorizing a kind of "algebraic structure" in general, he does not know a general definition of a coloured lax structure. To formulate this notion for a given kind of structure, seems essentially equivalent to theorizing the kind of structure.

Remark 0.15. On the other hand, we have an 'uncoloured theorization' as soon as we have a lax version of the kind of structure. However, unless we can further find a reasonable common generalization of this uncoloured "theorization" and the categorification, there might not be a reasonable notion of algebra over an uncoloured "theorized" object of the kind.

0.4. **A basic construction.** The idea of theorization was such that a categorified structure was an instance of the theorized form of the same structure. Let us suppose given a kind of structure and a theorization of it. Then, for a categorified structure $\mathcal{X}$, let us denote by $\Theta\mathcal{X}$, the theorized structure corresponding to $\mathcal{X}$. Concretely, $\Theta\mathcal{X}$ is obtained by replacing as needed, structure functors of $\mathcal{X}$ with bimodules copresented by them. The construction Θ generalizes the usual way to construct a multicategory from a monoidal category. Let us say that $\Theta\mathcal{X}$ is **represented** by $\mathcal{X}$.

Remark 0.16. Recall that a theorized structure in general could have colours. This flexibility is playing an important role here. Indeed, if the categorified structure $\mathcal{X}$ is structured on a family, say $\mathrm{Ob}\,\mathcal{X} = (\mathrm{Ob}_\lambda\,\mathcal{X})_{\lambda\in\Lambda}$ of categories, where Λ denotes a collection which is suitably specified in the chosen presentation of the form of the structure in question, then any object of $\mathrm{Ob}_\lambda\,\mathcal{X}$ for any $\lambda \in \Lambda$, is being a colour in the theorized structure $\Theta\mathcal{X}$.

Thus the coloured version of the notion of theorization is indeed necessary in order for every categorified structure to be an instance of a theorized structure.

As in the case of monoidal structure, Θ is usually only faithful, but not full. Indeed, for (families of) categories $\mathcal{X}, \mathcal{Y}$, each equipped with a categorified structure, a morphism $\Theta\mathcal{X} \to \Theta\mathcal{Y}$ is equivalent as data to a *lax* morphism $\mathcal{X} \to \mathcal{Y}$. See Section 2.5.3.

Example 0.17. Let $\mathcal{U}$ be a symmetric multicategory, and let $\mathcal{A}$ be a $\mathcal{U}$-monoidal category. Then a $\mathcal{U}$-algebra in $\mathcal{A}$, namely, a lax $\mathcal{U}$-monoidal functor $\mathbf{1} \to \mathcal{A}$, where $\mathbf{1}$ denotes the unit $\mathcal{U}$-monoidal category, is equivalent as data to a functor $\Theta\mathbf{1} \to \Theta\mathcal{A}$ of $\mathcal{U}$-graded multicategories, which is by definition, a $\mathcal{U}$-algebra in $\Theta\mathcal{A}$.

Remark 0.18. The functor Θ has a left adjoint (which in fact can be described in a concrete manner). In the example above, $\mathcal{U}$-algebra in $\mathcal{A}$ is thus equivalent to a $\mathcal{U}$-monoidal functor to $\mathcal{A}$ from the $\mathcal{U}$-monoidal category freely generated from the terminal $\mathcal{U}$-graded 1-theory $\Theta\mathbf{1}$ (which thus has a concrete description).

0.5. Theorization of category.

0.5.0. For illustration of the general definition, let us describe a natural theorization of the notion of category, which we shall call "categorical theory" here.

In order to define this, we first note that category can be understood as a kind of structure on family of sets. Indeed, for a category $\mathcal{X}$, there is the family $\mathrm{Map}_\mathcal{X} := \big(\mathrm{Map}_\mathcal{X}(x,y)\big)_{x,y}$ of sets of morphisms, parametrized by pairs x, y of objects of $\mathcal{X}$, so the structure of $\mathcal{X}$ can be understood as defined on this family $\mathrm{Map}_\mathcal{X}$ of sets, by the composition operations. Moreover, this presentation immediately leads to a generalization of the notion to the notion of category enriched in a symmetric monoidal category.

Now, if we choose and fix a collection as the collection of objects for our (enriched) categories, then 2-category with the same collection of objects can be considered as a categorification of those categories. Categorical theory will be a theorization of category whose associated categorification is 2-category.

The description of a **categorical theory** (enriched in sets) is as follows. Firstly, it, like a 2-category (our categorification) has objects, 1-morphisms, and sets of 2-morphisms. 1-morphisms do not compose, however. Instead, for every nerve $f\colon x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n$ of 1-morphisms and a 1-morphism $g\colon x_0 \to x_n$, one has the notion of (n**-ary**) **2-multimap** $f \to g$. The 2-morphisms, which were already mentioned, are just unary 2-multimaps. There are given unit 2-morphisms and associative composition for 2-multimaps, analogously to the similar operations for multimaps in a planar multicategory.

A 2-category in particular represents a categorical theory, in which a 2-multimap $f \to g$ is a 2-morphism $f_n \circ \cdots \circ f_1 \to g$ in the 2-category. Between two 2-categories, a natural map of the represented categorical theories is not precisely a functor, but is a lax functor of the 2-categories.

As we have also suggested, a categorical theory is also a generalization of a planar multicategory. Indeed, planar multicategory was a theorization of associative algebra. The relation between the notions of planar multicategory and of categorical theory, is parallel to the relation between the notions of associative monoid and of category. Namely, categorical theory is a 'many objects' (or "coloured") version of planar multicategory, where the word "object" here refers to one at a deeper level than the many objects which a multicategory (as a "coloured" operad) may already have are at.

One thing one should note then, is that, while the 2-multimaps in a categorical theory is generalizing the multimaps in a planar multicategory here, the 1-morphisms in a categorical theory is generalizing the *objects* of a planar multicategory, and no longer have the characteristic of operators like the 1-morphisms in a category. Indeed, a 1-morphism in a categorical theory and an object of a planar multicategory are both "colours", and we have also mentioned earlier that there is no operations of composition given for 1-morphisms in a categorical theory.

What we said above in comparison of the structures of a categorical theory and of a planar multicategory, is that a categorical theory $\mathcal{X}$ has one more layer of 'colouring' under the 1-morphisms, given by the collection of the objects of $\mathcal{X}$.

Example 0.19. Let $\mathcal{A}$ be an associative monoidal category. Then the categorical theory corresponding to the planar multicategory $\Theta\mathcal{A}$, is represented by the 2-category $B\mathcal{A}$. (To be technical, the former categorical theory is "simply coloured" in the sense that it has only one layer of colours, and it is equivalent to the 'simply coloured part' at the base object, of the categorical theory $\Theta B\mathcal{A}$. See Section 2.6.4 for an explanation in a similar situation.)

Example 0.20. The "Morita" 2-category due to Bénabou [1] of associative algebras and bimodules in a monoidal category $\mathcal{A}$ with nicely behaving colimits, is well-defined as a categorical theory when $\mathcal{A}$ is more generally, an arbitrary planar multicategory. There is a forgetful functor from this "Morita" categorical theory to $\mathcal{A}$ considered as a categorical theory.

0.5.1. Following the general pattern about theorization, there is a notion of category in a categorical theory, and, as an uncoloured version of it, monoid in a categorical theory, which generalizes monad in a 2-category. A monad in a 2-cateogry $\mathcal{X}$ was a lax functor to $\mathcal{X}$ from the unit 2-category (see Bénabou [1]), which can also be considered as a map between the categorical theories represented by these 2-categories. See Section 0.4. However, the latter is a monoid in the target categorical theory $\Theta\mathcal{X}$ by definition.

Example 0.21. Let $\mathcal{C}$ be a category enriched in groupoids, and let M be a monad *on* $\mathcal{C}$. Then there is a categorical theory as follows.

- An object is an object of $\mathcal{C}$.
- A map $x \to y$ is a map $Mx \to y$ in $\mathcal{C}$.
- Given a sequence of objects $x_0, \dots, x_n$ and maps $f_i \colon Mx_{i-1} \to x_i$ in $\mathcal{C}$ and $g \colon Mx_0 \to x_n$, the set $\mathrm{Mul}(f; g)$ of 2-multimaps $f \to g$, is the set of *commutative* diagrams

$$\begin{array}{ccccc} M^n x_0 & \xrightarrow{M^{n-1}f_1} & \cdots & \xrightarrow{f_n} & x_n \\ {\scriptstyle m}\downarrow & & & & \downarrow{\scriptstyle =} \\ Mx_0 & & \xrightarrow{\quad g \quad} & & x_n \end{array}$$

  in $\mathcal{C}$ (i.e., the set of isomorphisms filling the rectangle), where m denotes the multiplication operation on M.
- Composition is done in the obvious manner.

A monoid in this categorical theory is exactly an M-algebra in $\mathcal{C}$.

More generally, a category in a categorical theory can be described as a coloured version of a map of categorical theories. A basic example is a category in the categorical theory obtained by considering a planar multicategory $\mathcal{U}$ as a categorical theory. This is equivalent as data to a category enriched in the planar multicategory $\mathcal{U}$.

0.6. **Iterating theorization.**

0.6.0. In Section 0.5, we have theorized the notion of category by considering a category as a structure on the family of sets consisting of the sets of maps. Recall from Example 0.10 that a category was an 'initially graded' 1-theory. Section 0.3.2 shows thus that category is a theorization of Init-algebra, or bare, i.e., unstructured, object. (It is not difficult to see in the similar manner, that the versions enriched in a symmetric monoidal category, of the relevant notions also coincide.) For these reasons, we shall call a categorical theory also an "Init-graded 2-theory".

One can similarly consider the structure of a categorical theory as a structure on the sets of its 2-multimaps, and then try to theorize the notion of categorical theory after fixing the collections of objects and of 1-morphisms. It turns out that there is indeed an interesting theorization in this case, which in particular generalizes the notion of 3-category. One might call the resulting theorized object a categorical 2-theory or an initially graded 3-theory.

One might ask whether it is possible to iterate theorization in a similar manner here, or starting from some specific additional structure, rather than 'no structure at all'. We should of course ask possibility of interesting theorizations. One condition for this has been mentioned in Section 0.3.1. Other desirable things may include abundance of natural examples, and reasonable properties.

In the case where the answer to the question is affirmative, just as the original structure could be expressed as the structure of an algebra over the terminal object among the (unenriched) theorized objects of the same kind, the theorization similarly becomes the structure of an algebra over the terminal object among the twice theorized objects, and so on, so all the structures can be described using their iterated theorizations. Moreover, by considering an algebra over a non-terminal theory, an algebra over it, and so on, one obtains various general structures, which can all be treated in a unified manner.

The question asked above is non-trivial. However, we introduce the notion of n-theory in this work, which will inductively be an interesting theorization of $(n-1)$-theory. The hierarcy as n varies, of n-theories will be an infinite hierarchy of iterated theorizations which extends the various standard hierarchies of iterated categorifications, in particular, the hierarcy of n-categories in the "initially graded" case.

While our higher theories will be in general a completely new mathematical objects, we have already found very classical objects of mathematics among 2-theories. Namely, while we have seen that a categorical theory was an "initially graded" 2-theory, we have also noted in Section 0.5, that planar multicategories were among categorical theories.

We have also seen non-classical objects among 2-theories in Section 0.5. Less exotic examples of higher theories comes from the construction of Section 0.4. Namely, a higher categorified instance of a *lower* theorized structure leads to a higher theorized structure through the iterated application of the construction Θ (details of which can be found in Section 2.4). As an object, this is less interesting among the general higher theories for the very reason that it is represented by a lower theory. However, the *functors* between these theories are interesting in that it is much more general than functors which we consider between the original lower theories.

Namely, a functor between such higher theories amounts to highly *relaxed* functor between the higher categorified lower theories, as follows from an iteration of the remark in Section 0.4.

In Section 2.6 (as will be previewed in Section 0.7.2), we discuss a general construction which we call "delooping", through which we obtain an $(n+1)$-theory which normally fails to be representable by a categorified n-theory. Another construction, which is closely related to the classical Day convolution, will also be discussed in Section 4.4, and this also produces similar examples.

0.6.1. The notion of 2-theory immediately leads to a generalization of Proposition 0.0. Indeed, for a symmetric multicategory $\mathcal{U}$, one can define a theorization of $\mathcal{U}$-graded 1-theory, which we call *$\mathcal{U}$-graded 2-theory.* It follows from the general idea on theorization that the notion of $\mathcal{U}$-graded 1-theory makes sense naturally in a $\mathcal{U}$-graded 2-theory, and this notion gives a generalization of Proposition 0.0. Let us see this. (The discussion below will be slightly imprecise even though essentially correct. A more technical account can be found in Section 4.1.)

Given a $(\mathcal{U} \otimes E_1)$-monoidal category $\mathcal{A}$, one can categorically deloop $\mathcal{A}$ using the E_1-monoidal structure to obtain a $\mathcal{U}$-monoidal 2-category $B\mathcal{A}$, and hence a $\mathcal{U}$-graded 1-theory $\Theta B\mathcal{A}$ enriched in categories. Since this is a categorified $\mathcal{U}$-graded 1-theory, the notion of $\mathcal{U}$-graded operad in $\Theta B\mathcal{A}$ makes sense, which naturally generalizes from the case where $\mathcal{A}$ is symmetric monoidal, the notion of $\mathcal{U}$-graded operad in $\mathcal{A}$. Moreover, it is easy to check when $\mathcal{U}$ is one of the most familiar operads, that this coincides with the usual notion. However, this notion of operad, including the coloured cases of it, is nothing but the notion of 1-theory in the $\mathcal{U}$-graded 2-theory $\Theta(\Theta B\mathcal{A})$.

0.6.2. More generally, with appropriate notion of grading, an n-theory makes sense in an $(n+1)$-theory. Let us briefly discuss the notion of grading for higher theories. (The details will be found in Section 3.)

A starting point is that there is notion of algebra over each n-theory. We have mentioned that algebra over the terminal unenriched n-theory coincides with the notion of $(n-1)$-theory. For an n-theory $\mathcal{U}$ enriched in sets, we can theorize the notion of $\mathcal{U}$-algebra, generalizing from the case where $\mathcal{U}$ is terminal. We call our theorization **$\mathcal{U}$-graded n-theory**, where our term comes from the following. Compare with our discussion in Section 0.2.1.

Proposition 0.22 (Proposition 3.8)**.** *An unenriched $\mathcal{U}$-graded n-theory is equivalent as data to an unenriched **symmetric** n-theory $\mathcal{X}$ equipped with a functor $\mathcal{X} \to \mathcal{U}$ of symmetric n-theories.*

It would therefore seem natural to call a $\mathcal{U}$-algebra a **$\mathcal{U}$-graded $(n-1)$-theory**, and indeed, there is a natural notion of $\mathcal{U}$-graded 0-*theory* of which $\mathcal{U}$-algebra is an $(n-1)$-th theorization (Section 3.3.1). It is therefore also natural to refer to the intermediate theorizations as $\mathcal{U}$-graded **m-theory** for $1 \leq m \leq n-2$.

On the other hand, there is also a natural theorization of $\mathcal{U}$-graded n-theory, which we of course call **$\mathcal{U}$-graded $(n+1)$-theory**. We obtain the following fundamental results.

Theorem 0.23 (Theorems 3.17, 3.19)**.** *A $\mathcal{U}$-graded n-theory is equivalent as data to an algebra over the $(n+1)$-theory $\Theta\mathcal{U}$. A $\mathcal{U}$-graded $(n+1)$-theory is equivalent as data to a $\Theta\mathcal{U}$-graded $(n+1)$-theory.*

It follows that the natural notion of $\mathcal{U}$-graded m-theory for $m \geq n+2$, is simply $\Theta_n^m\mathcal{U}$-graded m-theory, where Θ_n^m denotes the $(m-n)$-fold 'iteration' of the construction Θ (Section 2.4).

We now obtain that $\mathcal{U}$-graded m-theory is a notion which naturally makes sense in a $\mathcal{U}$-graded $(m+1)$-theory. This is a more general enriched version of the notion, to be investegated in Section 4.

Example 0.24. Let us denote the terminal n-theory by $\mathbf{1}^n_{\mathrm{Com}}$, and let $\Delta\colon \mathcal{U} \to \mathbf{1}^n_{\mathrm{Com}}$ be the unique functor. Since every n-theory $\mathcal{V}$ is graded by $\mathbf{1}^n_{\mathrm{Com}}$, one obtains from this a $\mathcal{U}$-graded n-theory $\Delta^*\mathcal{V}$. A $\mathcal{U}$-algebra in $\Delta^*\mathcal{V}$ amounts to a (coloured) functor $\mathcal{U} \to \mathcal{V}$ of n-theories.

0.7. **Further developments.**

0.7.0. Let us preview a few more highlights of our work.

0.7.1. In addition to algebra over an n-theory enriched in sets, we also define the notion of algebra over a monoid (i.e., algebra enriched in sets) over an n-theory enriched in sets (if $n \geq 2$), algebra over unenriched such (if $n \geq 3$), and so on. For example, for an n-theory $\mathcal{U}$ enriched in sets, a $\mathcal{U}$-graded $(n-2)$-theory can be expressed as an algebra over the terminal $\mathcal{U}$-monoid, among which the terminal unenriched one is such that an algebra over it is exactly a $\mathcal{U}$-graded $(n-3)$-theory, and so on.

More generally, for a $\mathcal{U}$-graded m-theory $\mathcal{X}$ enriched in sets, we define the notion of $\mathcal{X}$-algebra, and more generally, of $\mathcal{X}$-graded ℓ-theory, as well as the notion of higher theory graded by unenriched such, and so on. We can actually give simple definitions of all these, using Theorem 0.23 as a general principle (Section 3.3). At the end, every structure (which is enriched in sets or groupoids) will come with a hierarchy of higher theories "graded" by it. We obtain a generalization of Proposition 0.22 with these new notions (Proposition 3.34). We also obtain a quite general enrichment of all the notions in Section 4.

0.7.2. An n-theory enriched in a symmetric monoidal category $\mathcal{A}$, can also be described as a (suitable) n-theory in the $(n+1)$-theory $\Theta^{n+1}B^n\mathcal{A}$ obtained by applying the construction Θ $n+1$ times to the symmetric monoidal $(n+1)$-category $B^n\mathcal{A}$ (the n-th iterated categorical deloop of $\mathcal{A}$). In Section 2.6, we generalize the categorical delooping construction for symmeric monoidal higher category, to a certain construction $\mathbb{B}$ which produces a symmetric n-theory from a symmetric $(n-1)$-theory. This is a generalization of the categorical delooping in such a manner that, for a symmetric monoidal category $\mathcal{A}$, there is a natural 'equivalence' $\Theta^{n+1}B^n\mathcal{A} = \mathbb{B}^n\Theta\mathcal{A}$ (or Corollary 2.15, to be more precise). It follows that a natural notion of n-theory enriched in a symmetric multicategory $\mathcal{M}$, which is not necessarily of the form $\Theta\mathcal{A}$, is n-theory in the $(n+1)$-theory $\mathbb{B}^n\mathcal{M}$, and, as n increases, the notion iteratively 'theorizes' the previous notions in a suitable sense. (See the remark after Definition 2.16.)

Incidentally, if $\mathcal{U}$ is not of the form $\Theta\mathcal{A}$, then $\mathbb{B}^n\mathcal{U}$ is usually *not* representable by a categorified n-theory.

0.7.3. For a symmetric monoidal n-category $\mathcal{C}$, we construct a certain $(n+1)$-theory $A_n\mathcal{C}$ and a functor $A_n\mathcal{C} \to \Theta^{n+1}B^n\mathrm{Set}$ of $(n+1)$-theories. The use of this is the following. Namely, while we have already mentioned that an n-theory $\mathcal{X}$ enriched in sets can be considered as an n-theory in $\Theta^{n+1}B^n\mathrm{Set}$, the construction above allows us to understand an $\mathcal{X}$-graded m-theory enriched in a symmetric monoidal category $\mathcal{A}$, where $0 \leq m \leq n-1$, as an appropriate lift of the theory to $A_nB^m\mathcal{A}$ (a more general statement being as Corollary 4.10, where $A_n = \Theta_n A_* \Theta^n_0$ in the notation there). There is also a version of this for $m = n$ (Corollary 4.12), which will have an application in our work.

0.7.4. We also touch on more topics, such as the following.

- Pull-back and push-forward constructions which changes gradings, and their properties (Sections 3.3, 4.3). The result Corollary 4.12 mentioned in Section 0.7.3, will be used for the construction of the push-forward 'on the right side'.
- Some other basic constructions such as a construction for higher theories related to Day's convolution [8]. The Day type construction leads to a notion of algebra over an *enriched* higher theory (Section 4.4).
- Hierarchies of iterated theorizations associated to more general systems of operations, with multiple inputs *and* multiple outputs, such as operations of 'shapes' of bordisms as in various versions of a topological field theory (Section 5). Examples also include iterated theorizations of the notion due to Vallette of coloured *properad* [18]. See Example 5.5.

The last topic leads to vast generalizations of the relevant versions of the notion of topological field theory. We obtain a simple but in a way exotic example (Section 5.2) in addition to examples of a more expected type.

0.7.5. We also enrich everything we consider in this work, in the Cartesian symmetric monoidal infinity 1-category of infinity groupoids, instead of in sets. Fortunately, this does not add any difficulty to the discussions. However, we invite the reader who does not wish to deal with homotopy theory, to Section 0.9.2.

0.8. **Outline.** We shall give a definition of the most unstructured version of a higher theory in **Section 1**. We shall follow up the definition in **Section 2** with discussions of simple subjects such as a planar variant, a less coloured variant, the construction Θ, and the generalized "delooping" construction. We shall then discuss algebra and graded higher theory over a higher theory in **Section 3**. This will include a discussion of "iterated monoid" over a higher theory. We shall then discuss in **Section 4**, a general manner for enriching the notion of higher theory, and various topics about this, such as a construction for higher theories which is related to the Day convolution. Finally, we shall discuss iterated theorizations of more general algebraic structure in **Section 5**. **Appendix** is for comparison of this work with a related important work [0] of Baez and Dolan.

0.9. **Notes for the reader.**

0.9.0. The first section here is about founding our treatment on homotopy theory. We suggest the reader who does not wish to deal with homotopy theory, to skip to *Section 0.9.2.*

0.9.1. For practical purposes, it seems best to build our theory in the framework of homotopy theory. Namely, we would like to let the infinity 1-category of infinity groupoids be the default place where we enrich any categorical or algebraic structure.

We shall simply choose this infinity 1-category to be our starting point, even though this is not actually necessary. This is not necessary since we could instead start from the category of sets, and find a fetus of higher category theory within what we build. In fact, the model for higher category theory which seems most convenient for this work is the one built by working on what amounts to a certain small part of what we build in this work; n-theories, which will be the subject of this work, will be much more general than n-categories. One model for higher category theory is actually just a few technical steps away from our work here.

However, we choose to separate construction of this model for higher category theory from the purpose of this work, to have two simpler and better focused expositions instead of one which would inevitably be more complicated and less focused. The other exposition will appear in [17]. We feel that the present approach keeps things simpler.

On the other hand, the subject of the present work does not select by any means, the mentioned model for higher category theory as the only acceptable model. Indeed, any other reasonable model for higher category theory of the reader's choice, would also be good as a foundation for this work. In order to communicate to as wide audience as we can, we shall try to make clear which data are being used from the theory of higher categories when we use them, so the hope is that even the reader who has no more than basic ideas on the essence of the higher category theory, would find our exposition largely accessible.

0.9.2. For the reader who does not wish to deal with homotopy theory, our terminology in the body will be such that it could be read as if we are working in the framework of the classical, discrete category theory. For example, we shall say "category" to in fact mean "infinity 1-category". So such a reader would be comfortable with interpreting what we write in the normal, classical manner, and then ignoring what is redundant in such an interpretation.

Remark 0.25. Here are two cautions. One is that, when we say "groupoid" (while in fact meaning infinity groupoid), this can often be interpreted as set in the classical context, but sometimes, it will be better to interpret it honestly as "groupoid" (i.e., 1-groupoid). The other is that there is fear that some of the examples we give may degenerate to trivialities in the non-homotopical interpretation.

While we also welcome the reader who does this, our method for theorization is by dealing with the structure of the coherence for higher associativity, which is also the key for higher category theory, so our expectation is that the reader who interprets our work in the classical context, would eventually find the "homotopical" interpretation more natural.

0.9.3. As we have mentioned in Section 0.9.2, we adopt the convention that all terms should be interpreted in homotopical/infinity 1-categorical sense. Namely, categorical terms are used in the sense enriched in the infinity 1-category of infinity groupoids, and algebraic terms are used freely in the sense generalized in accordance with the enriched categorical structures.

However, we do welcome the reader who prefers to work in the classical setting, to interpret the terms in the usual, non-homotopical manner. In this case, categorical terms (e.g., multicategory) should be understood in the sense enriched in the category of sets (or sometimes better groupoids) unless otherwise specified, and Remark 0.5 would continue in effect.

For example, by a 1-*category*, we officially mean an *infinity* 1-category, while also welcoming the classical interpretation. We often call a 1-category (namely an infinity 1-category) simply a **category**. More generally, for an integer $n \geq 0$, by an n-*category* (resp. *infinity* category), we mean an *infinity* n-category (resp. infinity infinity category).

0.9.4. Unless otherwise noted, we do not consider non-unital associative algebraic structures. Moreover, we normally treat unitality as part of associativity.

0.9.5. In our notations, we shall freely put a non-negative integer (or a variable, such as "n", for a non-negative integer) as a superscript to a letter in order to avoid excessive use of multiple subscripts. Other than the exceptions listed below, and unless otherwise noted, such a superscript will be a label just like a subscript, put on the right upper corner in order to preserve rooms for subscripts. In particular, there will be only few occasions where we need to take a power of a thing, in which case, we shall indicate so.

Major exceptions are as follows.

- "Δ^n", "d^i", "$\mathbb{R}^n$" will respectively denote the n-dimensional symplex, the i-th simplicial coface operator, n-dimensional Euclidean space.
- "f^{-1}" for a map or a morphism f, will denote the inverse of f, or the inverse image by the map f.
- "B^n" and "$\mathbb{B}^n$" will denote the n-fold applications of the ("delooping") constructions "B" and "$\mathbb{B}$" respectively (which will have been defined).
- "Θ^n_m" will denote the instance of a certain construction (defined in this work) which applies to an "m-theory", and produces an "n-theory".
- "$\mathbf{1}^n_{\mathcal{U}}$" and "$\mathbb{U}^n$" will respectively denote the terminal ($\mathcal{U}$-graded, unenriched uncoloured) n-theory and the n-dimensional "universal" monoid.

All other exceptions will be noted at the relevant places.

1. Symmetric higher theories

1.0. **Introduction.** After giving a small number of preliminary definitions, we shall give in this section, the definition of a *symmetric higher theory*, which will be a higher theory with least amount of structure.

1.1. The definition.

1.1.0. Let

- Ord denote the category (in the classical, discrete sense) of finite ordinals (including the empty set $\varnothing$),
- $\mathbf{\Delta}$ denote the category (again in the discrete sense) of combinatorial simplices, in other words, non-empty finite ordinals.

For example, we have objects $[0] = \{0\}$ and $[1] = \{0 < 1\}$ of $\mathbf{\Delta}$, and the maps in $\mathbf{\Delta}$ called the *coface operators* $d^i\colon [0] \to [1]$, for $i = 0, 1$, where $d^0(0) = 1$, $d^1(0) = 0$.

The following is about all of the combinatorics which we need for this work. Namely, there is a functor $[-]\colon \mathrm{Ord} \to \mathbf{\Delta}^{\mathrm{op}}$ defined as follows.

For an object $I \in \mathrm{Ord}$, we define

$$[I] := [1] \;{}^{d^0}\!\cup^{d^1}_{[0]} \cdots {}^{d^0}\!\cup^{d^1}_{[0]} [1] \qquad (I\text{-fold; e.g., } [\varnothing] = [0]).$$

In other words, $[I] = \bigcup_{i\in I}[1]$ is obtained by gluing for every pair $i < i+1$ of adjacent elements of I, 1 in the i-th component $[1]$, with 0 in the $(i+1)$-th component $[1]$.

For a map $\phi\colon I \to J$ in Ord, we note that $[I] = \bigcup_{j\in J}[\phi^{-1}j]$, where adjacent components are glued (similarly to before) at the respective maximal and minimal elements. We define $[\phi]\colon [I] \leftarrow [J]$ in $\mathbf{\Delta}$ to be the map obtained by gluing over $j \in J$, the maps $[1] \to [\phi^{-1}j]$ in $\mathbf{\Delta}$ preserving the minimum and the maximum.

Remark 1.0. Let

- Fin denote the category (in the discrete sense) of finite sets,
- Fin_* denote the category (discrete sense) of pointed finite sets.

Namely, an object of Fin_* is a finite set S equipped with a "base point" $* \in S$, and a morphism is a map f for which we have $f(*) = *$.

Then there is a commutative square

$$\begin{array}{ccc} \mathrm{Ord} & \xrightarrow{[-]} & \mathbf{\Delta}^{\mathrm{op}} \\ {\scriptstyle \mathrm{forget}}\downarrow & & \downarrow{\scriptstyle \Delta^1/\partial\Delta^1} \\ \mathrm{Fin} & \xrightarrow{(\ )_+} & \mathrm{Fin}_*, \end{array}$$

where Δ^1 denotes the simplicial 1-simplex $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Fin}$, with its boundary $\partial\Delta^1$, and $(\ )_+$ is the functor which externally adds a base point to every finite set.

Remark 1.1. The functor $[-]\colon \mathrm{Ord} \to \mathbf{\Delta}^{\mathrm{op}}$ has a right adjoint $\widetilde{\Delta}^1\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Ord}$, which, as a functor, is the lift of $\Delta^1\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Fin}$ obtained by putting the natural total order on the set of all faces of Δ^1 of each fixed dimension. In particular,

$$[I] = \mathrm{Hom}_{\mathbf{\Delta}}([0],[I]) \simeq \mathrm{Hom}_{\mathrm{Ord}}(I, \widetilde{\Delta}^1_0)$$

as a functor of $I \in \mathrm{Ord}$.

Notation 1.2. In the following, we usually write the elements of an ordinal I as $1 < 2 < \cdots$ in the ascending order, and then the elements of $[I]$ as $0 < 1 < 2 < \cdots$.

1.1.1. Next, we explain our terminology and notations conerning families, nerves and operations on them. Here we introduce only a minimal amount of it; more will be introduced later during various other definitions.

For $S \in \mathrm{Fin}$, we mean by an S-**family** a family of mathematical objects indexed by the elements of S.

Let $\phi\colon T \to S$ be a map in Fin. Then from an S-family $x = (x_s)_{s\in S}$, we obtain a T-family $\phi^*x := (x_{\phi t})_{t\in T}$.

For $I \in \mathrm{Ord}$, we mean by an I-**nerve** in a category, a pair consisting of

- a $[I]$-family of objects $x = (x_i)_{i\in[I]}$, and
- an I-family of maps $f = (f_i)_{i\in I}$, where $f_i\colon x_{i-1} \to x_i$.

Such an f is also called an I-nerve **connecting** the $[I]$-family x.

Let $\phi\colon I \to J$ be a map in Ord. Then from an $[I]$-family x, we obtain an $[J]$-family $\phi_!x := [\phi]^*x$, and from an I-nerve f as above connecting x, we obtain an J-nerve $\phi_!f$ connecting $\phi_!x$, defined by

$$(\phi_!f)_j = f_{[\phi](j)} \cdots f_{[\phi](j-1)+1}.$$

Note that $\{[\phi](j-1)+1 < \cdots < [\phi](j)\} = \phi^{-1}j \subset I$.

Definition 1.3. Let $I \in \mathrm{Ord}$. Then an $[I]$-family J in either Ord or Fin, is said to be **elemental** if $J_{[\pi](1)} = *$, the terminal object, where π denotes the unique map $I \to \{1\}$, so $[\pi](1)$ is the maximum of $[I]$.

1.1.2. The notion of multicategory is a theorization of the notion of commutative algebra. Indeed, a multicategory (enriched in groupoids) is a virtualized form of a op-lax symmetric monoidal category.

We can also formulate the notion of coloured lax commutative algebra as follows.

Definition 1.4. A **coloured lax commutative algebra** U in a symmetric monoidal 2-category $\mathcal{A}$ consists of the following data.

(0) A collection $\mathrm{Ob}\,U$, whose member is called an *object* of U, and, for every object $u \in \mathrm{Ob}\,U$, an object $U(u) \in \mathrm{Ob}\,\mathcal{A}$.

(1) For every finite set S, every S-family $u_0 = (u_{0s})_{s\in S}$ of objects of U, and every object u_1, a map $m^U_1(u_0; u_1)\colon U(u_0) \to U(u_1)$, where $U(u_0) := \bigotimes_{s\in S} U(u_{0s})$.

(2) Suppose given
  - a finite ordinal I,
  - an elemental $[I]$-family $S = (S_i)_{i\in[I]}$ of finite sets, and an I-nerve $\phi = (\phi_i)_{i\in I}$ in Fin connecting S,
  - for every $i \in [I]$, an S_i-family $u_i = (u_{is})_{s\in S_i}$ in $\mathrm{Ob}\,U$.

  Then a 2-morphism $m^U_2\colon \pi_!m^U_1[u] \to m^U_1(u_0; u_{[\pi](1)})$ in $\mathcal{A}$, where
  - $m^U_1[u] := \bigl(m^U_1(u_{i-1}; u_i)\bigr)_{i\in I}$, where $m^U_1(u_{i-1}; u_i) := \bigotimes_{s\in S_i} m^U_1(u_{i-1}|_s; u_{is})$, where $u_{i-1}|_s$ denotes the restriction of u_{i-1} to $(\phi_i)^{-1}s \subset S_{i-1}$, namely,

$m_1^U[u]$ is an I-nerve in $\mathcal{A}$ connecting the $[I]$-family $U(u) := \big(U(u_i)\big)_{i\in[I]}$ of objects,

– π denotes the unique map $I \to \{1\}$.

(∞) Data of *coherence* for the structure.

We shall not write down the details of (∞) since the explicit form of it is not so important here. A more general case is treated in Definition 1.6, in particular, Sections 1.8, 1.9, below.

Now, given a monoidal category $\mathcal{A}$, by its *categorical deloop*, we mean the 2-category $B\mathcal{A}$ with a chosen "base" object, in which all objects are equivalent, and the endomorphism monoidal category of the base object is given an equivalence with $\mathcal{A}$. (Note that this determines $B\mathcal{A}$ uniquely.)

For a symmetric monoidal category $\mathcal{A}$, we can consider a multicategory *enriched in* $\mathcal{A}$ as, by definition, a coloured lax commutative algebra $\mathcal{U}$ in the *symmetric monoidal* 2-category $B\mathcal{A}$ (with the induced symmetric monoidal structure) such that, for every object $u \in \mathrm{Ob}\mathcal{U}$, $\mathcal{U}(u)$ is the unit (i.e., the base) object of $B\mathcal{A}$. For a multicategory $\mathcal{U}$, we denote the object $m_1^{\mathcal{U}}(u_0; u_1) \in \mathcal{A}$ by $\mathrm{Mul}_{\mathcal{U}}(u_0; u_1)$. In the case where $\mathcal{A}$ is the Cartesian symmetric monoidal category Gpd of groupoids, $\mathrm{Mul}_{\mathcal{U}}(u_0; u_1)$ is the groupoid of multimaps $u_0 \to u_1$ in $\mathcal{U}$. For a general $\mathcal{A}$, the object $\mathrm{Mul}_{\mathcal{U}}(u_0; u_1)$ of $\mathcal{A}$ is made to behave as if it were formed by (generally fantastical) multimaps $u_0 \to u_1$.

For example, in the case where $\mathcal{A}$ is the Cartesian symmetric monoidal category of categories (with a fixed limit for the size), a multicategory enriched in $\mathcal{A}$ is the obvious categoried form of a multicategory. This is not a useless notion, and the notion of coloured lax commutative algebra can more generally be defined in a categorified multicategory, for example. A more general notion of enrichment will be the subject of Section 4.

1.1.3. Starting from commutative algebra and multicategory, there is an infinite hierarchy of iterated theorizations. The idea for its construction is simple. We consider the structure of a multicategory as given by an associative system of "composition" operations for multimaps. In general, we would like to produce from one kind of associative system of operations, another kind of associative system of operations. This can be done using the following simple observation on the inductivity of the structure of coherent associativity.

Suppose that we have a collection m of operations (e.g., m_1^U of Definition 1.4) which, if made coherently associative, would define (in perhaps a special case) n-th theorized version of a multicategory (e.g., a multicategory if $n = 0$). Suppose further that we actually have a collection m' of (at least lax) associativity maps for the operations m, but that we still do not have coherence data for these maps m', so m is not yet coherently (lax) associative. An example of m' is m_2^U for $m = m_1^U$ in Definition 1.4.

In this situation, data of coherence for the lax associativity has the following interpretation. Consider m' as itself a new collection of *operations*. Then the coherence data we are looking for amounts precisely to data of coherent *associativity* for the collection *m' of operations*.

A $(n+1)$-theorization of multicategory is then obtained by formalizing the structure given by the collection m' of operations and its coherent associativity. See Definition 1.6 below for the details.

Remark 1.5. As mentioned in Remark 0.4, the same idea leads (with relatively simple other ideas) to a model for higher category theory which has a certain convenient feature [17]. Indeed, since the structure with which we are dealing here is more general, the most crucial steps in the construction of this model for higher category theory, are also naturally embedded in this work, and most notably, in the following definition.

Definition 1.6. Let $n \geq 2$ be an integer. A **symmetric n-theory** $\mathcal{U}$ (which will often be called simply an "**n-theory**") enriched in a symmetric monoidal category $\mathcal{A}$, consists of data of the forms specified below as (0), (1), (2) (or just (0), (1) if $n = 2$), "(k)" for every integer k such that $3 \leq k \leq n-1$, (n), ($n+1$), ($n+2$), and "($n+\ell$)" for every integer $\ell \geq 3$.

We refer to a multicategory enriched in $\mathcal{A}$ also as a (symmetric) **1-theory** enriched in $\mathcal{A}$. We refer to a commutative algebra in $\mathcal{A}$ also as a (symmetric) **0-theory** ("enriched") in $\mathcal{A}$.

The case where $\mathcal{A}$ is the Cartesian symmetric monoidal category Gpd of groupoids, of n-theories, will play important roles, so we let Gpd be the default place where an n-theory is to be enriched, and consider an n-theory enriched in groupoids as an **unenriched** n-theory. An **n-theory** in the narrower sense will mean an "unenriched" n-theory.

An **n-theory** in the broader sense will mean an enriched (or unenriched) n-theory. Later, enrichment of an n-theory will be considered in a more general place than a symmetric monoidal category.

Specification of the forms of data will occupy the rest of this section.

1.2. **Objects of a higher theory.** The form of data (0) for Definition 1.6 is as follows.

(0) A collection Ob$\mathcal{U}$, whose member will be called an **object** of $\mathcal{U}$.

1.3. **Multimaps in a higher theory.** The form of data (1) for Definition 1.6 is as follows.

(1) Suppose given
  - (0′) an elemental [1]-family $I = (S, *)$ of finite sets, and a $\{1\}$-nerve ($\pi\colon S \to *$) in Fin connecting I, whole of which is determined by a free choice of S,
  - (0″) an S-family $u_0 = (u_{0s})_{s \in S}$ of objects of $\mathcal{U}$, and an object u_1.

  Then a collection $\mathrm{Mul}^{\pi}_{\mathcal{U}}(u_0; u_1)$ or $\mathrm{Mul}^{\pi}_{\mathcal{U}}[u]$ for short, whose member will be called an (S-ary) (**1-**)**multimap** $u_0 \to u_1$ in $\mathcal{U}$.

Observe that data of this form extend as follows. Suppose given

- $\psi\colon S \to T$ in Fin,
- u_0 as above,
- a T-family u_1 of objects of $\mathcal{U}$.

Then we let $\mathrm{Mul}^{\psi}_{\mathcal{U}}[u]$ denote the collection of all T-families $v = (v_t)_{t \in T}$, where v_t is a multimap $u_0|_t \to u_{1t}$ in $\mathcal{U}$, where the source here is the restriction of the S-family u_0 to $\psi^{-1}t \subset S$, so v_t is $\psi^{-1}t$-ary.

1.4. **2-multimaps in an n-theory, in the case $n \geq 3$.**

1.4.0. The form of data (2) for Definition 1.6 is as follows.

(2) Suppose given
  - (1′) an elemental [1]-family $I^1 = (I^1_0, \{1\})$ in Ord, and an $\{1\}$-nerve ($\pi\colon I^1_0 \to \{1\}$),
  - (0′) an elemental $[I^1_0]$-family $I^0 = (I^0_i)_{i \in [I^1_0]}$ in Fin, and a I^1_0-nerve ϕ^0 connecting I^0, namely, $\phi^0 = (\phi^0_i)_{i \in I^1_0}$, where $\phi^0_i\colon I^0_{i-1} \to I^0_i$,
  - (0″) an I^0-family u^0 in Ob$\mathcal{U}$, namely, $u^0 = (u^0_i)_{i \in [I^1_0]}$, where u^0_i is an I^0_i-family in Ob$\mathcal{U}$,

(1″) $*$ a ϕ^0-*nerve* u^1_0 of multimaps in $\mathcal{U}$, *connecting* u^0, which *by definition* means that $u^1_0 = (u^1_{0i})_{i\in I^1_0}$, where $u^1_{0i} \in \mathrm{Mul}^{\phi^0_i}_{\mathcal{U}}(u^0_{i-1}; u^0_i)$,
$*$ $u^1_1 \in \mathrm{Mul}^{\pi_!\phi^0}[\pi_! u^0]$.

Then a collection $\mathrm{Mul}^{\pi}_{\mathcal{U}}[u^0](u^1_0; u^1_1)$ or $\mathrm{Mul}^{\pi}_{\mathcal{U}}[u]$ for short, whose member will be called a **2-multimap** $u^1_0 \to u^1_1$ in $\mathcal{U}$.

1.4.1. We can extend data of this form from above for the similar input data with the requirement that the $[I^1_0]$-family I^0 be elemental discarded. The idea is to treat such input data as an $I^0_{[\pi(1)]}$-family of *elemental* data, to obtain an $I^0_{[\pi(1)]}$-family of outputs.

Thus, for input data similar to (1′) through (1″) above with I^0 *non*-elemental, we let $\mathrm{Mul}^{\pi}[u]_{\mathcal{U}}$ denote the collection whose member is an $I^0_{[\pi](1)}$-family $(v_s)_{s\in I^0_{[\pi](1)}}$ of 2-multimaps in $\mathcal{U}$, where $v_s \in \mathrm{Mul}^{\pi}_{\mathcal{U}}\left[u^0|_s\right]\left(u^1_0\big|_s; u^1_{1s}\right)$, where

- $u^0|_s := \left(u^0_i\big|_s\right)_{i\in[I^1_0]}$, where $u^0_i\big|_s$ is the restriction of u^0_i to $I^0_{is} := (\phi^0_{\leftarrow i})^{-1}s \subset I^0_i$, where $\phi^0_{\leftarrow i} := \phi^0_{[\pi](1)}\cdots\phi^0_{i+1}$,
- $u^1_0\big|_s := \left(\left(u^1_0\big|_s\right)_i\right)_{i\in I^1_0}$, where $\left(u^1_0\big|_s\right)_i := \left(u^1_{0it}\right)_{t\in I^0_{is}}$,

so $\left(u^1_0\big|_s\right)_i \in \mathrm{Mul}^{(\phi^0_{/s})_i}\left(u^0_{i-1}\big|_s; u^0_i\big|_s\right)$, where $(\phi^0_{/s})_i := \phi^0_i\big|_{I^0_{i-1,s}}\colon I^0_{i-1,s} \to I^0_{i,s}$, so $\phi^0_{/s}$ is an I^1_0-nerve in Fin connecting the *elemental* $[I^1_0]$-family $(I^0_{is})_{i\in[I^1_0]}$. In other words, $u^1_0\big|_s$ is a $\phi^0_{/s}$-nerve of multimaps in $\mathcal{U}$ connecting $u^0|_s$.

1.4.2. We can extend data (2) further by discarding the requirement that I^1 be elemental, in the similar way as above.

To do this, suppose given

- a map $\psi\colon I^1_0 \to I^1_1$ in Ord,
- not necessarily elemental data of the form (0′) and (0″) of (2),
- a ϕ^0-nerve u^1_0 and a $\psi_!\phi^0$-nerve u^1_1 respectively, of multimaps in $\mathcal{U}$ (which we mean to be interpreted following the previous step).

Then we let $\mathrm{Mul}^{\psi}_{\mathcal{U}}[u]$ denote the collection whose member is an I^1_1-family $v = (v_i)_{i\in I^1_1}$ of 2-multimaps in $\mathcal{U}$, where $v_i \in \mathrm{Mul}^{\pi_i}\left[u^0|_i\right]\left(u^1_0\big|_i; u^1_{1i}\right)$, where

- π_i denotes the unique map $I^1_{0i} := \psi^{-1}i \to *$, so $\psi = \sum_{i\in I^1_1}\pi_i$ (where $\sum_{i\in I^1_1}$ denotes the functor which takes the disjoint union equipped with the lexicographical order),
- $u^0|_i$ denotes the restriction of u^0 to $[I^1_{0i}] \subset [I^1_0]$,
- $u^1_0\big|_i$ denotes the restriction of u^1_0 to I^1_{0i},

so $u^1_0\big|_i$ is a $(\phi^0|_i)$-nerve connecting $u^0|_i$, where $\phi^0|_i$ denotes the restriction of ϕ^0 to I^1_{0i}.

1.5. k-multimaps in a higher theory.

1.5.0. The form of data (k) for $3 \le k \le n-1$ for Definition 1.6, is specified inductively as follows.

(k) Suppose given
($k-1'$) an elemental [1]-family $I^{k-1} = (I^{k-1}_0, \{1\})$ in Ord, and an $\{1\}$-nerve ($\pi\colon I^{k-1}_0 \to \{1\}$),
($k-2'$) an elemental $[I^{k-1}_0]$-family $I^{k-2} = (I^{k-2}_i)_{i\in[I^{k-1}_0]}$ in Ord, and an I^{k-1}_0-nerve $\phi^{k-2} = (\phi^{k-2}_i)_{i\in I^{k-1}_0}$ connecting I^{k-2},
– ($k-3'$) through ($k-3''$) of ($k-1$),

$(k-2'')$ a I^{k-2}-family $u^{k-2} = (u_i^{k-2})_{i\in[I_0^{k-1}]}$ of $(k-2)$-multimaps in $\mathcal{U}$, where the I_i^{k-2}-family u_i^{k-2} is in fact a $\phi_{\to i}^{k-2}{}_!\phi^{k-3}$-nerve (see $(k-1'')$ below) of $(k-2)$-multimaps in $\mathcal{U}$ (where $\phi_{\to i} := \phi_i \cdots \phi_1$), connecting $\phi_{\to i}^{k-2}{}_! u^{k-3}$,

$(k-1'')$ * a ϕ^{k-2}*-nerve* u_0^{k-1} of $(k-1)$-multimaps *connecting* u^{k-2} in $\mathcal{U}$, which *by definition* means that $u_0^{k-1} = (u_{0i}^{k-1})_{i\in I_0^{k-1}}$, where $u_{0i}^{k-1} \in \mathrm{Mul}_{\mathcal{U}}^{\phi_i^{k-2}}[u^{\leq k-2}|_i]$ (see below), where $u^{\leq k-2}|_i$ consists of

$$u^{\leq k-4} := (u^\nu)_{0\leq\nu\leq k-4},\ (\phi_{\to i-1}^{k-2})_! u^{k-3},\ u^{k-2}|_i,$$

where $u^{k-2}|_i$ denotes the restriction of u^{k-2} to $\{i-1, i\} \subset [I_0^{k-1}]$,

* $u_1^{k-1} \in \mathrm{Mul}^{\pi_! \phi^{k-2}}[\pi_! u^{\leq k-2}]$, where $\pi_! u^{\leq k-2}$ consists of

$$u^{\leq k-4},\ (\pi_! \phi^{k-2})_! u^{k-3},\ \pi_! u^{k-2}.$$

Then a collection $\mathrm{Mul}_{\mathcal{U}}^{\pi}[u^{\leq k-2}](u_0^{k-1}; u_1^{k-1})$ or $\mathrm{Mul}_{\mathcal{U}}^{\pi}[u]$ for short, whose member will be called a k**-multimap** $u_0^{k-1} \to u_1^{k-1}$ in $\mathcal{U}$.

Definition 1.7. We refer to data $(I; \pi, \phi)$, where $I := (I^\nu)_{0\leq\nu\leq k-1}$, $\phi := (\phi^\nu)_{0\leq\nu\leq k-2}$, of the form specified by $(k-1')$ through $(0')$ above, as the **arity of a** k**-multimap** in a symmetric higher theory.

We refer to data $u = (u^\nu)_{0\leq\nu\leq k-1}$ of the form specified by $(0'')$ through $(k-1'')$ above (by induction in k), as the **type of a** k**-multimap** in $\mathcal{U}$, of **arity** $(I; \pi, \phi)$.

Remark 1.8. Even though we have not yet specified the forms of the rest of data for $\mathcal{U}$, note that the notion of the type of a k-multimap "in $\mathcal{U}$" makes sense as soon as data of the forms (0) through $(k-1)$ are given "for $\mathcal{U}$".

Data of the form (k) above extend for the similar input data with the elementality requirements discarded. This can be done by induction, starting from the elemental case above, as follows.

1.5.1. Fix an integer ν such that $1 \leq \nu \leq k-1$, and suppose as an inductive hypothesis, that we have extended data (k) for input data similar to $(k-1')$ through $(k-1'')$ above, where the families I^0 through $I^{\nu-2}$ are allowed to be non-elemental. Then we extend data (k) for input data with the families up to $I^{\nu-1}$ allowed to be non-elemental, as follows.

Suppose given data similar to $(k-1')$ through $(k-1'')$ above, where the families I^0 through $I^{\nu-1}$ are allowed to be non-elemental (which we mean to be interpreted following the previous inductive step). Then we let $\mathrm{Mul}_{\mathcal{U}}^{\pi}[u]$ denote the collection whose member is an $I_{[\pi^\nu](1)}^{\nu-1}$-family $(v_i)_{i\in I_{[\pi^\nu](1)}^{\nu-1}}$, where π^ν denotes the unique map $I_0^\nu \to \{1\}$, and $v_i \in \mathrm{Mul}_{\mathcal{U}}^{\pi}[u|_i]$, where $u|_i$ consists of

$$u^{\leq\nu-4},\ (\phi_{\to i-1}^{\nu-2})_! u^{\nu-3},\ u^{\geq\nu-2}|_i := (u^\kappa|_i)_{\kappa\geq\nu-2},$$

where if $\nu \leq k-2$,

- $u^{\geq\nu-2,\leq k-3}|_i = (u^\kappa|_i)_{\nu-2\leq\kappa\leq k-3}$ are as already defined by the previous step of the induction on k (see the case $\nu = k-1$ below for $u^{\nu-2}|_i$ and $u^{\nu-1}|_i$, and the next point for $u^{\geq\nu,\leq k-3}|_i$),
- $u^{k-2}|_i := \left(u_j^{k-2}\big|_i\right)_{j\in[I_0^{k-1}]}$, where $u_j^{k-2}\big|_i$ is as already defined by the previous step of the induction on k (see below),

and if $\nu = k-1$,

- $u^{\nu-2}|_i$ denotes the restriction of $u^{\nu-2}$ to $[(\pi_!^\nu \phi^{\nu-1})^{-1} i] \subset [I_0^{\nu-1}]$,

- $u^{\nu-1}|_i := \big(u^{\nu-1}_j\big|_i\big)_{j\in[I^\nu_0]}$, where $u^{\nu-1}_j\big|_i$ denotes the $\big((\phi^{\nu-1}_{/i})_{\to j}\big)_!\big(\phi^{\nu-2}|_i\big)$-nerve connecting $\big((\phi^{\nu-1}_{/i})_{\to j}\big)_!\big(u^{\nu-2}|_i\big)$, obtained by restricting $u^{\nu-1}_j$ to $I^{\nu-1}_{ji} := (\phi^{\nu-1}_{\leftarrow j})^{-1}i \subset I^{\nu-1}_j$, where
  - $\phi^{\nu-1}_{/i}$ denotes the I^ν_0-nerve in Ord connecting the *elemental* $[I^\nu_0]$-family $(I^{\nu-1}_{ji})_{j\in[I^\nu_0]}$, obtained by restricting $\phi^{\nu-1}$,
  - $\phi^{\nu-2}|_i$ is the $I^{\nu-1}_{0i}$-nerve obtained by restricting $\phi^{\nu-2}$, connecting the $[I^{\nu-1}_{0i}]$-family $I^{\nu-2}|_i$ obtained by restricting $I^{\nu-2}$,

  so

$$\big((\phi^{\nu-1}_{/i})_{\to j}\big)_!\big(\phi^{\nu-2}|_i\big) = \big(\phi^{\nu-1}_{\to j}{}_!\phi^{\nu-2}\big)\big|_{I^{\nu-1}_{ji}},$$
$$\big((\phi^{\nu-1}_{/i})_{\to j}\big)_!\big(u^{\nu-2}|_i\big) = \big(\phi^{\nu-1}_{\to j}{}_!u^{\nu-2}\big)\big|_{[I^{\nu-1}_{ji}]},$$

and for any ν, $u^{k-1}_0\big|_i = (u^{k-1}_{0ji})_{j\in I^{k-1}_0}$, and $u^{k-1}_1\big|_i = (u^{k-1}_{1i})$ are as already specified by the previous step of the induction on k. Note (see below) that, by induction on k, u^{k-1}_{0j} for $j \in I^{k-1}_0$ is an $I^{k-2}_{[\pi](1)}$-family $(u^{k-1}_{0ji})_{i\in I^{k-2}_{[\pi^1](1)}}$, where $u^{k-1}_{0ji} \in \mathrm{Mul}^{\phi^{k-2}_i}[u^{\le k-2}|_{i,j}]$, and similarly for u^{k-1}_1.

1.5.2. Finally, data (k) extend for the input data with I^{k-1} non-elemental, as follows. Suppose given

- a map $\psi\colon I^{k-1}_0 \to I^{k-1}_1$ in Ord,
- not necessarily elemental data of the form $(k-2')$ through $(k-2'')$ above,
- a ϕ^{k-2}-nerve u^{k-1}_0 connecting u^{k-2}, and a $\psi_!\phi^{k-2}$-nerve u^{k-1}_1 connecting $\psi_!u^{k-2}$ respectively, of $(k-1)$-multimaps in $\mathcal{U}$.

Then we let $\mathrm{Mul}^\psi_{\mathcal{U}}[u]$ denote the collection whose member is an I^{k-1}_1-family $v = (v_i)_{i\in I^{k-1}_1}$ of k-multimaps in $\mathcal{U}$, where $v_i \in \mathrm{Mul}^{\pi_i}[u|_i]$, where

- π_i denotes the unique map $I^{k-1}_{0i} := \psi^{-1}i \to *$, so $\psi = \sum_{i\in I^{k-1}_1}\pi_i$,
- $u|_i$ consists of

$$u^{\le k-4},\ (\phi^{k-2}_{\to[\psi](i-1)})_!u^{k-3},\ u^{k-2}|_i,\ u^{k-1}|_i,$$

  where
  - $u^{k-2}|_i$ denotes the restriction of u^{k-2} to $[I^{k-1}_{0i}] \subset [I^{k-1}_0]$,
  - $u^{k-1}\big|_i := \big(u^{k-1}_j\big|_i\big)_{j\in[1]}$, where $u^{k-1}_0\big|_i$ denotes the restriction of u^{k-1}_0 to I^{k-1}_{0i}, and $u^{k-1}_1\big|_i := (u^{k-1}_{1i})$,

  so $u^{k-1}_0\big|_i$ is a $(\phi^{k-2}|_i)$-nerve connecting $u^{k-2}|_i$, where $\phi^{k-2}|_i$ denotes the restriction of ϕ^{k-2} to I^{k-1}_{0i}.

1.5.3. This allows us to go to the next inductive step.

1.6. **The object "formed by n-multimaps in an n-theory".** The form of data (n) for Definition 1.6 is as follows.

(n) Suppose given the type u of an n-multimap in $\mathcal{U}$ of arity given as $(I;\pi,\phi)$. Namely, suppose given a set of data similar to $(k-1')$ through $(k-1'')$ of "(k)" in Section 1.5, but with k substituted by n (where I^{k-2} should be a family in Fin if $n=2$), so these will be $(n-1')$ through $(n-1'')$ here. Then an object $\mathrm{Mul}^\pi_{\mathcal{U}}[u^{\le n-2}](u^{n-1}_0; u^{n-1}_1)$ of $\mathcal{A}$, or $\mathrm{Mul}^\pi_{\mathcal{U}}[u]$ for short. In the case where $\mathcal{A}$ is Gpd or some other category so that $\mathrm{Mul}^\pi_{\mathcal{U}}[u]$ can have its objects, then those objects will be called **n-multimaps** $u^{n-1}_0 \to u^{n-1}_1$ in $\mathcal{U}$. For a general $\mathcal{A}$, we shall call $\mathrm{Mul}^\pi_{\mathcal{U}}[u]$ the object "of *n-multimaps*".

Data of this form extend for non-elemental input data just as data "(k)" did in Section 1.5.

To see this, for the purpose of induction starting from the elemental case above, fix an integer ν such that $1 \le \nu \le n-1$, and suppose given data similar to $(n-1')$ through $(n-1'')$ above, where the families I^0 through $I^{\nu-1}$ are allowed to be non-elemental. Then we define $\mathrm{Mul}^{\pi}_{\mathcal{U}}[u] := \bigotimes_{i \in I^{\nu-1}_{[\pi^\nu](1)}} \mathrm{Mul}^{\pi}_{\mathcal{U}}[u|_i]$, which makes sense by induction on ν.

Suppose given next, instead of $\pi\colon I_0^{n-1} \to \{1\}$ above, a map $\psi\colon I_0^{n-1} \to I_1^{n-1}$ in Ord, and suppose u_1^{n-1} is now an $\psi_!\phi^{n-2}$-nerve of $(n-1)$-multimaps connecting $\psi_! u^{n-2}$. Then we let π_i for $i \in I_1^{n-1}$ denote the unique map $\psi^{-1}i \to *$ (so $\psi = \sum_i \pi_i$), and define $\mathrm{Mul}^{\psi}_{\mathcal{U}}[u] := \bigotimes_{i \in I_1^{n-1}} \mathrm{Mul}^{\pi_i}[u|_i]$.

1.7. **Composition of n-multimaps.** The form of data $(n+1)$ for Definition 1.6 is as follows.

$(n+1)$ Suppose given the arity $(I; \pi, \phi)$ of an $(n+1)$-multimap in a symmetric higher theory, namely

- $(k-1')$ and $(k-2')$ of "(k)" in Section 1.5, but with k substituted by $n+1$, so these will be (n') and $(n-1')$ here.
- $(n-2')$ through $(0')$ of (n) in Section 1.6,

as well as

- $(0'')$ through $(n-2'')$ of (n) in Section 1.6,
- $(k-2'')$ of "(k)" in Section 1.5, but with k substituted by $n+1$, so this will be $(n-1'')$ here.

Then a map $m_1^{\mathcal{U}}(\pi)\colon \mathrm{Mul}^{\phi^{n-1}}_{\mathcal{U}}[u] \to \mathrm{Mul}^{\pi_!\phi^{n-1}}_{\mathcal{U}}[\pi_! u]$ in $\mathcal{A}$, where

- the source of $m_1^{\mathcal{U}}(\pi)$ is the object $\bigotimes_{i \in I_0^n} \mathrm{Mul}^{\phi_i^{n-1}}[u|_i]$ of $\mathcal{A}$ ("of ϕ^{n-1}-nerves of n-multimaps connecting u^{n-1} in $\mathcal{U}$"), where $u|_i$ consists of

$$u^{\le n-3},\ (\phi^{n-1}_{\to i-1})_! u^{n-2},\ u^{n-1}|_i,$$

- $\pi_! u$ consists of $u^{\le n-2}$, $\pi_! u^{n-1}$.

The map $m_1^{\mathcal{U}}(\pi)$ will be called the **composition** operation for n-multimaps.

Definition 1.9. We refer to data $u = (u^\nu)_{0 \le \nu \le n-1}$ of the form specified by $(0'')$ through $(n-1'')$ above, as the **type of a ϕ^{n-1}-nerve of n-multimaps** in $\mathcal{U}$.

Data of the form $(n+1)$ above extend for non-elemental input data as follows.

To begin with, fix an integer ν such that $1 \le \nu \le n-1$, and suppose given data similar to (n') through $(n-1'')$ above, where the families I^0 through $I^{\nu-1}$ are allowed to be non-elemental. Then we define $m_1^{\mathcal{U}}(\pi)\colon \mathrm{Mul}^{\phi^{n-1}}_{\mathcal{U}}[u] \to \mathrm{Mul}^{\pi_!\phi^{n-1}}_{\mathcal{U}}[\pi_! u]$ as the monoidal product over $i \in I^{\nu-1}_{[\pi^\nu](1)}$ of the maps $m_1^{\mathcal{U}}(\pi)\colon \mathrm{Mul}^{\phi^{n-1}}_{\mathcal{U}}[u|_i] \to \mathrm{Mul}^{\pi_!\phi^{n-1}}[\pi_! u|_i]$, which makes sense by induction on ν.

Next, suppose given data similar to (n') through $(n-1'')$ above, where the families I^0 through I^{n-1} are allowed to be non-elemental. Then we define $m_1^{\mathcal{U}}(\pi)\colon \mathrm{Mul}^{\phi^{n-1}}_{\mathcal{U}}[u] \to \mathrm{Mul}^{\pi_!\phi^{n-1}}_{\mathcal{U}}[\pi_! u]$ as the monoidal product over $i \in I^{n-1}_{[\pi](1)}$ of the maps $m_1(\pi)\colon \mathrm{Mul}^{\phi^{n-1}_{/i}}[u|_i] \to \mathrm{Mul}^{\pi_!\phi^{n-1}_{/i}}[\pi_! u|_i]$.

Finally, suppose given, instead of $\pi\colon I_0^n \to \{1\}$ above, a map $\psi\colon I_0^n \to I_1^n$ in Ord. Then we let π_i for $i \in I_1^n$ denote the unique map $\psi^{-1}i \to *$ (so $\psi = \sum_i \pi_i$), and define $m_1^{\mathcal{U}}(\psi)\colon \mathrm{Mul}^{\phi^{n-1}}[u] \longrightarrow \mathrm{Mul}^{\psi_!\phi^{n-1}}[\psi_! u]$, where $\psi_! u$ consists of $u^{\le n-2}$, $\psi_! u^{n-1}$, as the monoidal product over $i \in I_1^n$ of the maps $m_1(\pi_i)\colon \mathrm{Mul}^{\phi^{n-1}|_i}[u|_i] \to \mathrm{Mul}^{(\psi_!\phi^{n-1})_i}[(\psi_! u)|_i]$, where $u|_i$ consists of

$$u^{\le n-3},\ (\phi^{n-1}_{\to[\psi](i-1)})_! u^{n-2},\ u^{n-1}|_i,$$

so $(\psi_! u)|_i$ consists of

$$u^{\le n-3},\ \big((\psi_!\phi^{n-1})_{\to i-1}\big)_! u^{n-2},\ \big(\psi_! u^{n-1}\big)\big|_i.$$

1.8. **The associativity isomorphism for the composition.** The form of data $(n+2)$ for Definition 1.6 is as follows.

$(n+2)$ Suppose given the arity $(I;\pi,\phi)$ of an $(n+2)$-multimap in a symmetric higher theory, and the type u of a ϕ^{n-1}-nerve of n-multimaps in $\mathcal{U}$. Then a 2-isomorphism $m_2^{\mathcal{U}}(\pi)\colon \pi_! m_1^{\mathcal{U}}(\phi^n) \xrightarrow{\sim} m_1^{\mathcal{U}}(\pi_!\phi^n)$ in $\mathcal{A}$, where $m_1^{\mathcal{U}}(\phi^n)$ denotes the I_0^{n+1}-nerve obtained by indexing with $i\in I_0^{n+1}$, the maps

$$m_1(\phi_i^n)\colon \operatorname{Mul}^{(\phi^n_{\to i-1})_!\phi^{n-1}}[(\phi^n_{\to i-1})_! u] \longrightarrow \operatorname{Mul}^{(\phi^n_{\to i})_!\phi^{n-1}}[(\phi^n_{\to i})_! u].$$

Data of this form extend for non-elemental input data as follows.

To begin with, fix an integer ν such that $1\le\nu\le n-1$, and suppose given data similar to $(n+1')$ through $(n-1'')$ above, where the families I^0 through $I^{\nu-1}$ are allowed to be non-elemental. Then we define $m_2^{\mathcal{U}}(\pi)\colon \pi_! m_1^{\mathcal{U}}(\phi^n) \xrightarrow{\sim} m_1^{\mathcal{U}}(\pi_!\phi^n)$ as the monoidal product over $i\in I^{\nu-1}_{[\pi^\nu](1)}$ of the 2-isomorphisms

$$m_2^{\mathcal{U}}(\pi)\colon \pi_! m_1(\phi^n) \xrightarrow{\ \sim\ } m_1(\pi^n)\colon \operatorname{Mul}^{\phi^{n-1}}[u|_i] \longrightarrow \operatorname{Mul}^{\pi^n{}_!\phi^{n-1}}[\pi^n{}_! u|_i],$$

which makes sense by induction on ν.

Next, suppose given data similar to $(n+1')$ through $(n-1'')$ above, where the families I^0 through I^{n-1} are allowed to be non-elemental. Then we define $m_2^{\mathcal{U}}(\pi)\colon \pi_! m_1(\phi^n) \xrightarrow{\sim} m_1(\pi_!\phi^n)$ as the monoidal product over $i\in I^{n-1}_{[\pi_!\phi^n](1)}$ of the 2-isomorphisms

$$m_2(\pi)\colon \pi_! m_1(\phi^n) \xrightarrow{\ \sim\ } m_1(\pi^n)\colon \operatorname{Mul}^{\phi^{n-1}_{/i}}[u|_i] \longrightarrow \operatorname{Mul}^{\pi^n{}_!\phi^{n-1}_{/i}}[\pi^n{}_! u|_i].$$

Next, suppose given data similar to $(n+1')$ through $(n-1'')$ above, where the families I^0 through I^n are allowed to be non-elemental. Then we write for $i\in I^n_{[\pi](1)}$, $\pi_i^n := \pi_!(\phi^n_{/i})$ (so $\pi_!\phi^n = \sum_i \pi_i^n$), and define $m_2(\pi)\colon \pi_! m_1(\phi^n) \xrightarrow{\sim} m_1(\pi_!\phi^n)$ as the monoidal product over $i\in I^n_{[\pi](1)}$ of the 2-isomorphisms

$$m_2(\pi)\colon \pi_! m_1(\phi^n_{/i}) \xrightarrow{\ \sim\ } m_1(\pi_i^n)\colon \operatorname{Mul}^{\phi^{n-1}|_i}[u|_i] \longrightarrow \operatorname{Mul}^{\pi^n_i{}_!(\phi^{n-1}|_i)}[(\pi^n{}_! u)|_i].$$

Finally, suppose given, instead of $\pi\colon I_0^{n+1}\to\{1\}$ above, a map $\psi\colon I_0^{n+1}\to I_1^{n+1}$ in Ord. Then we let π_i for $i\in I_1^{n+1}$ denote the unique map $\psi^{-1}i\to *$ (so $\psi = \sum_i \pi_i$), and define the isomorphism $m_2(\psi)\colon \psi_! m_1(\phi^n) \xrightarrow{\sim} m_1(\psi_!\phi^n)$ of I_1^{n+1}-nerves in $\mathcal{A}$ as the family indexed by $i\in I_1^{n+1}$ of the isomorphisms

$$m_2(\pi_i)\colon \pi_{i!} m_1\big(\phi^n|_i\big) \xrightarrow{\ \sim\ } m_1\big((\psi_!\phi^n)_i\big).$$

1.9. **Coherence for the associativity.**

1.9.0. The form of data $(n+\ell)$ for $\ell\ge 3$ for Definition 1.6, is specified inductively as follows.

$(n+\ell)$ Suppose given the arity $(I;\pi,\phi)$ of an $(n+\ell)$-multimap in a symmetric higher theory, and the type u of a ϕ^{n-1}-nerve of n-multimaps in $\mathcal{U}$. Then an ℓ-isomorphism

$$m_\ell^{\mathcal{U}}(\pi)\colon \pi_! m_{\ell-1}^{\mathcal{U}}(\phi^{n+\ell-2}) \xrightarrow{\ \sim\ } m_{\ell-1}^{\mathcal{U}}(\pi_!\phi^{n+\ell-2})$$

in $\mathcal{A}$, where $m_{\ell-1}^{\mathcal{U}}(\phi^{n+\ell-2})$ denotes the $I_0^{n+\ell-1}$-nerve of $(\ell-1)$-isomorphisms obtained by indexing with $i\in I_0^{n+\ell-1}$ the isomorphisms

$$m_{\ell-1}(\phi_i^{n+\ell-2})\colon m_{\ell-2}^{\mathcal{U}}(\phi^{n+\ell-2}_{\to i-1}{}_!\phi^{n+\ell-3}) \xrightarrow{\ \sim\ } m_{\ell-2}^{\mathcal{U}}(\phi^{n+\ell-2}_{\to i}{}_!\phi^{n+\ell-3})$$

of $(\ell-2)$-isomorphisms

$$\pi^{n+\ell-3}{}_! m_{\ell-3}(\phi^{n+\ell-4}) \xrightarrow{\sim} m_{\ell-3}(\pi^{n+\ell-3}{}_!\phi^{n+\ell-4})$$

(where $\pi^{n+\ell-3} := (\pi_!\phi^{n+\ell-2})_!\phi^{n+\ell-3}$) in $\mathcal{A}$, or $\mathrm{Mul}^{\phi^{n-1}}[u] \to \mathrm{Mul}^{\pi^n{}_!\phi^{n-1}}[\pi^n{}_!u]$ if $\ell = 3$.

Data of this form extend for non-elemental input data as follows.

The initial step is a similar induction as before. Fix an integer ν such that $1 \le \nu \le n+\ell-1$, and suppose given data similar to $(n+\ell-1')$ through $(n-1'')$ above, where the families I^0 through $I^{\nu-1}$ are allowed to be non-elemental. Then we define $m^{\mathcal{U}}_\ell(\pi)\colon \pi_! m_{\ell-1}(\phi^{n+\ell-2}) \xrightarrow{\sim} m_{\ell-1}(\pi_!\phi^{n+\ell-2})$ as

- if $\nu \le n+1$, the monoidal product over $I^{\nu-1}_{[\pi^\nu](1)}$,
- if $\nu \ge n+2$, the $(n+\ell+1-\nu)$-isomorphism of $I^{\nu-1}_{[\pi^\nu](1)}$-nerves of $(\nu-n-1)$-isomorphisms (or 1-morphisms if $\nu = n+2$) in $\mathcal{A}$, given by the family indexed by $I^{\nu-1}_{[\pi^\nu](1)}$,

of $m^{\mathcal{U}}_\ell(\pi)$ defined for each $i \in I^{\nu-1}_{[\pi^\nu](1)}$ as an instance of the previous inductive step.

For instance, as the case $\nu = n+\ell-1$, suppose given data similar to $(n+\ell-1')$ through $(n-1'')$ above, where the families I^0 through $I^{n+\ell-2}$ are allowed to be non-elemental. Then we write for $i \in I^{n+\ell-2}_{[\pi](1)}$, $\pi^{n+\ell-2}_i := \pi_!\phi^{n+\ell-2}_{/i}$ (so $\pi_!\phi^{n+\ell-2} = \sum_i \pi^{n+\ell-2}_i$), and define the isomorphism $m_\ell(\pi)\colon \pi_! m_{\ell-1}(\phi^{n+\ell-2}) \xrightarrow{\sim} m_{\ell-1}(\pi_!\phi^{n+\ell-2})$ of isomorphisms

$$\big(\pi_!\phi^{n+\ell-2}\big)_! m_{\ell-2}\big(\phi^{n+\ell-3}\big) \xrightarrow{\sim} m_{\ell-2}\big((\pi_!\phi^{n+\ell-2})_!\phi^{n+\ell-3}\big)$$

of $I^{n+\ell-2}_{[\pi](1)}$-nerves of $(\ell-2)$-isomorphisms (or 1-morphisms if $\ell = 3$) in $\mathcal{A}$, as given by the family indexed by $i \in I^{n+\ell-2}_{[\pi](1)}$ of

$$m_\ell(\pi)\colon \pi_! m_{\ell-1}(\phi^{n+\ell-2}_{/i}) \xrightarrow{\sim} m_{\ell-1}(\pi^{n+\ell-2}_i)\colon$$
$$\pi^{n+\ell-2}_i{}_! m_{\ell-2}\big(\phi^{n+\ell-3}|_i\big) \xrightarrow{\sim} m_{\ell-2}\Big(\big((\pi_!\phi^{n+\ell-2})_!\phi^{n+\ell-3}\big)_i\Big).$$

Finally, suppose given, instead of $\pi\colon I^{n+\ell-1}_0 \to \{1\}$ above, a map $\psi\colon I^{n+\ell-1}_0 \to I^{n+\ell-1}_1$ in Ord. Then we let π_i for $i \in I^{n+\ell-1}_1$ denote the unique map $\psi^{-1}i \to *$ (so $\psi = \sum_i \pi_i$), and define the isomorphism

$$m_\ell(\psi)\colon \psi_! m_{\ell-1}(\phi^{n+\ell-2}) \xrightarrow{\sim} m_{\ell-1}(\psi_!\phi^{n+\ell-2})$$

of $I^{n+\ell-1}_1$-nerves of $(\ell-1)$-isomorphisms in $\mathcal{A}$, as given by the family indexed by $i \in I^{n+\ell-1}_1$ of the isomorphisms

$$m_\ell(\pi_i)\colon \pi_{i!} m_{\ell-1}\big(\phi^{n+\ell-2}|_i\big) \xrightarrow{\sim} m_{\ell-1}\big((\psi_!\phi^{n+\ell-2})_i\big).$$

1.9.1. We can thus proceed to the next inductive step, and this completes Definition 1.6.

2. Simple variants and basic constructions

2.0. **Introduction.** In this section, we shall first discuss a planar variant of higher theories, which iteratevely theorize associative algebra. In particular, we shall find the structure of a planar $(n-1)$-theory formed by endomorphisms in a symmetric n-theory. This will lead to a discussion of other theorized structures similarly residing in a symmetric higher theory. We shall further discuss less coloured variants of higher theory, relation of higher theory to higher categorified structure, and a construction for a higher theory which generalizes the "delooping" construction for a symmetric monoidal category.

2.1. Planar theories.

2.1.0. The notion of symmetric multicategory had variants such as planar and braided. While a generalization of these will appear in Section 3, the notion of *planar* n-theory is particularly simple to describe, and turns out to be also fundamental, so we shall discuss it here.

The definition of a **planar n-theory** is the same as the definition of a symmetric n-theory except that one uses the category Ord instead of Fin. Namely, I^0 (and S) appearing in the definition 1.6 of an n-theory should now be in Ord, and everything else is as it makes sense under this modification.

In particular, one obtains a planar n-theory from a symmetric n-theory $\mathcal{U}$ by restricting the data defining $\mathcal{U}$, through the forgetful functor Ord $\to$ Fin.

2.1.1. The notion of planar n-theory is fundamental for the following reason. Given a symmetric n-theory $\mathcal{U}$, and its object x, the structure of a planar $(n-1)$-theory underlies the structure formed by endomorphisms (i.e., unary endomultimaps) of x.

The idea is that if one intends to take as the part $(0')$ of the input data for $\mathcal{U}$, the constant elemental I^1_0-nerve $* \to \cdots \to *$, and as the part $(0'')$, a constant family at x, then the rest of the required input data is of the same form as the form for the input data for a *planar* $(n-1)$-theory.

Thus, a planar $(n-1)$-theory $\mathcal{V} = \mathcal{M}ap_{\mathcal{U}}(x,x)$ is obtained as follows. Suppose inductively in k, that an input for the data (k) for the planar $(n-1)$-theory $\mathcal{V}$ is given by J^ν, ψ^ν, v^ν for all integers ν in the suitable range. Then one obtains an input for the data $(k+1)$ for $\mathcal{U}$ by letting

- $I^\nu := J^{\nu-1}$, $\phi^\nu := \psi^{\nu-1}$ for $\nu \geq 1$,
- I^0 and ϕ^0 constant as above, as well as u^0 constant at x,
- $u^\nu := v^{\nu-1}$ for $\nu \geq 1$,

so we use the output for this by $\mathcal{U}$ as the data (k) for $\mathcal{V}$ for the original input.

For example, the collection of the objects of $\mathcal{V}$ is $\mathrm{Mul}^\pi_{\mathcal{U}}(x;x)$, where π is the identity map of $I^0_1 = \{1\}$.

2.2. **More related higher theorizations.** In Section 2.1, we have found the structure of a planar $(n-1)$-theory within the structure of every n-theory. The case $n = 1$ of this is the associative algebra of endomorphisms within a multicategory, and by no accident, a planar n-theory is an n-th theorization of an associative algebra.

Since we can also find other structures within the structure of a multicategory, we can find higher theorized forms of them within the structure of an n-theory.

For example, if we focus on, instead of endormophisms on a selected object, all unary multimaps between arbitrary objects within an n-theory, then we find that they naturally form a structure which is an $(n-1)$-th theorization of the structure of a category. We have seen examples of theorized categories in Section 0.5. We have also noted there that theorized category was a 'more coloured' version of planar multicategory. Higher theorizations of category relates to planar higher theories in a similar manner.

For another example, if we fix one object of an n-theory to look at, but allow all endomultimaps of arbitrary arities (and arbitrary higher multimaps between them), then we find the structure of an $(n-1)$-th theorization of an uncoloured operad. The theorizations can have colours at 'shallower' levels, and these will be precisely defined as *$(n-1)$-tuply coloured* n-theories in Section 2.3.

2.3. **Restricting strata for colours.**

2.3.0. The data of a multicategory, or a coloured operad, enriched in groupoids, say, consists of collection of objects or "colours", and operations or "multimaps" which compose. In an n-theory, only the multimaps of dimension n are required to compose, so only these are really considered as operations, while the collections of lower dimensional multimaps are then considered as forming *strata of colours* whose role is to specify the types of the operations in the top dimension.

As we could consider uncoloured operad, we sometimes want to consider uncoloured, and only partially coloured, higher theories. Specifically, we would like to consider situations in which the data below some dimension are all fixed to be 'trivial'. We actually do this by simulating such a situation, instead of defining what we mean by the "trivial" data.

Definition 2.0. Let $n \geq 2$ be an integer. Then a (symmetric) **uncoloured n-theory** $\mathcal{U}$ enriched in a symmetric monoidal category $\mathcal{A}$, consists of data of the forms specified below as (n), $(n+1)$ and (∞).

(n) (The object "formed by *objects*".) Suppose given the arity $(I;\pi,\phi)$ of an n-multimap in a symmetric higher theory, namely, $(n-1')$ through $(0')$ of (n) in Section 1.6. Then an object $\mathrm{Ob}^\pi\,\mathcal{U}$ of $\mathcal{A}$, to be called the object of **objects** of $\mathcal{U}$ of the specified arity.

This extends for non-elemental input data just as data (n) of Section 1.6 did, and we use the resulting extended data in the specification of the next data form.

$(n+1)$ (*Composition* of objects.) Suppose given the arity $(I;\pi,\phi)$ of an $(n+1)$-multimap in a symmetric higher theory. Then a map $m_1^{\mathcal{U}}(\pi)\colon \mathrm{Ob}^{\phi^{n-1}}\,\mathcal{U} \to \mathrm{Ob}^{\pi_!\phi^{n-1}}\,\mathcal{U}$ in $\mathcal{A}$, where the source here is the object $\bigotimes_{i\in I_0^n} \mathrm{Ob}^{\phi_i^{n-1}}\,\mathcal{U}$. The map $m_1(\pi)$ will be called the **composition** operation for objects of $\mathcal{U}$.

(∞) Data of coherent associativity for the composition operations, corresponding to those for an n-theory described in Sections 1.8 and 1.9.

This completes Definition 2.0.

Definition 2.1. Let $n \geq 2$ be an integer. We say that an n-theory as defined in Definition 1.6, as **n-tuply coloured**.

Let m be an integer such that $1 \leq m \leq n-1$. Then a (symmetric) **m-tuply coloured n-theory** $\mathcal{U}$ enriched in a symmetric monoidal category $\mathcal{A}$, consists of data of the forms specified below as $(n-m)$, "(k)" for every integer k such that $n-m+1 \leq k \leq n-1$, (n), $(n+1)$ and (∞).

$(n-m)$ (*Object.*) Suppose given the arity $(I;\pi,\phi)$ of an $(n-m)$-multimap in a symmetric higher theory (or a finite set S with unique map $\pi\colon S \to *$, if $n-m=1$). Then a collection $\mathrm{Ob}^\pi\,\mathcal{U}$, whose member will be called an **object** of $\mathcal{U}$ of the specified arity.

This extends for non-elemental input data just as data (k) of Section 1.5 did for $k = n-m$, and we use the resulting extended data in the specification of the next data form.

(k) (*$(k-n+m)$-multimap*, inductively for $n-m+1 \leq k \leq n-1$.) Suppose given
- the arity $(I;\pi,\phi)$ of a k-multimap in a symmetric higher theory,
- if $k-3 \geq n-m$, then $(n-m'')$ through $(k-3'')$ of $(k-1)$ here,

and

$(k-2'')$ if $k-2 \geq n-m$, then an I^{k-2}-family $u^{k-2-n+m} = (u_i^{k-n+m-2})_{i\in[I_0^{k-1}]}$ of $(k-n+m-2)$-multimaps (or objects if $k-n+m-2=0$) in $\mathcal{U}$, where, if $k-n+m-2 \geq 1$, then the I_i^{k-2}-family $u_i^{k-n+m-2}$ is in fact a $\phi^{k-2}_{\to i}{}_!\phi^{k-3}$-nerve (see $(k-1'')$ below) of $(k-n+m-2)$-multimaps in $\mathcal{U}$, connecting $\phi^{k-2}_{\to i}{}_!u^{k-n-m-3}$,

$(k-1'')$ if $k-1=n-m$, then

- a ϕ^{k-2}-*nerve* u^0_0 of objects in $\mathcal{U}$, which *by definition* means that $u^0_0=(u^0_{0i})_{i\in I^{k-1}_0}$, where $u^0_{0i}\in \mathrm{Ob}^{\phi^{k-2}_i}\mathcal{U}$,
- an object $u^0_1\in \mathrm{Ob}^{\pi_!\phi^{k-2}}\mathcal{U}$;

if $k-1\geq n-m+1$, then

- a ϕ^{k-2}-*nerve* $u^{k-1-n+m}_0$ of $(k-n+m-1)$-multimaps in $\mathcal{U}$ *connecting* $u^{k-n+m-2}$, which *by definition* means that $u^{k-n+m-1}_0=(u^{k-n+m-1}_{0i})_{i\in I^{k-1}_0}$, where $u^{k-n+m-1}_{0i}\in \mathrm{Mul}^{\phi^{k-2}_i}[u^{\leq k-n+m-2}|_i]$ (see below),
- $u^{k-n+m-1}_1\in \mathrm{Mul}^{\pi_!\phi^{k-2}}[\pi_!u^{\leq k-n+m-2}]$.

Then a collection $\mathrm{Mul}^{\pi}_{\mathcal{U}}[u^{\leq k-n+m-2}](u^{k-n+m-1}_0;u^{k-n+m-1}_1)$ or $\mathrm{Mul}^{\pi}_{\mathcal{U}}[u]$ for short, whose member will be called a **$(k-n+m)$-multimap** $u^{k-n+m-1}_0\to u^{k-n+m-1}_1$ in $\mathcal{U}$.

This extends for non-elemental input data just as data (k) of Section 1.5 did, and we use the resulting extended data in the specification of the next data form.

(n) (The object "formed by *m-multimaps*".) Suppose given the arity $(I;\pi,\phi)$ of an n-multimap in a symmetric higher theory, and a set of data similar to $(0'')$ through $(k-1'')$ of "(k)" above, but with k substituted by n, so these will be $(0'')$ through $(n-1'')$ here. Then an object $\mathrm{Mul}^{\pi}_{\mathcal{U}}[u^{\leq m-2}](u^{m-1}_0;u^{m-1}_1)$ of $\mathcal{A}$, or $\mathrm{Mul}^{\pi}_{\mathcal{U}}[u]$ for short, to be called the object of **m-multimaps** $u^{m-1}_0\to u^{m-1}_1$ in $\mathcal{U}$.

This extends for non-elemental input data just as data (n) of Section 1.6 did, and we use the resulting extended data in the specification of the next data form.

$(n+1)$ (*Composition* of m-multimaps.) Suppose given

- the arity $(I;\pi,\phi)$ of an $(n+1)$-multimap in a symmetric higher theory,
- $(n-m'')$ through $(n-2'')$ of (n) above,
- $(k-2'')$ of (k) above, but with k substituted by $n+1$, so this will be $(n-1'')$ here.

Then a map $m^{\mathcal{U}}_1(\pi)\colon\ \mathrm{Mul}^{\phi^{n-1}}_{\mathcal{U}}[u]\to \mathrm{Mul}^{\pi_!\phi^{n-1}}_{\mathcal{U}}[\pi_!u]$ in $\mathcal{A}$, where the source here is the object $\bigotimes_{i\in I^n_0}\mathrm{Mul}^{\phi^{n-1}_i}[u|_i]$ (of ϕ^{n-1}-nerves of m-multimaps connecting u^{m-1} in $\mathcal{U}$). The map $m_1(\pi)$ will be called the **composition** operation for m-multimaps in $\mathcal{U}$.

(∞) Data of (coherent) associativity for the composition operations, corresponding to those for an n-theory described in Sections 1.8 and 1.9.

This completes Definition 2.1.

2.3.1. Any notion which makes sense for a general n-theory also makes sense for an n-theory with restricted strata of colours as above. Indeed, given the definition of a notion concerning an n-theory, one obtains the definition of the corresponding notion for a less coloured n-theory just by suppressing from the definition, every specification involving colours in the lower dimensions in the n-theory. For example, at a place where one needs to choose an object of an n-theory, one just does not need to make any choice with an m-tuply coloured n-theory if $m\leq n-1$. Similarly, at a place where one needs to make some choice for every object of the n-theory, one just make one choice with a not fully coloured n-theory.

2.4. **Forgetting categorifications to theorizations.**

2.4.0. For an integer $m \geq 0$, let us denote by Cat_m the Cartesian symmetric monoidal category of m-categories with a fix limit for the size, where we let a **0-category** mean a groupoid by convention.

While an n-theory enriched in Cat_m is an instance of an enriched n-theory, it is also *unenriched* in the sense that we can more generally consider m-categories enriched in a symmetric monoidal category. Thus it can be considered both as an "unenriched" instance of an *m-categorified* n-theory, and as an enriched "uncategorified" n-theory. Interpolating these two views, it can also be considered as an ℓ-categorified n-theory enriched in $(m-\ell)$-categories, for every integer ℓ such that $1 \leq \ell \leq m-1$.

2.4.1. Since $(n+1)$-theory is a theorization of n-theory, there are in particular, n-theories enriched in Cat_{m+1} among $(n+1)$-theories enriched in Cat_m. Indeed, a *op-lax* n-theory enriched in Cat_{m+1} can be characterized among $(n+1)$-theories enriched in Cat_m, as one in which every profunctor/distributor/bimodule (enriched in Cat_m) virtually giving the composition of *n-multimaps*, is corepresentable, and an n-theory can be characterized among op-lax n-theories as one in which every associativity map is an isomorphism.

Given an n-theory $\mathcal{U}$ enriched in $(m+1)$-categories, let us denote by $\Theta_n\mathcal{U}$, $(n+1)$-theory enriched in m-categories obtained by replacing each functor of composition operation for n-multimaps in $\mathcal{U}$, by the bimodule corepresented by it. We shall say that $\Theta_n\mathcal{U}$ is **represented** by $\mathcal{U}$.

Definition 2.2. Let $n \geq 0$, $m \geq 2$ be integers, and let $\mathcal{U}$ be an n-theory enriched in m-categories. Given an integer ℓ such that $0 \leq \ell \leq m$, we define an $(n+\ell)$-theory $\Theta_n^{n+\ell}\mathcal{U}$ enriched in $(m-\ell)$-categories, by the inductive relations

$$\Theta_n^{n+\ell}\mathcal{U} = \begin{cases} \mathcal{U} & \text{if } \ell = 0, \\ \Theta_{n+\ell-1}\Theta_n^{n+\ell-1}\mathcal{U} & \text{if } \ell \geq 1. \end{cases}$$

For example, we obtain from a symmetric monoidal n-category $\mathcal{A}$ an uncategorified n-theory $\Theta_0^n\mathcal{A}$. Even though this is an n-theory, it is true that this is not really a new mathematical object since it is essentially just a symmetric monoidal n-category $\mathcal{A}$. An n-theory which *fails* to be represented by an $(n-1)$-theory arises, for example, through the "delooping", as well as the "convolution", constructions, which we shall discuss in Sections 2.6 and 4.4 respectively.

However, considering symmetric monoidal n-categories as n-theories means considering very different *morphisms* between symmetric monoidal n-categories, since a functor of these n-theories turns out to be an *n-lax* version of a symmetric monoidal functor of the original symmetric monoidal n-categories. Here and everywhere, by "*n-lax*" we mean "relaxed n times". The way in which the structure is relaxed each time is actually quite interesting here, and the author also finds the structure resulting from iteration of these processes of relaxation fascinating.

Even though n-lax symmetric monoidal functor may still not be a *very* new notion, the construction above is certainly giving a new meaning to this notion, in a richer environment where many new and natural mathematical structures interact, as we shall show through this work.

In general, for m-categorified n-theories $\mathcal{U}, \mathcal{V}$, a functor $\Theta_n^{m+n}\mathcal{U} \to \Theta_n^{m+n}\mathcal{V}$ of uncategorified $(n+m)$-theories is equivalent as data to an m-lax functor $\mathcal{U} \to \mathcal{V}$, as follows similarly to Therem 2.10 below.

2.4.2. There are of course, less coloured versions of all the above. In particular, in the situation of Definition 2.2, if $\mathcal{U}$ is k-tuply coloured, where $0 \leq k \leq n-1$, then $\Theta_n^{n+\ell}\mathcal{U}$ is obtained as $(k+\ell)$-tuply coloured.

2.5. **Theorization and lax functors.**

2.5.0. The obvious notion of functor of n-theories has a reasonable lax version. Let us start with recording the definition of a functor.

Definition 2.3. Let $n \geq 2$ be an integer, and let $\mathcal{U}$ and $\mathcal{V}$ be n-theories enriched in a symmetric monoidal category $\mathcal{A}$. Then a **functor** $F\colon \mathcal{U} \to \mathcal{V}$ of n-theories consists of data of the forms specified below as (0), "(k)" for every integer k such that $1 \leq k \leq n-1$, (n), ($n+1$), ($n+2$), and "($n+\ell$)" for every integer $\ell \geq 3$.

Similarly to the data for an n-theory, data for a functor will be associated to input data satisfying the same elementality requirements as before. As before, each data form specified needs to be extended for non-elemental input data before the next data form is specified. The way how we do this is more or less the same as before, and we shall do this implicitly.

The forms of data are as follows.

(0) (*Action* on objects.) For every object u of $\mathcal{U}$, an object Fu of $\mathcal{V}$.

(k) (*Action* on k-mutimaps, inductively for $1 \leq k \leq n-1$.) Suppose given the type u of a k-multimap in $\mathcal{U}$ of arity given as $(I;\pi,\phi)$. Then for every k-multimap $x \in \mathrm{Mul}^{\pi}_{\mathcal{U}}[u]$, a k-multimap $Fx \in \mathrm{Mul}^{\pi}_{\mathcal{V}}[Fu]$.

(n) (*Action* on n-multimaps.) Suppose given the type u of an n-multimap in $\mathcal{U}$ of arity given as $(I;\pi,\phi)$. Then a map $\check{m}^F_1(\pi)\colon \mathrm{Mul}^{\pi}_{\mathcal{U}}[u] \to \mathrm{Mul}^{\pi}_{\mathcal{V}}[Fu]$ in $\mathcal{A}$.

($n+1$) (The isomorphism of *compatibility* with the composition.) Suppose given
 – the arity $(I;\pi,\phi)$ of an $(n+1)$-multimap in a symmetric higher theory,
 – the type u of a ϕ^{n-1}-nerve of n-multimaps in $\mathcal{U}$.

Then a 2-isomorphism

$$\check{m}^F_2(\pi)\colon \check{m}^F_1(\pi_!\phi^{n-1}) \circ m^{\mathcal{U}}_1(\pi) \xrightarrow{\sim} m^{\mathcal{V}}_1(\pi) \circ \check{m}^F_1(\phi^{n-1})$$

in $\mathcal{A}$, filling the square

$$\begin{array}{ccc} \mathrm{Mul}^{\phi^{n-1}}_{\mathcal{U}}[u] & \xrightarrow{m^{\mathcal{U}}_1(\pi)} & \mathrm{Mul}^{\pi_!\phi^{n-1}}_{\mathcal{U}}[\pi_!u] \\ {\scriptstyle \check{m}^F_1(\phi^{n-1})}\downarrow & & \downarrow{\scriptstyle \check{m}^F_1(\pi_!\phi^{n-1})} \\ \mathrm{Mul}^{\phi^{n-1}}_{\mathcal{V}}[Fu] & \xrightarrow{m^{\mathcal{V}}_1(\pi)} & \mathrm{Mul}^{\pi_!\phi^{n-1}}_{\mathcal{V}}[\pi_!Fu], \end{array}$$

where $\check{m}^F_1(\phi^{n-1})$ denotes the monoidal product over $i \in I^n_0$ of the maps

$$\check{m}^F_1(\phi^{n-1}_i)\colon \mathrm{Mul}^{\phi^{n-1}_i}[u|_i] \longrightarrow \mathrm{Mul}^{\phi^{n-1}_i}[Fu|_i].$$

($n+2$) (The isomorphism of *coherence* for the compatibility with the composition.) Suppose given
 – the arity $(I;\pi,\phi)$ of an $(n+2)$-multimap in a symmetric higher theory,
 – the type u of a ϕ^{n-1}-nerve of n-multimaps in $\mathcal{U}$.

Then an 3-isomorphism

$$\check{m}^F_3(\pi)\colon \check{m}^F_2(\pi_!\phi^n) \circ m^{\mathcal{U}}_2(\pi) \xrightarrow{\sim} m^{\mathcal{V}}_2(\pi) \circ \pi_!\check{m}^F_2(\phi^n)$$

in $\mathcal{A}$, filling the square

$$\begin{array}{ccc} \check{m}^F_1(\pi^n{}_!\phi^{n-1}) \circ \pi_! m^{\mathcal{U}}_1(\phi^n) & \xrightarrow{\pi_!\check{m}^F_2(\phi^n)} & \pi_! m^{\mathcal{V}}_1(\phi^n) \circ \check{m}^F_1(\phi^{n-1}) \\ {\scriptstyle m^{\mathcal{U}}_2(\pi)}\downarrow & & \downarrow{\scriptstyle m^{\mathcal{V}}_2(\pi)} \\ \check{m}^F_1(\pi^n{}_!\phi^{n-1}) \circ m^{\mathcal{U}}_1(\pi_!\phi^n) & \xrightarrow{\check{m}^F_2(\pi_!\phi^n)} & m^{\mathcal{V}}_1(\pi_!\phi^n) \circ \check{m}^F_1(\phi^{n-1}), \end{array}$$

where $\check{m}_2^F(\phi^n)$ denotes the I_0^{n+1}-nerve $\bigl(\check{m}_1^F(\phi_i^n)\bigr)_{i\in I_0^{n+1}}$ of 2-isomorphisms in $\mathcal{A}$ connecting in the reverse order the family indexed by j $(=[\pi](1)-i)$ $\in [I_0^{n+1}]$ of the maps

$$m_1^{\mathcal{V}}(\phi^n)_{\leftarrow j} \circ \check{m}_1^F\bigl((\phi^n_{\to j})_!\phi^{n-1}\bigr) \circ m_1^{\mathcal{U}}(\phi^n)_{\to j} \colon \operatorname{Mul}_{\mathcal{U}}^{\phi^{n-1}}[u] \longrightarrow \operatorname{Mul}_{\mathcal{V}}^{\pi_!\phi^{n-1}}[\pi_! Fu]$$

in $\mathcal{A}$.

$(n+\ell)$ (Higher coherence, inductively for $\ell \geq 3$.) Suppose given

- the arity $(I;\pi,\phi)$ of an $(n+\ell)$-multimap in a symmetric higher theory,
- the type u of a ϕ^{n-1}-nerve of n-multimaps in $\mathcal{U}$.

Then an $(\ell+1)$-isomorphism

$$\check{m}_{\ell+1}^F(\pi)\colon \check{m}_\ell^F(\pi_!\phi^{n+\ell-2}) \circ m_\ell^{\mathcal{U}}(\pi) \xrightarrow{\sim} m_\ell^{\mathcal{V}}(\pi) \circ \pi_!\check{m}_\ell^F(\phi^{n+\ell-2})$$

in $\mathcal{A}$, filling the square

$$\begin{array}{ccc}
\begin{array}{l}\check{m}_{\ell-1}^F(\pi^{n+\ell-2}{}_!\phi^{n+\ell-3})\circ \\ \qquad \pi_! m_{\ell-1}^{\mathcal{U}}(\phi^{n+\ell-2})\end{array} & \xrightarrow{\pi_!\check{m}_\ell^F(\phi^{n+\ell-2})} & \begin{array}{l}\pi_! m_{\ell-1}^{\mathcal{V}}(\phi^{n+\ell-2})\circ \\ \qquad \pi^{n+\ell-2}{}_!\check{m}_{\ell-1}^F(\phi^{n+\ell-3})\end{array} \\
\Big\downarrow {\scriptstyle m_\ell^{\mathcal{U}}(\pi)} & & \Big\downarrow {\scriptstyle m_\ell^{\mathcal{V}}(\pi)} \\
\begin{array}{l}\check{m}_{\ell-1}^F(\pi^{n+\ell-2}{}_!\phi^{n+\ell-3})\circ \\ \qquad m_{\ell-1}^{\mathcal{U}}(\pi_!\phi^{n+\ell-2})\end{array} & \xrightarrow{\check{m}_\ell^F(\pi_!\phi^{n+\ell-2})} & \begin{array}{l}m_{\ell-1}^{\mathcal{V}}(\pi_!\phi^{n+\ell-2})\circ \\ \qquad \pi^{n+\ell-2}{}_!\check{m}_{\ell-1}^F(\phi^{n+\ell-3}),\end{array}
\end{array}$$

where $\check{m}_\ell^F(\phi^{n+\ell-2})$ denotes the $I_0^{n+\ell-1}$-nerve $\bigl(\check{m}_{\ell-1}^F(\phi_i^{n+\ell-2})\bigr)_{i\in I_0^{n+\ell-1}}$ of ℓ-isomorphisms in $\mathcal{A}$ connecting in the reverse order the family indexed by j $(=[\pi](1)-i) \in [I_0^{n+\ell-1}]$ of the $(\ell-1)$-isomorphisms

$$\begin{aligned}
&m_{\ell-1}^{\mathcal{V}}(\phi^{n+\ell-2})_{\leftarrow j}\circ \\
&\quad \phi^{n+\ell-2}_{\leftarrow j}{}_!\check{m}_{\ell-1}^F(\phi^{n+\ell-2}_{\to j}{}_!\phi^{n+\ell-3}) \circ m_{\ell-1}^{\mathcal{U}}(\phi^{n+\ell-2})_{\to j}\colon \\
&\qquad \check{m}_{\ell-2}^F(\pi^{n+\ell-3}{}_!\phi^{n+\ell-4}) \circ \pi^{n+\ell-2}{}_! m_{\ell-2}^{\mathcal{U}}(\phi^{n+\ell-3}) \\
&\qquad\qquad \xrightarrow{\sim} m_{\ell-2}^{\mathcal{V}}(\pi^{n+\ell-3}) \circ \pi^{n+\ell-3}{}_!\check{m}_{\ell-2}^F(\phi^{n+\ell-4})
\end{aligned}$$

in $\mathcal{A}$.

This completes Definition 2.3.

2.5.1. Now a lax functor can be defined as follows.

Definition 2.4. Let $n \geq 2$ be an integer, and let $\mathcal{U}$ and $\mathcal{V}$ be n-theories enriched in a symmetric monoidal 2-category $\mathcal{A}$ (i.e., in the symmetric monoidal category underlying $\mathcal{A}$). Then a **lax functor** $F\colon \mathcal{U} \to \mathcal{V}$ consists of data of the forms

- (0) through (n) specified above for Definition 2.3 for the same value of "n",
- $(n+1)$ and (∞) below.

$(n+1)$ (The map of *compatibility* with the composition.) Suppose given

- the arity $(I;\pi,\phi)$ of an $(n+1)$-multimap in a symmetric higher theory,
- the type u of a ϕ^{n-1}-nerve of n-multimaps in $\mathcal{U}$.

Then a 2-map

$$m_2^F(\pi)\colon m_1^{\mathcal{V}}(\pi) \circ \check{m}_1^F(\phi^{n-1}) \longrightarrow \check{m}_1^F(\pi_!\phi^{n-1}) \circ m_1^{\mathcal{U}}(\pi)$$

in $\mathcal{A}$. See $(n+1)$ for Definition 2.3 above.

(∞) Data of coherence, similar to that for a functor.

This completes Definition 2.4.

2.5.2. The notion of lax functor can be used to define the notion of "*$(n+1)$-tuply*" *coloured lax n-theory*, which will allow us to describe an $(n+1)$-theory along the line discussed in Section 0.3.4, as to be done in Proposition 2.9. In order to do this, let us first understand an n-theory as a functor of higher theories.

In order to describe the source of the functor, we need the following definitions. Recall that an n-theory $\mathcal{U}$ consists by definition, of data (k) for all integers $k \geq 0$, of the forms specified in Section 1. Of these, the part $k \leq n-1$ does not involve the information of where the theory is enriched.

Definition 2.5. Let $n \geq 0$ be an integer. Then we refer to data of the forms (0) through $(n-1)$ as specified for Definition 1.6 of an n-theory, as a system of **colours up to dimension** $n-1$ for a higher theory, or a system of colours **for an n-theory**.

In particular, for every n-theory $\mathcal{U}$ and every integer m such that $0 \leq m \leq n-1$, we have a system of colours up to dimension m **underlying** $\mathcal{U}$, consisting of the data (0) through (m) for $\mathcal{U}$, so $\mathcal{U}$ consists of this system of colours and a structure on it.

Definition 2.6. Let $m \geq 0$ and $n \geq m+1$ be integers. Then for the system of colours up to dimension m consisting of data of the forms (0) through (m) as specified for Definition 1.6 of an n-theory, we refer to the rest of data for an n-theory, consisting of data of the forms (k) for $k \geq m+1$, as the **structure of an n-theory** on the system of colours.

In order to describe the target of the functor, we need to recall the *categorical delooping*, which associates to a monoid A (or more generally, a monoidal n-category), a category (or $(n+1)$-category in the general case) BA with a chosen "base" object, in which

- all objects are equivalent,
- the endomorphism monoid (or monoidal n-category) of the base object is given an equivalence with A.

Note that, if $\mathcal{A}$ is a *symmetric monoidal* n-category, then $B\mathcal{A}$ is canonically a symmetric monoidal $(n+1)$-category (with unit the base object) since the functor B preserves direct products.

Now the following gives an interpretation of a higher theory as a functor of higher theories.

Example 2.7. Choose and fixed a system of colours up to dimension $n-1$, and denote by $\mathcal{T}$ the terminal object among unenriched $(n+1)$-theories extending this system of lower colours. Explicitly, $\mathcal{T}$ is such that every collection of n-multimaps in it is non-empty, and every groupoid of $(n+1)$-multimaps in it is contractible.

Then the structure on the chosen system of colours, of an n-theory enriched in a symmetric monoidal category $\mathcal{A}$, is equivalent as data to a functor $\mathcal{T} \to \Theta_0^{n+1} B^n \mathcal{A}$ (where B^n indicates n-fold application of the delooping construction B) of $(n+1)$-theories which is **n-trapped** in the sense that it sends any object of $\mathcal{T}$ to the base object of $B^n\mathcal{A}$, and any k-multimap of $\mathcal{T}$ for $1 \leq k \leq n-1$, to the identity k-morphism or the base object of $B^{n-k}\mathcal{A}$.

Let us say that an $(n+1)$-theory $\mathcal{T}$ as in Example 2.7 is terminal **on** the chosen system of colours up to dimension $n-1$. Considering $\mathcal{T}$ as a coloured variant of the terminal unenriched uncoloured $(n+1)$-theory $\mathbf{1}_{\mathrm{Com}}^{n+1}$ (where $\mathrm{Com} = E_\infty$ denotes the commutative operad), one might say that an n-theory enriched in $\mathcal{A}$ is an n-trapped functor $\mathbf{1}_{\mathrm{Com}}^{n+1} \to \Theta_0^{n+1} B^n \mathcal{A}$ *with strata of colours* for an n-theory.

Definition 2.8. For an integer $n \geq 0$, an **$(n+1)$-tuply coloured lax n-theory** enriched in a symmetric monoidal 2-category $\mathcal{A}$, is an n-trapped lax functor $\mathbf{1}^{n+1}_{\mathrm{Com}} \to \Theta^{n+1}_0 B^n\mathcal{A}$ with strata of colours up to dimension n, namely, an n-trapped lax functor $F\colon \mathcal{T} \to \Theta^{n+1}_0 B^n\mathcal{A}$, where $\mathcal{T}$ is an unenriched $(n+1)$-theory, which is terminal on a system of colours for an $(n+1)$-theory.

An $(n+1)$-tuply coloured lax n-theory enriched in $\mathcal{A}$, defined by a lax functor F as above, is said to be **trapped** if F sends n-multimaps of $\mathcal{T}$ to the unit object of $\mathcal{A}$.

Proposition 2.9. *Let $\mathcal{A}$ be a symmetric monoidal category. Then, for an integer $n \geq 1$, an n-theory enriched in $\mathcal{A}$ is equivalent as data to a trapped n-tuply coloured lax $(n-1)$-theory enriched in $B\mathcal{A}$.*

This is a consequence of Theorem 2.10 below, as will be seen in Section 2.6.4.

2.5.3. The following is a key fact on the relation between theorization and categorification.

Theorem 2.10. *Let $n \geq 0$ be an integer, and $\mathcal{U}$ and $\mathcal{V}$ be categorified n-theories. Then a functor $\Theta_n\mathcal{U} \to \Theta_n\mathcal{V}$ of uncategorified $(n+1)$-theories, is equivalent as data to a lax functor $\mathcal{U} \to \mathcal{V}$.*

Proof. **0.** We first note that the form of data for the action of a functor $\Theta_n\mathcal{U} \to \Theta_n\mathcal{V}$ on the colours up to dimension $n-1$, is identical to the form of data for the action of a lax functor $\mathcal{U} \to \mathcal{V}$ on the colours up to the same dimension. Then for any fixed data of this form, we would like to prove that the categories (or 2-categories) naturally formed respectively by the structures on these same data, of functors (of $(n+1)$-theories), and of lax functors (of categorified n-theories), are equivalent.
1. The structure of a categorified n-theory is given on the categories of n-multimaps, by the (cohrently) associative composition functors. A lax functor $F\colon \mathcal{U} \to \mathcal{V}$ extending the chosen action, is then seen to be given by

(0) the action of F on the categories of n-multimaps through functors specified by the data $\check{m}^F_1$, and
(1) the (lax) compatibility specified by the data m^F_2 and $\check{m}^F_k$ for $k \geq 3$, of the mentioned action with the associative composition functors for n-multimaps in $\mathcal{U}$ and in $\mathcal{V}$.

2. Recall that we have obtained $\Theta_n\mathcal{U}$ by replacing the composition functors of $\mathcal{U}$ by the bimodules/distributors/profunctors represented by them, and similarly for $\mathcal{V}$.

Given data as (0) above for a lax functor F, of an action on n-multimaps, data of compatibility as (1) above, of this action with the composition functors for n-multimaps, can equivalently be described as an action on the (associative) composition bimodules for n-multimaps (which were represented by the composition functors) in $\Theta_n\mathcal{U}$ and in $\Theta_n\mathcal{V}$. Note that these bimodules (in $\Theta_n\mathcal{U}$ or in $\Theta_n\mathcal{V}$) are formed by $(n+1)$-multimaps, and the (coherent) associativity of the composition (for n-multimaps) can equivalently be described as the (coherently associative) composition operations for $(n+1)$-multimaps.

In this manner, we consider F as a structure between $\Theta_n\mathcal{U}$ and $\Theta_n\mathcal{V}$.

The difference of this structure between $\Theta_n\mathcal{U}$ and $\Theta_n\mathcal{V}$, from a functor $G\colon \Theta_n\mathcal{U} \to \Theta_n\mathcal{V}$ of $(n+1)$-theories, is that

- the action of G on n-multimaps does not (explicitly) include the data of functoriality included in $\check{m}^F_1$, and
- the action of G on $(n+1)$-multimaps (given by $\check{m}^G_1$) and its compatibility (given by $\check{m}^G_k$, $k \geq 2$) with the composition operations for $(n+1)$-multimaps

in $\Theta_n\mathcal{U}$ and in $\Theta_n\mathcal{V}$, do not explicitly include the data of compatibility with the bimodules structures, included in m_2^F and $\check{m}_{\geq 3}^F$.

Never the less, we obtain a functor $\Theta_n F\colon \Theta_n\mathcal{U} \to \Theta_n\mathcal{V}$ by forgetting these extra data (which will turn out in the next step to have been actually redundant).

3. Conversely, suppose given a functor $G\colon \Theta_n\mathcal{U} \to \Theta_n\mathcal{V}$ which extends the chosen actions up to dimension $n-1$. Then, from the discussions above, a lax functor $H\colon \mathcal{U} \to \mathcal{V}$ extending the same lower actions, such that $\Theta_n H = G$, is constructed if we equip the higher data for G with the following.

- A functoriality in the n-multimap $x \in \mathrm{Mul}_{\mathcal{U}}^{\pi}[u]$, on the n-multimap $Gx \in \mathrm{Mul}_{\mathcal{V}}^{\pi}[Hu]$.
- A compatibility of this functoriality with the data $\check{m}_i^G$, $i \geq 1$, of the action of G (in a manner compatible with the (associative) composition operations for $(n+1)$-multimaps in $\Theta_n\mathcal{U}$ and in $\Theta_n\mathcal{V}$) on $(n+1)$-multimaps.

In order to obtain a functoriality of the n-multimap Gx, one notes that a map in the category $\mathrm{Mul}_{\mathcal{U}}^{\pi}[u]$ is suitably 'unary' $(n+1)$-multimaps in $\Theta_n\mathcal{U}$. The desired functoriality is given by the restriction of the associative action of G to unary $(n+1)$-multimaps in $\Theta_n\mathcal{U}$.

A compatibility of this functoriality with the action of G means

- a naturality of every instance of the map

$$\check{m}_1^G(\pi)\colon \mathrm{Map}_{\mathrm{Mul}_{\mathcal{U}}^{\pi_!\phi^{n-1}}[\pi_! u^{\leq n-1}]}\big(m_1^{\mathcal{U}}(\pi)(u_0^n), u_1^n\big) \longrightarrow \mathrm{Map}_{\mathrm{Mul}_{\mathcal{V}}^{\pi_!\phi^{n-1}}[\pi_! Hu^{\leq n-1}]}\big(m_1^{\mathcal{V}}(\pi)(Gu_0^n), Gu_1^n\big)$$

  in u^n, and
- a compatibility of this naturality with the compatibility, given by $\check{m}_i^G$, $i \geq 2$, of these maps $\check{m}_1^G(\pi)$ with the composition operations in $\Theta_n\mathcal{U}$ and in $\Theta_n\mathcal{V}$.

We note that the functoriality in u^n of the target of $\check{m}_1^G(\pi)$ is the composite of the functoriality of $\mathrm{Map}_{\mathrm{Mul}_{\mathcal{V}}^{\pi_!\phi^{n-1}}[\pi_! Hu^{\leq n-1}]}\big(m_1^{\mathcal{V}}(\pi)(-), -\big)$ in its variables, with the functoriality of G. Moreover, the compatibility of this functoriality with the composition structures of $\Theta_n\mathcal{U}$ and of $\Theta_n\mathcal{V}$, is also induced by the same functoriality of G, from the compatibility of the functoriality of $\mathrm{Map}_{\mathrm{Mul}_{\mathcal{V}}^{\pi_!\phi^{n-1}}[\pi_! v]}\big(m_1^{\mathcal{V}}(\pi)(-), -\big)$ with the composition operations.

Now, the functoriality, together with its compatibility with the composition structure of $\Theta_n\mathcal{U}$, of $\mathrm{Map}_{\mathrm{Mul}_{\mathcal{U}}^{\pi_!\phi^{n-1}}[\pi_! u^{\leq n-1}]}\big(m_1^{\mathcal{U}}(\pi)(-), -\big) = \mathrm{Mul}_{\Theta_n\mathcal{U}}^{\pi}\big[u^{\leq n-1}\big]\big(-; -\big)$ can alternatively be considered as obtained by suitably restricting the composition structure of $\Theta_n\mathcal{U}$, and similarly for $\mathrm{Map}_{\mathrm{Mul}_{\mathcal{V}}^{\pi_!\phi^{n-1}}[\pi_! v]}\big(m_1^{\mathcal{V}}(\pi)(-), -\big)$. Since the functoriality of G on the category of n-multimaps is as constructed above, we see that the compatibility of the action of G on $(n+1)$-multimaps with the composition structures of $\Theta_n\mathcal{U}$ and of $\Theta_n\mathcal{V}$, also induces the desired data.

4. We have thus constructed a desired lax functor $H\colon \mathcal{U} \to \mathcal{V}$ naturally from G. Moreover, it is immediate to verify that this construction recovers in the case $G = \Theta_n F$, all the data "forgotten" in the construction $F \mapsto \Theta_n F$, so the lax functor "H" will be naturally equivalent to F. □

Remark 2.11. We have thus described for the $(n+1)$-theory $\mathcal{W} = \Theta_n\mathcal{V}$, a functor $G\colon \Theta_n\mathcal{U} \to \mathcal{W}$ as a lax functor $\mathcal{U} \to \mathcal{V}$. Similar arguments lead to a description of G for an arbitrary $(n+1)$-theory $\mathcal{W}$, as a structure generalizing a lax functor $\mathcal{U} \to \mathcal{V}$. One might call this structure between $\mathcal{U}$ and $\mathcal{W}$ a **functor** $\mathcal{U} \to \mathcal{W}$. Thus, a functor $\Theta_n\mathcal{U} \to \mathcal{W}$ will be equivalent as data to a functor $\mathcal{U} \to \mathcal{W}$.

2.5.4. Note that n-theories enriched in groupoids are also among n-theories enriched in categories since a groupoid can be considered as a category in which every morphism is invertible. This kind of unenriched categorified n-theories is special as a target of a functor in that there is no difference between an ordinary functor and a lax functor from an unenriched categorified n-theory to such a target.

Corollary 2.12. *The functor Θ_n is fully faithful on n-theories enriched in groupoids.*

2.6. **Delooping a higher theory.**

2.6.0. There is a construction, which we shall call the *delooping*, of a symmetric n-tuply coloured $(n+1)$-theory from a symmetric n-theory. We shall describe this construction, and then discuss its relation to the "categorical" delooping.

An $(n+1)$-theory which is obtained through this construction normally fails to be representable by an n-theory. Delooped theories will be conveniently used throughout this work as the targets of (possibly "coloured") functors of higher theories.

2.6.1. The delooping construction relies on the following construction.

Suppose first that the arity of a 2-multimap is specified as in $(1')$ and $(0')$ of (2) in Section 1.4. Then since our notation 1.2 chooses an embedding $I^1_0 \hookrightarrow [I^1_0]$, it makes sense to take the coproduct $\coprod_{I^1_0} I^0$. Then note that a ϕ^0-nerve u^0_0 as in (2) of Definition 2.1 in the case $m-n=1$, of objects of an $(n-1)$-tuply coloured n-theory, can be considered as a $(\coprod_{I^1_0} I^0)$-family, while an (I^0_0-ary) object u^0_1 is a $(\coprod_{I^1_1} \pi_! I^0)$-family, where $I^1_1 := \{1\}$.

Suppose next given the arity $(I;\pi,\phi)$ of a 3-multimap in a symmetric higher theory. Then for every $i \in I^2_0$ and $j \in I^1_i$, we have a map $\coprod_{(\phi^1_i)^{-1}j} (\phi^1_{\to i-1})_! I^0 \to I^0_{[\phi^1_{\to i}](j)}$ whose component for $k \in (\phi^1_i)^{-1}j$ is the composite

$$\phi^0_{[\phi^1_{\to i}](j)} \cdots \phi^0_{[\phi^1_{\to i-1}](k)+1} \colon I^0_{[\phi^1_{\to i-1}](k)} \longrightarrow I^0_{[\phi^1_{\to i}](j)}.$$

Taking the coproduct of these over j, we obtain a map $\coprod_{I^1_{i-1}} (\phi^1_{\to i-1})_! I^0 \to \coprod_{I^1_i} (\phi^1_{\to i})_! I^0$, which together form an I^2_0-nerve in Fin.

Let us denote this nerve by $\int_{I^1} \phi^0$, and the $[I^2_0]$-family of finite sets connected by it by $\int_{I^1} I^0$. Then data u^0_i of $(1'')$ of (3) for Definition 2.1 in the case $m = n-1$, can be considered as a $\left(\int_{I^1} I^0\right)_i$-family, so u^0 is identical as data to a $(\int_{I^1} I^0)$-family.

2.6.2. Suppose now given an n-theory $\mathcal{V}$. Then we wish to construct a new n-tuply coloured $(n+1)$-theory $\mathcal{U} = \mathbb{B}\mathcal{V}$ by precomposing the above constructions to the data for $\mathcal{V}$.

The construction of $\mathcal{U}$ is as follows.

(1) Given a finite set S, we let $\mathrm{Ob}^\pi \mathcal{U} = \mathrm{Ob}\,\mathcal{V}$, where π denotes the unique map $S \to *$.
(2) Given the arity of a 2-multimap as in the preliminary construction above, if $n = 0$, then we let $m^{\mathcal{U}}_1(\pi)\colon \mathrm{Ob}^{\phi^0} \mathcal{U} \to \mathrm{Ob}^{\pi_! \phi^0} \mathcal{U}$ be the multiplication map $(\mathrm{Ob}\,\mathcal{V})^{\otimes(\coprod_{I^1_0} I^0)} \to \mathrm{Ob}\,\mathcal{V}$. If $n \geq 1$, and further given a ϕ^0-nerve u^0_0 of objects of $\mathcal{U}$ and an I^0_0-ary object u^0_1 as in (2) of Definition 2.1 in the case $m = n-1$, then we let $\mathrm{Mul}^\pi_{\mathcal{U}}[u^0] = \mathrm{Mul}^{\pi'}_{\mathcal{V}}[u^0]$, where π' denotes the unique map $\coprod_{I^1_0} I^0 \to *$, and u^0 is considered as a $(\int_{I^1} I^0)$-family of objects in $\mathcal{V}$, by the preliminary construction above.

Next, if $n \geq 2$, suppose given input data for (3) for Definition 2.1 in the case $m = n-1$. Then we can construct a set of data of the form required of an input to (2) in Section 1.4 for $\mathcal{V}$, as follows. If we define

$$J_0^1 := I_0^2, \quad J^0 := \int_{I^1} I^0, \quad \psi^0 := \int_{I^1} \phi^0, \quad \psi^1 := (\pi\colon J_0^1 \to \{1\}),$$

then J, ψ, u give the required form of data. Using this, we let $\mathrm{Mul}^{\pi}_{\mathcal{U}}[u] = \mathrm{Mul}^{\pi}_{\mathcal{V}}[u]$.

If $n \geq 1$, we let $m^{\mathcal{U}}_{\nu}(\pi) = m^{\mathcal{V}}_{\nu}(\pi)$ for $\nu = 1$ or 2, in the similar manner, where the form of u is different in the case $n = 1$, and we are not given u in the case $n = 0$.

For every $k \geq 4$, data (k) for $\mathcal{U}$ are constructed in the similar manner from data $(k-1)$ for $\mathcal{V}$.

2.6.3. It is clear that if $\mathcal{U}$ is an m-tuply coloured n-theory, then its deloop $\mathbb{B}\mathcal{U}$ is obtained as an m-tuply coloured $(n+1)$-theory.

2.6.4. Our delooping construction for higher theories relates to the categorical delooping recalled in Section 2.5 for the formulation of Example 2.7. Let us describe one relation which will be convenient for us.

Specifically, let $n \geq 0$ be an integer, and $\mathcal{A}$ be a symmetric monoidal n-category. Then for an integer m such that $0 \leq m \leq n$, we would like to relate the $(m+1)$-theory $\mathbb{B}\Theta_0^m\mathcal{A}$ to the $(m+1)$-theory $\Theta_0^{m+1}B\mathcal{A}$, both enriched in $(n-m)$-categories.

An obvious issue for this is that $\Theta_0^{m+1}B\mathcal{A}$ is fully coloured whereas $\mathbb{B}\Theta_0^m\mathcal{A}$ is only m-tuply coloured. However, if we restrict the data for $\Theta_0^{m+1}B\mathcal{A}$ so the only object we consider is the base object of $B\mathcal{A}$, then the rest of the data for $\Theta_0^{m+1}B\mathcal{A}$ is of the form of data for an m-tuply coloured $(m+1)$-theory enriched in $(n-m)$-categories.

Proposition 2.13. *Let $n \geq 0$ be an integer, and $\mathcal{A}$ be a symmetric monoidal n-category. Then for every integer m such that $0 \leq m \leq n$, $\mathbb{B}\Theta_0^m\mathcal{A}$ is equivalent to the $(n-m)$-categorified m-tuply coloured $(m+1)$-theory obtained as above from $\Theta_0^{m+1}B\mathcal{A}$ by restricting objects to just the base object of $B\mathcal{A}$.*

Proof. The case $m = 0$ is immediate from the constructions, and the general case follows by induction, from Lemma 2.14 below. ☐

The lemma is as follows, and follows immediately from the definitions.

Lemma 2.14. *For an arbitrary times categorified n-theory $\mathcal{U}$, we have*

$$\mathbb{B}\Theta_n\mathcal{U} \simeq \Theta_{n+1}\mathbb{B}\mathcal{U}.$$

Proof of Proposition 2.9. Definition 2.8 and Proposition 2.13 implies that a trapped n-tuply coloured lax $(n-1)$-theory enriched in $B\mathcal{A}$ is equivalent as data to a lax functor $\mathbf{1}^n_{\mathrm{Com}} \to \mathbb{B}^n\mathcal{A}$ with strata of colours up to dimension $n-1$. Example 2.7 and Proposition 2.13 implies that an n-theory enriched in $\mathcal{A}$ is equivalent as data to a functor $\mathbf{1}^{n+1}_{\mathrm{Com}} \to \Theta_n\mathbb{B}^n\mathcal{A}$ with strata of colours up to dimension $n-1$.

Let $\mathcal{T}$ be an unenriched n-theory which is terminal on a system of colours up to dimension $n-1$. Then the $(n+1)$-theory $\Theta_n\mathcal{T}$ (where we have considered $\mathcal{T}$ as trivially categorified) is terminal on the same system of colours, so the result follows immediately from the equivalence of Theorem 2.10. ☐

We also obtain the following from Proposition 2.13.

Corollary 2.15. *Let $\mathcal{A}$ be a symmetric monoidal category. Then for an unenriched $(n+1)$-theory $\mathcal{U}$, an n-trapped functor $\mathcal{U} \to \Theta_0^{n+1}B^n\mathcal{A}$ is equivalent as data to a functor $\mathcal{U} \to \mathbb{B}^n\Theta_0\mathcal{A}$.*

Definition 2.16. Let $n \geq 0$ be an integer. Then an **n-theory enriched in a multicategory** $\mathcal{M}$, is a functor $\mathbf{1}^{n+1}_{\mathrm{Com}} \to \mathbb{B}^n\mathcal{M}$ with strata of colours for an n-theory.

Generalizing Proposition 2.9, this is equivalent as data to a functor $\mathbf{1}^n_{\mathrm{Com}} \to \mathbb{B}^n\mathcal{M}$ (Remark 2.11) with strata of colours up to dimension $n-1$ for a higher theory.

3. Graded higher theories

3.0. **Introduction.** The purpose of this section is to discuss grading of higher theory by a higher theory (enriched in groupoids). It turns out that this notion naturally arises by considering theorization of algebra over a higher theory. We shall also see how "graded" lower theory over a higher theory is related to *iterated monoid*, i.e., monoid (i.e., algebra in the category of groupoids) over a monoid over ... over a monoid, over a higher theory enriched in groupoids.

3.1. Algebras over a higher theory.

3.1.0. We have sought for the notion of n-theory as an interesting n-th theorization of the notion of commutative algebra. As have been discussed in Section 0, what we wished to get from this was for each n-theory to govern *algebras* over it, which naturally generalize $(n-1)$-theories.

Definition 3.0. Let $n \geq 2$ be an integer, and let $\mathcal{U}$ be an unenriched n-theory. Then a **$\mathcal{U}$-algebra** $\mathcal{X}$ enriched in a multicategory $\mathcal{M}$ consists of data of the forms specified below as (0) and (1) (or just (0) if $n=2$), "(k)" for every integer k such that $2 \leq k \leq n-2$, $(n-1)$, (n) and (∞).

Let us specify the forms of data for Definition.

Remark 3.1. As before, we implicitly extend each data form specified below for non-elemental input data before proceeding to specifying the next data form, in a more or less similar manner as before.

(0) (*Object.*) For every object u of $\mathcal{U}$, a collection $\mathrm{Ob}_u\,\mathcal{X}$, whose member will be called an **object** of $\mathcal{X}$ of **degree** u.

(1) (*Multimap*, in the case $n \geq 3$.) Suppose given $(0')$ and $(0'')$ of (1) in Section 1.3, and
   - (0°) an S-family x_0 of objects of $\mathcal{X}$ of degree u_0 (namely, $x_{0s} \in \mathrm{Ob}_{u_{0s}}\,\mathcal{X}$ for every $s \in S$), and an object x_1 of degree u_1.

   Then for every multimap $v \in \mathrm{Mul}^{\pi}_{\mathcal{U}}[u]$, a collection $\mathrm{Mul}^{\pi}_{\mathcal{X},v}(x_0; x_1)$ or $\mathrm{Mul}^{\pi}_{\mathcal{X},v}[x]$ for short, whose member will be called an (S-ary) (1-)**multimap** $x_0 \to x_1$ in $\mathcal{X}$ of **degree** v.

(k) (k-*multimap*, inductively for $2 \leq k \leq n-2$.) Suppose given (a), (0°), (b), $(k-2°)$ (or just (a), "(0°)" if $k=2$), and $(k-1°)$ below:
   - (a) the type u of a k-multimap in $\mathcal{U}$ of arity given as $(I; \pi, \phi)$.
   - (0°) an I^0-family x^0 of objects of $\mathcal{X}$, of degree u^0, namely, $x^0 = (x^0_i)_{i \in [I^1_0]}$, where x^0_i is an I^0_i-family of objects of $\mathcal{X}$, of degree u^0_i,
   - (b) if $k \geq 4$, then (1°) through $(k-3°)$ of $(k-1)$ here,
   - $(k-2°)$ (in the case $k \geq 3$) an I^{k-2}-family $x^{k-2} = (x^{k-2}_i)_{i \in [I^{k-1}_0]}$ of $(k-2)$-multimaps in $\mathcal{X}$, where x^{k-2}_i is an $\phi^{k-2}_{\to i\,!}\phi^{k-3}$-nerve of $(k-2)$-multimaps in $\mathcal{X}$, connecting $\phi^{k-2}_{\to i\,!}x^{k-3}$ of degree u^{k-2}_i,
   - $(k-1°)$
     - a ϕ^{k-2}-nerve x^{k-1}_0 of $(k-1)$-multimaps connecting x^{k-2}, of degree u^{k-1}_0 in $\mathcal{X}$, namely, $x^{k-1}_{0i} \in \mathrm{Mul}^{\phi^{k-2}_i}_{u^{k-1}_{0i}}[x^{\leq k-2}|_i]$ for every $i \in I^{k-1}_0$,
     - $x^{k-1}_1 \in \mathrm{Mul}^{\pi_!\phi^{k-2}}_{u^{k-1}_1}[\pi_! x^{\leq k-2}]$.

Then for every k-multimap $v \in \mathrm{Mul}^{\pi}_{\mathcal{U}}[u]$, a collection $\mathrm{Mul}^{\pi}_{\mathcal{X},v}[x^{\leq k-2}](x^{k-1}_0; x^{k-1}_1)$ or $\mathrm{Mul}^{\pi}_{\mathcal{X},v}[x]$ for short, whose member will be called a **k-multimap** $x^{k-1}_0 \to x^{k-1}_1$ in $\mathcal{X}$ of **degree** v.

Definition 3.2. We refer to data $x = (x^{\nu})_{0\leq\nu\leq k-1}$ of the form specified by (0°) through ($k-1$°) above (by induction in k), as the **type of a k-multimap** in $\mathcal{X}$ of **arity** $(I; \pi, \phi)$ and of **degree** u.

Remark 3.3. Even though we have not yet specified the forms of the rest of data for $\mathcal{X}$, note that the notion of the type of a k-multimap "in $\mathcal{X}$" makes sense as soon as data of the forms (0) through ($k-1$) are given "for $\mathcal{X}$".

($n-1$) (The object "formed by *($n-1$)-multimaps*".) Suppose given the type u of an ($n-1$)-multimap in $\mathcal{U}$ of arity given as $(I; \pi, \phi)$, and the type x of an ($n-1$)-multimap in $\mathcal{X}$ of the same arity of degree u, namely, a set of data similar to (0°) through ($k-1$°), of "(k)" above, but with k substituted by $n-1$, so these will be (0°) through ($n-2$°) here. Then for every ($n-1$)-multimap $v \in \mathrm{Mul}^{\pi}_{\mathcal{U}}[u]$, an object $\mathrm{Mul}^{\pi}_{\mathcal{X},v}[x^{\leq n-3}](x^{n-2}_0; x^{n-2}_1)$ of $\mathcal{M}$, or $\mathrm{Mul}^{\pi}_{\mathcal{X},v}[x]$ for short. In the case where $\mathcal{M}$ is Θ_0Gpd or some other multicategory so that $\mathrm{Mul}^{\pi}_{\mathcal{X},v}[x]$ can have its objects, then those objects will be called **($n-1$)-multimaps** $x^{n-2}_0 \to x^{n-2}_1$ in $\mathcal{X}$ of **degree** v. For a general $\mathcal{M}$, we shall call $\mathrm{Mul}^{\pi}_{\mathcal{X},v}[x]$ the object "of *($n-1$)-multimaps* in $\mathcal{X}$ of *degree* v".

(n) (*Action* of the n-multimaps of $\mathcal{U}$.) Suppose given
- the type u of an n-multimap in $\mathcal{U}$ of arity given as $(I; \pi, \phi)$,
- (0°) through ($n-3$°) of ($n-1$) above,
- ($k-2$°) of "(k)" above, but with k substituted by n, so this will be ($n-2$°) here.

Then a map

$$\check{m}^{\mathcal{X}}_1(\pi)\colon\ \mathrm{Mul}^{\pi}_{\mathcal{U}}\left[u\right] \longrightarrow \mathrm{Mul}_{\mathcal{M}}\left(\mathrm{Mul}^{\phi^{n-2}}_{\mathcal{X},u^{n-1}_0}[x]; \mathrm{Mul}^{\pi_!\phi^{n-2}}_{\mathcal{X},u^{n-1}_1}[\pi_! x]\right)$$

of groupoids, where $\mathrm{Mul}^{\phi^{n-2}}_{\mathcal{X},u^{n-1}_0}[x]$ denotes the $(\coprod_{I^{n-1}_0} I^{n-2})$-family $\coprod_{i\in I^{n-1}_0} \mathrm{Mul}^{\phi^{n-2}_i}_{\mathcal{X},u^{n-1}_{0i}}[x|_i]$ of objects of $\mathcal{M}$, and $\pi_! x$ consists of $x^{\leq n-3}$, $\pi_! x^{n-2}$. For v in the source of this map, the multimap

$$\check{m}_1(\pi)(v)\colon\ \mathrm{Mul}^{\phi^{n-2}}_{u^{n-1}_0}[x] \longrightarrow \mathrm{Mul}^{\pi_!\phi^{n-2}}_{u^{n-1}_1}[\pi_! x]$$

in $\mathcal{M}$ will be called the **composition** operation **along** v for ($n-1$)-multimaps in $\mathcal{X}$.

Definition 3.4. We refer to data $x = (x^{\nu})_{0\leq\nu\leq n-2}$ of the form specified by (0°) through ($n-2$°) above, as the **type of a ϕ^{n-2}-nerve of ($n-1$)-multimaps** in $\mathcal{X}$ of **degree** $u^{\leq n-2}$.

(∞) Data of (coherent) associativity for the action of n-multimaps, similar to those for a functor (Definition 2.3).

This completes Definition 3.0.

3.1.1. Similarly, one can define the notion of algebra over a less coloured n-theory. See Section 2.3.

Example 3.5. A $\mathbf{1}^n_{\mathrm{Com}}$-algebra enriched in a symmetric monoidal category $\mathcal{A}$, i.e., in the multicategory $\Theta_0\mathcal{A}$, is equivalent as data to an ($n-1$)-theory enriched in $\mathcal{A}$.

It follows from Example 2.7 and Corollary 2.15, that a $\mathbf{1}^n_{\mathrm{Com}}$-algebra enriched in the multicategory $\mathcal{M} = \Theta_0\mathcal{A}$, is a coloured functor $\mathbf{1}^n_{\mathrm{Com}} \to \mathbb{B}^{n-1}\mathcal{M}$, and this is true in fact, for every multicategory $\mathcal{M}$. This can further be generalized over an arbitrary n-theory $\mathcal{U}$ after we theorize the notion of $\mathcal{U}$-algebra, which we shall do next.

3.2. **Iterated theorizations of algebra.**

3.2.0. We would like to theorize the notion of algebra. Let us first relax the notion.

Definition 3.6. Let $n \geq 2$ be an integer, and let $\mathcal{U}$ be an unenriched n-theory. Then a **lax $\mathcal{U}$-algebra** $\mathcal{X}$ enriched in a categorified multicategory $\mathcal{M}$ consists of data of the forms (0) through (n), specified above for Definition 3.0 for the same value of "n", and data of coherent lax associativity for the action of n-multimaps, similar to those for a lax functor (Definition 2.4).

This essentially contains at least the unenriched version of a virtualization of the notion of op-lax categorified $\mathcal{U}$-algebra, namely, a *theorization* of the notion of $\mathcal{U}$-algebra. We shall write down an enriched version of the definition explicitly, in a form which will be convenient shortly. (The notion will be generalized in Section 4.) A natural name for the kind of thing will turn out to be "graded n-theory".

Definition 3.7. Let $n \geq 1$ be an integer, and let $\mathcal{U}$ be an n-theory enriched in groupoids. Then a **$\mathcal{U}$-graded n-theory** $\mathcal{X}$ enriched in a multicategory $\mathcal{M}$ consists of data of the forms specified below as (0) and (1) (or just (0) if $n = 2$), "(k)" for every integer k such that $2 \leq k \leq n-2$, $(n-1)$, (n), $(n+1)$, and (∞).

The forms of data are as follows. Remark 3.1 applies here again.

(0) (Object.) For every object u of $\mathcal{U}$, a collection $\mathrm{Ob}_u\,\mathcal{X}$, whose member will be called an **object** of $\mathcal{X}$ of **degree** u.

(1) (Multimap, in the case $n \geq 2$.) Suppose given $(0')$ and $(0'')$ of (1) in Section 1.3, and
 - (0°) an S-family x_0 of objects of $\mathcal{X}$ of degree u_0, and an object x_1 of degree u_1.

 Then for every multimap $v \in \mathrm{Mul}^{\pi}_{\mathcal{U}}[u]$, a collection $\mathrm{Mul}^{\pi}_{\mathcal{X},v}(x_0; x_1)$ or $\mathrm{Mul}^{\pi}_{\mathcal{X},v}[x]$ for short, whose member will be called an (S-ary) (**1-**)**multimap** $x_0 \to x_1$ in $\mathcal{X}$ of **degree** v.

(k) (k-multimap, inductively for $2 \leq k \leq n-1$.) Suppose given the type u of a k-multimap in $\mathcal{U}$ of arity given as $(I; \pi, \phi)$, and the type x of a k-multimap in $\mathcal{X}$ of the same arity of degree u, namely, (0°), (b), $(k-2^\circ)$ (or just "(0°)" if $k = 2$), and $(k-1^\circ)$ below:
 - (0°) an I^0-family x^0 of objects of $\mathcal{X}$, of degree u^0, namely, $x^0_i \in \mathrm{Ob}_{u^0_i}\,\mathcal{X}$ for every $i \in [I^1_0]$,
 - (b) if $k \geq 4$, then (1°) through $(k-3^\circ)$ of $(k-1)$ here,
 - $(k-2^\circ)$ (in the case $k \geq 3$) an I^{k-2}-family $x^{k-2} = (x^{k-2}_i)_{i \in [I^{k-1}_0]}$, where x^{k-2}_i is an $\phi^{k-2}_{\to i}{}_!\phi^{k-3}$-nerve of $(k-2)$-multimaps connecting $\phi^{k-2}_{\to i}{}_!x^{k-3}$ in $\mathcal{X}$, of degree u^{k-2}_i,
 - $(k-1^\circ)$ a ϕ^{k-2}-nerve x^{k-1}_0 of $(k-1)$-multimaps connecting x^{k-2}, of degree u^{k-1}_0 in $\mathcal{X}$, and $x^{k-1}_1 \in \mathrm{Mul}^{\pi_!\phi^{k-2}}_{u^{k-1}_1}[\pi_! x^{\leq k-2}]$.

 Then for every k-multimap $v \in \mathrm{Mul}^{\pi}_{\mathcal{U}}[u]$, a collection $\mathrm{Mul}^{\pi}_{\mathcal{X},v}[x^{\leq k-2}](x^{k-1}_0; x^{k-1}_1)$ or $\mathrm{Mul}^{\pi}_{\mathcal{X},v}[x]$ for short, whose member will be called a **k-multimap** $x^{k-1}_0 \to x^{k-1}_1$ in $\mathcal{X}$ of **degree** v.

(n) (Action of the n-multimaps of $\mathcal{U}$.) Suppose given the type x of an n-multimap in $\mathcal{X}$ of arity and degree given respectively as $(I;\pi,\phi)$ and u. Then a functor (to the underlying category of $\mathcal{M}$)

$$M_1^{\mathcal{X}}(\pi)[x^{\le n-2}](x_0^{n-1},x_1^{n-1})\colon \operatorname{Mul}_{\mathcal{U}}^{\pi}[u]\longrightarrow \mathcal{M}$$

which will also be denoted by $M_1^{\mathcal{X}}(\pi)[x]$ for short. For v in the source of this functor, we write

$$\operatorname{Mul}_{\mathcal{X},v}^{\pi}[x]:=\operatorname{Mul}_{\mathcal{X},v}^{\pi}[x^{\le n-2}](x_0^{n-1};x_1^{n-1}):=M_1^{\mathcal{X}}(\pi)[x](v).$$

In the case where $\mathcal{M}$ is Θ_0Gpd or some other multicategory so that $\operatorname{Mul}_{\mathcal{X},v}^{\pi}[x]$ can have its objects, then those objects will be called **n-multimaps** $x_0^{n-1}\to x_1^{n-1}$ in $\mathcal{X}$ of **degree** v. For a general $\mathcal{M}$, we shall call $\operatorname{Mul}_{\mathcal{X},v}^{\pi}[x]$ the object "of *n-multimaps* in $\mathcal{X}$ of *degree v*".

($n+1$) (Associativity map.) Suppose given
- the arity $(I;\pi,\phi)$ of an $(n+1)$-multimap in a symmetric higher theory,
- the type u of a ϕ^{n-1}-nerve of n-multimaps in $\mathcal{U}$,
- the type x of a ϕ^{n-1}-nerve of n-multimaps in $\mathcal{X}$ of degree $u^{\le n-1}$, namely, (0°) through ($k-2$°) of "(k)" above, but with k substituted by $n+1$, so this will be (0°) through ($n-1$°) here.

Then a multimap

$$M_2^{\mathcal{X}}(\pi)\colon M_1^{\mathcal{X}}(\phi^{n-1})[x]\longrightarrow M_1^{\mathcal{X}}(\pi_!\phi^{n-1})[\pi_!x]\circ m_1^{\mathcal{U}}(\pi)$$

of functors $\operatorname{Mul}_{\mathcal{U}}^{\phi^{n-1}}[u]\to\mathcal{M}$, where the source of this multimap is the $\coprod_{I_0^n} I^{n-1}$-family $\coprod_{i\in I_0^n}\operatorname{pr}_i{}^*M_1(\phi_i^{n-1})[x|_i]$, where pr_i denotes the projection $\operatorname{Mul}_{\mathcal{U}}^{\phi^{n-1}}[u]\to\operatorname{Mul}^{\phi_i^{n-1}}[u|_i]$. For v in the source of this functor, we write

$$m_1^{\mathcal{X}}(\pi)_v:=M_2^{\mathcal{X}}(\pi)(v)\colon \operatorname{Mul}_v^{\phi^{n-1}}[x]\longrightarrow \operatorname{Mul}_{\pi_!v}^{\pi_!\phi^{n-1}}[\pi_!x],$$

where $\pi_!v:=m_1^{\mathcal{U}}(\pi)(v)$, and call it the **composition** operation **along** v for n-multimaps.

(∞) Data of coherence for the associativity, corresponding to those for a lax $\mathcal{U}$-algebra (Definition 3.6).

This completes Definition 3.7.

3.2.1. Let us consider the unenriched case where $\mathcal{M}=\Theta_0$Gpd.

Note that in this case, $M_1(\pi)[x]$ above can be identified with the data of the canonical projection map

$$\operatorname*{colim}_{v\in\operatorname{Mul}_{\mathcal{U}}^{\pi}[u]}\operatorname{Mul}_{\mathcal{X},v}^{\pi}[x]\longrightarrow\operatorname{Mul}_{\mathcal{U}}^{\pi}[u]$$

of groupoids. The groupoid $\operatorname{colim}_v\operatorname{Mul}_{\mathcal{X},v}^{\pi}[x]$ of n-multimaps of arbitrary degrees, will further be the groupoid of n-multimaps in a symmetric n-theory.

Indeed, data M_2 induces "composition" operations $\operatorname{colim}_v m_1^{\mathcal{X}}(\pi)(v)$ on these groupoids, covering the composition operations $m_1^{\mathcal{U}}(\pi)$ in $\mathcal{U}$. Writing down the coherence data for the associativity in $\mathcal{X}$ (which is straightforward), we obtain the case $m=0$ of Proposition 3.8 below (hence our term for the notion).

The construction simply remains valid in the unenriched (higher) categorified case $\mathcal{M}=\Theta_0\mathrm{Cat}_m$.

Proposition 3.8. *Let $n\ge 1$ be an integer, and let $\mathcal{U}$ be an n-theory enriched in groupoids. Then for every integer $m\ge 0$, an unenriched m-categorified $\mathcal{U}$-graded n-theory (i.e., $\mathcal{U}$-graded n-theory enriched in $\Theta_0\mathrm{Cat}_m$) is equivalent as data to an unenriched m-categorified **symmetric** n-theory $\mathcal{Y}$ equipped with a functor $\mathcal{Y}\to\mathcal{U}$.*

In order to see this more precisely, let us introduce the following terminology, which will be justified shortly.

Definition 3.9. Let $n \geq 1$ be an integer, and $\mathcal{U}$ be an (unenriched) n-theory. Then, for an integer m such that $0 \leq m \leq n$, we refer to data of the forms (0) through $(m-1)$ specified for Definition 3.7, as a system of **colours** up to dimension $m-1$ **for a $\mathcal{U}$-graded higher theory**, or a system of colours **for a $\mathcal{U}$-graded m-theory**.

Suppose given a system $\mathcal{X}$ of colours up to dimension $n-2$ for a $\mathcal{U}$-graded higher theory. Then we obtain a system $\Delta_!\mathcal{X}$ (where $\Delta\colon \mathcal{U} \to \mathbf{1}^n_{\mathrm{Com}}$) of colours up to dimension $n-2$ for a symmetric higher theory, by inductively defining as follows.

We first define $\mathrm{Ob}\,\Delta_!\mathcal{X}$ as the collection whose member is a pair (u,x), where $u \in \mathrm{Ob}\mathcal{U}$ and $x \in \mathrm{Ob}_u\,\mathcal{X}$. If $n \geq 3$, then inductively for an integer k such that $1 \leq k \leq n-2$, the type of a k-multimap in $\mathcal{Y}$ will be an identical form of data as a pair (u,x), where u is the type of a k-multimap in $\mathcal{U}$, and x is the type of a k-multimap in $\mathcal{X}$ of degree u. We then inductively define $\mathrm{Mul}^{\pi}_{\Delta_!\mathcal{X}}[(u,x)]$ as the collection whose member is a pair (v,y) consisting of $v \in \mathrm{Mul}^{\pi}_{\mathcal{U}}[u]$ and $y \in \mathrm{Mul}^{\pi}_{\mathcal{X},v}[x]$. Moreover, one can associate to every member $(v,y) \in \mathrm{Mul}^{\pi}_{\Delta_!\mathcal{X}}[x]$ the member $v \in \mathrm{Mul}^{\pi}_{\mathcal{U}}[u]$.

Proposition 3.8 now follows since the category (or $(m+2)$-category) of extensions of the data $\Delta_!\mathcal{X}$ to an unenriched m-categorified symmetric n-theory equipped with a functor to $\mathcal{U}$, gets equated by the described construction, to the category (or $(m+2)$-category) of extensions of the data $\mathcal{X}$ to an unenriched m-categorified $\mathcal{U}$-graded n-theory.

Example 3.10. Let Init denote the initial uncoloured operad in groupoids.

- A category is equivalent as data to an Init-graded 1-theory.
- A planar multicategory is equivalent as data to an E_1-graded 1-theory.
- A braided multicategory (see Fiedorowicz [9]) is equivalent as data to an E_2-graded 1-theory.

Example 3.11. Since $\mathcal{U}$-graded n-theory is a theorization of $\mathcal{U}$-algebra, one obtains from a $\mathcal{U}$-monoidal category $\mathcal{X}$, a $\mathcal{U}$-graded n-theory by replacing the functors giving the composition operations by the bimodules/distributors/profunctors corepresented by them. We shall say that this n-theory is **represented** by $\mathcal{X}$, and shall denote it by $\Theta_{n-1}\mathcal{X}$, where the subscript comes from the fact that $\mathcal{U}$-algebra is a generalization of $(n-1)$-theory from the case $\mathcal{U} = \mathbf{1}^n_{\mathrm{Com}}$.

Example 3.12. Recall that we called a plain n-theory, i.e., an n-theory which is not considered with any grading, also a *symmetric* n-theory. Every symmetric n-theory is canonically graded by the terminal unenriched uncoloured n-theory $\mathbf{1}^n_{\mathrm{Com}}$, and there is no difference between a symmetric n-theory and a $\mathbf{1}^n_{\mathrm{Com}}$-graded n-theory.

3.2.2. For an unenriched $\mathcal{U}$-graded n-theory $\mathcal{X}$, let us denote the symmetric n-theory underlying $\mathcal{X}$ (which maps to $\mathcal{U}$; see Proposition 3.8) by $\Delta_!\mathcal{X}$, where Δ denotes the unique functor $\mathcal{U} \to \mathbf{1}^n_{\mathrm{Com}}$. For example, for the terminal unenriched uncoloured $\mathcal{U}$-graded n-theory $\mathbf{1}^n_{\mathcal{U}}$, we have that the canonical projection functor $\Delta_!\mathbf{1}^n_{\mathcal{U}} \to \mathcal{U}$ is an equivalence.

Using this, we can obtain a compact reformulation of the notion of algebra, as will be given now.

Suppose given a system of colours up to dimension $n-2$ for $\mathcal{U}$-graded higher theory, and let $\mathcal{T}$ denote the terminal unenriched $\mathcal{U}$-graded n-theory on this system of colours. Note that one can consider the structure of a $\mathcal{U}$-algebra on this system of colours. Indeed, the structure of a $\mathcal{U}$-algebra enriched in a multicategory $\mathcal{M}$, is equivalent as data to a functor $\Delta_!\mathcal{T} \to \mathbb{B}^{n-1}\mathcal{M}$ of n-theories (and equivalently therefore, an uncoloured $\Delta_!\mathcal{T}$-algebra enriched in $\mathcal{M}$). It is convenient to say that a $\mathcal{U}$-algebra enriched in $\mathcal{M}$ is a functor $\mathcal{U} \to \mathbb{B}^{n-1}\mathcal{M}$ *with strata of colours* up to dimension $n-2$, or colours for a $\mathcal{U}$-algebra.

For example, a $\mathcal{U}$-*monoid*, i.e., a $\mathcal{U}$-algebra enriched in groupoids, is naturally equivalent as data to a coloured functor $\mathcal{U} \to \mathbb{B}^{n-1}\Theta_0\mathrm{Gpd}$. In this sense, the symmetric $(n+1)$-theory $\mathbb{B}^{n-1}\Theta_0\mathrm{Gpd}$ classifies (uncoloured) monoids over unenriched symmetric n-theories, where the universal monoid is the uncoloured $\mathbb{B}^{n-1}\Theta_0\mathrm{Gpd}$-monoid $\mathbb{U}^{n-1}$ "classified" by the identity functor of $\mathbb{B}^{n-1}\Theta_0\mathrm{Gpd}$.

Proposition 3.13. *For the universal monoid $\mathbb{U}^{n-1}$, the projection functor of the $\mathbb{B}^{n-1}\Theta_0\mathrm{Gpd}$-graded n-theory $\Theta_{n-1}\mathbb{U}^{n-1}$ is equivalent to*

$$\mathbb{B}^{n-1}\Theta_0(\mathrm{Gpd}_*) \longrightarrow \mathbb{B}^{n-1}\Theta_0\mathrm{Gpd}$$

induced from the forgetful functor $\mathrm{Gpd}_* \to \mathrm{Gpd}$, where Gpd_* denotes the Cartesian symmetric monoidal category of pointed groupoids.

The proof is straightforward from the definitions.

3.2.3. Now the notion of algebra generalizes immediately as follows.

Definition 3.14. Let $n \geq 2$ be an integer, and $\mathcal{U}$ be an unenriched n-theory. Then a **$\mathcal{U}$-algebra in a symmetric n-theory** $\mathcal{V}$ is a *functor $\mathcal{U} \to \mathcal{V}$ with strata of colours* for a $\mathcal{U}$-algebra, namely, a functor $\Delta_!\mathcal{T} \to \mathcal{V}$, where $\mathcal{T}$ is the terminal unenriched $\mathcal{U}$-graded n-theory on a system of colours up to dimension $n-2$ for a $\mathcal{U}$-graded higher theory.

We generalize this as follows.

Definition 3.15. Let $n \geq 1$ be an integer, and $\mathcal{U}$ be an unenriched n-theory. Then an **n-tuply coloured lax $\mathcal{U}$-algebra** in a categorified symmetric n-theory $\mathcal{V}$, is a lax functor $\mathcal{U} \to \mathcal{V}$ with strata of colours up to dimension $n-1$, namely, a lax functor $\Delta_!\mathcal{T} \to \mathcal{V}$, where $\mathcal{T}$ is the terminal unenriched $\mathcal{U}$-graded n-theory on a system of colours up to dimension $n-1$ for a $\mathcal{U}$-graded higher theory.

We obtain from Definition 3.7 that a $\mathcal{U}$-graded n-theory enriched in a symmetric monoidal category $\mathcal{A}$, i.e., in $\Theta_0\mathcal{A}$, is (circularly) an n-tuply coloured lax $\mathcal{U}$-algebra in $\mathbb{B}^n\mathcal{A}$.

Remark 3.16. Using the definition mentioned in Remark 2.11, of a "functor", one can define an n-tuply coloured $\mathcal{U}$-algebra in a symmetric *$(n+1)$-theory* $\mathcal{W}$ as a functor $\mathcal{U} \to \mathcal{W}$ with strata of colours up to dimension $n-1$. In particular, a $\mathcal{U}$-graded n-theory enriched in a multicategory $\mathcal{M}$ will be equivalent as data to an n-tuply coloured $\mathcal{U}$-algebra in $\mathbb{B}^n\mathcal{M}$. Note the equivalence $\Theta_n\mathbb{B}^n\mathcal{A} = \mathbb{B}^n\Theta_0\mathcal{A}$ for a symmetric monoidal category $\mathcal{A}$, which follows from Lemma 2.14.

We obtain the following fundamental result.

Theorem 3.17. *Let $n \geq 1$ be an integer, and let $\mathcal{U}$ be an n-theory enriched in groupoids. Then a $\mathcal{U}$-graded n-theory enriched in a multicategory $\mathcal{M}$ is equivalent as data to a $\Theta_n\mathcal{U}$-algebra enriched in $\mathcal{M}$.*

Proof. We shall prove the case where $\mathcal{M}$ is represented by a symmetric monoidal category $\mathcal{A}$. The general case follows from essentially the same (and simpler) argument, but one needs to use Remark 3.16 and the remark after Definition 2.16, of which we have omitted the details.

Let us first note that systems of colours for a $\mathcal{U}$-graded n-theory and for a $\Theta_n\mathcal{U}$-algebra are identical forms of data. Choose and fix data of this form. Then we would like to show that the categories of the structures on this same system of colours, of a $\mathcal{U}$-graded n-theory and of $\Theta_n\mathcal{U}$-algebras, are equivalent. Let us prove this.

Let $\mathcal{T}$ denote the terminal unenriched $\mathcal{U}$-graded n-theory on the chosen system of colours. Then the structure of a $\mathcal{U}$-graded n-theory on those strata of colours could be described as a lax functor $\Delta_!\mathcal{T} \to \mathbb{B}^n\mathcal{A}$, which Theorem 2.10 and Lemma 2.14 equates with the data of a functor $\Theta_n\Delta_!\mathcal{T} \to \mathbb{B}^n\Theta_0\mathcal{A}$.

Let next $\mathcal{J}$ denote the terminal unenriched $\Theta_n\mathcal{U}$-graded $(n+1)$-theory on the same system of colours. Then the structure of a $\Theta_n\mathcal{U}$-algebra on those strata of colours can be described as a functor $\Delta'_!\mathcal{J} \to \mathbb{B}^n\Theta_0\mathcal{A}$, where $\Delta' \colon \Theta_n\mathcal{U} \to \mathbf{1}^{n+1}_{\mathrm{Com}}$.

However, it is immediate that the strata of colours up to dimension $n-1$, of $\Theta_n\Delta_!\mathcal{T}$, and of $\Delta'_!\mathcal{J}$, are identical, and these two $(n+1)$-theories on the same strata of colours are in fact equivalent. The result follows. $\square$

Definition 3.18. Let A be a 0-theory in groupoids, i.e., a commutative monoid. Then an **A-graded** 0-theory is a $\Theta_0 A$-algebra.

In particular, an unenriched A-graded 0-theory is equivalent as data to a commutative monoid X equipped with a morphism $X \to A$.

3.2.4. The notion of $\mathcal{U}$-graded n-theory can be theorized in almost the same way as how we have theorized the notion of algebra over an n-theory. We might call the resulting object a **$\mathcal{U}$-graded $(n+1)$-theory**.

However, this is not really a new notion as should be expected from Theorem 3.17.

Theorem 3.19. *Let $n \geq 0$ be an integer, and let $\mathcal{U}$ be an n-theory enriched in groupoids. Then a $\mathcal{U}$-graded $(n+1)$-theory is equivalent as data to a $\Theta_n\mathcal{U}$-graded $(n+1)$-theory.*

The proof is also similar to the proof of Theorem 3.17. We leave the details to the interested reader.

Definition 3.20. Let $n \geq 0$ be an integer, and let $\mathcal{U}$ be an n-theory enriched in groupoids. For an integer $m \geq n+2$, a **$\mathcal{U}$-graded m-theory** is a $\Theta^m_n\mathcal{U}$-graded m-theory, or equivalently, a $\Theta^{m+1}_n\mathcal{U}$-algebra.

Thus, for every $\mathcal{U}$, $\mathcal{U}$-graded m-theories are iterated theorizations of $\mathcal{U}$-algebra for $m \geq n$.

Example 3.21. An E_1-graded n-theory is equivalent as data to a planar n-theory defined in Section 2.1.

More generally, for every multicategory $\mathcal{U}$ enriched in groupoids, there is a similar description of a $\mathcal{U}$-graded n-theory. Recall that the notion of planar n-theory was defined by replacing the category Fin in the definition of a symmetric n-theory, with the category Ord. For a similar description of the notion of $\mathcal{U}$-graded n-theory, we would like to replace Fin by a category which is an analogue for $\mathcal{U}$, of Ord.

In order to construct this category, note that the forgetful functor Θ_0 from $\mathcal{U}$-monoidal categories to $\mathcal{U}$-graded multicategories, has a left adjoint $L_{\mathcal{U}}$, and that the unique functor $\Delta \colon \mathcal{U} \to \mathrm{Com}$ of symmetric multicategories, where $\mathrm{Com} = E_\infty$ denotes the commutative operad (i.e., the terminal unenriched symmetric operad), induces a functor $\Delta_* \colon L_{\mathcal{U}}\mathbf{1}^1_{\mathcal{U}} \to \Delta^* L_{\mathrm{Com}}\mathbf{1}^1_{\mathrm{Com}} = \Delta^*\mathrm{Fin}$ of $\mathcal{U}$-monoidal categories, where $\mathbf{1}^1_{\mathcal{U}}$ denotes the terminal unenriched $\mathcal{U}$-graded multicategory.

For simplicity, suppose first that $\mathcal{U}$ is uncoloured. Then we obtain a description of a $\mathcal{U}$-graded n-theory by replacing the category Fin in the definition 1.6 of a symmetric n-theory, with $\mathcal{L} := L_{\mathcal{U}}\mathbf{1}^1_{\mathcal{U}}$, where

- by a family *indexed by* $J \in \mathcal{L}$, we mean a family indexed by $\Delta_*J \in \mathrm{Fin}$, and
- for $I \in \mathrm{Ord}$, we say that a $[I]$-family J of objects of $\mathcal{L}$ is *elemental* if Δ_*J is an elemental $[I]$-family in Fin.

If $\mathcal{U}$ is instead a coloured multicategory, then the description is similar except that we would need to have objects of the "theory" to have "degrees" in $\mathcal{U}$. We leave the details to the reader.

3.3. **Iterated monoids.**

3.3.0. Let $\mathcal{U}$ be an n-theory enriched in groupoids, and $\mathcal{X}$ be a $\mathcal{U}$-monoid. Then since the structure of $\mathcal{X}$ is a generalization of the structure of an $(n-1)$-theory, it is natural to consider the notion of $\mathcal{X}$-algebra if $n \geq 2$.

The notion can actually be reduced to the same notion in the special case where $\mathcal{X}$ is the unit (or terminal uncoloured) $\mathcal{U}$-monoid. Indeed, if we denote by Δ the unique functor $\mathcal{U} \to \mathbf{1}^n_{\mathrm{Com}}$, then an $\mathcal{X}$-algebra will simply be an algebra over the unit $\Delta_!\Theta_{n-1}\mathcal{X}$-monoid. (In particular, the dependence of the notion on $\mathcal{U}$ will be only through the presentation of the symmetric n-theory $\mathcal{Y} = \Delta_!\Theta_{n-1}\mathcal{X}$ as underlying $\Theta_{n-1}\mathcal{X}$.)

3.3.1. Let now $\mathcal{X}$ be the unit $\mathcal{U}$-monoid (so $\mathcal{U} = \Delta_!\Theta_{n-1}\mathcal{X}$). Then the notion of $\mathcal{X}$-algebra will be such that the notion of $\mathcal{U}$-algebra theorizes it. In fact, the notion of $\mathcal{U}$-algebra 'detheorizes' more times, and the most detheorized notion will at the end be equivalent to the notion of algebra over the unit monoid over ... over the unit monoid over $\mathcal{U}$, where we should have $n-1$ unit monoids in the expression.

We can directly define all iteratively detheorized notions as follows.

Definition 3.22. Let $n \geq 1$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids. Then for an integer m such that $0 \leq m \leq n-1$, a **$\mathcal{U}$-graded m-theory** enriched in a multicategory $\mathcal{M}$, is a functor $\mathcal{U} \to \Theta^n_{m+1}\mathbb{B}^m\mathcal{M}$ with strata of colours for a $\mathcal{U}$-graded m-theory.

To be explicit, in the case $m = 0$, there is no colours added. In the case $m \geq 1$, given a system of colours for a $\mathcal{U}$-graded m-theory $\mathcal{X}$, the structure on it, of a $\mathcal{U}$-graded m-theory, is a functor $\mathcal{T} \to \Theta^n_{m+1}\mathbb{B}^m\mathcal{M}$, where $\mathcal{T}$ is the symmetric n-theory described as follows.

(0) $\mathrm{Ob}\,\mathcal{T}$ is the collection whose member is a pair (u, x), where $u \in \mathrm{Ob}\,\mathcal{U}$ and $x \in \mathrm{Ob}_u\,\mathcal{X}$.

By induction, for k such that $1 \leq k \leq m$, the type of a k-multimap in $\mathcal{T}$ of a given arity $(I; \pi, \phi)$, is specified by

- the type u of a k-multimap in $\mathcal{U}$ of arity $(I; \pi, \phi)$,
- the type x of a k-multimap in $\mathcal{X}$ of the same arity of degree u.

(k) (Inductively for k such that $1 \leq k \leq m-1$.) Suppose given the arity $(I; \pi, \phi)$ of a k-multimap, and the type (u, x) of a k-multimap in $\mathcal{T}$ of arity $(I; \pi, \phi)$. Then $\mathrm{Mul}^\pi_{\mathcal{T}}[(u, x)]$ is the collection whose member is a pair (v, y), where $v \in \mathrm{Mul}^\pi_{\mathcal{U}}[u]$ and $y \in \mathrm{Mul}^\pi_{\mathcal{X},v}[x]$.

(m) Suppose given data similar to the input data for "(k)" above, but with k substituted by m. Then $\mathrm{Mul}^\pi_{\mathcal{T}}[(u, x)] = \mathrm{Mul}^\pi_{\mathcal{U}}[u]$.

(ℓ) (Inductively for ℓ such that $m+1 \leq \ell \leq n$.) Suppose given the arity $(I; \pi, \phi)$ of a ℓ-multimap, and the type of an ℓ-multimap in $\mathcal{T}$ of arity $(I; \pi, \phi)$, which by induction, will be specified by
  - the type u of a ℓ-multimap in $\mathcal{U}$ of arity $(I; \pi, \phi)$,
  - the type x of a ϕ^{m-1}-nerve of m-multimaps in $\mathcal{X}$ of degree $u^{\leq m-1}$.

  Then $\mathrm{Mul}^\pi_{\mathcal{T}}[(u, x)] = \mathrm{Mul}^\pi_{\mathcal{U}}[u]$.

The composition is given by the composition in $\mathcal{U}$.

Thus, the notion of $\mathcal{U}$-graded $(n-1)$-theory coincides with the notion of $\mathcal{U}$-algebra, and for all $m \geq 0$ (which may be $\geq n$), the notion of $\mathcal{U}$-graded m-theory is iteratively an m-th theorization of the notion of $\mathcal{U}$-graded 0-theory.

Lemma 3.23. *A $\mathcal{U}$-graded m-theory is equivalent as data to a $\Theta_n\mathcal{U}$-graded m-theory.*

The proof of this is direct from Corollary 2.12.

Definition 3.24. Let $n \geq 0$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids. Then we refer to a $\mathcal{U}$-graded 1-theory also as a $\mathcal{U}$-graded **multicategory**.

Definition 3.25. Let $n \geq 1$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids. Let $m \geq 0$ and $\ell \geq 1$ be integers. Then, for an ℓ-categorified $\mathcal{U}$-graded m-theory $\mathcal{X}$, we denote by $\Theta_m\mathcal{X}$, the $(\ell-1)$-categorified $\mathcal{U}$-graded $(m+1)$-theory *represented* by $\mathcal{X}$, namely, obtained by replacing the structure functors of $\mathcal{X}$ by the bimodules/distributors/profunctors corepresented by them.

If $\mathcal{X}$ is enriched in ℓ-categories, then it can be regarded as k-categorified for any $k \geq \ell$. However, resulting $\Theta_m\mathcal{X}$ is independent of k. In particular, $\Theta_m\mathcal{X}$ is also defined for an $\mathcal{U}$-graded m-theory $\mathcal{X}$ enriched in groupoids, and it is an uncategorified $\mathcal{U}$-graded $(m+1)$-theory, which can be considered as as highly categorified (trivially) as one wishes to.

Definition 3.26. Let $n \geq 1$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids. Let $m \geq 0$ be an integer, and $\mathcal{X}$ be a $\mathcal{U}$-graded m-theory which is possibly higher categorified. Then, for an integer $\ell \geq 0$, we define a (less or uncategorified) $\mathcal{U}$-graded $(m+\ell)$-theory $\Theta_m^{m+\ell}\mathcal{X}$ by the inductive relations

$$\Theta_m^{m+\ell}\mathcal{X} = \begin{cases} \mathcal{X} & \text{if } \ell = 0, \\ \Theta_{m+\ell-1}\Theta_m^{m+\ell-1}\mathcal{X} & \text{if } \ell \geq 1. \end{cases}$$

3.3.2. The following definition essentially achieves (more than) our initial goal of the discussions here.

Definition 3.27. Let $n \geq 0$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids. Let $m \geq 0$ be an integer, and $\mathcal{X}$ be a $\mathcal{U}$-graded m-theory enriched in groupoids. Then for an integer $\ell \geq 0$, an **$\mathcal{X}$-graded** ℓ-theory is a $\Delta_!\Theta_m^N\mathcal{X}$-graded ℓ-theory, where $N := \max\{m, n\}$, and $\Delta\colon \Theta_n^N\mathcal{U} \to \mathbf{1}_{\mathrm{Com}}^N$.

Example 3.28. Let $\mathcal{X}$ be the terminal uncoloured $\mathcal{U}$-graded m-theory enriched in groupoids. Then we have $\Delta_!\Theta_m^N\mathcal{X} = \Theta_n^N\mathcal{U}$, so an $\mathcal{X}$-graded ℓ-theory is equivalent as data to a $\mathcal{U}$-graded ℓ-theory, as had been predicted.

3.3.3. In order to analyse the notions further, we shall next consider how gradings can be altered.

Given a functor $F\colon \mathcal{V} \to \mathcal{U}$ of n-theories enriched in groupoids, it is clear that a $\mathcal{U}$-graded m-theory $\mathcal{X}$ gets pulled back by F. Let us denote the resulting $\mathcal{V}$-graded m-theory by $F^*\mathcal{X}$. In the case where $m = n$, and $\mathcal{X}$ is unenriched, the projection $(\Delta_\mathcal{V})_!F^*\mathcal{X} \to \mathcal{V}$ (where $\Delta_\mathcal{V}\colon \mathcal{V} \to \mathbf{1}_{\mathrm{Com}}^n$) is the base change of the projection $(\Delta_\mathcal{U})_!\mathcal{X} \to \mathcal{U}$ by F in a suitable sense. In the case where $m \leq n-1$, one has ${}^\mathcal{V}\Theta_m^n(F^*\mathcal{X}) = F^*({}^\mathcal{U}\Theta_m^n\mathcal{X})$, where the superscripts to Θ_m^n indicate the gradings considered. If $m \geq n+1$, then F^* is the pull-back by $\Theta_n^m F\colon \Theta_n^m\mathcal{V} \to \Theta_n^m\mathcal{U}$.

Example 3.29. For an n-theory $\mathcal{V}$ enriched in groupoids, and an uncoloured $\mathcal{V}$-monoid $\mathcal{X}$ defined by a functor $F\colon \mathcal{V} \to \mathbb{B}^{n-1}\Theta_0\mathrm{Gpd}$, we have an equivalence $\Theta_{n-1}\mathcal{X} = F^*(\Theta_{n-1}\mathbb{U}^{n-1})$ of (simply coloured) $\mathcal{V}$-graded n-theories, where $\Theta_{n-1}\mathbb{U}^{n-1}$ has been described in Proposition 3.13.

Let again $\mathcal{U}$ be an n-theory enriched in groupoids. For an integer $m \geq 0$, suppose that $\mathcal{V}$ is a $\mathcal{U}$-graded m-theory enriched in groupoids, and denote by P, the projection $\Delta_!\Theta_m^N\mathcal{V} \to \Theta_n^N\mathcal{U}$, where $N \geq m, n$, and $\Delta\colon \mathcal{U} \to \mathbf{1}_{\mathrm{Com}}^n$. Then also clearly, one obtains from an unenriched (possibly higher categorified) $\mathcal{V}$-graded m-theory $\mathcal{Y}$, its push forward $P_!\mathcal{Y}$ as a $\mathcal{U}$-graded m-theory, generalizing the construction $\Delta_!$.

In the case $m = n$, $P_!\mathcal{Y}$ has the same underlying symmetric n-theory as that of $\mathcal{Y}$, with projection to $\mathcal{U}$ given by the projection to $\Delta_!\mathcal{V}$ composed with $P\colon \Delta_!\mathcal{V} \to \mathcal{U}$.

In the case $m \le n-1$, the description of $P_!\mathcal{Y}$ is as follows.

(0) For an object $u \in \mathrm{Ob}\,\mathcal{U}$, $\mathrm{Ob}_u\, P_!\mathcal{Y}$ is the collection whose member is a pair (v, y), where $v \in \mathrm{Ob}_u\,\mathcal{V}$ and $y \in \mathrm{Ob}_v\,\mathcal{Y}$.

Let k be an integer such that $1 \le k \le m$, and suppose given the type u of a k-multimap in $\mathcal{U}$ whose arity is given as $(I; \pi, \phi)$. Then by induction, the type of a k-multimap in $P_!\mathcal{X}$ of the same arity of degree u, is specified by

- the type v of a k-multimap in $\mathcal{V}$ of the same arity of degree u,
- the type y of a k-multimap in $\mathcal{Y}$ of the same arity of degree v.

(k) (Inductively for $1 \le k \le m-1$.) Suppose given
  - a k-multimap u^k in $\mathcal{U}$ whose arity and type are given respectively as $(I; \pi, \phi)$ and $u^{\le k-1} = (u^\nu)_{0\le \nu \le k-1}$,
  - the type (v, y) of a k-multimap in $P_!\mathcal{X}$ of the same arity of degree $u^{\le k-1}$.

  Then $\mathrm{Mul}^\pi_{P_!\mathcal{Y},u^k}[(v, y)]$ is the collection whose member is a pair (w, z), where $w \in \mathrm{Mul}^\pi_{\mathcal{V},u^k}[v]$ and $z \in \mathrm{Mul}^\pi_{\mathcal{Y},w}[y]$.

(m) Suppose given data similar to the input data for "(k)" above, but with k substituted by m. Then

$$\mathrm{Mul}^\pi_{P_!\mathcal{Y},u^m}[(v, y)] = \operatorname*{colim}_{w\in \mathrm{Mul}^\pi_{\mathcal{V},u^m}[v]} \mathrm{Mul}^\pi_{\mathcal{Y},w}[y].$$

The composition is given by the composition in $\mathcal{Y}$.

It is immediate to verify that we have an equivalence ${}^{\mathcal{U}}\Theta^n_m(P_!\mathcal{Y}) \simeq P_!{}^{\mathcal{V}}\Theta^n_m\mathcal{Y}$.

In the case $m \ge n+1$, the description above applies after $\mathcal{U}$ is replaced by $\Theta^m_n\mathcal{U}$.

In general, between suitable categories, the construction $P_!$ gives a left adjoint to the functor P^*.

Remark 3.30. A construction P_*, which suitably gives a *right* adjoint of P^*, will be considered later in Section 4.3. This construction is not as obvious, and will in general, only produce an $(m+1)$-theory from an m-theory.

Example 3.31. $P_!\mathbf{1}^m_{\mathcal{V}} = \mathcal{V}$ for the terminal unenriched uncoloured $\mathcal{V}$-graded m-theory $\mathbf{1}^m_{\mathcal{V}}$.

Example 3.32. In the case where $\mathcal{V}$ is the terminal uncoloured $\mathcal{U}$-graded m-theory enriched in groupoids, P is an equivalence, and P^* and $P_!$ are the mutually inverse equivalences giving the identification of Example 3.28.

Example 3.33. Consider the case where $\mathcal{U}$ is an initially graded 1-theory enriched in groupoids, and $m = 0$. In this case, a $\mathcal{U}$-monoid $\mathcal{V}$ is a groupoid-valued functor on $\mathcal{U}$, and the projection $P\colon (\Delta_{\mathcal{U}})_!\Theta_0\mathcal{V} \to \mathcal{U}$ is the corresponding op-fibration (fibred in groupoids). For a $\mathcal{V}$-graded 0-theory $\mathcal{X}$, the op-fibration

$$(\Delta_{\mathcal{U}})_! P_!\Theta_0 X \longrightarrow \mathcal{U}$$

describing the $\mathcal{U}$-monoid $P_!\mathcal{X}$ is the composite of the op-fibrations $(\Delta_{\mathcal{V}})_!\Theta_0 X \to \Theta_0\mathcal{V}$ and P, agreeing with the alternative description of the $\mathcal{U}$-module $P_!\mathcal{X}$ as the left Kan extension along P of the $(\Delta_{\mathcal{U}})_!\Theta_0\mathcal{V}$-module X.

The version of this connection for $\mathcal{U} = \mathbf{1}^n_{\mathrm{Com}}$, has been concretely expressed in Proposition 3.13.

Proposition 3.8 generalizes as follows.

Proposition 3.34. *Let $n \geq 0$ be an integer, $\mathcal{U}$ be an n-theory enriched in groupoids, $m \geq 0$ be an integer, and $\mathcal{V}$ be a $\mathcal{U}$-graded m-theory enriched in groupoids. Then an unenriched $\mathcal{V}$-graded m-theory is equivalent as data to an unenriched $\mathcal{U}$-graded m-theory $\mathcal{X}$ equipped with a "projection" functor $\mathcal{X} \to \mathcal{V}$.*

Proof. We may assume $m \leq n$ without loss of generality, and the case $m = n$ may not be tautologous, but is obviously true.

For the case $m \leq n-1$, denote by P, the projection $(\Delta_{\mathcal{U}})_!\Theta^n_m\mathcal{V} \to \mathcal{U}$, where $\Delta_{\mathcal{U}} \colon \mathcal{U} \to \mathbf{1}^n_{\mathrm{Com}}$. Then, from an unenriched $\mathcal{V}$-graded m-theory $\mathcal{Y}$, we obtain a $\mathcal{U}$-graded m-theory $P_!\mathcal{Y}$ equipped with the functor

$$P_!\Delta_{\mathcal{Y}} \colon P_!\mathcal{Y} \longrightarrow P_!\mathbf{1}^m_{\mathcal{V}} = \mathcal{V},$$

where $\Delta_{\mathcal{Y}}$ denotes the unique functor $\mathcal{Y} \to \mathbf{1}^m_{\mathcal{V}}$.

Conversely, given an unenriched $\mathcal{U}$-graded m-theory $\mathcal{X}$ with projection $Q \colon \mathcal{X} \to \mathcal{V}$, one obtains a $\mathcal{V}$-graded m-theory $\big((\Delta_{\mathcal{U}})_!\Theta^n_m Q\big)_!\mathbf{1}^m_{\mathcal{X}}$.

We would like to show that these constructions are inverse to each other.

The functor $P_!\Delta \colon P_!\big((\Delta_{\mathcal{U}})_!\Theta^n_m Q\big)_!\mathbf{1}^m_{\mathcal{X}} \to P_!\mathbf{1}^m_{\mathcal{V}}$ is equivalent to $Q \colon \mathcal{X} \to \mathcal{V}$.

It therefore, suffices to show a natural equivalence $\big((\Delta_{\mathcal{U}})_!\Theta^n_m P_!\Delta_{\mathcal{Y}}\big)_!\mathbf{1}^m_{P_!\mathcal{Y}} \simeq \mathcal{Y}$. However, the functor $(\Delta_{\mathcal{U}})_!\Theta^n_m P_!\Delta_{\mathcal{Y}}$ can be identified with the functor

$$(\Delta_{\mathcal{U}} \circ \Theta^n_m P)_!{}^{\mathcal{V}}\Theta^n_m\Delta_{\mathcal{Y}} \colon (\Delta_{\mathcal{U}} \circ \Theta^n_m P)_!{}^{\mathcal{V}}\Theta^n_m\mathcal{Y} \longrightarrow (\Delta_{\mathcal{U}} \circ \Theta^n_m P)_!{}^{\mathcal{V}}\Theta^n_m\mathbf{1}^m_{\mathcal{V}},$$

so we obtain

$$\begin{aligned} {}^{\mathcal{V}}\Theta^n_m\big((\Delta_{\mathcal{U}})_!\Theta^n_m P_!\Delta_{\mathcal{Y}}\big)_!\mathbf{1}^m_{P_!\mathcal{Y}} &= \big((\Delta_{\mathcal{U}})_!\Theta^n_m P_!\Delta_{\mathcal{Y}}\big)_!{}^{P_!\mathcal{Y}}\Theta^n_m\mathbf{1}^m_{P_!\mathcal{Y}}, \\ &= \big((\Delta_{\mathcal{U}} \circ \Theta^n_m P)_!{}^{\mathcal{V}}\Theta^n_m\Delta_{\mathcal{Y}}\big)_!\mathbf{1}^n_{\mathcal{Y}} \\ &= {}^{\mathcal{V}}\Theta^n_m\mathcal{Y}, \end{aligned}$$

from which the result follows. ☐

4. Enrichment of higher theories

4.0. **Introduction.** The subject of this section will be a general notion of enrichment for graded higher theories. We shall show how some previous notions such as grading by a graded higher theory, can be compactly understood using the new notions introduced in this section. We shall also show how the new framework helps with considering push-forward construction 'on the right side', along some functors of higher theories. We shall also discuss a construction for higher theories which is related to the Day convolution for symmetric monoidal categories, and leads to a notion of algebra over an enriched higher theory.

4.1. Enriched theories.

4.1.0. Given a kind of algebraic structure, one motivation for theorizing it was to generalize it to similar structure definable in a theorized form of the same kind of algebraic structure. Specifically, we would like to define the kind in question of algebraic structure in a theorized structure $\mathcal{V}$, as a (coloured) morphism to $\mathcal{V}$ from the terminal unenriched theorized structure.

For example, for an n-theory $\mathcal{U}$ enriched in groupoids, we would like to define the notion of $\mathcal{U}$-algebra in a $\mathcal{U}$-graded n-theory, since $\mathcal{U}$-graded n-theory is the kind of structure which theorize the structure of a $\mathcal{U}$-algebra. More generally, the following definition seems reasonable, which can be considered as giving a quite general manner of *enrichment*, of the notion of graded higher theory.

Definition 4.0. Let $n \geq 0$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids. Let $m \geq 0$ and $M \geq m+1$ be integers, and $\mathcal{V}$ be a $\mathcal{U}$-graded M-theory. Then an **m-theory in $\mathcal{V}$** is a functor $\mathbf{1}_{\mathcal{U}}^{M} \to \mathcal{V}$ of $\mathcal{U}$-graded M-theories with strata of colours for a $\mathcal{U}$-graded m-theory. Namely, it consists of

- a system of colours up to dimension $m-1$ for a $\mathcal{U}$-graded higher theory,
- a functor $\Theta_{m+1}^{M}\mathcal{T} \to \mathcal{V}$, where $\mathcal{T}$ denotes the terminal unenriched $\mathcal{U}$-graded $(m+1)$-theory on the chosen system of colours.

The system of colours will be called the system of colours **of** the m-theory defined.

In the case $m = n-1$, m-theory will also be called an **algebra**.

Remark 4.1. In order to avoid redundancy, we call the defined kind of object simply an "m-theory" (or "algebra") instead of a "$\mathcal{U}$-graded m-theory" (or $\mathcal{U}$-algebra) when it is understood that a $\mathcal{U}$-grading is contained is the data $\mathcal{V}$. We may explicitly refer to the m-theory as a $\mathcal{U}$-graded m-theory when ($M \geq n$ and) $\mathcal{V}$ is a priori, just a symmetric M-theory, and different higher theories may be being considered over which we would like to grade $\mathcal{V}$, perhaps including the commutative operad Com. This would clarify that a $\mathcal{U}$-grading is considered for $\mathcal{V}$. Note however, that this would be still ambiguous if more than one $\mathcal{U}$-gradings are being considered for $\mathcal{V}$.

Similarly to the definitions in Section 2.3, there are also less coloured versions of the notion of enriched graded theory. These are the cases where the $\mathcal{U}$-graded theory $\mathcal{T}$ which were considered in Definition 4.0, are less coloured.

There is also a lax generalization.

Definition 4.2. Let $n \geq 0$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids. Let $m \geq 0$ and $M \geq m+1$ be integers, and let $\mathcal{V}$ be a possibly higher categorified $\mathcal{U}$-graded M-theory. Then for an integer $\ell \geq 0$, an **ℓ-lax** m-theory in $\mathcal{V}$ is an m-theory in $\Theta_{M}^{M+\ell}\mathcal{V}$.

$\mathcal{V}$ needs to be at least ℓ-categorified in order for this to be properly more general than the $(\ell-1)$-lax notion.

If $\mathcal{V}$ is represented by a once more categorified $\mathcal{U}$-graded $(M-1)$-theory $\mathcal{W}$, then an ℓ-lax m-theory in $\mathcal{V} = \Theta_{M-1}\mathcal{W}$, is also an "$m$-tuply coloured" $(\ell+1)$-lax $(m-1)$-theory in $\mathcal{W}$.

4.1.1. As we have discussed in Section 0, we obtain the following in a low "theoretic" level of algebra. For a functorial formulation of the following, see Proposition 0.0.

Proposition 4.3. *For every coloured symmetric operad $\mathcal{U}$ in groupoids, the notion of coloured $\mathcal{U}$-graded operad in a symmetric monoidal category has a generalization in a ($\mathcal{U} \otimes E_1$)-monoidal category. Namely, there is a notion of coloured $\mathcal{U}$-graded operad in a ($\mathcal{U} \otimes E_1$)-monoidal category, such that the notion of coloured $\mathcal{U}$-graded operad in a symmetric monoidal category coincides with the notion of coloured $\mathcal{U}$-graded operad in its underlying ($\mathcal{U} \otimes E_1$)-monoidal category.*

Indeed, for a $(\mathcal{U} \otimes E_1)$-monoidal category $\mathcal{A}$, it sufficed to define a coloured $\mathcal{U}$-graded operad in $\mathcal{A}$ as a 1-theory in the $\mathcal{U}$-graded 2-theory $\Theta_0^2 B\mathcal{A}$, where $B\mathcal{A}$ denotes the $\mathcal{U}$-monoidal 2-category obtained by categorically delooping $\mathcal{A}$ using the E_1-monoidal structure (or a suitably "trapped" (See Section 2.5) functor $\mathbf{1}_{\mathcal{U}}^{2} \to \Theta_0^2 B\mathcal{A}$, to be more precise).

In the case where $\mathcal{U} = \mathrm{Init}, E_1, E_2$, this coincides with the familiar notion. See Example 3.10.

Remark 4.4. It seems natural that the notion of $\mathcal{U}$-graded multicategory can further be generalized to be enriched in a $(\mathcal{U} \otimes E_1)$-graded multicategory. We hope to come back to this in a sequel to this work.

4.1.2. We can more generally consider the similarly general enrichement of the notion of $\mathcal{U}$-graded higher theory in the case where $\mathcal{U}$ is graded by an N-theory, where $N \geq n+1$. In this case, a $\mathcal{U}$-graded higher theory in the previous sense was simply a $\Delta_! \Theta_n^N \mathcal{U}$-graded higher theory.

The notion is therefore a special case of the notion defined in Definition 4.0. To be explicit, we have the following. (There will be changes in the notation.)

Let

- $n \geq 0$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids,
- $m \geq 0$ be an integer, and $\mathcal{X}$ be a $\mathcal{U}$-graded m-theory enriched in groupoids,
- $\ell \geq 0$ and $L \geq \ell + 1$ be integers, and $\mathcal{Y}$ be an $\mathcal{X}$-graded L-theory.

Then $\mathcal{Y}$ is a $\Delta_! \Theta_m^N \mathcal{X}$-graded L-theory, where $N := \max\{m, n\}$, and $\Delta \colon \mathcal{U} \to \mathbf{1}_{\mathrm{Com}}^n$, so an ℓ-theory *in* $\mathcal{Y}$ makes sense according to Definition 4.0.

Definition 4.5. Let

- $n \geq 0$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids,
- $m \geq 0$ be an integer, and $\mathcal{X}$ be a $\mathcal{U}$-graded m-theory enriched in groupoids,
- $\ell \geq 0$ and $L \geq \ell + 1$ be integers, and $\mathcal{V}$ be a $\mathcal{U}$-graded L-theory.

Then an **$\mathcal{X}$-graded** ℓ-theory in $\mathcal{V}$ is an ℓ-theory in the $\mathcal{X}$-graded L-theory $P^* \mathcal{V}$, where P denotes the projection $\Delta_! \Theta_m^N \mathcal{X} \to \Theta_n^N \mathcal{U}$, where $N := \max\{m, n\}$, and $\Delta \colon \mathcal{U} \to \mathbf{1}_{\mathrm{Com}}^n$.

In the case $\ell = m - 1$, an $\mathcal{X}$-graded ℓ-theory will also be called an **$\mathcal{X}$-algebra**.

Thus, a $\mathcal{U}$-graded m-theory in $\mathcal{V}$ is an uncoloured $\mathcal{T}$-algebra in $\mathcal{V}$ for an unenriched $\mathcal{U}$-graded $(m+1)$-theory $\mathcal{T}$ which is terminal on a system of colours up to dimension $m - 1$.

Remark 4.6. For the notion of Definition 4.5, we are refraining from saying "*$\mathcal{U}$-graded* $\mathcal{X}$-graded" theory when we do not intend to emphasize the $\mathcal{U}$-grading, and this is part of our convention on the terminology. See Remark 4.1.

Example 4.7. For $\mathcal{U}$ and $\mathcal{X}$ as in Definition 4.5, a $\mathcal{X}$-graded ℓ-theory enriched in a multicategory $\mathcal{M}$, is equivalent as data to an $\mathcal{X}$-graded ℓ-theory in the $\mathcal{U}$-graded $(\ell+1)$-theory $\Delta^* \mathbb{B}^\ell \mathcal{M}$, where $\Delta \colon \mathcal{U} \to \mathbf{1}_{\mathrm{Com}}^n$.

Example 4.8. In Definition 4.5, if $\mathcal{X}$ is the terminal uncoloured $\mathcal{U}$-graded m-theory enriched in groupoids, then P is an equivalence, and an $\mathcal{X}$-graded ℓ-theory in $\mathcal{V}$ is equivalent as data to an ℓ-theory in $\mathcal{V}$.

4.2. **Graded theories as lifts of an algebra.**

4.2.0. Using Definition 4.5, an $\mathcal{X}$-graded ℓ-theory in $\mathcal{V}$ in the notation there, can be written as an appropriate coloured functor of $\mathcal{U}$-graded higher theories. On the other hand, $\mathcal{X}$ itself may be defined by a coloured functor $F \colon \mathbf{1}_{\mathcal{U}}^{m+1} \to \Delta^* \mathbb{B}^m \Theta_0 \mathrm{Gpd}$ of $\mathcal{U}$-graded theories. In this situation, one might wish to describe a higher theory graded by $\mathcal{X}$, directly in terms of F. We shall show that it can indeed be described as a coloured lift of F.

Note that we may assume $m = L (\geq \ell + 1)$ without loss of generality. We shall first discuss a result in this situation (*Proposition 4.9*).

For application in Section 4.3, we shall also give an analogous result *Proposition 4.11* for the case "$\ell = m$".

4.2.1. Let us first recall common notation.

If $\mathcal{C}$ is a category and x is an object of $\mathcal{C}$, then we denote by $\mathcal{C}_{/x}$, the category of objects of $\mathcal{C}$ lying over x, i.e., equipped with a map to x. More generally, if a category $\mathcal{D}$ is equipped with a functor to $\mathcal{C}$, then we define $\mathcal{D}_{/x} := \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/x}$. Note here that $\mathcal{C}_{/x}$ is mapping to $\mathcal{C}$ by the functor which forgets the structure map to x. Note that the notation is abusive in that the name of the functor $\mathcal{D} \to \mathcal{C}$ is dropped from it. In order to avoid this abuse from causing any confusion, we shall use this notation only when the considered functor $\mathcal{D} \to \mathcal{C}$ is clear from the context.

4.2.2. To get on the task now, let us denote by CAT the very large category of large categories. Consider this as symmetric monoidal by the Cartesian structure, and give the functor category $\mathrm{Fun}(\mathrm{Gpd}^{\mathrm{op}}, \mathrm{CAT})$ the structure of a multicategory by the Day convolution. Namely, we consider the structure of a multicategory underlying (or "represented" by) the symmetric monoidal structure given by the Day convolution. Then the Grothendieck construction defines a functor $G\colon \mathrm{Fun}(\mathrm{Gpd}^{\mathrm{op}}, \mathrm{CAT}) \to \Theta_0(\mathrm{CAT}_{/\mathrm{Gpd}})$ of multicategories, where $\mathrm{CAT}_{/\mathrm{Gpd}}$ is made symmetric monoidal by the structure induced from the (Cartesian) symmetric monoidal structure of Gpd.

Furthermore, considering Gpd as a symmetric monoidal (full) subcategory of CAT, we obtain the composite

$$\Theta_0\mathrm{Gpd} \xrightarrow{\text{Yoneda}} \mathrm{Fun}(\mathrm{Gpd}^{\mathrm{op}}, \mathrm{Gpd}) \xrightarrow{\text{inclusion}} \mathrm{Fun}(\mathrm{Gpd}^{\mathrm{op}}, \mathrm{CAT}) \xrightarrow{G} \Theta_0(\mathrm{CAT}_{/\mathrm{Gpd}}),$$

where the structures of multicategories on the functor categories are by the Day convolution. We denote this functor by $T\colon \Theta_0\mathrm{Gpd} \to \Theta_0(\mathrm{CAT}_{/\mathrm{Gpd}})$, $X \mapsto T_X$, so T_X is defined by the forgetful functor $AX := \mathrm{Gpd}_{/X} \to \mathrm{Gpd}$.

Now let $n \geq 0$ be an integer, and $\mathcal{U}$ be an n-theory. Then for an integer $m \geq 0$, we obtain from a $\mathcal{U}$-graded m-theory $\mathcal{V}$ enriched in Gpd, a $\mathcal{U}$-graded m-theory $T_*\mathcal{V}$ enriched in $\mathrm{CAT}_{/\mathrm{Gpd}}$, by postcomposition with $\mathbb{B}^m T\colon \mathbb{B}^m\Theta_0\mathrm{Gpd} \to \mathbb{B}^m\Theta_0(\mathrm{CAT}_{/\mathrm{Gpd}})$.

$T_*\mathcal{V}$ can be considered as a $\mathcal{U}$-graded m-theory $A_*\mathcal{V}$ enriched in CAT, equipped with a functor $A_*\mathcal{V} \to \Delta^*\mathbb{B}^m\mathrm{Gpd}$, where $\Delta\colon \mathcal{U} \to \mathbf{1}^n_{\mathrm{Com}}$. In particular, we obtain a functor $\Theta_m A_*\mathcal{V} \to \Delta^*\mathbb{B}^m\Theta_0\mathrm{Gpd}$ by Lemma 2.14.

Proposition 4.9. *Let $n \geq 0$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids. Let $m \geq 1$ be an integer, and $\mathcal{T}$ be a $\mathcal{U}$-graded $(m+1)$-theory enriched in groupoids. Suppose that an uncoloured $\mathcal{T}$-monoid $\mathcal{X}$ is defined by a functor $F\colon \mathcal{T} \to \Delta^*\mathbb{B}^m\Theta_0\mathrm{Gpd}$, where $\Delta\colon \mathcal{U} \to \mathbf{1}^n_{\mathrm{Com}}$, of $\mathcal{U}$-graded $(m+1)$-theories (e.g., these data may be specifying a $\mathcal{U}$-graded m-theory).*

Then, for an integer ℓ such that $0 \leq \ell \leq m-1$, an $\mathcal{X}$-graded ℓ-theory in a $\mathcal{U}$-graded m-theory $\mathcal{V}$ enriched in groupoids, is equivalent as data to a lift with strata of colours up to dimension $\ell - 1$, of F to $\Theta_m A_*\mathcal{V}$.

Proof. Since $\mathcal{X}$ is uncoloured, a system of colours for an $\mathcal{X}$-graded ℓ-theory (up to dimension $\ell-1$) is equivalent as data to a system of colours for an $\mathcal{T}$-graded ℓ-theory. If $\mathcal{J}$ is the terminal unenriched $\mathcal{T}$-graded $(\ell+1)$-theory on a system of colours up to dimension $\ell - 1$, then, from the definitions, a correspondence is immediate between the structures on this system of colours, of $\mathcal{X}$-graded ℓ-theories in $\mathcal{V}$, and lifts of F to functors $P_!\Theta^{m+1}_{\ell+1}\mathcal{J} \to \Theta_m A_*\mathcal{V}$, where P denotes the projection $\Delta_!\Theta^N_{m+1}\mathcal{T} \to \Theta^N_n\mathcal{U}$, $N \geq m+1, n$. ☐

Corollary 4.10. *Let F, $\mathcal{X}$ be as in Proposition. Then, for an integer ℓ such that $0 \leq \ell \leq m-1$, an $\mathcal{X}$-graded ℓ-theory enriched in a symmetric multicategory $\mathcal{M}$ is equivalent as data to a lift with strata of colours up to dimension $\ell - 1$, of F to $\Delta^*\mathbb{B}^\ell\Theta_{m-\ell}A_*\Theta_1^{m-\ell}\mathcal{M}$.*

Proof. An $\mathcal{X}$-graded ℓ-theory enriched in a symmetric multicategory $\mathcal{M}$, is an $\mathcal{X}$-graded ℓ-theory in $\Delta^*\mathcal{W}$, where $\mathcal{W}$ denotes the symmetric $(\ell+1)$-theory $\mathbb{B}^\ell\mathcal{M}$. Proposition identifies this with a coloured lift of F to $\Delta^*\Theta_m A_*\Theta^m_{\ell+1}\mathcal{W}$. Moreover, there is an equivalence

$$\Theta_m A_*\Theta^m_{\ell+1}\mathcal{W} = \mathbb{B}^\ell\Theta_{m-\ell}A_*\Theta_1^{m-\ell}\mathcal{M}.$$

of unenriched $(m+1)$-theories lying over $\mathbb{B}^m\Theta_0\mathrm{Gpd} = \mathbb{B}^\ell\mathbb{B}^{m-\ell}\Theta_0\mathrm{Gpd}$. □

4.2.3. In Section 4.3, we shall use a natural analogue of Proposition 4.9 for $\ell = m$. The way how we obtain it will be by restricting the context.

Let us denote by **CAT** the very large 2-*category* of large categories extending the 1-category CAT. In order to formulate the result, we first extend T to the composite

$$T\colon \Theta_0\mathbf{CAT} \xrightarrow{\text{Yoneda}} \mathbf{Fun}(\mathbf{CAT}^{\mathrm{op}}, \mathbf{CAT}) \xrightarrow{\text{restriction}} \mathbf{Fun}(\mathrm{Gpd}^{\mathrm{op}}, \mathbf{CAT}) \xrightarrow{G} \Theta_0(\mathbf{CAT}_{/\mathrm{Gpd}})$$

of functors of categorified multicategories, where **Fun** indicates the 2-categories of functors extended to categorified multicategories by the Day convolution, and Gpd is considered as a symmetric monoidal subcategory of **CAT**. Thus, for $\mathcal{C} \in \mathbf{CAT}$, the object $T_\mathcal{C} \in \mathbf{CAT}_{/\mathrm{Gpd}}$ is defined by the forgetful functor $A\mathcal{C} := \mathrm{Gpd}_{/\mathcal{C}} \to \mathrm{Gpd}$.

Now let $n \geq 0$ be an integer, and $\mathcal{U}$ be an n-theory. Then as before, for an integer $m \geq 0$, we obtain from a $\mathcal{U}$-graded m-theory $\mathcal{V}$ enriched in **CAT**,

- a $\mathcal{U}$-graded m-theory $A_*\mathcal{V}$ enriched in **CAT**,
- a functor $\Theta_m A_*\mathcal{V} \to \Delta^*\mathbb{B}^m\Theta_0\mathrm{Gpd}$,

by considering $T_*\mathcal{V}$.

Proposition 4.11. *For $\mathcal{U}$, F, $\mathcal{X}$ as in Proposition 4.9, an $\mathcal{X}$-graded m-theory in the $(m+1)$-theory represented by a $\mathcal{U}$-graded m-theory $\mathcal{V}$ enriched in* **CAT***, is equivalent as data to a lift with strata of colours up to dimension $m-1$, of F to $\Theta_m A_*\mathcal{V}$.*

The proof is similar to the proof of Proposition 4.9.

Corollary 4.12. *For F, $\mathcal{X}$ as in Proposition 4.9, an $\mathcal{X}$-graded m-theory enriched in a symmetric monoidal category $\mathcal{A}$, is equivalent as data to a lift with strata of colours up to dimension $m-1$, of F to $\Delta^*\mathbb{B}^m\Theta_0 A_*\mathcal{A}$.*

Proof. This is the case $\mathcal{V} = \Delta^*\mathbb{B}^m\mathcal{A}$ of Proposition since we have an equivalence

$$A_*\mathbb{B}^m\mathcal{A} = \mathbb{B}^m A_*\mathcal{A}$$

of $(m+1)$-theories enriched in CAT, lying over $\mathbb{B}^m\mathrm{Gpd}$. □

4.3. **The right adjoint of the restriction of degrees.**

4.3.0. In Section 3.3, we have considered the left adjoint to the functor 'restricting' the degrees for graded theories. We would like to consider the *right* adjoint in the following situation.

Let

- $n \geq 0$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids,
- $m \geq 0$ be an integer, and $\mathcal{V}$ be an $\mathcal{U}$-graded m-theory enriched in groupoids,
- $\mathcal{X}$ be an unenriched $\mathcal{V}$-graded m-theory.

Denote by P the projection functor $\Delta_!\Theta^N_m\mathcal{V} \to \Theta^N_n\mathcal{U}$, where $N \geq m, n$, and $\Delta\colon \mathcal{U} \to \mathbf{1}^n_{\mathrm{Com}}$. Then it turns out that we can construct a $\mathcal{U}$-graded *$(m+1)$-theory* $P_*\mathcal{X}$ having an appropriate universal property.

We shall do this construction in two steps. The key observation is that we can reduce the situation above into two simpler situations according to the factorization of P as

$$\Delta_! \Theta^N_m \mathcal{V} \xrightarrow{R} \Delta_! \Theta^N_m \mathcal{T} \xrightarrow{Q} \Theta^N_n \mathcal{U},$$

where $\mathcal{T}$ denotes the $\mathcal{U}$-graded m-theory enriched in groupoids which is terminal on the system of colours of $\mathcal{V}$, so $\mathcal{V}$ can be identified with an uncoloured $\mathcal{T}$-graded m-theory.

The two cases which we shall treat separately below, will imply respectively that we obtain a $\mathcal{T}$-graded $(m+1)$-theory $R_* \mathcal{X}$, and that we obtain from this, a $\mathcal{U}$-graded $(m+1)$-theory $Q_* R_* \mathcal{X}$. Moreover, it will follow that the construction $P_* := Q_* R_*$ has a universal property desired of the right push-forward.

Remark 4.13. As we shall see in the construction, Q_* will produce an $(m+1)$-theory in which the groupoids of $(m+1)$-multimaps may not necessarily be small, even if the groupoids of m-multimaps are all small in $\mathcal{X}$.

Let us see the constructions Q_* and R_*.

4.3.1. In order to construct R_* above, it suffices to construct a $\mathcal{U}$-graded $(m+1)$-theory $P_* \mathcal{X}$ in the spacial case of the original situation where $\mathcal{V}$ is an *uncoloured* $\mathcal{U}$-graded m-theory. This can be done with the following construction at the universal level.

Recall from Section 3.2 and Corollary 4.12 that $\mathcal{V}$ corresponds to a functor $F_{\mathcal{V}} \colon \Theta^N_n \mathcal{U} \to \Theta^N_{m+1} \mathbb{B}^m \Theta_0 \mathrm{Gpd}$, where $N \geq n, m+1$, and $\mathcal{X}$ corresponds then to a functor $F_{\mathcal{X}} \colon \mathcal{J} \to F_{\mathcal{V}}{}^* \mathbb{B}^m \Theta_0 A_* \mathrm{Gpd}$ of unenriched $\mathcal{U}$-graded $(m+1)$-theories, where $\mathcal{J}$ is terminal on the system of colours of $\mathcal{X}$.

Let **Cocorr** denote the symmetric monoidal 2-category of groupoids and cocorrespondences. Thus, its object is a groupoid, and for groupoids X, Y, the category $\mathrm{Map}_{\mathbf{Cocorr}}(X, Y)$ is the category formed naturally by the diagrams of the form

$$X \longrightarrow M \longleftarrow Y$$

in Gpd, where the groupoid M is allowed to vary arbitrarily. Composition is done by the obvious push-out operation. The symmetric monoidal structure is induced from the Cartesian product in Gpd.

Then there is a symmetric monoidal lax functor $\Gamma \colon A_* \mathrm{Gpd} \to \mathbf{Cocorr}$ which sends

- an object of $A_* \mathrm{Gpd}$ given by $X \in \mathrm{Gpd}$ and a functor $F \colon X \to \mathrm{Gpd}$, to the groupoid $\lim_X F$,
- a map $(X, F) \to (Y, G)$ in $A_* \mathrm{Gpd}$ given by a map $f \colon X \to Y$ and a map $\widetilde{f} \colon F \to f^* G$, to the map in **Cocorr** corresponding to the diagram

  $$\lim_X F \xrightarrow{\widetilde{f}} \lim_X f^* G \xleftarrow{f^*} \lim_Y G$$

  in Gpd,

and extends these data naturally. From this, we obtain the induced functor $\Gamma \colon \Theta^2_0 A_* \mathrm{Gpd} \to \Theta^2_0 \mathbf{Cocorr}$ of 2-theories, and hence

$$\mathbb{B}^m \Gamma \colon \Theta_{m+1} \mathbb{B}^m \Theta_0 A_* \mathrm{Gpd} = \mathbb{B}^m \Theta^2_0 A_* \mathrm{Gpd} \longrightarrow \mathbb{B}^m \Theta^2_0 \mathbf{Cocorr}.$$

Denote by $\mathbf{Cocorr}_*$ the symmetic monoidal 2-category of co-correspondences in the category Gpd_* of pointed groupoids. The forgetful functor $\mathrm{Gpd}_* \to \mathrm{Gpd}$ (which preserves push-outs and direct products) induces a symmetric monoidal functor $\mathbf{Cocorr}_* \to \mathbf{Cocorr}$. In particular, we can consider the $(m+2)$-theory $\Theta_{m+1} \mathbb{B}^m \Theta_0 \mathbf{Cocorr}_*$ as graded by $\Theta_{m+1} \mathbb{B}^m \Theta_0 \mathbf{Cocorr} = \mathbb{B}^m \Theta^2_0 \mathbf{Cocorr}$. This $\Theta_{m+1} \mathbb{B}^m \Theta_0 \mathbf{Cocorr}$-graded $(m+2)$-theory is in fact representable by a $(m+1)$-theory. This will be an instance of the following.

Lemma 4.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be n-theories enriched in* CAT*, and suppose given a functor $P\colon \mathcal{D} \to \mathcal{C}$ of categorified n-theories. Then the $\Theta_n\mathcal{C}$-graded $(n+1)$-theory corresponding to the induced functor $P\colon \Theta_n\mathcal{D} \to \Theta_n\mathcal{C}$ of symmetric $(n+1)$-theories, is representable by a categorified $\Theta_n\mathcal{C}$-graded n-theory if the functors induced by P on the categories of n-multimaps, are all op-fibrations.*

Moreover, the representing $\Theta_n\mathcal{C}$-graded n-theory in this situation, is enriched in fact in groupoids if and only if the op-fibrations are all fibred in groupoids.

Proof. Let y be the type of an n-multimap in $\mathcal{D}$, and x be an n-multimap in $\mathcal{C}$ of type Py. Then the category of n-multimaps in the representing n-theory, of type y of degree x, will be the fibre of the op-fibration over x in the category of n-multimaps of type y in $\mathcal{D}$. We would like to let $(n+1)$-multimaps in $\Theta_n\mathcal{C}$ act on these categories, but an $(n+1)$-multimap in $\Theta_n\mathcal{C}$ is a morphism in the category of appropriate n-multimaps in $\mathcal{C}$, which acts on the fibres of the op-fibrations. It is straightforward to check that this extends to the structure of a categorified $\Theta_n\mathcal{C}$-graded n-theory which represents the $\Theta_n\mathcal{C}$-graded $(n+1)$-theory $\Theta_n\mathcal{D}$.

The construction also shows the second statement. □

Remark 4.15*.* The condition is also necessary.

To see this, note that the category of n-multimaps in $\mathcal{C}$ of a given arity can be recovered from $\Theta_n\mathcal{C}$, as the category formed by n-multimaps in $\Theta_n\mathcal{C}$ of the same arity, under the unary $(n+1)$-multimaps between them in $\Theta_n\mathcal{C}$. Moreover, we obtain a similar description of the category of n-multimaps in $\mathcal{D}$ of a given arity, as a category *lying over* the corresponding category for $\mathcal{C}$, in view of the noted description of the latter category. On the other hand, for a categorified $\Theta_n\mathcal{C}$-graded n-theory $\mathcal{E}$, the category formed by n-multimaps in $\Delta_!\Theta_n\mathcal{E}$ of a given arity (where $\Delta\colon \Theta_n\mathcal{C} \to \mathbf{1}^{n+1}_{\mathrm{Com}}$) under the unary $(n+1)$-multimaps between them in $\Delta_!\Theta_n\mathcal{E}$, can be seen to be lying over the corresponding category in $\mathcal{F} := \Theta_n\mathcal{C}$, as the op-fibration corresponding to the action of the unary $(n+1)$-multimaps in $\mathcal{F}$ between those n-multimaps, on the categories of n-multimaps in $\mathcal{E}$ of the same arity of appropriate degrees. Therefore, if $\Theta_n\mathcal{D}$ corresponds to the $\mathcal{F}$-graded $(n+1)$-theory $\Theta_n\mathcal{E}$, then we conclude that the assumption of Proposition is satisfied by $\mathcal{D}$.

We can indeed apply Lemma to the forgetful functor $\mathbb{B}^m\Theta_0\mathbf{Cocorr}_* \to \mathbb{B}^m\Theta_0\mathbf{Cocorr}$ of categorified $(m+1)$-theories, since the forgetful functor $\mathbf{Cocorr}_* \to \mathbf{Cocorr}$ is such that the functors induced on the categories of 1-morphisms are easily seen to be op-fibrations fibred in groupoids.

Let us identify $\mathbb{B}^m\Theta_0^2\mathbf{Cocorr}_*$ with the $\mathbb{B}^m\Theta_0^2\mathbf{Cocorr}$-graded $(m+1)$-theory enriched in groupiods representing it. From this, we obtain a simply coloured $\mathcal{J}$-graded *$(m+1)$-theory* $F_{\mathcal{X}}{}^*(\mathbb{B}^m\Gamma)^*(\mathbb{B}^m\Theta_0^2\mathbf{Cocorr}_*)$.

Definition 4.16. Let $\mathcal{U}$, $\mathcal{X}$, P, $\mathcal{J}$, $F_{\mathcal{X}}$ be as above. Denote by Q the projection $\Delta_!\Theta^N_{m+1}\mathcal{J} \to \Theta^N_n\mathcal{U}$, where $N := \max\{m+1, n\}$, and $\Delta\colon \mathcal{U} \to \mathbf{1}^n_{\mathrm{Com}}$.

Then we define a $\mathcal{U}$-graded $(m+1)$-theory $P_*\mathcal{X}$ as $Q_!F_{\mathcal{X}}{}^*(\mathbb{B}^m\Gamma)^*(\mathbb{B}^m\Theta_0^2\mathbf{Cocorr}_*)$.

Example 4.17. In the case where $\mathcal{V}$ is the terminal unenriched uncoloured $\mathcal{U}$-graded m-theory $\mathbf{1}^m_{\mathcal{U}}$, $\mathcal{X}$ can be identified with a $\mathcal{U}$-graded m-theory, and it follows from Proposition 3.13 that $P_*\mathcal{X} = \Theta_m\mathcal{X}$.

4.3.2. In order to do the other construction, let us introduce a notation.

Suppose given a collection Λ, and a family $X = (X_\lambda)_{\lambda\in\Lambda}$ of groupoids parametrized by Λ. Then by $\prod_\Lambda X = \prod_{\lambda\in\Lambda} X_\lambda$, we denote the not necessarily small groupoid whose truncated n-type is $\prod_\Lambda X^{\leq n}$ naturally formed by associations σ to every member $\lambda \in \Lambda$, of an object $\sigma(\lambda) \in X^{\leq n}_\lambda$, where $X^{\leq n}_\lambda$ denotes the truncated n-type of X_λ.

Note that $\prod_\Lambda X^{\leq n}$ may not be small, but is well-defined as a homotopy n-type by induction on n. For example, we define $\sigma, \tau \in \prod_\Lambda X^{\leq 0}$ to be *equal* if and only if $\sigma(\lambda) = \tau(\lambda)$ for every $\lambda \in \Lambda$, and then this equality relation is an equivalence relation, so the members of $\prod_\Lambda X^{\leq 0}$ form a possibly large 0-type under this relation of equality.

4.3.3. In order to construct Q_* of Section 4.3.0, it suffices to construct a $\mathcal{U}$-graded $(m+1)$-theory $P_*\mathcal{X}$ in the original situation *modified* as follows.

- $\mathcal{V}$ is terminal on the system of colours for a $\mathcal{U}$-graded m-theory.
- $\mathcal{X}$ is a $\mathcal{V}$-graded *$(m+1)$-theory*.

The construction is as follows. If necessary, Theorem 3.17 allows us to replace $\mathcal{U}$ by $\Theta^N_n\mathcal{U}$ for any $N > n$, so we shall assume without loss of generality, that the dimension of $\mathcal{U}$ is sufficiently high.

For an object $u \in \mathrm{Ob}\mathcal{U}$, we let an object $\sigma \in \mathrm{Ob}_u P_*\mathcal{X}$ of $P_*\mathcal{X}$ of degree u be an association to every $v \in \mathrm{Ob}_u \mathcal{V}$ of an object $\sigma(v) \in \mathrm{Ob}_v \mathcal{X}$.

Let k be an integer such that $1 \leq k \leq m-1$. Then the collections of k-multimaps in $P_*\mathcal{X}$ will inductively be as follows. Suppose given a k-multimap u^k in $\mathcal{U}$ of arity and type given respectively as $(I;\pi,\phi)$ and $u^{\leq k-1} = (u^\nu)_{0\leq\nu\leq k-1}$, and the type $\sigma = (\sigma^\nu)_{0\leq\nu\leq k-1}$ of a k-multimap in $P_*\mathcal{X}$ of the same arity of degree $u^{\leq k-1}$ (see Definition 3.7). Then, we let a k-multimap $\tau \in \mathrm{Mul}^\pi_{P_*\mathcal{X},u^k}[\sigma]$ be an association to every k-multimap v^k in $\mathcal{V}$ of arity $(I;\pi,\phi)$ and degree u^k, of a k-multimap $\tau(v^k) \in \mathrm{Mul}^\pi_{\mathcal{X},v^k}[\sigma(v^{\leq k-1})]$, where $v^{\leq k-1} = (v^\nu)_{0\leq\nu\leq k-1}$ is the type of v^k (of degree $u^{\leq k-1}$), and $\sigma\big(v^{\leq k-1}\big) := \big(\sigma^\nu(v^\nu)\big)_{0\leq\nu\leq k-1}$ (where $\sigma^\nu\big(v^\nu\big) := \big(\sigma^\nu_i(v^\nu_i)\big)_{i\in[I^{\nu+1}_0]}$ if $\nu \leq k-2$, etc.) is by induction, the type of a k-multimap in $\mathcal{X}$ of the same arity of degree $v^{\leq k-1}$.

The collections of m-multimaps in $P_*\mathcal{X}$ will be as follows. Suppose given input data similar to above, but with k replaced by m. Then we let an m-multimap $\tau \in \mathrm{Mul}^\pi_{P_*\mathcal{X},u^m}[\sigma]$ be an association to every one of the types v of m-multimaps in $\mathcal{V}$ of arity $(I;\pi,\phi)$ and degree $u^{\leq m-1}$, of an m-multimap $\tau[v] \in \mathrm{Mul}^\pi_{P_!\mathcal{X},u^m}[\sigma(v)]$.

The groupoids of $(m+1)$-multimaps in $P_*\mathcal{X}$ will be as follows. Suppose given a $(m+1)$-multimap u^{m+1} in $\mathcal{U}$ of arity and type given respectively as $(I;\pi,\phi)$ and $u^{\leq m}$, and the type σ of a $(m+1)$-multimap in $P_*\mathcal{X}$ of the same arity of degree $u^{\leq m}$. Then we let

$$\mathrm{Mul}^\pi_{P_*\mathcal{X},u^{m+1}}[\sigma] = \prod_v \mathrm{Mul}^\pi_{P_!\mathcal{X},u^{m+1}}[\sigma^{\leq m-1}(v)](\sigma^m_0[v];\sigma^m_1[\pi_!v]),$$

where v runs through all the types of ϕ^{m-1}-nerves of m-multimaps in $\mathcal{V}$ of degree $u^{\leq m-1}$.

The action of $(m+2)$-multimaps of $\mathcal{U}$ on the groupoids of $(m+1)$-multimaps in $P_*\mathcal{X}$, will be as follows. Suppose given a $(m+2)$-multimap u^{m+2} in $\mathcal{U}$ of arity and type given respectively as $(I;\pi,\phi)$ and $u^{\leq m+1}$, and the type σ of a ϕ^m-nerve of $(m+1)$-multimaps in $P_*\mathcal{X}$ of degree $u^{\leq m}$. Then we let the action $\mathrm{Mul}^{\phi^m}_{P_*\mathcal{X},u^{m+1}_0}[\sigma] \to \mathrm{Mul}^{\pi_!\phi^m}_{P_*\mathcal{X},u^{m+1}_1}[\pi_!\sigma]$ of u^{m+2} be given by composing the following two maps, namely,

- the map

$$\prod_{i\in I_0^{m+1}}\prod_{w_i}\mathrm{Mul}^{\phi_i^m}_{P_!\mathcal{X},u_{0i}^{m+1}}[\sigma^{\leq m-2}(w_i^{\leq m-2})]$$
$$\big[\big((\phi^m_{\to i-1})_!\sigma^{m-1}\big)\big(w_i^{m-1}\big)\big]\big(\sigma^m_{i-1}[w_i];\sigma^m_i[(\phi^m_i)_!w_i]\big)$$

$$\longrightarrow\prod_{v}\prod_{i\in I_0^{m+1}}\mathrm{Mul}^{\phi_i^m}_{P_!\mathcal{X},u_{0i}^{m+1}}\big[\sigma^{\leq m-2}(v^{\leq m-2})\big]\big[(\phi^m_{\to i-1})_!\big(\sigma^{m-1}(v^{m-1})\big)\big]$$
$$\big(\sigma^m_{i-1}[(\phi^m_{\to i-1})_!v^{m-1}];\sigma^m_i[(\phi^m_{\to i})_!v^{m-1}]\big),$$

where
  - the source is simply the result of expanding the factors of the product $\mathrm{Mul}^{\phi^m}_{P_*\mathcal{X},u_0^{m+1}}[\sigma]=\prod_{i\in I_0^{m+1}}\mathrm{Mul}^{\phi_i^m}_{P_*\mathcal{X},u_{0i}^{m+1}}[\sigma|_i]$,
  - v runs through all the types of ϕ^{m-1}-nerves of m-multimaps in $\mathcal{V}$ of degree $u^{\leq m-1}$,
  - the map is induced from the correspondence $v\mapsto w_i=(\phi^m_{\to i-1})_!v$, and
- the map

$$\prod_v\mathrm{Mul}^{\phi^m}_{P_!\mathcal{X},u_0^{m+1}}\big[\sigma^{\leq m-1}(v)\big]\big[\big(\sigma^m_i[(\phi^m_{\to i})_!v^{m-1}]\big)_{i\in[I^m+1_0]}\big]$$
$$\longrightarrow\prod_v\mathrm{Mul}^{\pi_!\phi^m}_{P_!\mathcal{X},u_1^{m+1}}[\sigma^{\leq m-1}(v)](\sigma^m_0[v];\sigma^m_1[\pi_!v]),$$

where v runs through the same range, and the map is given by the action of u^{m+2} in $P_!\mathcal{X}$.

It is straightforward to extend these data naturally to the full data for a $\mathcal{U}$-graded $(m+1)$-theory $P_*\mathcal{X}$.

4.3.4. From these constructions, we in particular have obtained the constructions R_* and Q_* in the situation of Section 4.3.0. Therefore, we can extend the previous constructions to this situation by defining $P_*:=Q_*R_*$. Note Example 4.17. P_* will have the following universal property.

Proposition 4.18. *Let*
- $n\geq 0$ *be an integer, and* $\mathcal{U}$ *be an* n*-theory enriched in groupoids,*
- $m\geq 0$ *be an integer, and* $\mathcal{V}$ *be a* $\mathcal{U}$*-graded* m*-theory enriched in groupoids,*
- $\mathcal{X}$ *be an unenriched* $\mathcal{V}$*-graded* m*-theory.*

Denote by P *the projection functor* $\Delta_!\Theta^N_m\mathcal{V}\to\Theta^N_n\mathcal{U}$*, where* $N\geq m,n$*, and* $\Delta\colon\mathcal{U}\to\mathbf{1}^n_{\mathrm{Com}}$*.*

Then, for a $\mathcal{U}$*-graded* $(m+1)$*-theory* $\mathcal{Z}$*, a functor* $\mathcal{Z}\to P_*\mathcal{X}$ *of* $\mathcal{U}$*-graded* $(m+1)$*-theories is naturally equivalent as data to a functor* $P^*\mathcal{Z}\to\Theta_m\mathcal{X}$ *of* $\mathcal{V}$*-graded* $(m+1)$*-theories.*

Indeed, this is an immediate consequence of the similar universal properties of the constructions R_* and Q_*, which can be verified easily from the constructions.

Example 4.19. Let $\mathcal{U}$, $\mathcal{V}$ be as in Proposition, but suppose moreover that $\mathcal{V}$ is uncoloured. Then a system of colours for a $\mathcal{V}$-graded higher theory is the same as a system of colours for a $\mathcal{U}$-graded higher theory. Let $\ell\geq 0$ be an integer, and suppose given such a system of colours up to dimension $\ell-1$. Let $\mathcal{T}$ denote the terminal unenriched $\mathcal{V}$-graded $(\ell+1)$-theory on this system of colours. We would like to consider for an integer $L\geq\ell+1,m$, Proposition for an unenriched $\mathcal{V}$-graded L-theory $\mathcal{X}$ and the $\mathcal{U}$-graded $(L+1)$-theory $\mathcal{Z}:=P_!\Theta^{L+1}_{\ell+1}\mathcal{T}$, where P is as in Proposition.

We obtain that a functor $\mathcal{Z} \to P_*\mathcal{X}$ is equivalent as data to a functor $\Theta^L_{\ell+1}\mathcal{T} = P^*P_!\Theta^L_{\ell+1}\mathcal{T} \to \mathcal{X}$, which simply defines the structure of an ℓ-theory in $\mathcal{X}$, on the considered system of colours.

Corollary 4.20. *In addition to $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$ of Proposition, suppose given a $\mathcal{U}$-graded m-theory $\mathcal{W}$ enriched in groupoids. Let $\widetilde{P}\colon P_!P^*\mathcal{W} \to \mathcal{W}$ be the counit for the adjunction, and $Q\colon \Delta_!\Theta^N_{m+1}\mathcal{W} \to \Theta^N_n\mathcal{U}$ and $\widetilde{Q}\colon P_!P^*\mathcal{W} \to \mathcal{V}$ be respective projections.*

Then there is a natural equivalence $Q^*P_*\mathcal{X} \simeq \widetilde{P}_*\widetilde{Q}^*\mathcal{X}$ of $\mathcal{W}$-graded $(m+1)$-theories.

Proof. This follows immediately from Proposition and the following lemma. □

Lemma 4.21. *Let*

- *$n \geq 0$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids,*
- *$m \geq 0$ be an integer, and $\mathcal{V}$ and $\mathcal{W}$ be $\mathcal{U}$-graded m-theories enriched in groupoids.*

Let P, $\widetilde{P}$, Q, $\widetilde{Q}$ be similar to those in Proposition and Corollary 4.20.

Then, for a $\mathcal{W}$-graded m-theory $\mathcal{Y}$, there is a natural equivalence $P^*Q_!\mathcal{Y} \simeq \widetilde{Q}_!\widetilde{P}^*\mathcal{Y}$ of $\mathcal{V}$-graded m-theories.

Proof. Straightforward from the definitions. □

4.4. **Convolution for higher theories.** In Example 4.19, if $\mathcal{X}$ is of the form $P^*\mathcal{Y}$ for a $\mathcal{U}$-graded L-theory $\mathcal{Y}$, then we obtain that a $\mathcal{V}$-graded ℓ-theory in $\mathcal{Y}$ can equivalently be written as an ℓ-theory in the $\mathcal{U}$-graded $(L+1)$-theory $P_*P^*\mathcal{Y}$.

There is also a coloured generalization of this. We change the notations, and consider the following situation.

Let

- $n \geq 0$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids,
- $m \geq 0$ be an integer, and $\mathcal{T}$ be a $\mathcal{U}$-graded $(m+1)$-theory enriched in groupoids, which is terminal on a system of colours up to dimension $m-1$,
- $\mathcal{X}$ be an uncoloured $\mathcal{T}$-monoid, so together with $\mathcal{T}$, this is defining a $\mathcal{U}$-graded m-theory,
- $\ell \geq 0$ and $L \geq \ell+1, m$ be integers, and $\mathcal{Y}$ be an unenriched $\mathcal{U}$-graded L-theory.

For these, let $P\colon \Delta_!\Theta^N_{m+1}\mathcal{T} \to \Theta^N_n\mathcal{U}$ (where $N \geq m+1, n$, and $\Delta\colon \mathcal{U} \to \mathbf{1}^n_{\mathrm{Com}}$) and $R\colon P_!\Theta_m\mathcal{X} \to \mathcal{T}$ be the respective projections.

Then the $\mathcal{T}$-graded $(L+1)$-theory $R_*R^*P^*\mathcal{Y}$ is such that an $\mathcal{X}$-graded ℓ-theory in $\mathcal{Y}$ is equivalent as data to an ℓ-theory in $R_*R^*P^*\mathcal{Y}$. Therefore, $R_*R^*P^*\mathcal{Y}$, considered as a construction between the $\mathcal{U}$-graded m-theories $\mathcal{X}$ and $\mathcal{Y}$, is in a way analogous to the Day convolution for monoidal categories [8].

Remark 4.22. In the case $\ell = 0$, we obtain that the data of an $\mathcal{X}$-graded 0-theory in $\mathcal{Y}$ is further equivalent to the data of a 0-theory in $(PR)_*(PR)^*\mathcal{Y}$. This may be closer to the conventional contexts for the Day convolution.

The purpose of this section is to obtain an enriched generalization of this for the case $\ell = m-1$, where enrichment is in a symmetric monoidal higher category.

Let us start with the following.

Definition 4.23. Suppose given

- an integer $d \geq 0$, and an d-theory $\mathcal{T}$ enriched in groupoids,
- an integer k such that $0 \leq k \leq d$, and a symmetric monoidal k-category $\mathcal{A}$,
- functors $F, G\colon \mathcal{T} \to \mathbb{B}^{d-k}\Theta^k_0\mathcal{A}$, or equivalently, $(F, G)\colon \mathcal{T} \to \mathbb{B}^{d-k}\Theta^k_0(\mathcal{A} \times \mathcal{A})$.

Then we define a $\mathcal{T}$-graded d-theory $\mathrm{Fun}(F,G)$ as $(F,G)^*\mathbb{B}^{d-k}\mathrm{Fun}_{\mathcal{A}}$, where $\mathrm{Fun}_{\mathcal{A}}$ denotes $\Theta_0^k\mathrm{Fun}([1],\mathcal{A})$ (where the k-category $\mathrm{Fun}([1],\mathcal{A})$ is given the object-wise symmetric monoidal structure) considered as a $\Theta_0^k(\mathcal{A}\times\mathcal{A})$-graded k-theory via the symmetric monoidal functor

$$(d_1,d_0)\colon \mathrm{Fun}([1],\mathcal{A}) \longrightarrow \mathcal{A}\times\mathcal{A}.$$

induced from the simplicial coface operators $d^1,d^0\colon[0]\to[1]$.

We would like to generalize the construction "Fun" for *coloured* theories. The following definition includes this.

Definition 4.24. Let

- $n\geq 0$ be an integer, and $\mathcal{U}$ be an n-theory enriched in groupoids with the unique functor $\Delta\colon\mathcal{U}\to\mathbf{1}^n_{\mathrm{Com}}$,
- $m\geq 0$ be an integer, and $\mathcal{T},\mathcal{J}$ be $\mathcal{U}$-graded $(m+1)$-theory enriched in groupoids,
- k be an integer such that $0\leq k\leq m+1$, and $\mathcal{A}$ be a symmetric monoidal k-category,
- $F\colon\mathcal{T}\to\Delta^*\mathbb{B}^{m+1-k}\Theta_0^k\mathcal{A}$ and $G\colon\mathcal{J}\to\Delta^*\mathbb{B}^{m-k+1}\Theta_0^k\mathcal{A}$ be functors of $\mathcal{U}$-graded $(m+1)$-theories.

For these, let $P\colon\Delta_!\Theta^N_{m+1}\mathcal{T}\to\Theta^N_n\mathcal{U}$ (where $N:=\max\{m+1,n\}$) and $\widetilde{Q}\colon P_!P^*\mathcal{J}\to\mathcal{T}$ be the respective projections, and $\widetilde{P}\colon P_!P^*\mathcal{J}\to\mathcal{J}$ be the counit for the adjunction.

Then we define the $\mathcal{T}$-graded $(m+1)$-theory $\mathrm{Fun}\big((\mathcal{T},F),(\mathcal{J},G)\big)$ as $\widetilde{Q}_!\mathrm{Fun}(F\widetilde{Q},G\widetilde{P})$, where

$$F\widetilde{Q},G\widetilde{P}\colon\Delta_!\Theta^N_{m+1}P_!P^*\mathcal{J}\longrightarrow\Theta^N_{m-k+1}\mathbb{B}^{m-k+1}\mathcal{A}$$

are the indicated functors of symmetric N-theories.

This indeed gives the desired enriched generalization of the previous construction done using the right push-forward construction. Namely, we obtain the following in the special case where $k=1$ and $\mathcal{A}=\mathrm{Gpd}$ of the construction here.

Proposition 4.25. *Let*

- $n\geq 0$ *be an integer, and* $\mathcal{U}$ *be an* n*-theory enriched in groupoids with the unique functor* $\Delta\colon\mathcal{U}\to\mathbf{1}^n_{\mathrm{Com}}$,
- $m\geq 0$ *be an integer, and* $\mathcal{T}$ *and* $\mathcal{J}$ *be* $\mathcal{U}$*-graded* $(m+1)$*-theories enriched in groupoids, each of which is terminal on a system of colours up to dimension* $m-1$,
- $F\colon\mathcal{T}\to\Delta^*\mathbb{B}^m\Theta_0\mathrm{Gpd}$ *and* $G\colon\mathcal{J}\to\Delta^*\mathbb{B}^m\Theta_0\mathrm{Gpd}$ *be functors of* $\mathcal{U}$*-graded* $(m+1)$*-theories.*

Denote by $\mathcal{X}$ *and* $\mathcal{Y}$*, the* $\mathcal{U}$*-graded* m*-theories defined by* F *and* G*.*

Then $\mathrm{Fun}\big(\mathcal{X},\mathcal{Y}\big):=\mathrm{Fun}\big((\mathcal{T},F),(\mathcal{J},G)\big)$ *is equivalent to* $R_*R^*P^*\mathcal{Y}$*, where* $R\colon\Theta_m\mathcal{X}\to\mathcal{T}$ *and* $P\colon\Delta_!\Theta^N_{m+1}\mathcal{T}\to\Theta^N_n\mathcal{U}$ *(where* $N\geq m+1,n$*) are the respective projections.*

The proof is straightforward by direct inspection of the constructions.

It follows that, if $m\geq 1$, an $\mathcal{X}$-algebra in $\mathcal{Y}$ is equivalent as data to an $(m-1)$-theory in $\mathrm{Fun}(\mathcal{X},\mathcal{Y})$. Since the latter notion makes sense for any symmetric monoidal k-category $\mathcal{A}$, where $0\leq k\leq m+1$, we can think of it as the definition in such an enriched context, of an $\mathcal{X}$-algebra in $\mathcal{Y}$.

5. Higher theorization of symmetric monoidal functor

5.0. The definition.

5.0.0. We have so far considered iterated theorizations of algebra over a multicategory. One might wonder whether there are iterative theorizations of symmetric monoidal functor on a fixed symmetric monoidal category $\mathcal{B}$, to a varying target symmetric monoidal category.

This is actually a generalization of the previous case. Indeed, the functor Θ_0 from symmetric monoidal categories to multicategories, has a left adjoint L, so, for a multicategory $\mathcal{U}$, a symmetric monoidal functor on $L\mathcal{U}$ is the same as a $\mathcal{U}$-algebra, which we have already theorized.

We would like to show here that, if a symmetric monoidal category $\mathcal{B}$ admits a certain concrete form of description, then we indeed obtain iterated theorizations of symmetric monoidal functor on $\mathcal{B}$, by replacing in Definition 1.6 of a symmetric higher theory, the coCartesian symmetric monoidal category Fin by $\mathcal{B}$.

It turns out that $\mathcal{B}$ may more generally be a symmetric monoidal infinity category satisfying certain conditions.

Remark 5.0. When the dimension of the category $\mathcal{B}$ is at least 2, then a source of difficulty for having interesting symmetric monoidal functors $\mathcal{B} \to \mathcal{A}$, where $\mathcal{A}$ is a symmetric monoidal category, is that such a functor must invert all maps in $\mathcal{B}$ in dimensions ≥ 2. However, this restriction will be discarded in one dimension at a time as theorization (in particular, relaxation) of the notion is iterated. Indeed, in the n-dimensional theory which we define below, the inversion of maps of $\mathcal{B}$ will be forced only in dimensions $\geq n + 2$.

Let $\mathcal{B}$ be a symmetric monoidal infinity category, and denote the underlying infinity category of $\mathcal{B}$ by $\mathcal{C}$, so $\mathcal{B}$ is $\mathcal{C}$ equipped with a symmetric monoidal structure. The conditions we would like to impose on $\mathcal{B}$ are (0) and "(k)" below for all integers $k \geq 1$.

(0) The groupoid $\mathrm{Ob}\,\mathcal{C}$ of objects of $\mathcal{C}$ is free as a commutative monoid on a groupoid of generators.

(k) Suppose given data of the forms $(k - 1')$ through $(1')$ of (k) in Section 1.5, or equivalently, the arity of a k-multimap in an Init-graded higher theory. Then the groupoid formed by all k-multimap of the specified arity in the Init-graded k-theory $\Theta_1^k\mathcal{C}$, is free on a groupoid as a commutative monoid under the symmetric monoidal structure of $\mathcal{B}$.

Note that the groupoid of free generators is then the full subgroupoid consisting of *indecomposable* objects (from which we exclude the unit(s) by definition).

Remark 5.1. If $\mathcal{B}$ is in fact a symmetric monoidal *d-category* for a finite value of d, then the groupoid of $(d + 1)$-multimaps, and more gerenerally, of ϕ^d-nerves of $(d{+}1)$-multimaps in $\Theta_1^{d+1}\mathcal{C}$ (see Definition 1.6), will be equivalent to the groupoid of ϕ^{d-1}-nerves of d-morphisms in $\Theta_1^d\mathcal{C}$. In particular, the conditions (k) for all integers $k \geq 0$ hold in this case if the conditions hold for all $k \leq d$, and the groupoid of ϕ^{d-1}-nerves of d-morphisms is free as a commutative monoid for every specification of the arity.

In addition to $\mathcal{B} = \mathrm{Fin}$, the following are examples of $\mathcal{B}$ satisfying these conditions. The symmetric monoidal structures are all given by the "disjoint union".

- The 2-category **Corr**(Fin) of correspondences of finite sets, and its underlying 1-category Corr(Fin).
- The category Bord_1 of compact 0-dimensional manifolds and (the groupoids of) 1-dimensional bordisms between them. Any choice of tangential structure on manifolds.
- The category $\mathrm{End}^{d-1}_{\mathrm{Bord}_d}(\varnothing)$ of closed $(d-1)$-manifolds and (the groupoids of) d-dimensional bordisms. Any choice of tangential structure.

- The *fully extended cobordism d-category* Bord_d of bordisms up to dimension d. Any choice of tangential structure.
- For an integer k such that $2 \leq k \leq d-1$, the k-category $\mathrm{End}^{d-k}_{\mathrm{Bord}_d}(\varnothing)$ of endomorphisms in Bord_d of the empty $(d-k-1)$-dimensional cobordism.
- The $(d+1)$-category of d-th iterated cocorrespondences in Fin, and its underlying d-category. See e.g., Lurie [15, Section 3.2] for the idea, and Ben-Zvi and Nadler [2, Remark 1.17] for an explicit discussion of a definition which applies readily here. See also Calaque [5].

Remark 5.2. One can also define versions of the fully extended cobordism category where each bordism is given a codimension n embedding into the Euclidean space. These bordisms will form an E_n-monoidal d-category.

While our technique so far does not seem to apply directly for theorizing the notion of topological field theory on such a category, this kind of category seems close to satisfying an E_n analogue of the assumption required for our technique. We hope to treat theorization of these topological field theories in a sequel to this work.

Remark 5.3. Some other non-embedded (symmetric monoidal) variants of the cobordism category, such as discussed by Lurie in [15, Section 4], also satisfy the conditions of Remark 5.1.

5.0.1. Let now $\mathcal{B}$ be a symmetric monoidal infinity category satisfying the conditions (k) above for every integer $k \geq 0$. We shall obtain an n-th theorization of symmetric monoidal functor (to a variable symmetric monoidal 1-category) on $\mathcal{B}$. (See Remark 5.0.) Our n-theorized objects will be called "$\mathcal{B}$-graded n-theories".

A **$\mathcal{B}$-graded n-theory** will consist of data similar to the data (k) for $k \geq 0$, for a symmetric (i.e., "Fin-graded") n-theory (see Section 1), but with appropriate modifications applied as follows.

Firstly, the form of data (0) (in the case $n \geq 1$) will be as follows.

(0) For every indecomposable object b of $\mathcal{B}$, a collection $\mathrm{Ob}_b\,\mathcal{U}$, whose member will be called an **object** of $\mathcal{U}$ of **degree** b.

We extend this for an arbitrary object $b \in \mathcal{B}$ as follows. Namely, we let $\mathrm{Ob}_b\,\mathcal{U}$ be the collection of *b-families* of objects of $\mathcal{U}$, defined as follows.

Definition 5.4. Let b be an object of $\mathcal{B}$. Then a **b-family** of objects of a $\mathcal{B}$-graded higher theory $\mathcal{U}$, is a pair consisting of

- a decomposition $b \simeq \bigotimes_S c$, where S is a finite set, and $c = (c_s)_{s\in S}$ is an S-family of indecomposable objects of $\mathcal{B}$, and
- a *c-family* $u \in \mathrm{Ob}_c\,\mathcal{U}$, by which we mean that u is an S-family $(u_s)_{s\in S}$, where $u_s \in \mathrm{Ob}_{c_s}\,\mathcal{U}$.

Let us next describe the type of a 1-multimap. This is where the true difference of a general $\mathcal{B}$-graded theory from the case $\mathcal{B} = \mathrm{Fin}$ is seen. Namely, for a general $\mathcal{B}$, a multimap in a $\mathcal{B}$-graded theory will in general, not only accept multiple inputs, but also emit multiple outputs.

Thus, the *type of a multimap in a $\mathcal{B}$-graded higher theory $\mathcal{U}$* consists of

(0′) a map $b^1 \colon b^0_0 \to b^0_1$ in $\mathcal{B}$ which is indecomposable with respect to the commutative monoid structure,

(0″) a b^0-family u of objects of $\mathcal{U}$, namely, $u = (u_i)_{i=0,1}$, where u_i is a b^0_i-family of objects of $\mathcal{U}$.

In general, we modify Definition 1.6 of a symmetric n-theory in the following two respects.

- We modify the form of data (k) for every $k \geq 1$ as follows. We

  – keep the forms of input data $(k-1')$ through $(1')$ unchanged,
  – replace the nerve in Fin in $(0')$ with a k-multimap in $\Theta_1^k\mathcal{C}$ (where $\mathcal{C}$ denotes the infinity category underlying $\mathcal{B}$ as before) of the specified arity, which is moreover, indecomposable with respect to the commutative monoid structure,
  – modify the form of the rest of input data accordingly,
  – let the similar data as associated in (k) of Definition 1.6, be associated to these modified form of input data.
- We generalize the processes of extension of data (k) to processes of extension from indecomposable to arbitrary k-multimaps, using the free decomposition of k-multimaps just similarly to before.

Example 5.5. For the symmetric monoidal 1-category $\mathcal{B} = \mathrm{Cocorr}(\mathrm{Fin})$ underlying the symmetric monoidal 2-category of cocorrespondences in Fin, the notion of $\mathcal{B}$-graded 1-theory enriched in a symmetric monoidal category $\mathcal{A}$, coincides with the notion of coloured *properad* of Vallette in $\mathcal{A}$ [18]. Thus, the notion of (coloured) properad is a theorization of the notion of $\mathcal{B}$-algebra, and $\mathcal{B}$-graded higher theories give further theorizations.

5.1. **Symmetric monoidal functors as algebras in a theory.**

5.1.0. We would like to see that, if $\mathcal{B}$ is a symmetric monoidal d-category which satisfies our assumptions, then symmetric monoidal functors $\mathcal{B} \to \mathcal{A}$ with $\mathcal{A}$ any symmetric monoidal d-category, are indeed included in our framework.

5.1.1. Let us start with the following. Let $\mathcal{B}$ be a symmetric monoidal d-category which satisfies our assumptions (k) for all integers $k \geq 0$. Then we would like to construct a $\mathcal{B}$-graded d-theory from a symmetric monoidal d-category $\mathcal{E}$ equipped with a symmetric monoidal functor $P\colon \mathcal{E} \to \mathcal{B}$. The d-theory, which we shall denote by $\mathcal{E}^\theta$, is as follows.

For a indecomposable object $b \in \mathcal{B}$, an object of $\mathcal{E}^\theta$ of degree b is an object of $\mathcal{E}$ in the fibre over b.

For every integer k such that $1 \leq k \leq d-1$, data (k) which we specify for $\mathcal{E}^\theta$ is inductively as follows. Denote by $\mathcal{C}$ the d-category, i.e., $(d-1)$-categorified Init-graded 1-theory, underlying $\mathcal{B}$. Suppose given

- an indecomposable k-multimap b^k in $\Theta_1^k\mathcal{C}$ of arity specified by data of the forms $(k-1')$ through $(1')$ of (k) in Section 1.5, and of type given as $b^{\leq k-1} = (b^\nu)_{0\leq\nu\leq k-1}$,
- a type $e = (e^\nu)_{0\leq\nu\leq k-1}$ of a k-multimap in the $\mathcal{B}$-graded theory $\mathcal{E}^\theta$ of the same arity of degree $b^{\leq k-1}$.

By induction, e_i^{k-1} (where $i = 0, 1$) will be a family consisting of lifts in $\mathcal{E}$ of the factors/components (in the unique decomposition in $\mathcal{B}$) of the (nerve of) $(k-1)$-multimaps (or objects if $k = 1$) b_i^{k-1}, so $\bigotimes e_i^{k-1}$ in $\mathcal{E}$ lifts b_i^{k-1}, where $\bigotimes$ indicates taking the monoidal product of the members of the family (which is e_i^{k-1} here) in $\mathcal{E}$. Moreover, if $k \geq 2$, then the $(k-1)$-morphisms $\pi_!\big(\bigotimes e_0^{k-1}\big)$ and $\bigotimes e_1^{k-1}$ in $\mathcal{E}$ have common source and target by induction.

Given these data, we define a k-multimap $e_0^{k-1} \to e_1^{k-1}$ in $\mathcal{E}^\theta$ of degree b^k, to be a lift of b^k to a k-morphism $\pi_!\big(\bigotimes e_0^{k-1}\big) \to \bigotimes e_1^{k-1}$ (or $\bigotimes e_0^0 \to \bigotimes e_1^0$ if $k = 1$) in $\mathcal{E}$, completing the induction.

Similarly, the groupoids of n-multimaps in $\mathcal{E}^\theta$ will be the groupoids of similar lifts, and n-multimaps in $\mathcal{E}^\theta$ compose by the composition of n-multimaps in $\Theta_0^n\mathcal{E}$.

Thus we have constructed a $\mathcal{B}$-graded d-theory $\mathcal{E}^\theta$.

Remark 5.6. This construction is not faithful in $\mathcal{E}$ (equipped with $P\colon \mathcal{E} \to \mathcal{B}$). Instead, the construction $(\ )^\theta$ gives (non-trivial) right localization functors of suitable categories.

5.1.2. Let us denote by $\mathbf{1}^d_{\mathcal{B}}$, the terminal unenriched uncoloured $\mathcal{B}$-graded d-theory. Inspecting the construction above, it is easy to see that a 0-theory in $\mathcal{E}^\theta$, or a functor $\mathbf{1}^d_{\mathcal{B}} \to \mathcal{E}^\theta$, is equivalent as data to a section to the symmetric monoidal functor $P\colon \mathcal{E} \to \mathcal{B}$, which commutes with the symmetric monoidal structures, but is $(d-1)$-lax as a functor.

Let now $\mathcal{A}$ be a symmetric monoidal d-category. Then we have the projection functor $\mathcal{A} \times \mathcal{B} \to \mathcal{B}$, which is symmetric monoidal. It follows that a 0-theory in $(\mathcal{A} \times \mathcal{B})^\theta$ is a symmetric monoidal $(d-1)$-lax functor $\mathcal{B} \to \mathcal{A}$.

Remark 5.7. Even though the construction above has thus captured symmetric monoidal functors $\mathcal{B} \to \mathcal{A}$ for every symmetric monoidal 1-*category* $\mathcal{A}$, this is not the most interesting target if $d \geq 2$. However, in the case where $d \geq 2$ and the dimension of $\mathcal{A}$ is also d, the reader may be unsatisfied for the laxness which has crept in. (In relation to Remark 5.6, this laxness is due to the way how $\mathcal{B}$ as the terminal one among symmetric monoidal d-categories lying over $\mathcal{B}$, can fail to be local with respect to the right localization if $d \geq 2$.) Compared with d-lax symmetric monoidal functors $\mathcal{B} \to \mathcal{A}$, which could be captured in the framework of $\Theta^d_0\mathcal{B}$-graded d-theories in the previous approach, our new approach here has only eliminated the laxness with respect to the symmetric monoidal structure. However, in order to deal with the remaining laxness with a similar technique, we would need to assume a finer version of the unique decomposition, which would be more difficult to be satisfied.

5.2. **An example of different nature.** In the case $\mathcal{B} = \mathrm{Bord}_1$, there is an example of a 1-theory which is associated to a category rather than a symmetric monoidal category. An algebra in it will appear very different from a 1-dimensional field theory in the usual sense. Let us sketch these. The tangential structure we consider is framing, or equivalently, orientation.

Let $\mathcal{C}$ be a category. Then we can use it to construct a Bord_1-graded 1-theory as follows.

Firstly, we need to associate to every indecomposable object of Bord_1, a collection to be the collection of objects of that degree. To every 0-dimensional manifold consisting of one point pt with any framing of $\mathrm{pt} \times \mathbb{R}^1$, we associate the collection $\mathrm{Ob}\,\mathcal{C}$.

Next, we need to associate to every indecomposable map in Bord_1, a groupoid to be the groupoid of 1-multimaps of that degree.

- If the bordism is diffeomorphic to the interval as a manifold, then to every object $x \in \mathrm{Ob}\,\mathcal{C}$ at the incoming (relatively to the orientation) end point, and every object $y \in \mathrm{Ob}\,\mathcal{C}$ at the outgoing end point, we associate the groupoid $\mathrm{Map}_{\mathcal{C}}(x, y)$.
- If the bordism is diffeormorphic to the circle, then, for simplicity, we associate the terminal groupoid (but note Remark 5.8 below).

Finally, we need to define the composition operations. This can be given by the composition of maps in $\mathcal{C}$ (and its associativity).

Thus, we have sketched a construction of a Bord_1-graded 1-theory. Let us denote this theory by $\mathcal{Z}_{\mathcal{C}}$.

Note that, in the case where $\mathcal{C}$ is the unit category $\mathbf{1}$, we obtain $\mathcal{Z}_{\mathbf{1}} = \mathbf{1}^1_{\mathrm{Bord}_1}$, the terminal Bord_1-graded 1-theory. It follows that, in general, any object $x \in \mathcal{C}$, or equivalently, a functor $\mathbf{1} \to \mathcal{C}$, induces a functor $\mathcal{Z}_x\colon \mathbf{1}^1_{\mathrm{Bord}_1} \to \mathcal{Z}_{\mathcal{C}}$, which, by definition, is a *field theory in* $\mathcal{Z}_{\mathcal{C}}$.

The contractibility of the diffeomorphism group of the framed interval implies that $\mathcal{Z}_x$ for $x \in \mathcal{C}$ exhaust all field theories in $\mathcal{Z}_{\mathcal{C}}$.

Remark 5.8. There is another version of $\mathcal{Z}_{\mathcal{C}}$, in which the groupoid which we associate to the circle is the Hochschild homology $\mathrm{HH}_\bullet \mathcal{C} \simeq \int^{x \in \mathcal{C}} \mathrm{Map}_{\mathcal{C}}(x, x)$. All the claims above also hold for this version of $\mathcal{Z}_{\mathcal{C}}$, as a result of the following observation (and simple computations).

The observation is as follows. Let $\uparrow, \downarrow$ denote the two 1-framed points of opposite framings, and let I denote the terminal category (i.e., Init-graded 1-theory) on the two colours "$\uparrow$" and "$\downarrow$" (so, as a category, I is a contractible groupoid). Let Th_I and $\mathrm{Th}_{\mathrm{Bord}}$ respectively denote the categories of I-graded and of Bord_1-graded 1-theories. There is an obvious adjunction $\Delta_! \colon \mathrm{Th}_I \rightleftarrows \mathrm{Cat} : \Delta^*$, where $\Delta \colon \{\uparrow, \downarrow\} \to *$. The contractibility of the diffeomorphism group of the framed interval implies that there is also a functor $\mathrm{Th}_{\mathrm{Bord}} \to \mathrm{Th}_I$ with left adjoint $\overline{\mathcal{Z}}$ satisfying $\mathcal{Z} = \overline{\mathcal{Z}} \circ \Delta^*$.

Appendix A. A comparison to the work of Beaz and Dolan

A.0. Resemblance has been pointed out between our work and the beautiful pioneering work as early as about two decades ago, of Baez and Dolan [0]. Since some of our purposes overlap theirs, and the methods also has great similarity, we think that a comparison of two works would be worthwhile.

Specifically, Baez and Dolan introduce what they call the "slice operad" construction for the purpose of defining the notion of "opetopic set", which they use to give a definition of an n-category. The slice operad construction is not just interesting and powerful, but some of the ideas which they have developed for this construction, are quite close to some of the ideas which we have used for our work. There seems to be no doubt therefore, that our work was shaped by the great influences from some ideas which go back to their work (or which at least were popularized perhaps through their work).

The "slice" construction constructs from a multicategory O, a new multicategory O^+. This looks close to our ΘO, even though ΘO is a 2-theory, rather than a 1-theory. Since Baez and Dolan constructed O^+ as a multicategory, they did not need to introduce a new concept like our concept of higher theory. Moreover, iteration of their construction is automatic, unlike iteration of the process of theorization, which was the first main theme of our work. We recognize this as a great advantage of their construction.

A.1. This does not mean, however, that staying in the world of multicategories is necessary or desirable in all respects. We have succeeded after all, in defining all the higher notions of theory, and the new framework accommodates simpler approaches to some issues. For example, the case $n = 1$ of our Theorem 3.17 implies that the 2-theory ΘO is such that an uncoloured ΘO-algebra is precisely the same as an O^+-algebra (described in the quotation below). Moreover, the construction of ΘO from O was direct and immediate.

The notion of theorization also clarify the work of Baez and Dolan conceptually. Let us first hear the description of the slice operad in the inventors' own words. We shall quote from [0]. In the context at hand, their term "operad" means coloured operad (in sets, over which "algebras" are also considered in sets), and "type" means colour in our terminology.

> "We define the "slice operad" O^+ of an operad O in such a way that an algebra of O^+ is precisely an operad over O, i.e., an operad with the same set of types as O, equipped with an operad homomorphism to O. Syntactically, it turns out that:
>
> 1. The types of O^+ are the operations of O.

2. The operations of O^+ are the reduction laws of O.
3. The reduction laws of O^+ are the ways of combining reduction laws of O to give other reduction laws.

This gets at the heart of the process of "categorification," in which laws are promoted to operations and these operations satisfy new coherence laws of their own. Here the coherence laws arise simply from the ways of combining the the old laws." [*sic*]

(John C. Baez and James Dolan [0, Section 1])

In their work, they observe the points (1), (2), (3) from the actual construction of O^+, but do not explicitly discuss the conceptual reason for why O^+ had to be related to categorification. The notion of theorization sheds light on this. Indeed, "an operad over O" in their definition, is precisely an (uncoloured) theorized O-monoid (or algebra "enriched" in sets), as those authors may have known in some formulation.

A.2. There are also other advantages in employing higher theories. For example, recall that the ultimate goal of Baez and Dolan was to give a definition of an n-category. For this, they needed a few more steps after defining the slice operad. On the other hand, a version of n-categories are already among the n-theories. To examine the difference closely, we generally consider an n-theory formed not just by the n-multimaps, but with strata of colours consisting of objects to $(n-1)$-multimaps (which is the usual "colour" in an operad in the case $n = 1$), and in a special case, the structure of an n-category is formed by these objects as objects, and the unary higher multimaps as higher morphisms. Contrary to this, Baez and Dolan do not consider an algebraic structure having more than one layer of colours since they consider only multicategories. This is the reason why they needed to find another route which might appear like a detour from the point of view of higher theories. (However, some opetopic sets appear to be modeling a version of initially graded higher theories, so their method merely does not appear as direct as one can wish to make it.)

The flexibility coming from the rooms for strata of colours, is also important for considering enrichment, since possibility for more interesting enrichment requires more strata of colours. Even though the purposes of Baez and Dolan did not motivate them to consider a very general notion of enrichment, their framework as built may not support a very interesting notion of enrichment, either. Note also that, even if one intends to work only with mutlicategories, enrichment of multicategories is most generally done along 2-theories.

A.3. From the quotation of Baez and Dolan's words above, the idea expressed in the final two sentences is remarkable. Indeed, it is exactly the idea which we have described in Section 1.1, and used in our definition of an n-theory, except for two differences.

One difference is that we see the same, more generally at the heart of theorization. The other is that we have a simpler understanding of the "new coherence law", in terms of the theorized form of the structure.

Now, our version of their idea has led to a process which keeps the complexity of structures from increasing rapidly by instead raising the theoretic order, and the resulting simplicity helped us enormously with various constructions concerning higher theories. (In those constructions, roles were also played by the flexibility from the rooms for strata of colours.)

Our version of their idea also helped us with treating some systems of operations with multiple inputs and multiple outputs.

A.4. To summarize, our work has benefited from the fruits of the developments which were initiated by such prominent works as Beaz and Dolan's.

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