UNIVERSAL WEAK SEMISTABLE REDUCTION

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Abstract. We show that a toroidal morphism can be reduced to a weakly semistable one in a universal way if we allow families to be modified to Deligne-Mumford stacks instead of schemes.

1. Introduction

The semistable reduction theorem of [KKMSD73] is an essential step in the construction of compactifications of many moduli spaces. Roughly, the main result of [KKMSD73] is that given a flat family $X \to \operatorname{Spec} R$ over the spectrum of a discrete valuation ring, where the total space of X is smooth and projective, there exists a finite base change $\operatorname{Spec} R' \to \operatorname{Spec} R$ and a modification X' of the fiber product $X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ such that the central fiber of X' is a divisor with normal crossings which is reduced.

Extensions of this result to the case where the base of $X \to \operatorname{Spec} R$ has higher dimensions are studied in the work [AK00] of Abramovich and Karu. In [AK00] the authors show that given a surjective morphism $X \to S$ of projective complex varieties with geometrically integral generic fiber, then one can find an alteration $S' \to S$ and a modification X' of $X \times_S S'$ which is weakly semistable, rather than semistable – in other words, such that the family $X' \to S'$ is flat and has reduced fibers (and where S' is non-singular). The proof of [AK00] has two steps: in the first step, the morphism $X \to S$ is replaced by a morphism of toroidal embeddings, in the sense of [KKMSD73]; in the second step, the theorem is proved explicitly for morphisms $X \to S$ of toroidal embeddings.

In this paper we study only the final step of [AK00], that is, we assume from the offset that our morphisms are morphisms of toroidal embeddings. The main result of this paper is that in this context, if we relax the hypotheses to allow families $X \to S$ of Deligne-Mumford stacks rather than schemes, weak semistable reduction can be done "universally". Specifically, we show

Theorem 1.0.1 (Universal Weak Semistable Reduction). *Let $X \to S$ be any proper,surjective, log smooth morphism of toroidal embeddings. Then, there exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{S} & \longrightarrow & S \end{array}$$

where $\mathcal{X} \to \mathcal{S}$ is a weakly semistable morphism of Deligne-Mumford stacks, such that given any diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

with $T \to S$ a toroidal alteration and Y a modification of the fiber product $X \times_S T$ such that $Y \to T$ is weakly semistable, the morphism $Y \to T$ factors uniquely through $\mathcal{X} \to \mathcal{S}$. Furthermore, $\mathcal{X} \times_{\mathcal{S}} T \to T$ is weakly semistable.

2. The Toric Case

We begin by studying the toric case first. We do this because the exposition is simpler in this case, yet all the essential ideas of the proof are already present, and the general case reduces to the toric case.

We may identify a toric variety by a pair (F, N) of a lattice N and a fan F in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. We usually denote the toric variety associated to (F, N) by $\mathbb{A}(F, N)$ or simply $\mathbb{A}(F)$. We will blur the distinction between (F, N) and $\mathbb{A}(F, N)$ and refer to either as a toric variety. A morphism of toric varieties $(F, N) \to (G, Q)$ is a homomorphism of lattices $p : N \to Q$, such that $p_{\mathbb{R}} : N_{\mathbb{R}} \to Q_{\mathbb{R}}$ takes each cone $\sigma \in F$ into a cone $\kappa \in G$. For the convenience of the reader, we briefly recall the combinatorial descriptions of the key notions we will use throughout the paper:

Definition 2.0.2. The support $\operatorname{Supp} F$ of a fan F is the set of vectors in $N_{\mathbb{R}}$ that belong to some cone in F.

Definition 2.0.3. A morphism of toric varieties $p : (F, N) \to (G, Q)$ is called proper if $p^{-1}(\operatorname{Supp} G) = \operatorname{Supp} F$.

Definition 2.0.4. A morphism of toric varieties $i : (F', N) \to (F, N)$ is called a modification or subdivision if $i : N \to N$ is the identity and $\operatorname{Supp}(F') = \operatorname{Supp}(F)$.

Definition 2.0.5. A morphism of toric varieties $j : (G', Q') \to (G, Q)$ is called an alteration if $j : Q' \to Q$ is a finite index injection and $\operatorname{Supp} G' = \operatorname{Supp} G$.

Remark 2.0.6. Note that if $j : Q' \to Q$ is an injection with finite cokernel, then the condition $\operatorname{Supp} G' = \operatorname{Supp} G$ is equivalent to saying the morphism $(G', Q') \to (G, Q)$ is proper. Furthermore, for any homomorphism $j : Q' \to Q$ we get a "pull-back" fan $j_{\mathbb{R}}^{-1}(G)$ which is isomorphic to G, since $j_{\mathbb{R}}$ is an isomorphism. Thus, any toric alteration can be factored as a modification $(G', Q') \to (G, Q')$ composed with a finite index inclusion $(G, Q') \to (G, Q)$.

Remark 2.0.7. The pullback fan of remark 2.0.6 is a special case of the following more general construction:

2.1. Minimal Modification.

Let $p : N \to Q$ be a fixed homomorphism of lattices, and suppose F, G are fans in the lattices N, Q respectively.

Lemma 2.1.1 ([GM], Lemma 4)**.** *There exists a minimal modification (F', N) of (F, N) which maps to (G, Q) inducing $p : N \to Q$.*

Proof. The fan F' is simply defined as $F' := \{p^{-1}(\kappa) \cap \sigma : \kappa \in G, \sigma \in F\}$. For the proof that this is a fan and satisfies the universal property refer to [GM] or [AM14]. □

Definition 2.1.2. We call a morphism $p : (F, N) \to (G, Q)$ of toric varieties *weakly semistable* if

(1) Every cone σ in F surjects onto a cone $\kappa \in G$

(2) Whenever we have $p(\sigma) = \kappa$, we have an equality of monoids $p(N \cap \sigma) = Q \cap \kappa$

Remark 2.1.3. Our definition of weak semistability is slightly different than the definition in [AK00]: we do not demand that (G, Q) is smooth.

The above definitions are justified since a morphism of toric varieties satisfies the above combinatorial conditions if and only if the morphism of geometric realizations satisfies the analogous geometric property. We discuss what weak semistability means in practice: it is shown in [AK00] that a morphism that satisfies (1) is equidimensional and a morphism that satisfies (2) has reduced fibers. Furthemore, it is shown in Appendix C of [Wis14] that the geometry of the central fiber of such a morphism resembles the geometry of a semistable morphism as in [KKMSD73]: a node in the central fiber is the intersection of precisely two irreducible components. This justifies the terminology "semistable". Further justification is provided by the following lemma:

Lemma 2.1.4. *If $p : (F, N) \to (G, Q)$ is weakly semistable and F, G are smooth, then p is semistable.*

The condition of weak semistability is crucial in the study of families of toric varieties. First of all, it includes the condition that a family should be flat:

Theorem 2.1.5. *A weakly semistable morphism of toric varieties is flat and saturated (in the terminology of* [Tsu97]*).*

Proof. The statement is local, so we may assume we are in the situation where a single cone σ in N maps into a single cone κ in Q. By assumption, we have that faces of σ map onto faces of κ, and whenever τ maps onto λ, we have $N \cap \tau$ mapping onto $\lambda \cap Q$.

Consider the dual monoids $Q_\kappa^\vee = \kappa^\vee \cap Q^\vee$ and $N_\sigma^\vee = \sigma^\vee \cap N^\vee$ in the dual lattices. Since $N \cap \sigma$ surjects onto $Q \cap \kappa$, the dual map $Q_\kappa^\vee \to N_\sigma^\vee$ is injective and saturated. To see flatness, we will verify that this dual map is an integral map of monoids in the sense of Kato. We use Kato's equational criterion for integrality [Kat89]. Suppose we are given

$$p_1 + q_1 = p_2 + q_2$$

where $p_i \in N_\sigma^\vee$ and $q_i \in Q_\kappa^\vee$. We want to show that $p_1 = w + r_1$, $p_2 = w + r_2$, where $w \in N_\sigma^\vee$, $r_i \in Q_\kappa^\vee$, and $q_1 + r_1 = q_2 + r_2$. Since the map $Q_\kappa^\vee \to N_\sigma^\vee$ is injective and saturated, we certainly have a (non-canonical) splitting of lattices $N_\sigma^{\vee,\mathrm{gp}} = Q_\kappa^{\vee,\mathrm{gp}} \oplus L$. So, we may identify any p_1, p_2 with $(w, r_1), (w, r_2)$ and we must have $q_1 + r_1 = q_2 + r_2$. The point however is that this splitting may not respect the monoids, i.e w may not be positive on $\sigma \cap N$. To fix this, we will carefully choose a particular splitting. Pick the face τ of σ which maps isomorphically onto κ and on which the values of p_1 are minimal. To see that this is possible, let $v_1, \cdots, v_m$ be the extremal rays of κ, and let u_k denote lifts of the rays v_i in σ. Among the u_k, choose $u_1, u_2, \cdots u_m$ such that $u_i \mapsto v_i$ and such that $p_1(u_i)$ is minimal along all possible lifts of v_i to an extremal ray of σ. The face τ of σ generated by the u_i is the desired face. By assumption, we have $\tau \cap N = Q \cap \kappa$. Using this splitting $N_\sigma^{\vee,\mathrm{gp}} = N_\tau^{\vee,\mathrm{gp}} \oplus L = Q_\kappa^{\vee,\mathrm{gp}} \oplus L$, we see that we may write $p_1 + q_1 = p_2 + q_2$ in the form $(w, r_1 + q_1) = (w, r_2 + q_2)$. We may identify κ with τ, and thus the projection $\sigma \to \kappa$ gives us a map $p : \sigma \to \tau$. Every element x of σ can be written uniquely as $x = p(x) + v, v \in \operatorname{Ker} p$. Note that by construction, $w(x) = p_1(v)$, and $r_1(x) = p_1(p(x))$. To check that w is non-negative on σ, it suffices to check it is non-negative on its extremal rays. For such a ray x in $\sigma \cap \operatorname{Ker} p$, the result is clear since then $w(x) = p_1(x) \geq 0$ by assumption. For an extremal ray not in $\operatorname{Ker} p$, we write $x = p(x) + v$, where $p(x)$ is an extremal ray on τ. We then have that $p_1(p(x)) \leq p_1(x)$ by choice of τ; hence $w(x) = p_1(v) = p_1(x - p(x)) \geq 0$, which completes the proof of integrality.

The fact that the morphism of monoids is saturated follows by theorem 4.2 in [Tsu97].

□

Observe that in general for any toric monoid σ in the lattice N, we have that $\overline{N_\sigma^\vee} = \operatorname{Hom}(N \cap \sigma, \mathbb{N})$ – the notation $\overline{M}$ indicates the sharpening of M, i.e the quotient of M by the subgroup of units. For example, if σ has full dimension in N, $N_\sigma^\vee = \operatorname{Hom}(N \cap \sigma, \mathbb{N})$, as $N_\sigma^\vee$ then has no non-trivial units. Let M be any integral monoid, and consider the exact sequence

$$0 \to U \to M \to \overline{M} \to 0$$

Applying the functor $M \mapsto M^{\mathrm{gp}}$, we get a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & U & \longrightarrow & M^{\mathrm{gp}} & \longrightarrow & \overline{M}^{\mathrm{gp}} & \longrightarrow & 0 \end{array}$$

The middle map is injective, as the monoid M is integral by assumption; the map from U to $U^{\mathrm{gp}} = U$ is the identity, since M injects into M^{gp}. Exactness of the diagram at $M^{\mathrm{gp}} \to \overline{M}^{\mathrm{gp}}$ follows, since the associated group functor is left adjoint to the inclusion (Mon $\to$ Gp), so preserves colimits, hence quotients. It is not hard to verify that the diagram is also Cartesian. Let now P be any saturated monoid with a map $Q_\kappa^\vee \to P$. Since $Q_\kappa^\vee \to N_\sigma^\vee$ is integral and saturated, we have that $M := P \oplus_{Q_\kappa^\vee} N_\sigma^\vee$ is a saturated submonoid of M^{gp}. Passing to the sharpening, we get as well:

Corollary 2.1.6. *The morphism* $\operatorname{Hom}(\kappa \cap Q, \mathbb{N}) \to \operatorname{Hom}(N \cap \sigma, \mathbb{N})$ *is saturated.*

2.2. Fiber Products.

The category of toric varieties posseses fiber products:

Definition 2.2.1 (Toric Fiber Products)**.** The toric fiber product of

$$\begin{array}{ccc} & & (F, N) \\ & & \downarrow \\ (H, L) & \longrightarrow & (G, Q) \end{array}$$

is the toric variety with fan $F \times_G H = \{\sigma \times_\kappa \lambda : \sigma \in F, \lambda \in H, \kappa \in G, \sigma \to \kappa, \lambda \to \kappa\}$ in the lattice $N \times_Q L$.

It is straightforward to verify that this collection forms a fan and that it satisfies the universal property of the fiber product with respect to toric maps. However, the toric fiber product is an ill-behaved construction, as it does not in general agree with the fiber product of the associated toric varieties in the category of schemes:

$$\mathbb{A}(F \times_G H) \neq \mathbb{A}(F) \times_{\mathbb{A}(G)} \mathbb{A}(H)$$

Example 2.2.2. Consider the diagram

$$\begin{array}{ccc} & & (\mathbb{R}_+^2, \mathbb{Z}^2) \\ & & \downarrow \\ (\mathbb{R}_+^2, \mathbb{Z}^2) & \longrightarrow & (\mathbb{R}_+^2, \mathbb{Z}^2) \end{array}$$

where the morphisms are $(a, b) \mapsto (a, a + b)$ and $(c, d) \mapsto (c + d, d)$ respectively (these are the two charts of the blowup of $\mathbb{A}^2$ at the origin). We have $\mathbb{R}_+^2 \times_{\mathbb{R}_+^2} \mathbb{R}_+^2 = \{(a, b, c, d); a+b = c, a = c+d\} = \mathbb{R}_+^2 \subset \mathbb{R}^2$. On the other hand, the fiber product of these two morphisms in the category of schemes is the variety $\{(x, y, z, w) : xy = z, x = zw\}$ which is reducible.

Example 2.2.3. What fails in the previous example is that the morphisms considered are not flat. However, flatness does not suffice to ensure toric fiber products agree with schematic fiber products. For instance, given the diagram

$$\begin{array}{ccc} & & (\mathbb{R}_+, \mathbb{Z}) \\ & & \downarrow \\ (\mathbb{R}_+, \mathbb{Z}) & \longrightarrow & (\mathbb{R}_+, \mathbb{Z}) \end{array}$$

where the two morphisms are $a \mapsto 2a$, $b \mapsto 3b$ respectively, the toric fiber product is $\{(a, b) : 2a = 3b\} \cong \mathbb{R}_+(3, 2) \subset (\mathbb{Z}(3, 2))_{\mathbb{R}}$ whose geometric realization is $\mathbb{A}^1$, whereas the schematic fiber product is $\{(x, y) : x^2 = y^3\}$.

Remark 2.2.4 (Colimits of Lattices)**.** Given a diagram of lattices, we may take the limit or colimit of the diagram in the category of abelian groups. The limit of such a diagram is always a lattice, hence coincides with the limit in the category of lattices as well. In general, colimits of lattices are not lattices. However, given a finitely generated abelian group L, we can form the associated lattice $\overline{L} = L/L^{\text{tor}}$. The functor $L \mapsto \overline{L}$ is a left adjoint, and thus, the colimit $\varinjlim L_i$ in the category of lattices coincides with $\overline{\varinjlim L_i}$, where the second direct limit is understood as the colimit in the category of abelian groups.

Remark 2.2.5 (Double Dual of a Lattice)**.** Note that for any lattice L, we have a natural isomorphism $L \cong (L^\vee)^\vee$, where$L^\vee = \mathrm{Hom}(L, \mathbb{Z})$ as usual. Since we have

$$(\varinjlim L_i)^\vee := \mathrm{Hom}(\varinjlim L_i, \mathbb{Z}) = \varprojlim L_i^\vee$$

by the defining property of a colimit, it follows that

$$(\varprojlim L_i)^\vee = (\varinjlim L_i^\vee)^\vee)^\vee \cong \varinjlim L_i^\vee$$

where the colimits are understood in the category of lattices, not abelian groups, as in the preceeding remark. In particular, we have that

$$(N \times_Q L)^\vee = N^\vee \oplus_{Q^\vee} L^\vee$$

where, again, the coproduct is the coproduct in lattices, i.e the coproduct in groups divided by any potential torsion.

In general, for cones $\sigma \in F, \kappa \in G, \lambda \in H$, we get maps $\sigma^\vee \cap N^\vee = N_\sigma^\vee \to N^\vee$, $Q_\kappa^\vee \to Q^\vee, L_\lambda^\vee \to L^\vee$ and so a map $N_\sigma^\vee \oplus_{Q_\kappa^\vee} L_\lambda^\vee \to N^\vee \oplus_{Q^\vee} L^\vee \cong (N \times_Q L)^\vee$ when the coproduct of the lattices is taken in the category of lattices. It is clear that vectors in the image of $N_\sigma^\vee \oplus_{Q_\kappa^\vee} L_\lambda^\vee$ are non-negative on $\sigma \times_\kappa \lambda \subset N \times_Q L$, and so we get a map $N_\sigma^\vee \oplus_{Q_\kappa^\vee} L_\lambda^\vee \to (\sigma \times_\kappa \lambda)^\vee$. On the one hand,

$$\mathbb{C}[(\sigma \times_\kappa \lambda)^\vee \cap (N \times_Q L)^\vee]$$

are the affine charts for the fiber product of $(F, N), (G, Q), (H, L)$ in the category of toric varieties. On the other hand, since the functor **Mon** $\to$ **$\mathbb{C}$-alg**, $M \to \mathbb{C}[M]$ is left adjoint to the inclusion of $\mathbb{C}$-algebras into monoids (considered as monoids via multiplication), the functor preserves colimits, so

$$\mathbb{C}[N_\sigma^\vee \oplus_{Q_\kappa^\vee} L_\lambda^\vee] \cong \mathbb{C}[N_\sigma^\vee] \otimes_{\mathbb{C}[Q_\kappa^\vee]} \mathbb{C}[L_\lambda^\vee]$$

which are the affine charts of the fiber product in the category of schemes. Thus, we see that the toric fiber product and the usual schematic fiber product coincide if and only if

$$N_\sigma^\vee \oplus_{Q_\kappa^\vee} L_\lambda^\vee \to (\sigma \times_\kappa \lambda)^\vee \cap (N \times_Q L)^\vee$$

is an isomorphism for all cones $\{\sigma \times_\kappa \lambda\}$ in $F \times_G H$.

Weak semistability thus reveals itself through the following lemma:

Lemma 2.2.6. *Suppose*

$$\begin{array}{c} (F, N) \\ \downarrow p \\ (G, Q) \end{array}$$

weakly semistable. Then, for a map $(G', Q') \to (G, Q)$ of toric varieties, the geometric realization of the diagram

$$\begin{array}{ccc} F_{G'} = F \times_G G' & \longrightarrow & F \\ \downarrow p_{G'} & & \downarrow p \\ G' & \longrightarrow & G \end{array}$$

is cartesian in the category of schemes, and $p_{G'}$ is also weakly semistable.

Proof. The statement is local on F, so we may replace F by a single cone σ, G by a single cone κ, and G' by a single cone λ. Since p is weakly semistable, theorem 2.1.5 together with theorem 4.2 in [Tsu97] imply that the pushout

$$N_\sigma^\vee \oplus_{Q_\kappa^\vee} L_\lambda^\vee$$

is saturated in its associated group. This associated group is actually a lattice, since by choosing a face of τ of σ which maps isomorphically to κ, and with $N \cap \tau \cong Q \cap \kappa$, we obtain a splitting $N_\sigma^\vee \cong Q_\kappa^\vee \oplus N'$ – c.f the proof of 2.1.5. Thus,

$$N_\sigma^\vee \oplus_{Q_\kappa^\vee} L_\lambda^\vee$$

is identified with the intersection of a cone C in $(N \times_Q L)^\vee_{\mathbb{R}}$ with $(N \times_Q L)^\vee$. The same is true for $(\sigma \cap \lambda) \cap (N \times_Q L)$. However, the dual of each of these cones is isomorphic to $\sigma \times_\kappa \lambda$, in the first case by the defining property of the colimit, and in the second by the relation $(C^\vee)^\vee = C$ for the double dual of a cone in a fixed lattice. Applying the dual again, we see that $N_\sigma^\vee \oplus_{Q_\kappa^\vee} L_\lambda^\vee$ must be isomorphic to $(\sigma \cap \lambda) \cap (N \times_Q L)$, and the result follows by the discussion preceeding the lemma. □

Sometimes, when the morphism $G' \to G$ has additional structure, the condition that $F \to G$ is weakly semistable may be relaxed. For example, in a situation relevant to the paper we have

Lemma 2.2.7. *Suppose $p : (F, N) \to (G, Q)$ is arbitrary and $j : (G', Q') \to (G, Q)$ is an alteration with $G = G'$. Then the fiber product $\mathbb{A}(F) \times_{\mathbb{A}(G,Q)} \mathbb{A}(G, Q')$ is a toric variety, and its fan is given by the minimal modification of $(F, N \times_Q Q')$ mapping to (G, Q').*

Proof. The question is again local on F, so we may assume $F = \sigma$, $G = G' = \kappa$ is a single cone. Since $(\sigma \cap N) \times_{(\kappa \cap Q)} (\kappa \cap Q') = \sigma \cap N'$ is evidently saturated in N', where $N' = N \times_Q Q'$, the proof of lemma 2.2.1 will remain valid, provided we can show that $\mathrm{Hom}(\sigma \cap N, \mathbb{N}) \oplus_{\mathrm{Hom}(\kappa \cap Q, \mathbb{N})} \mathrm{Hom}(\kappa \cap Q', \mathbb{N})$ is saturated. But the functor $P \mapsto P^{\mathrm{sat}}$ is a left adjoint, thus preserves direct limits, and the result follows. □

2.3. **Toric Stacks.** In what follows, we will need the notion of a *toric stack*. For us, a toric stack will be always given by the data of a "KM" fan, i.e a triple $(F, N, \{N_\sigma\}_{\sigma \in F})$, where (F, N) is the usual data of a toric variety, and N_σ is a collection of sublattices of N, one for each $\sigma \in F$, with the properties

- $N_\sigma \subset N \cap \mathrm{Span}\, \sigma$ is a finite index inclusion.
- $N_\sigma \cap \mathrm{Span}\, \tau = N_\tau$ for a face τ of σ.

A morphism of toric stacks $(F, N, \{N_\sigma\}) \to (G, Q, \{Q_\kappa\})$ is a morphism $(F, N) \to (G, Q)$ such that whenever $\sigma \mapsto \kappa$, $N_\sigma \to N \to Q$ factors through Q_κ.

The data of a KM fan (F, N, N_σ) has a geometric realization into a normal, separated DM stack $\mathbb{A}(F, N, N_\sigma)$, which comes with an open dense torus acting on the stack in a way compatible with the action of the torus on itself by multiplication. The coarse moduli space of the stack is the toric variety (F, N). The data of a morphism has a realization into a morphism of these stacks, and the associated morphism between coarse spaces is simply $(F, N) \to (G, Q)$. The reader interested in this geometric realization and the properties of toric stacks is referred to the paper [GM15].

Any toric variety can be regarded as a toric stack, by taking $N_\sigma = N \cap \mathrm{Span}\, \sigma$ for each cone σ – note that there is no additional information in the N_σ in this case. Under this identification, the category of toric varieties becomes a full subcategory of the category of toric stacks. We will use this identification in what follows and keep denoting a toric variety (F, N) by (F, N) even when the context makes it clear that it is considered as a toric stack. We have from [GM15]:

Lemma 2.3.1. *A morphism of toric stacks* $p : (F, N, \{N_\sigma\}) \to (G, Q, \{Q_\kappa\})$ *is representable if and only if* $p^{-1}(Q_\kappa) = N_\sigma$ *whenever* $\sigma \mapsto \kappa$*.*

2.4. **The Main Construction.** We now fix the morphism $p : (F, N) \to (G, Q)$ which is *surjective and proper.*

Definition 2.4.1. Let $\mathcal{C}$ be the category whose objects are diagrams

$$\begin{array}{ccc} (\Phi, N') & \xrightarrow{j} & (F, N) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ (\Gamma, Q') & \xrightarrow[i]{} & (G, Q) \end{array}$$

such that

- The map i is an alteration.
- N' is the fiber product $N \times_Q Q'$.
- Φ is a modification of $j^{-1}(F)$.
- π is weakly semistable.

A morphism in $\mathcal{C}$ is a commutative diagram

$$\begin{array}{ccc} (\Phi'', N'') & \longrightarrow & (\Phi', N') \\ \downarrow & & \downarrow \\ (\Gamma'', Q'') & \longrightarrow & (\Gamma', Q') \end{array}$$

which commutes with the morphisms to $p : (F, N) \to (G, Q)$.

Theorem 2.4.2. *The category* $\mathcal{C}$ *has a terminal object which is a DM toric stack. In other words, there is a diagram*

$$\begin{array}{ccc} (F', N, N_\sigma) & \longrightarrow & (F, N) \\ \downarrow & & \downarrow \\ (G', Q, Q_\kappa) & \longrightarrow & (G, Q) \end{array}$$

such that every diagram

$$\begin{array}{ccc} (\Phi, N') & \xrightarrow{j} & (F, N) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ (\Gamma, Q') & \xrightarrow[i]{} & (G, Q) \end{array}$$

factors uniquely as

$$\begin{array}{ccccc} (\Phi, N') & \longrightarrow & (F', N, N_\sigma) & \longrightarrow & (F, N) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle p} \\ (\Gamma, Q') & \longrightarrow & (G', Q, Q_\kappa) & \longrightarrow & (G, Q) \end{array}$$

Proof. We first construct G'. Let $p(F)$ denote the collection of images of cones of F. Note that though every cone $p(\sigma)$ is contained in a cone of G, thus is convex, $p(F)$ is in general not a fan, as cones may not intersect along faces. We define G' as the subdivision of G determined by the cones in $p(F)$. Explicitly, this means the following: For every vector w in G, we look at the collection

$$N_0(w) = \{\sigma \in F : p(v) = w \text{ for some } v \text{ in the interior of } \sigma\}$$

The cones κ of G' are precisely the cones such that for any two w, w' in the interior of κ, we have $N_0(w) = N_0(w')$. Next, for a cone $\kappa \in G'$, we take

$$Q_\kappa = \cap_{\sigma \in N_0(\kappa)} p(N \cap \sigma)$$

In the interior of κ, this has the following description: $w \in Q_\kappa$ if and only if there exist $v_i \in \sigma_i$ with $p(v_i) = w$ for every cone $\sigma_i \in N_0(\kappa)$. This completes the construction of the base (G', Q, Q_κ).

At this point we need to verify that this construction actually yields a fan. The difficult part is verifying that the cones are strictly convex. So fix a cone $\kappa \in G'$, and pick two interior vectors $w, w' \in \kappa$. We will show that the whole line segment connecting w to w' must also be in the interior of κ. Suppose there exists a $t \in (0, 1)$ for which $N_0(tw + (1-t)w')$ is different from $N_0(w) = N_0(w')$. Take for simplicity the smallest such t – this makes sense since the condition $N_0(w) = N_0(u)$ is an open condition on u– and denote the point $tw + (1-t)w'$ by w'' to ease the notation. Certainly, since every cone σ_i in $N_0(w)$ is strictly convex, the line segment between two lifts of w, w' in σ_i is also in σ_i, so $N_0(w) \subset N_0(w'')$. So take a cone $\sigma \in N_0(w'') - N_0(w)$, and a lift v'' of w'' in $N_0(w'')$. We look at the fiber of the map of vector space $N_\mathbb{R} \to Q_\mathbb{R}$ over the interval $[w, w']$ in $Q_\mathbb{R}$. Call this fiber $N_{[w,w']}$, and let $F_{[w,w']}$ be the intersection of $N_{[w,w']}$ with the fan F. Then $F_{[w,w']}$ is a polyhedral decomposition of $N_{[w,w']}$. The cone σ restricts to the vertex v'' in $F_{[w,w']}$ since the relative dimension of σ under $F \to G$ is 0 by assumption, and similarily cones in $N_0(w)$ correspond to edges in $F_{[w,w']}$. Since $(F, N) \to (G, Q)$ is surjective and proper, the support of $F_{[w,w']}$ is all of $N_{[w,w']}$. In particular, the star of v'' in $F_{[w,w']}$ must intersect $F_{[w,w'')}$ non-trivially; so, in particular, there is an edge in the star of v'' in $F_{[w,w']}$ which maps to a vector $sw + (1-s)w'$ with $s < t$. By assumption on t, we have $N_0(sw + (1-s)w') = N_0(w)$, so in fact the edge corresponds to a cone in $N_0(w)$ and thus contains a lift of w. Since σ is by choice not in $N_0(w)$, v'' is an extreme point of the edge. But this is a contradiction, since the edge must extend to contain a lift of w' as well, as we assumed $N_0(w) = N_0(w')$. Thus we must have $N_0(w) = N_0(sw + (1-s)w')$ for all $s \in [0, 1]$, and convexity follows.

To construct (F', N, N_σ), we simply take the minimal subdivision of (F, N) that maps to (G', Q, Q_κ), as in 2.1.1. This means that F' is the fan $\{p^{-1}(\kappa) \cap \sigma : \kappa \in G', \sigma \in F\}$ and the sublattice corresponding to $\sigma' := p^{-1}(\kappa) \cap \sigma$ is $L_{\sigma'} := p^{-1}(Q_\kappa \cap \sigma')$. As mentioned in 2.1.1, a proof that this construction yields a fan can be found in [GM],[AM14]. In [AM14],[GM] it is also shown that such a morphism is weakly semistable: essentially, cones of F' map onto cones of G' by construction, and N_σ maps onto Q_κ whenever $\sigma \mapsto \kappa$ by construction again.

Suppose now we are given a diagram

$$\begin{array}{ccc} (\Phi, N') & \xrightarrow{j} & (F, N) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ (\Gamma, Q') & \xrightarrow[i]{} & (G, Q) \end{array}$$

where i is an alteration, N' the fiber product $N \times_Q Q'$, and Φ a subdivision of $j^{-1}F$. Assume furthermore that π is semistable. Let w, w' be two lattice points in the interior of a cone γ of the fan Γ. Suppose that w maps into a cone $\kappa \in G'$; we show that w' maps to the same cone as well. Consider lifts $v_1, \cdots, v_n$ of $v = i(w)$ to cones $\sigma_i \in N_0(w) \subset F$. Since Φ subdivides $j^{-1}F$, there are cones $g_1, \cdots, g_n$ in Φ such that $j(g_i) \subseteq \sigma_i$; so we may find lifts $w_1, \cdots, w_n$ of w in g_i. But then each cone g_i maps to γ under the projection π, and hence maps onto γ and $\gamma \cap Q' = \pi(g_i \cap N)$ from conditions (1), (2) in the definition of semistability. Since w' is in $\gamma \cap Q'$ as well, this means that there exists $w'_1, \cdots, w'_n \in g_i \cap N$ that map to w' as well – and hence there are $v'_1 = j(w'_1), \cdots, v'_n = j(w'_n)$ in the cones $\sigma_i \in N_0(w)$ that map to w' as well. It follows that $N_0(w) \subset N_0(w')$, thus, by symmetry, $N_0(w) = N_0(w')$; hence w, w' belong to the same cone κ of G'. Furthermore, they are in the image of the lattice N_{σ_i} for each cone in $N_0(w)$, thus in fact in the monoid Q_κ. Thus (Γ, Q') factors through (G', Q, Q_κ). The fact that (Φ, N) must factor through (F, N, N_σ) factors through the universal property defining (F, N, N_σ) automatically.

□

Remark 2.4.3. It might be worth pointing out that this proof goes through without assuming that the map $i : (\Gamma, Q') \to (G, Q)$ is an alteration. All that is required is that the kernel of $N \to Q$ and that the kernel of $N' \to Q'$ coincide.

Using the notation of definition 2.4.1, we have as a corollary:

Corollary 2.4.4. *The minimal modification Φ of $j^{-1}(F)$ such that*

$$\begin{array}{ccc} (\Phi, N') & \xrightarrow{j} & (F, N) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ (\Gamma, Q') & \xrightarrow[i]{} & (G, Q) \end{array}$$

commutes and π is weakly semistable is given by the fiber product of

$$\begin{array}{ccc} & & (F, N, N_\sigma) \\ & & \downarrow{\scriptstyle p} \\ (\Gamma, Q') & \xrightarrow[i]{} & (G', Q, Q_\kappa) \end{array}$$

Its geometric realization coincides with $\mathbb{A}(\Gamma, Q') \times_{\mathbb{A}(G', Q, Q_\kappa)} \mathbb{A}(F, N, N_\sigma)$.

Proof. This follows immediately by combining 2.4.2, 2.2.6, 2.3.1. □

3. Globalizing

We are now ready to discuss the changes necessary to generalize the above construction in the toroidal case. Recall the relevant definitions from [KKMSD73]. To any toroidal embedding (X, U) there is associated a stratification, the strata being determined by the irreducible components of the divisor $X - U$. To each stratum Y, we associate a lattice

$$\begin{aligned} M^Y &= \text{Divisors on } StarY \\ M^Y_+ &= \text{Effective Divisors on } StarY \\ N^Y &= \mathrm{Hom}(M_Y, \mathbb{Z}) \\ \sigma^Y &= \{v \in N^Y : v \text{ is non-negative on } M^Y_+\} \end{aligned}$$

The collection of the cones σ^Y is a cone complex, which we denote by $C(X)$; the only contrast with the toric theory is that the cones do not all inhabit a single (canonical) lattice N. Subdivisions of this cone complex correspond to birational modifications of (X, U) which are the identity on U.

We make heavy use of the following observation, explained in [KKMSD73]. We consider morphisms

$$\lambda : \operatorname{Spec} \mathbb{C}[[\mathbb{N}]] \to X$$

which take the generic point η of $\operatorname{Spec} \mathbb{C}[[\mathbb{N}]]$ to U, and the closed point 0 to Y. Then, for a divisor D in M^Y, we get a pairing

$$\langle \lambda, D \rangle = \mathrm{ord}_0 \lambda^* D$$

This way we obtain a map $\mathrm{Hom}\,(\operatorname{Spec} \mathbb{C}[[\mathbb{N}]], X) \to \sigma^Y$, whose image is the interior of σ^Y. So we may identify an interior $v \in \sigma^Y$ with an equivalence class of maps $\operatorname{Spec} \mathbb{C}[[\mathbb{N}]]$ to X, two maps being equivalent if and only if their order of intersection with each divisor is the same. We will abbreviate this equivalence class of maps by v as well. Similarily, the $\mathbb{R}_+$ span of the cone σ^Y can be identified with the image of $\mathrm{Hom}\,(\operatorname{Spec} \mathbb{C}[[\mathbb{R}_+]], X)$. This is best explained through the following three observations.

Remark 3.0.5. Suppose V is an affine toric variety, corresponding to the cone σ in the lattice N. Denote the dual lattice of N by M as usual, and denote by $\sigma^\vee$ the dual cone of σ, i.e $\{u \in M_\mathbb{R} : \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}$, so that $V = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. An element $v \in \sigma \cap N$ is the same thing as a homomorphism of monoids $\sigma^\vee \cap M \to \mathbb{N}$. We have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Mon}}(\sigma^\vee \cap M, \mathbb{N}) &= \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathbb{C}[\sigma^\vee \cap M], \mathbb{C}[\mathbb{N}]) = \\ &= \mathrm{Hom}_{\mathrm{Schemes}}(\operatorname{Spec} \mathbb{C}[\mathbb{N}] = \mathbb{A}^1, V) \end{aligned}$$

If v in in the interior of σ, the image of 0 under $\mathbb{A}^1 \to V$ is precisely the torus fixed point of V. Composing $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\mathbb{N}]$ with the completion $\mathbb{C}[\mathbb{N}] \to \mathbb{C}[[\mathbb{N}]]$ gives a morphism $\operatorname{Spec} \mathbb{C}[[\mathbb{N}]] \to V$ which defines precisely the same homomorphism $\sigma^\vee \cap M \to \mathbb{N}$ as v. Thus, for an affine toric variety each equivalence class of morphisms $\operatorname{Spec} \mathbb{C}[[\mathbb{N}]] \to V$ has a canonical representative, obtained by completing the homomorphism $\mathbb{C}[\mathbb{N}] \to V$ correpsonding to v.

Remark 3.0.6. Suppose $V = V(\sigma)$ is the affine toric variety associated to the cone σ in N, as in the preceeding remark. A similar description as the one given in the preceeding remark can in fact be given for any vector $v \in \sigma$ rather than just the integral ones. A vector v in σ corresponds to a homomorphism $\sigma^\vee \to \mathbb{R}_+$, which induces (and is determined by) by restriction to a homomorphism $\sigma^\vee \cap M \to \mathbb{R}_+$. This is the same data as a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\mathbb{R}_+]$ as above, which induces a map $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[[\mathbb{R}_+]]$, i.e a map $\lambda : \operatorname{Spec} \mathbb{C}[[\mathbb{R}_+]] \to V(\sigma)$. When v is in the interior of σ, we have that under this morphism the closed point maps to the torus fixed point of $V(\sigma)$. Note that for any toric divisor D of $V(\sigma)$, i.e any element $u \in M$, we have $\langle v, u\rangle = ord_{t=0}\lambda^* u$ by construction. We may thus identify vectors $v \in \sigma$ with morphisms

$$\lambda : \operatorname{Spec} \mathbb{C}[[\mathbb{R}_+]] \to V(\sigma)$$

such that $\lambda(\eta) \in$ torus, and $\lambda(0) =$ torus fixed point, up to the equivalence relation that $ord_{t=0}\lambda^*$ induces the same homomorphism on M. We also have the analogue of the observation in the preceeding remark, that every equivalence class of a morphism $\operatorname{Spec} \mathbb{C}[[\mathbb{R}_+]] \to V(\sigma)$ has a unique representative obtained by completing $\operatorname{Spec} \mathbb{C}[\mathbb{R}_+] \to V(\sigma)$.

Remark 3.0.7. We can now combine the two remarks above with the fact that every toroidal embedding is etale locally (hence formally locally) isomorphic to a toric variety, to obtain that every interior vector $v \in \sigma^Y$, not necessarily integral, corresponds to an equivalence class of morphisms

$$\lambda : \operatorname{Spec} \mathbb{C}[[\mathbb{R}_+]] \to V$$

such that $\lambda(\eta) \in U$, $\lambda(0) \in Y$. Of course, there is no longer a canonical representative for a morphism in this equivalence class. Here, the ring $\mathbb{C}[[\mathbb{R}_+]]$ is the completion of $\mathbb{C}[\mathbb{R}_+]$ with respect to the valuation on $\mathbb{C}[\mathbb{R}_+]$ which takes a polynomial $\sum_{\alpha\in\mathbb{R}_+} c^\alpha x^\alpha$ to $\inf\{\alpha : c_\alpha \neq 0\}$. The completion $\mathbb{C}[[\mathbb{R}_+]]$ is naturally a valuation ring as well; we denote its fraction field by $\mathbb{C}[[\mathbb{R}]]$. The field $\mathbb{C}[[\mathbb{R}]]$ is algebraically closed. A proof of this fact, using an appropriate version of Hensel's lemma is given in appendix 1 of [FOOO10]. It is also possible to give a direct proof by observing that the proof of the Newton-Puiseaux theorem extends to the case of real exponents as well.

3.1. **Toroidal Stacks.** We would now like to transport the main points of the theory of toric stacks to toroidal embeddings. Though it is possible to give analogous definitions by working étale locally and modifying the appropriate results of [GM15], we prefer not to work from scratch, and use the construction of the "root stack" of Borne and Vistoli, [BV12], section 4. In order to use this result, we have to work a little bit with logarithmic structures rather than toroidal embeddings. Note that a toroidal embedding (X, U) carries a canonical structure of a log scheme (X, M), by setting

$$M(V) = \{f \in \mathcal{O}_X(V) : f \in \mathcal{O}_X^*(V \cap U)\}$$

Conversely, by the chart criterion of log smoothness of [Kat89], it follows that a toroidal embedding without self intersection is the same thing as a log smooth log scheme (X, M) with Zariski log structure. The sheaf M has the following characteristic monoid $\overline{M}$ at the generic point η_Y of a stratum Y:

$$\overline{M}_{\eta_Y} = (\sigma^Y)^\vee \cap M^Y$$

Given a log scheme (X, M), the construction of Borne and Vistoli asserts that for any map of monoids $\overline{M} \to \overline{M}'$ which is *Kummer*, i.e injective with finite cokernel, there exists a DM log stack $(\mathcal{X}, M')$ mapping to (X, M), with the morphism $M \to M'$ inducing the given map $\overline{M} \to \overline{M'}$. Furthremore, the morphism is log smooth and the terminal object of the category of log schemes $(Y, N) \to (X, M)$ such that $M \to N$ factors through $M \to M'$.

Since the stack $(\mathcal{X}, M')$ is log smooth with Zariski log structure, we will refer to it as a toroidal stack without self intersection. A method analogous to the toric case for producing toroidal stacks from a toroidal embedding (X, U) is to assign for each cone σ^Y in $C(X)$ a sublattice $N_{\sigma^Y} \subset N^Y \cap \operatorname{Span}(\sigma^Y)$ which is injective with finite cokernel, and with the property that $N_\tau = \operatorname{Span} \tau \cap N_\sigma$ for a face τ of σ – on the level of log structures, this is precisely a Kummer extension $\overline{M}'$ of the sheaf $\overline{M}$, where M is the canonical log structure. Thus, a compatible triple $(C(X), N^Y, N_{\sigma^Y})$, as Y ranges through the strata of (X, U) produces a toroidal stack.

3.2. **The Main Construction.** The toric construction explained in section 2 carries over to the toroidal case with minimal changes, by replacing the fans of X and S with the cone complexes $C(X), C(S)$. The morphism $X \to S$ induces a morphism $p : C(X) \to C(S)$ by composing a map $\mathbb{C}[[\mathbb{R}_+]] \to X$ with $X \to S$. So suppose a cone $\kappa \in C(S)$, and a point $w \in \kappa$ are given. We consider

$$N_0(w) = \{\sigma : \exists! v \in \sigma^o \text{ such that } p(v) = w\}$$

Lemma 3.2.1. *The cones* $\{w : N_0(w) = \text{ fixed}\}$ *are strictly convex, and form a subdivision of* κ*.*

Proof. We try to mimic the proof of the toric case 2.4.2. The question is local on S, so we may assume that $C(S)$ is a single cone κ. We take two vectors w, w' in κ for which $N_0(w) = N_0(w')$, and try to show that we have $N_0(tw+(1-t)w') = N_0(w)$ for all $t \in [0, 1]$. As above, we may assume that this condition fails for some t and derive a contradiction, and even take for t the minimal element of $[0, 1]$ for which the condition fails (note that the condition can only fail in a closed subset of $[0, 1]$). So we may replace κ by the interval $[w, w']$, and $C(X)$ by its fiber over $[w, w']$, which we will denote by $C(X)_{[w,w']}$. Put $w'' = tw + (1 - t)w'$. As in the toric proof, we can choose an element in $N_0(w'') - N_0(w)$, which corresponds to a vertex v'' in $C(X)_{[w,w']}$. The key step in the toric proof is that the star of v'' in $C(X)_{[w,w']}$ intersects the fiber of $[w, w'')$ in $C(X)$, which follows from the properness of the map. This statement is not clear in the toroidal situation, but we claim it is nevertheless still correct. To see this, pick a family of maps

$$\operatorname{Spec} \mathbb{C}[[\mathbb{R}_+]] \times [0, 1] \to S$$

corresponding to the interval $[w, w']$ in κ, and a lift

$$\operatorname{Spec} \mathbb{C}[[\mathbb{R}_+]] \to X$$

of the map corresponding to w'', which represents the vector $v'' \in C(X)$. We abusively denote this map by v'' as well. Let $x = v''(0) \in X$. Since $X \to S$ is log smooth, we may choose a chart

$$\begin{array}{ccc} (X,x) & \xrightarrow{f} & (V,v) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ (S,p(x)) & \xrightarrow[g]{} & (W,\pi(v)) \end{array}$$

for the morphism p, where: the horizontal morphisms f, g are étale; V, W are toric varieties, π is a toric morphism, and $v, \pi(v)$ are special points in the torus orbits; and the morphism $N_{\mathbb{R}} \to Q_{\mathbb{R}}$ is surjective, where N, Q are the lattices of V and W respectively. Let $[z, z']$ denote the interval corresponding to $[w, w']$ in $Q_{\mathbb{R}}$ under g, and let $y'' \in N_{\mathbb{R}}$ denote the element corresponding to v'' under f. Since $N_{\mathbb{R}} \to Q_{\mathbb{R}}$ is surjective, we may lift $[z, z']$ to an interval $[y, y'] \in N_{\mathbb{R}}$ lying over $[z, z']$, with $y'' \in [y, y']$. In other words, if we denote by $T(V), T(W)$ the tori of V and W respectively, we get that the family

$$\operatorname{Spec} \mathbb{C}[\mathbb{R}] \times [0,1] \to T(W) \subset W$$

corresponding to $[z, z']$ lifts to a family of maps

$$\operatorname{Spec} \mathbb{C}[\mathbb{R}] \times [0,1] \to T(V) \subset V$$

which under π projects to $[z, z']$, and with y'' the map over z'', i.e such that the morphism at $t \in [0,1]$ is y''. Denote the cone of $C(V)$ that contains y'' in its interior by ρ, and the cone of $C(X)$ that contains v'' in its interior by σ, and let η_ρ, η_σ be the generic points of the strata in V and X corresponding to ρ and σ respectively. Since $y''(0)$ is in the interior of the stratum corresponding to ρ, the valuation of $y''^{*}(D)$ is positive for every divisor containing the stratum, and the same is true for every vector in the interval $[y, y']$ which is sufficiently close to y''. Consequently, for such vectors, the induced map $\mathcal{O}_{V,\eta_\rho} \to \mathbb{C}[\mathbb{R}]$ is continuous with respect to the Krull topology on $\mathcal{O}_{V,\eta_\rho}$. Thus, the family of maps

$$y_s : \operatorname{Spec} \mathbb{C}[\mathbb{R}] \times [0,1] \to T(V) \subset V$$

extends to a family of maps

$$y_s : \operatorname{Spec} \mathbb{C}[[\mathbb{R}]] \times (t - \epsilon, t + \epsilon) \to \operatorname{Spec} \hat{\mathcal{O}}_{V,\eta_\rho}$$

for sufficiently small ϵ – to be precise, ϵ small enough that $y_s^*(D) > 0$ for all D containing the stratum corresponding to ρ, and all $s \in (t - \epsilon, t + \epsilon)$. Since the map $X \to V$ is étale, and $\operatorname{Spec} \mathbb{C}[[\mathbb{R}]]$ is algebraically closed, there is a lift of this family to a family of maps

$$\operatorname{Spec} \mathbb{C}[[\mathbb{R}]] \times (t - \epsilon, t + \epsilon) \to X$$

which under p composes to the original family $\operatorname{Spec} \mathbb{C}[[\mathbb{R}]] \times (t-\epsilon, t+\epsilon) \to S$ corresponding to $[w, w']$. Now, for each $s \in (t - \epsilon, t + \epsilon)$, the map $\operatorname{Spec} \mathbb{C}[[\mathbb{R}]] \to X$ extends to a map $\operatorname{Spec} \mathbb{C}[[\mathbb{R}_+]] \to X$ by properness of $X \to S$. But such a map corresponds to an element of $C(X)$, so we get a family of elements in $C(X)$ which at $s = t$ specialize to v''. Thus, the star of v'' in $C(X)$ contains a lift of the line segment $[w, w'')$, and the same argument as in the toric case goes through. $\square$

Theorem 2.4.2 now carries through without any change in the proof, once we consider the appropriate generalization of the category $\mathcal{C}$ in the toroidal setting. We fix a proper, surjective toroidal morphism $X \to S$, which gives a cone complex morphism $C(X) \to C(S)$. We consider the subdivision of S determined by the cones $\{w \in C(S) : N_0(w) = \text{ constant}\}$, and the subdivision of $C(X)$ given by $p^{-1}(\{w \in C(S) : N_0(w) = \text{ constant}\}) \cap \sigma$. For a cone κ whose interior is given by the collection $\{w : N_0(w) = \{\sigma_i\}_1^n\}$, we take the sublattice $Q'_\kappa$of Q_κ to be the lattice generated by the elements in $\cap_{i=1}^n p(\sigma_i \cap N_{\sigma_i})$, and for a cone $\sigma_i \cap p^{-1}(\kappa)$ we take the sublattice $N'_{\sigma_i \cap p^{-1}(\kappa)} = p^{-1}(Q'_\kappa)$. This construction yields a toroidal morphism of toroidal stacks $\mathcal{X} \to \mathcal{S}$, according to [BV12].

By an alteration, we mean an alteration $T \to S$ which is also a morphism of toroidal embeddings. Concretely, this means that when $\kappa' \in C(T) \to \kappa \in C(S)$, the map $\kappa' \to \kappa$ is an alteration in the toric sense. Then, we consider

Definition 3.2.2. Let $\mathcal{C}_t$ be the category whose objects are diagrams

$$\begin{array}{ccc} Y & \xrightarrow{j} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow[i]{} & S \end{array}$$

such that

- Y, T are toroidal embeddings and j, π, i are morphisms of toroidal embeddings.
- The map i is an alteration.
- Y is a modification of the fiber product $X \times_S T$.
- π is weakly semistable.

A morphism in $\mathcal{C}_t$ is a commutative diagram

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

which commutes with the morphisms to $p : X \to S$.

Then we have

Theorem 3.2.3. *The family $\mathcal{X} \to \mathcal{S}$ is the terminal object of C_t.*

Proof. The proof of the toric case in 2.4.2 carries through in exactly the same way. □

As a corollary, we get

Theorem 1.0.1 (Universal Weak Semistable Reduction)**.** Let $X \to S$ be any proper, surjective, log smooth morphism of toroidal embeddings. Then, there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{S} & \longrightarrow & S \end{array}$$

where $\mathcal{X} \to \mathcal{S}$ is a weakly semistable morphism of Deligne-Mumford stacks, such that given any diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

with $T \to S$ a toroidal alteration and Y a modification of the fiber product $X \times_S T$ such that $Y \to T$ is weakly semistable, the morphism $Y \to T$ factors uniquely through $\mathcal{X} \to \mathcal{S}$. Furthermore, $\mathcal{X} \times_{\mathcal{S}} T \to T$ is weakly semistable.

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