Entanglement thermodynamics for charged black holes

Pankaj Chaturvedi *[1], Vinay Malvimat†[2], and Gautam Sengupta‡[3]

[1,2,3] Department of Physics,
Indian Institute of Technology Kanpur,
Kanpur 208016, INDIA

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Abstract

The holographic quantum entanglement entropy for an infinite strip region of the boundary for the field theory dual to charged black holes in $\mathcal{A}dS_{3+1}$ is investigated. In this framework we elucidate the low and high temperature behavior of the entanglement entropy pertaining to various limits of the black hole charge. In the low temperature regime we establish a first law of entanglement thermodynamics for the boundary field theory.

*E-mail: cpankaj@iitk.ac.in
†E-mail: vinaymm@iitk.ac.in
‡E-mail: sengupta@iitk.ac.in

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1 Introduction

Quantum correlations play an important role in studying various aspects of many-body physics pertaining to condensed matter systems, statistical mechanics, quantum information and quantum gravity. In this regard entanglement entropy acts as a key quantity, providing the measure for the quantum correlations in a bipartite quantum system.

A bipartite quantum system can be described as the union of the subsystem A under observation which is correlated with the rest of the system defined by its complement A_c. For this bipartite quantum system the full Hilbert space $\mathcal{H}$ can be expressed as a tensor product of the Hilbert spaces of A and A_c as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A_c}$ in the case when there is no entanglement between subsystems A and A_c. However, if the subsystems A and A_c are entangled then the full Hilbert space can not be expressed as the tensor product of the Hilbert spaces of A and A_c, which implies that the density matrix (ρ) of the full system also cannot be expressed as the tensor product of ρ_A and ρ_{A_c} i.e. $\rho \neq \rho_A \otimes \rho_{A_c}$. In such a scenario, at zero temperature the whole system can be represented by a pure state $\mid \psi\rangle \; \epsilon \; \mathcal{H}$, with the density matrix given by $\rho =\mid \psi\rangle\langle\psi \mid$. The reduced density matrix (ρ_A) for the subsystem A is obtained via tracing ρ over the degrees of freedom of A_c giving $\rho_A = Tr_{A_c}\rho$ and the entanglement entropy of the subsystem A is defined as the corresponding Von Neumann entropy, $S_A = -Tr_A(\rho_A \log \rho_A)$. More interestingly, if the two subsystems A and A_c are spatially partitioned from each other by a geometric boundary (∂A) then in d-spatial dimensions it is observed that the entanglement entropy follows an area law, $S_A \propto (\partial A)/a^{d-1} + \cdots$, where the expansion parameter a is some small UV cutoff and the dots represents higher order terms in power of a [1]. In the case of two dimensional conformal field theories, the entanglement entropy is observed to depict a logarithmic behavior which is in contrast to the area law discussed before. In particular, for two dimensional conformal field theories one can write $S_A = (c/3)\log(l/a) + \cdots$, where l is the length of interval corresponding to subsystem A and c is the central charge of the theory [2–5]. The entanglement entropy also follows two basic properties (i) $S_A{=}S_{A_c}$, where A_c is the compliment of the region A and (ii) $S_{A_1} + S_{A_2} \geq S_{A_1} \cup S_{A_2} + S_{A_1} \cap S_{A_2}$, which is the strong subadditivity condition for two different regions A_1 and A_2 of a quantum system [6–9] (See [10–13] for applications).

The method to obtain analytically the entanglement entropy for a CFT is the replica trick [2, 5, 14] which incorporates first the computation of $Tr(\rho_A^n)$ for some positive integral value of n and then taking the limit $n \to 1$ in the expression $S_A = -\partial_n(Tr\rho_A^n) \mid_{n=1}$ (See [15] for recent reviews and applications). Further advancement in the understanding and computation of entanglement entropy was due to the works of Calabrese and Cardy, who used the methods of $(1 + 1)$-dimensional CFT to study entanglement in numerous static as well as dynamic scenarios [16–20]. However, in higher dimensions only the of quasi free fermions and bosons has been studied rigorously studied in [21–26]. It was observed that for the bosonic case on a lattice structure, the entanglement entropy for a subsystem A satisfies an *"Area law"* i.e. $S_A = l^{d-1}/a^{d-1}$ with l being the length of the susbystem A measured in lattice units and a being the UV cutoff whereas, in the fermionic case a logarithmic behavior of entanglement entropy was observed i.e. $S_A = l^{d-1} \log l/a$. Thus it is evident from the above discussions that the "Area law" for the entanglement entropy seems to be persistent even for the quantum field theories in the higher dimensions.

Further development in understanding of the entanglement entropy for quantum field theories came with the advent of AdS/CFT correspondence which relates (d)-dimensional strongly coupled boundary quantum field theory to a theory of weakly coupled gravity in $(d+1)$-dimensional bulk space-time [27–31]. Using this duality Ryu and Takyanagi proposed a conjecture to holographically evaluate the entanglement entropy of a (d)-dimensional quantum field theory [32,33] which was later generalized for time dependent backgrounds by Hubeny-Rangamani-Takayanagi (HRT formula)in [34]. This holographic prescription for obtaining the entanglement entropy S_A for a region A (enclosed by the boundary ∂A) in the (d)-dimensional boundary field theory involves computation of the area of the extremal surface (denoted by γ_A) extending from the boundary ∂A of the region A into the $(d+1)$-dimensional bulk such that S_A is given by $Area(\gamma_A)/(4G_N^{(d+1)})$, where $G_N^{(d+1)}$ is the gravitational constant of the bulk. The holographic prescription of computing entanglement entropy due Ryu and Takyanagi reproduced the aforementioned *"Area law of entanglement entropy"* for a (d)-dimensional boundary CFT dual to $(d+1)$-dimensional AdS vacuum solution in the bulk and also confirmed the validity of the strong subadditivity inequalities [35]. Besides this, it also confirmed the logarithmic behavior of the entanglement entropy for a $(1+1)$-dimensional boundary CFT in AdS_{2+1}/CFT_{1+1} setup [35]. This remarkable success of Ryu and Takyanagi conjecture embarked the study of entanglement entropy and related phenomenon for different classes of (d)-dimensional boundary field theories dual to different solutions of classical gravity in a $(d+1)$-dimensional bulk (See [35, 36] and the references therein). One such application of the conjecture involves the study of the entanglement entropy for a boundary CFT at finite temperature which is dual to a Schwarzschild black hole solution in the bulk [37–40]. Furthermore, for such a boundary CFT at finite temperature the authors in [40] have used analytic series expansion technique to derive an expression entanglement entropy in terms of temperature and subsystem length and have used it to study low and high temperature behavior of entanglement entropy. Such studies related to a boundary CFT at finite temperature lead to a proposal of an analogous *"First law of thermodynamics"* like relation for entanglement entropy by the authors in [41] (See [42] for other studies related to entanglement thermodynamics).

Another class of boundary field theories that have been studied using the Ryu and Takyanagi conjecture are the ones which are dual to charged black holes in the bulk [39,43,44]. In particular, the authors in [43] have studied the entanglement entropy for the boundary field theory dual to Reissner-Nordstrom black holes in arbitrary $(d+1)$-dimensional AdS spacetime. For such a boundary field theory at finite temperature and charge density, the issue of entanglement thermodynamics was also addressed in [45, 46]. Despite, such attempts to study entanglement entropy for boundary field theories dual to *Reissner-Nordstrom* black holes in the AdS bulk a complete description of the temperature dependence of entanglement entropy is still lacking and the entanglement thermodynamics ("First law of thermodynamics" like relation for entanglement entropy) still remains underexplored for such boundary field theories. Thus in this article we attempt to comprehensively investigate the temperature dependence of the entanglement entropy for a strip like region in the boundary field theory which is dual to RN black holes in AdS_4/CFT_3 setup. We also comprehensively explore the entanglement thermodynamics pertaining to such boundary field theories at finite temperature and charge density.

This article is organized as follows. In section 1 we review the Ryu-Takyanagi conjucture and describe the setup for computing the holographic entanglement entropy for boundary field theory dual to planar black holes. In section 2 we state the holographic prescription for computing the entanglement entropy for boundary field theories dual to Reissner-Nordstrom black holes in AdS_4/CFT_3 setup. In section 3 we compute the entanglement entropy for boundary field theory dual to extremal and non-extremal charged black holes in the small charge regime and obtain the "First law of entanglement thermodynamics" at low temperatures. In the same section we also study the low and high temperature behavior of entanglement entropy in the small charge regime for the case of non-extremal black holes in the AdS_4 bulk spacetime. In section 4 we present the entanglement entropy for boundary field theory dual to extremal and non-extremal charged black holes in the large charge regime. Finally in section 5 we summarize out results and findings.

2 Review of holographic entanglement entropy of planar black holes

The AdS/CFT correspondence suggests that a black hole solution in the bulk AdS spacetime is dual to a boundary CFT at finite temperature. In order to compute the entanglement entropy for such a strongly coupled boundary field theory at finite temperature one has to implement the Ryu-Takayangi prescription as described in introduction. For our purpose we have geometrically partitioned the boundary field theory into a subsystem(A) which is geometrically a long strip and is entangled with rest of the system as shown in fig.(1). According to this prescription, entanglement entropy(S_A) for the subsystem(A) is given by the area of the extremal surface(γ_A) which extends into the bulk and is anchored to the boundary of the subsystem(A) [32]. Following the Ryu-Takayangi conjecture, in the general AdS_{d+1}/CFT_d setup the entanglement entropy (S_A) for the subsystem A may be given as

$$S_A = \frac{Area(\gamma_A)}{4G_N^{d+1}} \tag{1}$$

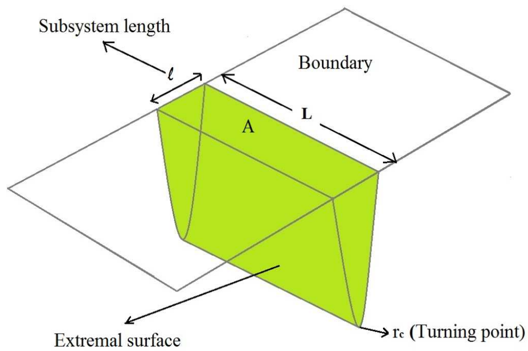

Figure 1: Schematic of extremal surface anchored on the subsystem that lives on the boundary

In a simpler setup the behavior of the entanglement entropy with temperature for a boundary field theory dual to a Schwarschild black hole in the bulk AdS_{d+1} spacetime was first studied in [40]. Here we review the method adopted in [40] for computation of entanglement entropy when a black hole with planar horizon is present in the bulk. We will be working in the set up of AdS_4/CFT_3 while restricting ourselves to the Poincaré patch of $AdS-4$ as in the Poincaré coordinates the black hole in the bulk has a planar horizon $R^{(2,1)}$ and the dual field theory lives on the conformal boundary of AdS_4 spacetime.

In Poincaré coordinates, the metric for a black hole with a planar horizon in AdS_{3+1} spacetime can be given as

$$ds^2 = -\frac{r^2}{L^2}f(r)dt^2 + \frac{L^2dr^2}{r^2f(r)} + \frac{r^2}{L^2}d\vec{x}^2 \tag{2}$$

Where L is the AdS length scale L and the components of the vector $\vec{x}$ corresponds to $\{x, y\}$. In the boundary field theory side, we have chosen the subsystem (A) to be confined geometrically by a long strip $x \in \left[-\frac{l}{2}, \frac{l}{2}\right]$, $y \in \left[\frac{L}{2}, \frac{L}{2}\right]$ as shown in fig.(1). The area of the surface anchored on the boundary of the subsystem (A) may be given as

$$\mathcal{A} = L\int dr\ r\sqrt{r^2x'^2 + \frac{1}{r^2f(r)}} \tag{3}$$

Extremizing the area functional given by eq.(3) leads to the Euler-Lagrange equation which on integrating with respect to radial coordinate (r) yields following relation

$$\frac{dx}{dr} = \pm\frac{r_c^2}{r^4\sqrt{f(r)(1-\frac{r_c^4}{r^4})}}, \tag{4}$$

here, r_c is the constant of integration and represents the turning point of the extremal surface in the higher dimensional AdS_4 bulk spacetime. Integrating the above equation once again with the boundary condition $x(\infty) = \pm\frac{l}{2}$ and ($r = r_c$ at $x = 0$), a relation between the parameters l and r_c can be written down as follows

$$\frac{l}{2} = \int_{r_c}^{\infty} \frac{r_c^2 dr}{r^4\sqrt{f(r)(1-\frac{r_c^4}{r^4})}} \tag{5}$$

From eq.(3) it may be seen that the area integral becomes divergent as, $r \to r_c$ and has to be regularized by introducing an UV cutoff (a) of the boundary field theory. The holographic dictionary relates the UV cutoff of the boundary field theory to an infrared cutoff (r_b) in the bulk AdS spacetime and both are inversely related through AdS length scale as, $r_b = \frac{L^2}{a}$. Using equations (3), (4) and (5), the modified form of area functional $\mathcal{A}$ for the extremal surface may be obtained as follows

$$\mathcal{A} = 2L\int_{r_c/r_b}^{\infty} \frac{dr}{\sqrt{f(r)(1-\frac{r_c^4}{r^4})}} \tag{6}$$

Furthermore the entanglement entropy can also be bifurcated into a divergent part and a finite part as

$$S_A = S_A^{divergent} + S_A^{finite} \tag{7}$$

$$S_A^{finite} = \frac{\mathcal{A}^{finite}}{4G_N^{3+1}} \tag{8}$$

The explicit expressions for the finite and the divergent parts of the entanglement entropy can be found in [40]. The divergent part of entanglement entropy is subtracted out by adding appropriate counter terms which amounts to holographic renormalization of entanglement entropy of the boundary field theory. The finite part of entanglement entropy can then be used to study the high and low temperature behavior of entanglement entropy for the boundary field theory which is dual to Schwarzschild-AdS planar black hole as covered in [40]. In the same reference the authors use various approximations in order to get an analytic expression for the entanglement entropy which is due to the fact that the integrals mentioned in equations (3), (4) and (5) have no known analytic expression for the case of boundary field theory dual to Schwarzschild-AdS black hole. In later section we will see that this holds true for the case of boundary field theory dual to Reissner-Nordstrom black holes in higher dimensional bulk AdS spacetime. To elucidate this further we would like to state that the authors in [40] have used an expansion technique which involves the use of Gamma functions and thus simplifies the analytic computation of the area integral in eq.(6). We will also be incorporating the same expansion technique in order to derive an analytic expression of entanglement entropy for our case of boundary field theory dual to Reissner-Nordstrom black holes in higher dimensional bulk AdS spacetime. Given below is an useful relation that was used in [40] to express the analytic form of entanglement entropy in the case of boundary field theory dual to Schwarzschild-AdS black hole

$$\frac{1}{\sqrt{1-x}} = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}x^n \tag{9}$$

As stated earlier in this section that although the authors in [40] have computed entanglement entropy for a boundary field theory dual to Schwarzschild-AdS black hole in general $(d+1)$-dimensions, their results can also be summarized in the AdS_4/CFT_3 setup to which we address next.

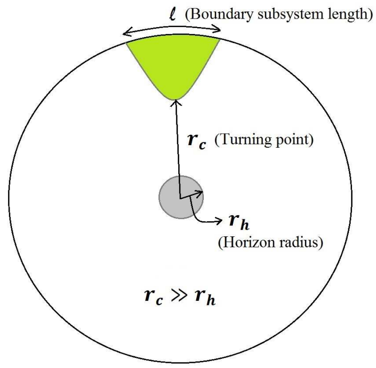

Figure 2: Schematic of extremal surface when the horizon radius is small

It is to be noted that the state space of the boundary field theory dual to Schwarzschild black hole in AdS_4, is described only by its temperature which is related to Hawking temperature of the black hole in the bulk. Thus the ground state of the boundary field theory will correspond to pure AdS solution of Einstein equations in $(3+1)$-dimensional bulk whereas, an excited state will correspond to Schwarzschild black hole in AdS_4 bulk. Thus the low temperature limit of the boundary field theory will correspond to a black hole with small radius of horizon which amounts to working in the limit, $r_h \lll r_c$ where r_c represents the turning point of the extremal surface in the bulk as shown in fig.(2). In this limit the authors in [40] showed that the leading contribution to entanglement entropy S_A of the subsystem (A) comes from S_A^{AdS} which corresponds to the entanglement entropy of the subsystem (A) of the boundary field theory dual to pure AdS_{3+1} bulk. The explicit form of S_A in low temperature approximation can be written down as follows

$$S_A = S_A^{AdS} + k(r_h l)^3 \tag{10}$$

The low temperature behavior for Schwarzschild case was also studied in [41] where the authors derive a first law like relation for the dual boundary field theory which is also known as “First law of entanglement thermodynamics”. This law states that for a subsystem A of the boundary field theory, the difference between the entanglement entropy of an excited state at a small non-zero temperature and the zero-temperature ground state is proportional to the change in the internal energy of the subsystem (A) as

$$\Delta S_A = \frac{1}{T_{ent}} \Delta E_A \tag{11}$$

$$\Delta S_A = S_A^{Temp \neq 0} - S_A^{Temp=0}, \quad \Delta E = \int_A d^d x \; (T_{tt}^{Temp \neq 0} - T_{tt}^{Temp=0}) \tag{12}$$

where, T_{tt} is the time component of the stress-energy tensor of boundary field theory which can be calculated in AdS_4/CFT_3 setup using the prescription given in [47]. The proportionality constant T_{ent} is known as entanglement temperature and was shown to be inversely related to the subsystem length (l) as

$$T_{ent} = c \, l^{-1} \tag{13}$$

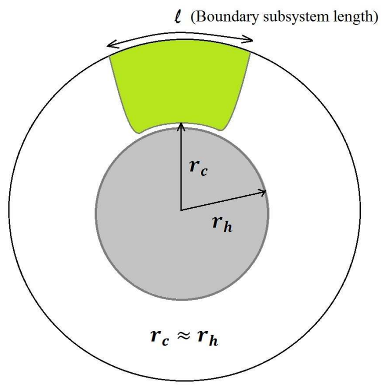

Figure 3: Schematic of extremal surface when the horizon radius is large

In [41] it was also shown that this "First law of thermodynamics" like relation for entanglement entropy remains valid at low temperatures and for small subsystem length (l). On the other hand in [40], the high temperature behavior (corresponding to the limit, $rh \to r_c$ as shown in fig.(3)) of the entanglement entropy for boundary field theory dual to Schwarzschild-AdS black hole was shown to follow the form given below

$$S_A = c_0 LlT^2 + TL(c_1 + c_2\epsilon) + O[\epsilon^2], \quad \epsilon \propto \exp(-\sqrt{3}Tl) \tag{14}$$

The first term in the expression of entanglement entropy at high temperature given by eq.(14) corresponds to the extensive thermal entropy of the subsystem (A) as it scales with the area of the subsystem in the $(2+1)$-dimensional conformal boundary and as volume in higher dimensional AdS_4 bulk spacetime. The ϵ corrections decrease exponentially with temperature and they scale as the length of the boundary of subsystem (A) and therefore correspond to entanglement between subsystem (A) and the rest of the system.

3 Entanglement entropy of charged planar black holes

In this section, using the Ryu-Takyanagi prescription we establish the framework for computing the entanglement entropy of a strip like region (referred to as subsystem (A)) in the boundary field theory which is dual to RN black holes in AdS_4/CFT_3 setup. The metric of Reissner Nordstrom black hole in AdS_4 spacetime with a planar horizon can be given as

$$ds^2 = -\frac{r^2}{L^2}f(r)dt^2 + \frac{L^2}{r^2 f(r)}dr^2 + \frac{r^2}{L^2}(dx^2 + dy^2), \tag{15}$$

$$f(r) = 1 - \frac{M}{r^3} + \frac{Q^2}{r^4} \tag{16}$$

whereas, the Hawking-temperature of planar RN black hole in AdS_r may be given as

$$T = \frac{f'(r)}{4\pi}\bigg|_{r=r_h} = \frac{3r_h}{4\pi}(1 - \frac{Q^2}{3r_h^4}) \tag{17}$$

Now using equations (5), (6) with the lapse function ($f(r)$) and metric given by the equations (16) and (15) the length and the area integral for the subsystem (A) of the boundary field theory dual to RN black hole in AdS_4 spacetime can be written down as

$$\frac{l}{2} = \int_{r_c}^{\infty} \frac{r_c^2 dr}{r^4\sqrt{(1-\frac{r_c^4}{r^4})}}\Big(1-\frac{M}{r^3}+\frac{Q^2}{r^4}\Big)^{-\frac{1}{2}}, \tag{18}$$

$$\mathcal{A} = 2L\int_{r_c}^{\infty} \frac{dr}{\sqrt{(1-\frac{r_c^4}{r^4})}}\Big(1-\frac{M}{r^3}+\frac{Q^2}{r^4}\Big)^{-\frac{1}{2}} \tag{19}$$

The equations (18) and (19) can then be used to determine analytically the entanglement entropy of the subsystem (A) of the $(2+1)$-dimensional boundary field theory. It may be seen from eq.(18) the subsystem length l can be obtained as a function of r_c. We then invert the relation between l and r_c to obtain r_c as a function of subsystem length l and substitute it in expression for the area of the extremal surface given by the eq.(19). This procedure determines the form of area of the extremal surface solely in terms of subsystem length l, black hole charge Q and the black hole mass M. However, it is to be noted that in the bulk theory there are only two parameters namely the charge Q and the mass M of the black hole which are related to each other by the radius of horizon (r_h) of the black hole as follows

$$f(r_h) = 0 \Rightarrow M = \frac{r_h^4 + Q^2}{r_h} \tag{20}$$

The condition $f(r_h) = 0$, implies that the lapse function vanishes at the horizon ($r = r_h$). The using the relation (20) we re-express the lapse function ($f(r)$) in terms of the radius of horizon r_h and the charge Q of the black hole as follows

$$f(r) = 1 - \frac{r_h^3}{r^3} - \frac{Q^2}{r^3 r_h} + \frac{Q^2}{r^4} \tag{21}$$

Thus with this reparametrization of the lapse function $f(r)$ it can be said that the bulk theory is now characterized effectively by the charge Q and the radius of horizon r_h. Next, in order to evaluate the integrals we make a change of variable from r to $u = \frac{r_c}{r}$ in expressions of subsystem length l and area $\mathcal{A}$ given by equations (18) and (19) which gives us the following modified forms of the integrals

$$l = \frac{2}{r_c}\int_0^1 \frac{u^2\left(1-\frac{rh^3u^3}{r_c{}^3}-\frac{Q^2u^3}{r_c{}^3rh}+\frac{Q^2u^4}{r_c{}^4}\right)^{-\frac{1}{2}}}{\sqrt{1-u^4}}du, \tag{22}$$

$$\mathcal{A} = 2Lr_c\int_0^1 \frac{\left(1-\frac{rh^3u^3}{r_c{}^3}-\frac{Q^2u^3}{r_c{}^3rh}+\frac{Q^2u^4}{r_c{}^4}\right)^{-\frac{1}{2}}}{u^2\sqrt{1-u^4}}du \tag{23}$$

The expression for subsystem length l and extremal area $\mathcal{A}$ given by equations (22) and (23) can then be used to compute the entanglement entropy for the subsystem A and study its behavior with temperature and charge of the RN black hole which we will address in later sections. Furthermore, it is also important to note that for a boundary field theory which is dual to charged black holes in bulk spacetime the state space depends on two parameters in the bulk namely the temperature T (which is related to radius of horizon r_h) and the charge Q of the black hole. Thus it becomes vital to consider the bulk theory of gravity (which is a charged black hole in AdS spacetime) in a particular ensemble. Here we choose to work in the canonical ensemble for which the charge Q of the black hole remains fixed. Unlike the case of boundary field theory dual to Schwarschild black hole in the AdS_4 bulk where the low and high temperature behavior of the entanglement entropy of subsystem A is controlled only by the temperature (T) of the black hole, here in the case of the boundary field theory dual to charged black hole in the AdS_4 bulk the low and high temperature of the entanglement entropy is also controlled by the charge Q of the black hole. This may be seen from the condition of extremality for RN black holes in AdS_4 space obtained from eq.(17) as

$$r_h \geq \frac{\sqrt{Q}}{3^{\frac{1}{4}}} \tag{24}$$

Where the equality is satisfied when the black hole is at zero temperature (i.e extremal) and non-equality stands for non zero temperature of the charged black hole. Therefore the second factor in the expression for temperature given by eq.(17) decides whether the black hole is close to extremality or not (at low or high temperatures). Hence this factor shows up at several places in our calculations of finite temperature entanglement entropy as we will show in the later sections. It is also to be noted that for the case of boundary field theory dual to charged black holes the ground state is dual to extremal black holes. Thus in order to obtain a "first law of entanglement thermodynamics" we also study the entanglement entropy of the boundary field theory dual to extremal black holes.

From the inequality in eq.(24), it may be observed that the horizon radius (r_h) is bounded from below by a quantity which proportional to the charge Q of the black hole. Thus the range of possible values for the horizon radius is decided by the value of charge Q. To say more precisely, if the charge of the black hole is small the radius of horizon (r_h) can assume both small (Low temperature regime) and large values (High temperature regime). However, if the charge of the black hole is large then r_h can only assume large values (Only high temperature regime can be explored). In contrast to the case of non-extremal black holes, the radius of horizon (r_h) for extremal black holes is directly related to the charge Q of the black hole which may be seen from the equality condition in eq.(24). Thus in the case of extremal black holes which are at zero temperature small or large charge will imply small or large radius of horizon (r_h) respectively. So, in the light of facts mentioned above in this section we will explore the behavior of entanglement entropy in different regimes of the value of the charge of the planar RN black hole in AdS_4 spacetime. To elucidate this further we will first explore the entanglement entropy of the subsystem A of the boundary field theory dual to extremal and the Non-extremal (charged) black holes in the small charge regime and then we go on to the large charge regime and do the same.

4 Small charge regime

In this section we explore the low and high temperature behavior of the entanglement entropy for the boundary subsystem A in the small charge regime for the non-extremal RN black holes in AdS_4 bulk spacetime. However, in case of boundary field theory dual to charged black holes the ground state is dual to the "extremal black hole". So, in the small charge regime we also study the entanglement entropy of subsystem A of the boundary field theory dual to extremal black holes. At low temperatures we show that it is possible to obtain a first law like relation for the boundary field theory dual to RN black holes in AdS_4 bulk using extremal black holes as the ground state.

4.1 Extremal black hole (Zero temperature)

The gauge/gravity duality says that for boundary field theories dual to charged black holes in the AdS bulk spacetime the ground state corresponds to "extremal black holes". Thus here we address the computation of entanglement entropy for the subsystem A of the boundary field theory dual to extremal extremal black holes in AdS_4 bulk spacetime. As pointed out earlier through eq.(17) that by solving $T = 0$, the horizon radius for extremal black holes may be obtained as

$$r_h = \frac{\sqrt{Q}}{3^{\frac{1}{4}}} \tag{25}$$

Thus in the case of extremal black holes the small charge limit also means small horizon radius ($l\frac{\sqrt{Q}}{3^{\frac{1}{4}}} \leq lr_h << 1$). If we put the extremality condition (25) in the expression of lapse function given by eq.(21) then for extremal black holes $f(r)$ takes the following form

$$f(r) = 1 - \frac{4r_h^3}{r^3} + \frac{3r_h^4}{r^4} \tag{26}$$

With the above form of lapse function the integral for the subsystem length l and the extremal area $\mathcal{A}$ given by equations (22) and (23)becomes

$$\begin{aligned} l &= \frac{2}{r_c}\int_0^1 \frac{u^2\left(1-\frac{4rh^3}{{r_c}^3}u^3+\frac{3rh^4}{{r_c}^4}u^4\right)^{-\frac{1}{2}}}{\sqrt{1-u^4}}du, & (27)\\ \mathcal{A} &= 2Lr_c\int_0^1 \frac{\left(1-\frac{4rh^3}{{r_c}^3}u^3+\frac{3rh^4}{{r_c}^4}u^4\right)^{-\frac{1}{2}}}{u^2\sqrt{1-u^4}}du. & (28)\end{aligned}$$

As horizon radius (r_h) is small, the black hole remains deep inside the bulk and therefore far away from the extremal surface i.e $r_h << r_c$. Thus in this limit, we Taylor expand the quantity $f(u)^{-1/2}$ around $\frac{r_h}{r_c} = 0$ and keep the non-vanishing terms up to $O[(\frac{r_h}{r_c})^3u^3]$ as

$$f(u)^{-\frac{1}{2}} \approx 1 + 2\frac{r_h^3}{r_c^3}u^3. \tag{29}$$

Using this approximation we evaluate the integral in eq.(27) to be as follows

$$l \approx \frac{2}{r_c}\int_0^1 \left(\frac{u^2}{\sqrt{1-u^4}} + \frac{2u^5(\frac{r_h}{r_c})^3}{\sqrt{1-u^4}}\right)du. \tag{30}$$

The relation in eq.(30) can be inverted to obtained r_c in terms of boundary subsystem length l as follows

$$r_c = \frac{\pi {r_h}^3}{2l{r_c}^3} + \frac{2\sqrt{\pi}\Gamma(\frac{3}{4})}{l\Gamma(\frac{1}{4})} + O[\frac{r_h^4}{r_c^4}]. \tag{31}$$

Solving the above equation perturbatively in terms of (r_hl) we obtain

$$r_c = \frac{1}{l}\left[\frac{2\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{l^3{r_h}^3\Gamma(\frac{1}{4})^3}{16\sqrt{\pi}\Gamma(\frac{3}{4})^3} + O[(r_hl)^4]\right] \tag{32}$$

Similarly, for obtaining an analytic expression for the extremal area we use the same approximated form of the quantity $f(u)^{-1/2}$ from eq.(29) in eq.(28) to obtain following form for the extremal area

$$\begin{aligned} \mathcal{A} &\approx 2Lr_c\int_0^1 \frac{1+2u^3(\frac{rh}{rc})^3}{u^2\sqrt{1-u^4}}du \\ &\approx 2Lr_c\left(\int_0^1 \frac{1}{u^2\sqrt{1-u^4}}du + \int_0^1 \frac{2u^3(\frac{rh}{rc})^3}{u^2\sqrt{1-u^4}}du\right) & (33)\end{aligned}$$

From the expression of $\mathcal{A}$ in eq.(33) it is observed that the first term is same as the pure AdS and is divergent. Therefore, we include the UV cutoff $1/r_b$ in the integral for $\mathcal{A}$ and add a counter term $(-2Lr_b)$ in order to obtain the finite part of the extremal area as

$$\mathcal{A}^{finite} \approx 2Lr_c\int_{\frac{r_c}{r_b}}^{1}\frac{1}{u^2\sqrt{1-u^4}}du - 2Lr_b + 2Lr_c\int_0^1\frac{2u(\frac{rh}{rc})^3}{\sqrt{1-u^4}}du$$

$$\approx Lr_c\left[\frac{\sqrt{\pi}\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{4})} + \frac{\pi {r_h}^3}{{r_c}^3}\right] \tag{34}$$

If we substitute for r_c from eq.(32)in eq.(34) and keep terms up to $O(r_h^3 l^3)$ then we get the following approximated form of the finite part of extremal area

$$\mathcal{A}^{finite} = \frac{L}{l}\left[\frac{\pi\Gamma(\frac{3}{4})\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{4})^2} + \frac{l^3{r_h}^3\Gamma(\frac{1}{4})^2(8\Gamma(\frac{3}{4})+\Gamma(-\frac{1}{4}))}{32\Gamma(\frac{3}{4})^3} + O[(r_h l)^4\right] \tag{35}$$

Since the renormalized entanglement entropy (S_A^{finite}) is related to the finite part of the extremal area as

$$S_A^{finite} = \frac{\mathcal{A}^{finite}}{4G_N^{(3+1)}} \tag{36}$$

The explicit expression for the entanglement entropy of subsystem A in a boundary theory dual to extremal black hole in small charge regime may be written down as follows

$$S_A^{finite} = \frac{1}{4G_N^{3+1}}\frac{L}{l}\left[-\frac{4\pi\Gamma(\frac{3}{4})^2}{\Gamma(\frac{1}{4})^2} + \frac{l^3{r_h}^3\Gamma(\frac{1}{4})^2}{8\Gamma(\frac{3}{4})^2} + O[(r_h l)^4\right] \tag{37}$$

For extremal black holes we also have, $r_h^3 = \frac{M^{ext}}{4}$ which follows from equations (20) and (25). Thus replacing r_h^3 by M^{ext} (Mass of the extremal black hole) in eq.(37) we obtain following form of renormalized entanglement entropy

$$S_A^{finite} \approx S_A^{AdS} + kM^{ext}Ll^2, \quad k = \frac{1}{4G_N^{3+1}}\frac{\Gamma(\frac{1}{4})^2}{32\Gamma(\frac{3}{4})^2}, \tag{38}$$

where, S_A^{AdS} is the entanglement entropy of the subsystem (A) when the bulk theory is pure AdS [40]. We see that when charge is small, the leading contribution to entanglement entropy of subsystem (A) in a boundary field theory dual to extremal black holes comes from AdS. We will see in later subsections how this sub-leading correction term in the above equation becomes important in defining the first law like relation.

4.2 Non-extremal black hole (low temperature)

We now consider the subsystem A of boundary field theory dual to non-extremal black hole in AdS_4 with small charge and at low temperature. One can see from the extremality bound in (24) that when temperature and charge both are small, the horizon radius is small. This implies that $Q/r_h^2 \sim 1$ and r_h is small such that, $r_h << r_c$ is satisfied. For a non-extremal black hole the form of the lapse function $f(u)$ may be given as

$$f(u) = 1 - (\frac{r_h}{r_c})^3u^3 - \frac{Q^2}{r_h^4}\left((\frac{r_h}{r_c})^3u^3 - (\frac{r_h}{r_c})^4u^4\right) \tag{39}$$

We define a new parameter $\alpha = \frac{Q^2}{r_h^4}$ to substitute for black hole charge Q in eq.(39) and then Taylor expand the quantity $f(u)^{-1/2}$ around $\frac{r_h}{r_c} = 0$ while keeping the non vanishing terms up to $O[(\frac{r_h}{r_c})^3u^3]$ as

$$f(u)^{-\frac{1}{2}} \approx 1 + \frac{1+\alpha}{2}(\frac{r_h}{r_c})^3u^3 \tag{40}$$

Using the approximated form of the lapse function given by eq.(40) in the integral (22) for subsystem length l we obtain

$$l \approx \frac{2}{r_c}\int_0^1 \frac{u^2}{\sqrt{1-u^4}}\Big(1+\frac{1+\alpha}{2}(\frac{r_h}{r_c})^3 u^3\Big) \tag{41}$$

The relation in eq.(41) can be inverted to obtained r_c in terms of boundary subsystem length l which is then solved perturbatively in terms of $(r_h l)$ to obtain the relation between r_c and l as follows

$$r_c = \frac{1}{l}\Big[\frac{2\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{(\pi\alpha+\pi)l^3 {r_h}^3\Gamma(\frac{1}{4})^3}{64\pi^{3/2}\Gamma(\frac{3}{4})^3} + O[(r_h l)^4]\Big]. \tag{42}$$

Similarly, Using the approximated form of the lapse function given by eq.(40)in eq.(22) we obtain the following form for the extremal area

$$\begin{aligned}\mathcal{A} &\approx 2Lr_c\int_0^1 \frac{1}{u^2\sqrt{1-u^4}}\Big(1+\frac{1}{2}(\frac{r_h}{r_c})^3 u^3(1+\alpha)\Big)\\ &\approx 2Lr_c\Big[\int_0^1 \frac{1}{u^2\sqrt{1-u^4}} + \int_0^1 \frac{1+\alpha}{u^2\sqrt{1-u^4}}\Big(\frac{r_h^3 u^3}{2r_c^3}\Big)\Big]\end{aligned} \tag{43}$$

The first term in the integral (43) for extremal area is divergent and is same as the entanglement entropy of subsystem A of the boundary field theory dual to bulk AdS_4 space time. Therefore, regularizing the integral for $\mathcal{A}$ in the same way as done for the extremal black hole case in the previous subsection, we obtain the finite part of the extremal area as

$$\mathcal{A}^{finite} \approx L\Big[\frac{r_c\sqrt{\pi}\Gamma(-\frac{1}{4})}{2\Gamma(\frac{1}{4})} + \frac{\pi {r_h}^3}{4{r_c}^2} + \frac{\pi\alpha {r_h}^3}{4{r_c}^2}\Big] \tag{44}$$

Substituting the expression for r_c from eq.(42) in eq.(44) and keeping terms up to $O(r_h^3)$ the finite part of the extremal area and the entanglement entropy may be written down as

$$\mathcal{A}^{finite} = -\frac{4L\pi\Gamma(\frac{1}{4})^2}{l\Gamma(\frac{3}{4})^2} + \frac{l^2 L{r_h}^3\Gamma(\frac{1}{4})^2}{32\Gamma(\frac{3}{4})^2} + \frac{\alpha l^2 L{r_h}^3\Gamma(\frac{1}{4})^2}{32\Gamma(\frac{3}{4})^2} + O(r_h^4 l^3), \tag{45}$$

$$S_A^{finite} = \frac{1}{4G}\frac{L}{l}\Big[-\frac{4\pi\Gamma(\frac{1}{4})^2}{\Gamma(\frac{3}{4})^2} + \frac{{r_h}^3 l^3\Gamma(\frac{1}{4})^2}{32\Gamma(\frac{3}{4})^2} + \frac{\alpha {r_h}^3 l^3\Gamma(\frac{1}{4})^2}{32\Gamma(\frac{3}{4})^2} + O(r_h^4 l^4)\Big] \tag{46}$$

For non-extremal RN black holes in AdS_4 we also have a constraint relation between radius of horizon (r_h), black hole charge (Q) and mass (M) which follows from eq.(20) as

$$r_h^3(1+\alpha) = r_h^3(1+\frac{Q^2}{r_h^4}) = M \tag{47}$$

Using the constraint relation given by eq.(47) in the expression of entanglement entropy in eq.(46) we obtain

$$S_A^{finite} \approx S_A^{AdS} + kMLl^2, \quad k = \frac{1}{4G}\frac{\Gamma(\frac{1}{4})^2}{32\Gamma(\frac{3}{4})^2} \tag{48}$$

We see that when charge and temperature both are small, the leading contribution in the entanglement entropy for the case of non-extremal black holes comes from AdS just like for the case of extremal black holes.

4.3 Entanglement thermodynamics of charged black holes

From AdS/CFT correspondence it may be observed that when there is a Reissner-Nordstrom black hole present in the bulk then one has to consider extremal black hole as dual to the ground state (zero temperature state) of the boundary field theory. Thus in order to obtain the "first law of entanglement thermodynamics" we subtract equation (38) from (48) to obtain

$$\Delta S_A = \frac{1}{T_{ent}} \Delta E_A, \tag{49}$$

where,

$$\Delta S_A = S_A - S_A^{ext},$$

$$\Delta E_A = \int_A dxdy\ T_{tt}^{Temp\neq 0} - \int_A dxdy\ T_{tt}^{Temp=0} = \frac{Ll}{8\pi G_N}(M - M^{ext}),$$

$$T_{ent} = \pi \frac{\Gamma(\frac{3}{4})^2}{16\Gamma(\frac{1}{4})^2} \frac{1}{l}, \tag{50}$$

here, T_{ent} is known as the entanglement temperature which matches with the Schwarzschild case. This kind of first law of entaglement thermodynamics for boundary field theories dual to charged black holes in the bulk was also derived in [45]. There the authors have considered the grand canonical ensemble with the pure AdS as the ground-state. This differs from our results as we have considered the canonical ensemble for which the ground state of the boundary field theory is dual to the extremal AdS black hole in the bulk. Furthermore, the first law like relation given by eq.(49) for the entanglement entropy may be extended to include a work term due to pressure and volume as studied in [42]. However, here we are only interested in studying the dependence of the quantity ΔS_A on the quantity ΔE_A as studied in [41]

4.4 Non-extremal black hole (high temperature)

In this section we explore the high temperature behavior of entanglement entropy when the black hole in the bulk has small charge. From the extremality bound one can see that when the temperature is high and charge is small, the horizon radius is very large ($r_h l >> 1$). As a result the quantity $\frac{Q}{\sqrt{3}r_h^2} << 1$. We will call this quantity $\delta = \frac{Q}{\sqrt{3}r_h^2}$ and Taylor expand $f(u)^{-\frac{1}{2}}$ around $\delta = 0$.

$$f(u)^{-\frac{1}{2}} \approx \frac{1}{\sqrt{1 - \frac{{r_h}^3 u^3}{{r_c}^3}}} + \frac{3}{2}(\frac{r_h}{r_c})^3 \frac{\delta^2 u^3 (1 - \frac{r_h u}{r_c})}{(1 - \frac{{r_h}^3 u^3}{{r_c}^3})^{3/2})} \tag{51}$$

Using the approximated form of the lapse function given by eq.(51) in the integral (22) for subsystem length l we obtain

$$l = \frac{2}{r_c} \int_0^1 \frac{u^2}{\sqrt{1-u^4}} \left(\frac{1}{\sqrt{1 - \frac{{r_h}^3 u^3}{{r_c}^3}}} + \frac{3\delta^2}{2}(\frac{r_h}{r_c})^3 \frac{u^3 (1 - \frac{r_h u}{r_c})}{(1 - \frac{{r_h}^3 u^3}{{r_c}^3})^{3/2})} \right) \tag{52}$$

One can use the following expansions to simplify the above integral

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} x^n, \qquad \frac{1}{(1-x)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} \frac{2\Gamma(n+\frac{3}{2})}{\sqrt{\pi}\Gamma(n+1)} x^n \tag{53}$$

After using these two expansions eq.(52) becomes

$$\frac{lr_c}{2}=\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)}(\frac{r_h}{r_c})^{3n}\int_0^1\frac{u^{3n+2}}{\sqrt{1-u^4}}+\delta^2\sum_{n=0}^{\infty}\frac{3\Gamma(n+\frac{3}{2})}{\sqrt{\pi}\Gamma(n+1)}(\frac{r_h}{r_c})^{3n+3}\int_0^1\frac{u^{3n+5}(1-\frac{r_h}{r_c}u)}{\sqrt{1-u^4}} \tag{54}$$

These integrals can now be evaluated through well known identity in terms of Gamma functions

$$\int_0^1 x^{\mu-1}(1-x^{\lambda})^{\nu-1}=\frac{B(\frac{\mu}{\lambda},\nu)}{\lambda},\ \ where\ B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{55}$$

Using this identity, we evaluate integrals in eq.(52)

$$\begin{aligned} lr_c = &\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{2\Gamma(n+1)}\frac{\Gamma(\frac{3n+3}{4})}{\Gamma(\frac{3n+5}{4})}(\frac{r_h}{r_c})^{3n}+\frac{3\delta^2}{2}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{3n+6}{4})}{\Gamma(\frac{3n+8}{4})}(\frac{r_h}{r_c})^{3n+3}\\ &-\frac{3\delta^2}{2}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{3n+7}{4})}{\Gamma(\frac{3n+9}{4})}(\frac{r_h}{r_c})^{3n+4} \end{aligned} \tag{56}$$

The first series goes as $\sim \frac{x^n}{n}$ for large n and the other two series go as $\sim x^n$. Isolating the divergent terms, we see that the divergences of the last two series cancel and we get

$$\begin{aligned} lr_c = &\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}+\sum_{n=1}^{\infty}\left(\frac{\Gamma(n+\frac{1}{2})}{2\Gamma(n+1)}\frac{\Gamma(\frac{3n+3}{4})}{\Gamma(\frac{3n+5}{4})}-\frac{1}{\sqrt{3}n}\right)(\frac{r_h}{r_c})^{3n}\\ &+\frac{3\delta^2}{2}\sum_{n=0}^{\infty}\left(\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{3n+6}{4})}{\Gamma(\frac{3n+8}{4})}-\frac{2}{\sqrt{3}}\right)(\frac{r_h}{r_c})^{3n+3}\\ &-\frac{3\delta^2}{2}\sum_{n=0}^{\infty}\left(\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{3n+7}{4})}{\Gamma(\frac{3n+9}{4})}-\frac{2}{\sqrt{3}}\right)(\frac{r_h}{r_c})^{3n+4}\\ &+\sqrt{3}\delta^2\frac{(\frac{r_h}{r_c})^3}{(1+\frac{r_h}{r_c}+(\frac{r_h}{r_c})^2)}-\frac{1}{\sqrt{3}}\log[1-(\frac{r_h}{r_c})^3] \end{aligned} \tag{57}$$

As observed in [39] extremal surface can never penetrate horizon in an asymptomatically-AdS static black hole background, which implies that r_c is always greater than r_h. As the horizon radius is large for the case being studied, r_h approaches very close to the extremal surface, $r_h \sim r_c$. Therefore substituting $r_c = r_h(1+\epsilon)$, we expand the above equation in ϵ and isolate the leading term in ϵ

$$lr_h = -\frac{1}{\sqrt{3}}\log[3\epsilon]+c_1+\delta^2c_2+O[\epsilon] \tag{58}$$

$$c_1 = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}+\sum_{n=1}^{\infty}\left(\frac{\Gamma(n+\frac{1}{2})}{2\Gamma(n+1)}\frac{\Gamma(\frac{3n+3}{4})}{\Gamma(\frac{3n+5}{4})}-\frac{1}{\sqrt{3}n}\right)$$

$$c_2 = \frac{1}{\sqrt{3}}-\frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{3n+6}{4})}{\Gamma(\frac{3n+8}{4})}-\frac{2}{\sqrt{3}}\right)+\frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{3n+7}{4})}{\Gamma(\frac{3n+9}{4})}-\frac{2}{\sqrt{3}}\right)$$

Where

$$\epsilon=\varepsilon_{ent}e^{-\sqrt{3}(lr_h-\delta^2c_2)},\ \ \varepsilon_{ent}=\frac{1}{3}e^{\sqrt{3}c_1} \tag{59}$$

Using the approximated form of the lapse function given by eq.(51) in the integral (23) for extremal surface area $\mathcal{A}$ we obtain

$$\mathcal{A} = 2Lr_c \int_0^1 \frac{u^2}{\sqrt{1-u^4}} \left(\frac{1}{\sqrt{1-\frac{{r_h}^3 u^3}{{r_c}^3}}} + \frac{3\delta^2}{2} (\frac{r_h}{r_c})^3 \frac{u^3(1-\frac{r_h u}{r_c})}{(1-\frac{{r_h}^3 u^3}{{r_c}^3})^{3/2})} \right) \tag{60}$$

One can see in the above equation that only the first term has the divergence and it is same as the Pure AdS divergence

$$\mathcal{A}^{finite} = 2Lr_c \int_{\frac{r_c}{r_b}}^1 \frac{u^2}{\sqrt{1-u^4}} \left(\frac{1}{\sqrt{1-\frac{{r_h}^3 u^3}{{r_c}^3}}} + \frac{3\delta^2}{2} (\frac{r_h}{r_c})^3 \frac{u^3(1-\frac{r_h u}{r_c})}{(1-\frac{{r_h}^3 u^3}{{r_c}^3})^{3/2})} \right) - 2Lr_b \tag{61}$$

We can evaluate the above integrals after using the expansions given in (53)

$$\begin{aligned} \mathcal{A}^{finite} = 2Lr_c \Big[& \sqrt{\pi}\frac{\Gamma(-\frac{1}{4})}{4\Gamma(\frac{1}{4})} + \sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} (\frac{r_h}{r_c})^{3n} \int_0^1 \frac{u^{3n-2}}{\sqrt{1-u^4}} \\ & + \delta^2 \sum_{n=0}^{\infty} \frac{3\Gamma(n+\frac{3}{2})}{\sqrt{\pi}\Gamma(n+1)} (\frac{r_h}{r_c})^{3n+3} \int_0^1 \frac{u^{3n+1}(1-\frac{r_h}{r_c}u)}{\sqrt{1-u^4}} \Big] \end{aligned} \tag{62}$$

We can now evaluate the integrals using the identity in eq.(55) to obtain

$$\begin{aligned} \mathcal{A}^{finite} = \; & Lr_c \Big[\sqrt{\pi}\frac{\Gamma(-\frac{1}{4})}{2\Gamma(\frac{1}{4})} + \sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{2\Gamma(n+1)} \frac{\Gamma(\frac{3n-1}{4})}{\Gamma(\frac{3n+1}{4})} (\frac{r_h}{r_c})^{3n} \\ & + \frac{3\delta^2}{2} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{3n+2}{4})}{\Gamma(\frac{3n+4}{4})} (\frac{r_h}{r_c})^{3n+3} - \frac{3\delta^2}{2} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{3n+3}{4})}{\Gamma(\frac{3n+5}{4})} (\frac{r_h}{r_c})^{3n+4} \Big] \end{aligned} \tag{63}$$

We use the identity $\Gamma(n+1) = n\Gamma(n)$ and write the above equation in the following form

$$\begin{aligned} \mathcal{A}^{finite} = \; & Lr_c \Big[\sqrt{\pi}\frac{\Gamma(-\frac{1}{4})}{2\Gamma(\frac{1}{4})} + \sum_{n=1}^{\infty} \left(1 + \frac{2}{3n-1}\right) \frac{\Gamma(n+\frac{1}{2})}{2\Gamma(n+1)} \frac{\Gamma(\frac{3n+3}{4})}{\Gamma(\frac{3n+5}{4})} (\frac{r_h}{r_c})^{3n} \\ & + \frac{3\delta^2}{2} \sum_{n=0}^{\infty} \left(1 + \frac{2}{3n+2}\right) \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{3n+6}{4})}{\Gamma(\frac{3n+8}{4})} (\frac{r_h}{r_c})^{3n+3} \\ & - \frac{3\delta^2}{2} \sum_{n=0}^{\infty} \left(1 + \frac{2}{3n+3}\right) \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{3n+7}{4})}{\Gamma(\frac{3n+9}{4})} (\frac{r_h}{r_c})^{3n+4} \Big] \end{aligned} \tag{64}$$

We see that we can now use equation (54) and simplify the above equation

$$\begin{aligned} \mathcal{A}^{finite} = \; & Lr_c \Big[\sqrt{\pi}\frac{\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{4})} + lr_c + \sum_{n=1}^{\infty} \left(\frac{2}{3n-1}\right) \frac{\Gamma(n+\frac{1}{2})}{2\Gamma(n+1)} \frac{\Gamma(\frac{3n+3}{4})}{\Gamma(\frac{3n+5}{4})} (\frac{r_h}{r_c})^{3n} \\ & + \frac{3\delta^2}{2} \sum_{n=0}^{\infty} \left(\frac{2}{3n+2}\right) \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{3n+6}{4})}{\Gamma(\frac{3n+8}{4})} (\frac{r_h}{r_c})^{3n+3} - \frac{3\delta^2}{2} \sum_{n=0}^{\infty} \left(\frac{2}{3n+3}\right) \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{3n+7}{4})}{\Gamma(\frac{3n+9}{4})} (\frac{r_h}{r_c})^{3n+4} \Big] \end{aligned}$$

(65)

We see that the first series in the above equation goes as $\sim \frac{x^n}{n^2}$ for large n, where as the last two go as $\sim \frac{x^n}{n}$. All of these series diverge at the $O[\epsilon]$

$$
\begin{aligned}
\mathcal{A}^{finite} &= Lr_c\Bigg[\sqrt{\pi}\frac{\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{4})} + lr_c + \sum_{n=1}^{\infty}\left(\frac{1}{3n-1}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{3n+3}{4})}{\Gamma(\frac{3n+5}{4})} - \frac{2}{3\sqrt{3}n^2}\right)(\frac{r_h}{r_c})^{3n} \\
&+ 3\delta^2\sum_{n=1}^{\infty}\left(\frac{1}{3n+2}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{3n+6}{4})}{\Gamma(\frac{3n+8}{4})} - \frac{2}{3\sqrt{3}n}\right)(\frac{r_h}{r_c})^{3n+3} \\
&- 3\delta^2\sum_{n=1}^{\infty}\left(\frac{1}{3n+3}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{3n+7}{4})}{\Gamma(\frac{3n+9}{4})} - \frac{2}{3\sqrt{3}n}\right)(\frac{r_h}{r_c})^{3n+4} \qquad (66)
\end{aligned}
$$

$$
\begin{aligned}
&+ 3\delta^2\left((\frac{r_h}{r_c})^3\frac{(\Gamma(\frac{3}{2}))^2}{2} - (\frac{r_h}{r_c})^4\frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{4})}{3\Gamma(\frac{9}{4})}\right) \\
&+ \frac{2}{3\sqrt{3}}Li_2[(\frac{r_h}{r_c})^3] - \frac{2\delta^2}{\sqrt{3}}(\frac{r_h}{r_c})^3(1-\frac{r_h}{r_c})\log[1-(\frac{r_h}{r_c})^3]\Bigg]
\end{aligned}
$$

If we now put $r_c = r_h(1+\epsilon)$ and now expand up to $O[\epsilon]$. Then we obtain

$$
\mathcal{A}^{finite} = Llr_h^2 + Lr_h(k_1+\delta^2 k_2) + Lr_h\epsilon\Big[k_3 + \delta^2(k_4 + k_5\log[\epsilon])\Big] \qquad (67)
$$

$$
\begin{aligned}
k_1 &= \frac{\pi^2}{9\sqrt{3}} + \frac{\sqrt{\pi}\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{4})} + \sum_{n=1}^{\infty}\left(\frac{1}{3n-1}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{3n+3}{4})}{\Gamma(\frac{3n+5}{4})} - \frac{2}{3\sqrt{3}n^2}\right) \\
k_3 &= \frac{\pi^2}{9\sqrt{3}} - \frac{3\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \\
k_4 &= \frac{3\sqrt{\pi}\Gamma(\frac{7}{4})}{\Gamma(\frac{9}{4})} - \frac{\log(9)}{\sqrt{3}} \\
k_5 &= \frac{4}{\sqrt{3}} - 2\sqrt{3} \qquad (68)
\end{aligned}
$$

$$
\begin{aligned}
k_2 = \frac{3\pi}{8} - \frac{3\sqrt{\pi}\Gamma(\frac{7}{4})}{2\Gamma(\frac{9}{4})} &+ 3\sum_{n=1}^{\infty}\left(\frac{1}{3n+2}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{3n+6}{4})}{\Gamma(\frac{3n+8}{4})} - \frac{1}{3\sqrt{3}n}\right) \\
&- 3\sum_{n=1}^{\infty}\left(\frac{2}{3n+3}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{3n+7}{4})}{\Gamma(\frac{3n+9}{4})} - \frac{2}{3\sqrt{3}n}\right)
\end{aligned}
$$

Therefore the renormalized entanglement entropy($S_A^{finite} = \frac{\mathcal{A}^{finite}}{4G}$) of non-extremal black hole at high temperature in the small charge regime is as follows

$$
S_A^{finite} \approx LlS_{BH} + \frac{L}{4G}r_h(k_1+\delta^2 k_2) + \frac{L}{4G}r_h\epsilon\Big[k_3 + \delta^2(k_4 + k_5\log[\epsilon])\Big] \qquad (69)
$$

Where $S_{BH} = \frac{r_h^2}{4G_N}$ is the entropy density of the planar black hole.

$$
\epsilon \approx \varepsilon_{ent} e^{-\frac{4\pi}{\sqrt{3}}Tl(1+\delta^2)} \qquad (70)
$$

We observe that the first term in the eq(69) scales with the area of the subsystem in 2+1 dimensional CFT and increases with temperature. Therefore we deduce that it corresponds to the thermal part. The second and latter terms are proportional to the length of the boundary separating subsystem(A) and its compliment. Therefore they contain the information about the entanglement at high temperatures. The ϵ corrections decay exponentially with temperature just as they did for Schwarschild case, except that there is a small δ^2 term due to the presence of charge.

5 Large charge regime

In this section we explore the behavior of entanglement entropy in the large charge regime of the black hole in the bulk. The extremality bound given in (24) shows that when the charge is large, horizon radius is large too ($r_h l >> 1$). So we evaluate the leading contribution to the extremal surface area by expanding f(u) near $u_0 = \frac{r_c}{r_h}$, which in terms of r coordinates means a near horizon expansion. This kind of expansion was also done to evaluate the entanglement entropy for charged black holes in [43].

5.1 Extremal black hole

Now we evaluate the entanglement entropy for the extremal black hole when the charge is large. In order for the above discussed near horizon expansion to hold, we have to show that the term $u - u_0$ is small, so that we can neglect the higher order terms. In the integral for area, u goes from $\frac{r_c}{r_b}$ to 1. For large charge case $r_c \sim r_h$ as a result $u_0 \sim 1$. Since r_c and r_b are both large, u is close to u_0 through out the integral and our near-horizon expansion is valid. Taylor expanding f(u) around $u_0 = \frac{r_c}{r_h}$ gives the following form of the lapse function

$$f(u) = 6(\frac{r_h}{r_c})^2(u - u_0)^2 + O[(u - u_0)^3] \tag{71}$$

$$f(u) \approx 6(1 - \frac{r_h}{r_c}u)^2 \tag{72}$$

Using the approximated form of the lapse function given by eq.(72) in the integral (22) for subsystem length l, we obtain

$$l = \frac{2}{r_c}\int_0^1 \frac{u^2}{\sqrt{1-u^4}}\frac{du}{\sqrt{6}(1-\frac{r_h}{r_c}u)} \tag{73}$$

As already discussed in section 4.4, r_c will always remain greater than r_h and $\frac{r_h}{r_c}u < 1$. We expand$(1 - \frac{r_h}{r_c}u)^{-1}$ binomially to evaluate the integral .

$$\frac{lr_c}{2} = \frac{1}{\sqrt{6}}(\frac{r_h}{r_c})^n\int_0^1 \frac{u^{2+n}}{\sqrt{1-u^4}}du \tag{74}$$

$$lr_c = \sum_{n=0}^{\infty}\sqrt{\frac{\pi}{6}}\frac{\Gamma(\frac{n+3}{4})}{2\Gamma(\frac{n+5}{4})}(\frac{r_h}{r_c})^n \tag{75}$$

We see that the series goes like $\sim \frac{x^n}{\sqrt{n}}$ for large n and diverges as $r_c \to r_h$. So we isolate the divergent term.

$$lr_c = \sqrt{\frac{\pi}{6}}\frac{\Gamma(\frac{3}{4})}{2\Gamma(\frac{5}{4})} + \sqrt{\frac{\pi}{6}}\sum_{n=1}^{\infty}\left(\frac{\Gamma(\frac{n+3}{4})}{2\Gamma(\frac{n+5}{4})} - \frac{1}{\sqrt{n}}\right)(\frac{r_h}{r_c})^n + \sqrt{\frac{\pi}{6}}Li_{\frac{1}{2}}[\frac{r_h}{r_c}] \tag{76}$$

Where Li is the polylog function. We use $r_c = r_h(1+\epsilon)$, do an expansion in ϵ keep the leading term

$$lr_h = \sqrt{\frac{\pi}{6}}\frac{\Gamma(\frac{3}{4})}{2\Gamma(\frac{5}{4})} + \sqrt{\frac{\pi}{6}}\sum_{n=1}^{\infty}\left(\frac{\Gamma(\frac{n+3}{4})}{2\Gamma(\frac{n+5}{4})} - \frac{1}{\sqrt{n}}\right) + \sqrt{\frac{\pi}{6}}[\zeta\left(\frac{1}{2}\right) + \frac{\sqrt{\pi}}{\sqrt{\epsilon}}] + O[\epsilon^{\frac{1}{2}}] \tag{77}$$

Where $\zeta(x)$ is the zeta function

$$\frac{1}{\sqrt{\epsilon}} = \frac{\sqrt{6}}{\pi}\, lr_h - \frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{3}{4})}{2\Gamma(\frac{5}{4})} - \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\left(\frac{\Gamma(\frac{n+3}{4})}{2\Gamma(\frac{n+5}{4})} - \frac{1}{\sqrt{n}}\right) - \frac{1}{\sqrt{\pi}}\zeta(\frac{1}{2}) \tag{78}$$

$$\begin{aligned} \epsilon &= \frac{\pi^2}{6(lr_h - k_l)^2} \\ k_l &= \sqrt{\frac{\pi}{6}}\left[\frac{\Gamma(\frac{3}{4})}{2\Gamma(\frac{5}{4})} + \sum_{n=1}^{\infty}\left(\frac{\Gamma(\frac{n+3}{4})}{2\Gamma(\frac{n+5}{4})} - \frac{1}{\sqrt{n}}\right) + \zeta(\frac{1}{2})\right] \end{aligned} \tag{79}$$

Using the approximated form of the lapse function given by eq.(72) in the integral (23) for extremal surface area $\mathcal{A}$, we obtain

$$\mathcal{A} = 2Lr_c\int_0^1 \frac{1}{u^2\sqrt{1-u^4}}\frac{1}{\sqrt{6}(1-\frac{r_h}{r_c}u)}du \tag{80}$$

We expand$(1-\frac{r_h}{r_c}u)^{-1}$ binomially to evaluate the above integral using the identity in eq.(55)

$$\mathcal{A} = 2Lr_c\sum_{n=0}^{\infty}\int_0^1 \frac{u^{n-2}}{\sqrt{1-u^4}}(\frac{r_h}{r_c})^n \tag{81}$$

The terms corresponding to $n=0$ and $n=1$ in the above equation seem to be divergent. Both of them can be handled by introducing a cut-off (r_b). Finite part of the $n=0$ term is given by

$$\mathcal{A}_0^{finite} = \frac{2Lr_c}{\sqrt{6}}\int_{\frac{r_c}{r_b}}^1 \frac{u^{-2}}{\sqrt{1-u^4}} - \frac{2Lr_b}{\sqrt{6}} = 2Lr_c\left[\frac{\sqrt{\frac{\pi}{6}}\Gamma(-\frac{1}{4})}{4\Gamma(\frac{1}{4})}\right] \tag{82}$$

The term corresponding to $n=1$ in eq.(81) is

$$\mathcal{A}_1 = 2Lr_h\int_0^1 \frac{u^{-1}}{\sqrt{1-u^4}} \tag{83}$$

$$= 2Lr_h\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)}\int_0^1 u^{-1+4k} \tag{84}$$

We see that $k=0$ term has to be regulated.

$$\mathcal{A}_1 = \frac{2Lr_h}{\sqrt{6}}\int_{\frac{r_c}{r_b}}^1 \frac{1}{u} + \frac{2Lr_h}{\sqrt{6}}\sum_{k=1}^{\infty}\frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)}\int_0^1 u^{-1+4k} \tag{85}$$

$$= \frac{2Lr_h}{\sqrt{6}}\left[-\log[\frac{r_c}{r_b}] + \frac{\log[4]}{4}\right] \tag{86}$$

We know that $r_c \sim r_h$ and r_h is large. Since r_c and r_b both are large, the quantity $-\log[\frac{r_c}{r_b}] \sim 0$. Therefore this term can be ignored.

$$\mathcal{A}_1^{finite} \approx \frac{2Lr_h}{\sqrt{6}}\left[\frac{\log[4]}{4}\right] \tag{87}$$

Substituting the finite parts of $n=0$ and $n=1$ terms given by eq.(82) and eq.(87) in eq.(81) we get the finite part of the area of extremal surface

$$\mathcal{A}^{finite} = \frac{2Lr_c}{\sqrt{6}}\left[\frac{\sqrt{\pi}\Gamma(-\frac{1}{4})}{4\Gamma(\frac{1}{4})} + \frac{r_h}{r_c}\frac{\log[4]}{4} + \sum_{n=2}^{\infty}\frac{\sqrt{\pi}\Gamma(\frac{n-1}{4})}{4\Gamma(\frac{n+1}{4})}(\frac{r_h}{r_c})^n\right] \tag{88}$$

The series in the above expression goes as $\sim \frac{x^n}{\sqrt{n}}$ for large n and diverges as $r_c \rightarrow r_h$. This can be slightly re-arranged using the identity $\Gamma(n+1) = n\Gamma(n)$ to avoid divergence and get the leading contribution.

$$\mathcal{A}^{finite} = \frac{2Lr_c}{\sqrt{6}}\left[\frac{\sqrt{\pi}\Gamma(-\frac{1}{4})}{4\Gamma(\frac{1}{4})} + \frac{r_h}{r_c}\frac{\log[4]}{4} + \sum_{n=2}^{\infty}(1+\frac{2}{n-1})\frac{\sqrt{\pi}\Gamma(\frac{n+3}{4})}{4\Gamma(\frac{n+5}{4})}(\frac{r_h}{r_c})^n\right] \tag{89}$$

We can now use eq.(75) for the first term in the series of the above equation and this gives us

$$\mathcal{A}^{finite} = \frac{2Lr_c}{\sqrt{6}}\left[-2\frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{r_h}{r_c}\frac{\log[4]}{4} + \sqrt{6}\frac{lr_c}{2} - \sqrt{\pi}\frac{1}{4\Gamma(\frac{3}{2})} + \sum_{n=2}^{\infty}(\frac{1}{n-1})\frac{\sqrt{\pi}\Gamma(\frac{n+3}{4})}{2\Gamma(\frac{n+5}{4})}(\frac{r_h}{r_c})^n\right] \tag{90}$$

The series is now convergent as it goes as $\sim \frac{x^n}{n\sqrt{n}}$ for large n. So the leading term is given by just putting $r_c = r_h$

$$\mathcal{A}^{finite} = Llr_h^2 + Lr_hC \tag{91}$$

Where

$$C = \frac{2}{\sqrt{6}}\left[-2\frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{\log[4]}{4} - \frac{1}{2} + \sum_{n=2}^{\infty}(\frac{1}{n-1})\frac{\sqrt{\pi}\Gamma(\frac{n+3}{4})}{2\Gamma(\frac{n+5}{4})}\right] \tag{92}$$

To find the sub-leading term we put $r_c = r_h(1+\epsilon)$ and expand the finite part of the area in ϵ, keeping the terms up to $O[\epsilon]$. Even though the series in equation(90) is finite at the leading order, it is divergent at the sub-leading order and hence we have to isolate the divergent term.

$$\begin{aligned}\mathcal{A}^{finite} &= \frac{2L}{\sqrt{6}}\Bigg[-2\frac{\sqrt{\pi}\Gamma(\frac{3}{4})r_c}{\Gamma(\frac{1}{4})} + r_h(\frac{\log[4]}{4}) + \sqrt{6}\frac{lr_c^2}{2} \\ &- \sqrt{\pi}\frac{r_c}{4\Gamma(\frac{3}{2})} + \frac{\sqrt{\pi}r_c}{2}\sum_{n=2}^{\infty}(\frac{1}{n-1}\frac{\Gamma(\frac{n+3}{4})}{\Gamma(\frac{n+5}{4})} - \frac{2}{n\sqrt{n}})(\frac{r_h}{r_c})^n - \sqrt{\pi}r_h + \sqrt{\pi}r_cLi_{\frac{3}{2}}[\frac{r_h}{r_c}]\Bigg]\end{aligned}$$

Expanding the above equation in ϵ and simplifying, we obtain

$$\begin{aligned}\mathcal{A}^{finite} &= Llr_h^2 + Lr_h(K_1 + K_2\sqrt{\epsilon} + K_3\epsilon + O[\epsilon^{\frac{3}{2}}]) \qquad (93)\\ K_1 &= \frac{2}{\sqrt{6}}\left[-2\frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{\log[4]}{4} - \frac{1+2\sqrt{\pi}}{2} + \sqrt{\pi}\zeta\left(\frac{3}{2}\right) + \frac{\sqrt{\pi}}{2}\sum_{n=2}^{\infty}(\frac{1}{n-1}\frac{\Gamma(\frac{n+3}{4})}{\Gamma(\frac{n+5}{4})} - \frac{2}{n\sqrt{n}})\right] \\ K_2 &= -\frac{2\pi}{\sqrt{6}} \\ K_3 &= \frac{2}{\sqrt{6}}\left[\frac{1}{2} - \sqrt{\pi} + \sqrt{\pi}\zeta\left(\frac{3}{2}\right)\right]\end{aligned}$$

$$\epsilon = \frac{\pi^2}{6(lr_h - k_l)^2} = \frac{\sqrt{3}\pi^2}{6(l\sqrt{Q} - 3^{\frac{1}{4}}k_l)^2} \tag{94}$$

Therefore the renormalized entanglement entropy of the extremal charged black hole in the large charge regime is given by

$$S_A^{finite} = LlS_{BH} + \frac{Lr_h}{4G}(K_1 + K_2\sqrt{\epsilon} + K_3\epsilon + O[\epsilon^{\frac{3}{2}}]) \tag{95}$$

The first term in the above equation scales with area of the subsystem and is extensive. Since the black hole is at zero temperature the entire contribution comes from charge.The ϵ corrections does not decrease exponentially with charge in this case but in a power law form as given by (94) .

5.2 Non-extremal black hole

In this section we explore the large charge regime of the non-extremal black hole. Extremality condition puts a bound on horizon radius[$r_h > \frac{\sqrt{Q}}{3^{\frac{1}{4}}}$]. Therefore when charge is large horizon radius is large too ($r_h l >> 1$). The entire argument done in previous section for the extremal black hole with large charge goes through for non-extremal case also. So we again do the near horizon expansion for f(u) i.e around $u_0 = \frac{r_c}{r_h}$ and calculate the leading contribution for the area integral.

$$f(u) = (-3 + \frac{Q^2}{r_h^4})\frac{r_h}{r_c}(u - u_0) + O[(u - u_0)^2] \tag{96}$$

$$f(u) \approx (3 - \frac{Q^2}{r_h^4})(1 - \frac{r_h}{r_c}u) \tag{97}$$

Note that the prefactor appearing in the above equation is the same that comes in temperature. We call this factor $\delta = (3 - \frac{Q^2}{r_h^4})$. When $\delta \to 0$ temperature is low and when $\delta \to 3$ temperature is high. Using the approximated form of the lapse function given by eq.(97) in the integral (22) for subsystem length l, we obtain

$$l = \frac{2}{r_c\sqrt{\delta}} \int_0^1 \frac{u^2}{\sqrt{1-u^4}} \frac{1}{\sqrt{1 - \frac{r_h}{r_c}u}} \tag{98}$$

Using the expansion given in eq.(53) with $x = \frac{r_h}{r_c}u$, the integral can be evaluated

$$lr_c = \frac{1}{2\sqrt{\delta}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{n+3}{4})}{\Gamma(\frac{n+5}{4})} (\frac{r_h}{r_c})^n \tag{99}$$

The series in the above equation goes as $\sim \frac{x^n}{n}$ and therefore diverges as $r_c \to r_h$. We isolate the divergent term. As the horizon radius is large because of the large charge, we put $r_c = r_h(1+\epsilon)$, expand in ϵ to obtain the leading term. We get

$$\sqrt{\delta} l r_h = -\log[\epsilon] + k + O[\epsilon]$$

$$\epsilon = \varepsilon_{ent} e^{-\sqrt{\delta} l r_h} \tag{100}$$

Where

$$\varepsilon_{ent} = e^k, \qquad k = \frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{2\Gamma(\frac{5}{4})} + \sum_{n=1}^{\infty} \left(\frac{\Gamma(n+\frac{1}{2})}{2\Gamma(n+1)} \frac{\Gamma(\frac{n+3}{4})}{\Gamma(\frac{n+5}{4})} - \frac{1}{n} \right)$$

Using the approximated form of the lapse function given by eq.(97) in the integral (23) for extremal surface area $\mathcal{A}$, we obtain

$$\mathcal{A} = \frac{2Lr_c}{\sqrt{\delta}} \int_0^1 \frac{1}{u^2\sqrt{1-u^4}} \frac{1}{\sqrt{1 - \frac{r_h}{r_c}u}} \tag{101}$$

Using the expansion for $\frac{1}{\sqrt{1-x}}$ given in eq.(53) with $x = \frac{r_h}{r_c}u$

$$\mathcal{A} = \frac{2Lr_c}{\sqrt{\delta}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} \int_0^1 \frac{u^{n-2}}{\sqrt{1-u^4}} (\frac{r_h}{r_c})^n \tag{102}$$

We see that terms corresponding to $n = 0$ and $n = 1$ are divergent. We isolate the divergences of these terms and regulate them with the cutoff (r_b). First the $n = 0$ term

$$\mathcal{A}_0 = \frac{2Lr_c}{\sqrt{\delta}} \int_0^1 \frac{1}{u^2\sqrt{1-u^4}} du \tag{103}$$

$$\mathcal{A}_0 = \frac{2Lr_c}{\sqrt{\delta}} \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)} \int_0^1 u^{-2+4k} du$$

Divergence is in the term corresponding to $k = 0$ and has to be regulated with cutoff r_b .

$$\mathcal{A}_0 = \frac{2Lr_c}{\sqrt{\delta}} \int_{\frac{r_c}{r_b}}^1 u^{-2} du + \frac{2Lr_c}{\sqrt{\delta}} \sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)} \frac{1}{4k-1} \tag{104}$$

The finite part of the above expression is given by

$$\mathcal{A}_0^{finite} = \mathcal{A}_0 - \frac{2Lr_b}{\sqrt{\delta}} = \frac{2Lr_c}{\sqrt{\delta}} \Big[- \frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \Big] \tag{105}$$

Now consider the term corresponding to n=1 in eq.(102)

$$\begin{aligned} \mathcal{A}_1 &= \frac{Lr_h}{\sqrt{\delta}} \int_0^1 \frac{u^{-1}}{\sqrt{1-u^4}} du \\ &= \frac{Lr_h}{\sqrt{\delta}} \Big[\int_{\frac{r_c}{r_b}}^1 \frac{1}{u} + \sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{1}{2}}{\sqrt{\pi}\Gamma(k+1)} \frac{1}{4k} \Big] \\ &= \frac{Lr_h}{\sqrt{\delta}} \Big[\log[\frac{r_b}{r_c}] + \frac{\log[4]}{4} \Big] \end{aligned} \tag{106}$$

Just like large charge case of extremal black hole, the quantity $\log[\frac{r_b}{r_c}] \sim 0$. This is because both r_b and r_c are both large. Therefore we ignore it.

$$\mathcal{A}_1^{finite} = \frac{Lr_h}{\sqrt{\delta}} \Big[\frac{\log[4]}{4} \Big] \tag{107}$$

Substituting the finite parts of the terms corresponding to $n = 0$ and $n = 1$ as given by (105) and eq.(107) in eq.(102) we obtain the finite part of the area to be

$$\mathcal{A}^{finite} = \mathcal{A}_0^{finite} + \mathcal{A}_1^{finite} + \frac{2Lr_c}{\sqrt{\delta}} \sum_{n=2}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} \int_0^1 \frac{u^{n-2}}{\sqrt{1-u^4}} (\frac{r_h}{r_c})^n \tag{108}$$

$$= \frac{2Lr_c}{\sqrt{\delta}} \Big[- \frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{r_h}{2r_c} (\frac{\log[4]}{4}) + \frac{1}{4} \sum_{n=2}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{n-1}{4})}{\Gamma(\frac{n+1}{4})} (\frac{r_h}{r_c})^n \Big] \tag{109}$$

Using $\Gamma(n+1) = n\Gamma(n)$, we can write the above equation as follows

$$\mathcal{A}^{finite} = \frac{2Lr_c}{\sqrt{\delta}} \Big[- \frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{r_h}{2r_c} (\frac{\log[4]}{4}) + \frac{1}{4} \sum_{n=2}^{\infty} (1 + \frac{2}{n-1}) \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{n+3}{4})}{\Gamma(\frac{n+5}{4})} (\frac{r_h}{r_c})^n \Big] \tag{110}$$

We can now use eq.(99) for the first term of the series in the above equation and simplify to get

$$\mathcal{A}^{finite} = \frac{2Lr_c}{\sqrt{\delta}} \Big[- \frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{r_h}{2r_c} (\frac{\log[4]}{4}) + \frac{lr_c\sqrt{\delta}}{2} - \frac{\sqrt{\pi}\Gamma(\frac{3}{4}}{4\Gamma(\frac{5}{4})} - \frac{1}{4} \tag{111}$$

$$+ \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n-1} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{n+3}{4})}{\Gamma(\frac{n+5}{4})} (\frac{r_h}{r_c})^n \Big] \tag{112}$$

The series in the above equation goes as $\sim \frac{x^n}{n^2}$ for large n and hence converges as $r_c \to r_h$. Therefore the leading term can be found just by putting $r_c = r_h$.

$$\mathcal{A}^{finite} = Llr_h^2 + \frac{Lr_h}{\sqrt{\delta}}C \tag{113}$$

Where

$$C = -\frac{4\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{\log[4]-2}{4} + \sum_{n=2}^{\infty}\frac{1}{n-1})\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{n+3}{4})}{\Gamma(\frac{n+5}{4})} \tag{114}$$

The series in eq.(112) is convergent up to the leading term but diverges if we consider $O[\epsilon]$ terms. In order to obtain the area up to $O[\epsilon]$, we isolate the divergent part in eq.(112) .

$$\mathcal{A}^{finite} = \frac{2L}{\sqrt{\delta}}\Bigg[-\frac{2r_c\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + r_h\frac{\log[4]}{8} + \frac{lr_c^2\sqrt{\delta}}{2} - \frac{r_c}{4} + \tag{115}$$

$$\frac{r_c}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{n+3}{4})}{\Gamma(\frac{n+5}{4})} - \frac{2}{n^2}\right)(\frac{r_h}{r_c})^n - r_h + r_c\, Li_2[\frac{r_h}{r_c}]\Bigg] \tag{116}$$

Now we put $r_c = r_h(1+\epsilon)$, expand in ϵ and keep the terms up to $O[\epsilon]$. The finite part of the area is given by

$$\begin{aligned}
\mathcal{A}^{finite} &= Llr_h^2 + \frac{2Lr_h}{\sqrt{\delta}}\Big[K_1 + K_2\epsilon + O[\epsilon^2]\Big] \\
K_1 &= -\frac{2\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{\log[4]-10}{8} + \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{\Gamma(\frac{n+3}{4})}{\Gamma(\frac{n+5}{4})} - \frac{2}{n^2}\right) + \frac{\pi^2}{6} \\
K_2 &= \frac{\pi^2}{6} - \frac{7}{4}
\end{aligned}$$

The renormalized entanglement entropy($S_A^{finite} = \frac{\mathcal{A}^{finite}}{4G}$) for the nonextremal black hole in the large charge regime, therefore comes out to be as follows

$$S_A^{finite} = LlS_{BH} + \frac{Lr_h}{2G\sqrt{\delta}}\Big[K_1 + K_2\epsilon + O[\epsilon^2]\Big] \tag{117}$$

$$\epsilon = \varepsilon_{ent}e^{-\sqrt{\delta}lr_h} = \varepsilon_{ent}e^{-\frac{4\pi Tl}{\sqrt{\delta}}} \tag{118}$$

The first term scales with the area of the subsystem and is extensive. $\delta < 3$ as the quantity $\frac{Q^2}{r_h^4}$ is always positive. Therefore ϵ corrections are high when the temperature is low and they decay exponentially with temperature when the temperature is high.

6 Summary and Conclusions

In summary we have studied the entanglement entropy of a strip like region denoted by subsystem A in the boundary field theory dual to charged black holes in AdS_4/CFT_3 setup. Here, we have focused mainly on two aspects of the holographic entanglement entropy: first is the behavior of the entanglement entropy with the charge of the black hole and then its temperature dependence. For this we have obtained an analytic expression for the holographic entanglement entropy using the analytic techniques adopted in [40]. In the small charge regime we have obtained the low and high temperature behavior of the holographic entanglement entropy in the case of Reissner-Nordstrom black hole in the AdS_4 bulk spacetime. As stated earlier that when there is a Reissner-Nordstrom black hole present in the bulk then one has to consider extremal black hole as dual to the ground state (zero temperature state) of the boundary field theory. Thus in the small charge and low temperature regime we have obtained "First law of entanglement thermodynamics" in the canonical ensemble with extremal AdS black hole as the ground state of the boundary field theory. This differs from the work in [45] where the authors established the first law of entanglement thermodynamics for boundary field theory dual to charged black holes in the grand canonical ensemble with the pure AdS as the ground state. We have also studied the holographic entanglement entropy in the large charge regime in the case of both non-extremal and extremal black holes in the AdS_4 bulk spacetime. Specifically, the large charge of the black hole in the bulk forces the boundary field theory to have a high temperature in the case of non-extremal black hole whereas, in the case of extremal black hole large charge of the black hole implies large radius of horizon. We also establish that at high temperatures the holographic entanglement entropy shows a characteristic exponential dependence on the temperature T and the extremality parameter $\delta = Q/\sqrt{3}r_h^2$ both in the small and the large charge regimes. This exponential dependence of holographic entanglement entropy on temperature was also established in [40] for the case of Schwarschild black hole in the the AdS bulk spacetime.

Our work leads to extremely interesting future directions for investigations. One possible avenues for this is to to investigate the holographic entanglement entropy for the boundary field theories dual to rotating and charged rotating black holes in four or higher dimensions in the bulk. It would be also interesting to extend the analytic approach adopted here for the case of boundary field theories dual to charged "Gauss-Bonnet" black holes in the bulk and study the temperature dependence of holographic entanglement entropy.

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