Variable Besov spaces: continuous version

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January 5, 2016

Abstract

We introduce Besov spaces with variable smoothness and integrability by using the continuous version of Calderòn reproducing formula. We show that our space is well-defined, i.e., independent of the choice of basis functions. We characterize these function spaces by so-called Peetre maximal functions and we obtain the Sobolev embeddings for these function spaces. We use these results to prove the atomic decomposition for these spaces.

MSC 2010: 46E35.

Key words and phrases. Atom, Embeddings, Besov space, variable exponent.

1 Introduction

Function spaces play an important role in harmonic analysis, in the theory of differential equations and in almost every other field of applied mathematics. Some of these function spaces are Besov spaces. The theory of these spaces has been developed in detail in [35] and [36] (and continued and extended in the more recent monographs [37] and [38]), but has a longer history already including many contributors; we do not want to discuss this here. For general literature on function spaces we refer to [1, 5, 19, 28, 34, 39] and references therein.

Based on continuous characterizations of Besov spaces, we introduce new family of function spaces of variable smoothness and integrability. These type of function spaces, initially appeared in the paper of A. Almeida and P. Hästö [4], where several basic properties were shown, such as the Fourier analytical characterisation, Sobolev embeddings and the characterization in terms of Nikolskij representations involving sequences of entire analytic functions. Later, [13] characterized these spaces by local means and established the atomic characterization. Afterwards, Kempka and Vybíral [24] characterized these spaces by ball means of differences and also by local means, see [27] and [29] for the duality of $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ spaces.

The paper is organized as follows. First we give some preliminaries where we fix some notations and recall some basics facts on function spaces with variable integrability and we give some key technical lemmas needed in the proofs of the main statements. We then define the Besov spaces $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. We prove an useful characterization of these

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spaces based on the so called local means. The theorem on local means that proved for Besov spaces of variable smoothness and integrability is highly technical and its proved required new techniques and ideas. Using the results from Sections 3 and 4, we prove in Section 5 the atomic decomposition for $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

2 Preliminaries

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. The expression $f \lesssim g$ means that $f \leq c\, g$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$. As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x.

By supp$f$ we denote the support of the function f , i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function.

The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$. We denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. The Fourier transform of a Schwartz function f is denoted by $\mathcal{F}f$ while its inverse transform is denoted by $\mathcal{F}^{-1}f$.

By c we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g. $c(p)$ means that c depends on p, etc.). Further notation will be properly introduced whenever needed.
The variable exponents that we consider are always measurable functions p on $\mathbb{R}^n$ with range in $[c, \infty[$ for some $c > 0$. We denote the set of such functions by $\mathcal{P}_0$. The subset of variable exponents with range $[1, \infty[$ is denoted by $\mathcal{P}$. We use the standard notation $p^- := \operatorname*{ess\text{-}inf}_{x\in\mathbb{R}^n} p(x)$, $\quad p^+ := \operatorname*{ess\text{-}sup}_{x\in\mathbb{R}^n} p(x)$.
The variable exponent modular is defined by $\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \varrho_{p(x)}(|f(x)|)dx$, where $\varrho_p(t) = t^p$. The variable exponent Lebesgue space $L^{p(\cdot)}$ consists of measurable functions f on $\mathbb{R}^n$ such that $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg (quasi)-norm on this space by the formula $\|f\|_{p(\cdot)} := \inf\Big\{\lambda > 0 : \varrho_{p(\cdot)}\Big(\frac{f}{\lambda}\Big) \leq 1\Big\}$. A useful property is that $\|f\|_{p(\cdot)} \leq 1$ if and only if $\varrho_{p(\cdot)}(f) \leq 1$, see [11], Lemma 3.2.4.
Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}_{>}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^{q(\cdot)}_{>}(L^{p(\cdot)})}((f_v)_v) := \sum_{v=1}^{\infty} \inf\Big\{\lambda_v > 0 : \varrho_{p(\cdot)}\Big(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\Big) \leq 1\Big\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}_{>}(L^{p(\cdot)})} := \inf\Big\{\mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(\frac{1}{\mu}(f_v)_v\Big) \leq 1\Big\}. \qquad (1)$$

If $q^+ < \infty$, then we can replace (1) by the simpler expression $\varrho_{\ell_>^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_{v=1}^{\infty} \left\| |f_v|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}$. The case $p := \infty$ can be included by replacing the last modular by $\varrho_{\ell_>^{q(\cdot)}(L^\infty)}((f_v)_v) := \sum_{v=1}^{\infty} \left\| |f_v|^{q(\cdot)} \right\|_\infty$.

We say that $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous*, abbreviated $g \in C_{\rm loc}^{\log}(\mathbb{R}^n)$, if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x-y|)} \tag{2}$$

for all $x, y \in \mathbb{R}^n$. We say that g satisfies the log-*Hölder decay condition*, if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\log} > 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is *globally*-log-*Hölder continuous*, abbreviated $g \in C^{\log}$, if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The constants $c_{\log}(g)$ and $c_{\log}$ are called the *locally* log-*Hölder constant* and the log-*Hölder decay constant*, respectively. We note that all functions $g \in C_{\rm loc}^{\log}(\mathbb{R}^n)$ always belong to L^∞.

We define the following class of variable exponents

$$\mathcal{P}^{\log}(\mathbb{R}^n) := \left\{ p \in \mathcal{P} : \frac{1}{p} \in C^{\log} \right\},$$

were introduced in [12, Section 2]. We define $1/p_\infty := \lim_{|x|\to\infty} 1/p(x)$ and we use the convention $\frac{1}{\infty} = 0$. Note that although $\frac{1}{p}$ is bounded, the variable exponent p itself can be unbounded. Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $\varphi \in L^1$ and $\Psi(x) := \sup_{|y| \geq |x|} |\varphi(y)|$. We suppose that $\Psi \in L^1$. Then it was proved in [11, Lemma 4.6.3] that

$$\|\varphi_\varepsilon * f\|_{p(\cdot)} \leq c \|\Psi\|_1 \|f\|_{p(\cdot)}$$

for all $f \in L^{p(\cdot)}$, where $\varphi_\varepsilon := \frac{1}{\varepsilon^n} \varphi\left(\frac{\cdot}{\varepsilon}\right)$, $\varepsilon > 0$. We refer to the papers [7] and [9], where various results on maximal function on variable Lebesgue spaces were obtained.

Recall that $\eta_{t,m}(x) := t^{-n}(1 + t^{-1}|x|)^{-m}$, for any $x \in \mathbb{R}^n$, $t > 0$ and $m > 0$. Note that $\eta_{t,m} \in L^1$ when $m > n$ and that $\|\eta_{t,m}\|_1 = c_m$ is independent of t, where this type of function was introduced in [20] and [11]. We refer to the recent monograph [8] for further properties, historical remarks and references on variable exponent spaces.

2.1 Some technical lemmas

In this subsection we present some results which are useful for us. The following lemma is from [10, Lemma 6.1], see also [24, Lemma 19].

Lemma 1 *Let $\alpha \in C_{\rm loc}^{\log}(\mathbb{R}^n)$, $m \in \mathbb{N}_0$ and let $R \geq c_{\log}(\alpha)$, where $c_{\log}(\alpha)$ is the constant from* (2) *for α. Then*

$$t^{-\alpha(x)} \eta_{t,m+R}(x-y) \leq c\, t^{-\alpha(y)} \eta_{t,m}(x-y)$$

for any $0 < t \leq 1$ with $c > 0$ independent of $x, y \in \mathbb{R}^n$ and t.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all, namely we can move the term inside the convolution as follows:

$$t^{-\alpha(x)}\eta_{t,m+R} * f(x) \leq c\ \eta_{t,m} * (t^{-\alpha(\cdot)}f)(x).$$

Lemma 2 *Let $r, N > 0$, $m > n$ and $\theta, \omega \in \mathcal{S}(\mathbb{R}^n)$ with $\mathrm{supp}\mathcal{F}\omega \subset \overline{B(0,1)}$. Then there exists $c = c(r,m,n) > 0$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$, we have*

$$|\theta_N * \omega_N * g(x)| \leq c(\eta_{N,m} * |\omega_N * g|^r(x))^{1/r}, \quad x \in \mathbb{R}^n, \tag{3}$$

where $\theta_N(\cdot) = N^n\theta(N\cdot)$, $\omega_N(\cdot) = N^n\omega(N\cdot)$ and $\eta_{N,m} := N^n(1+N|\cdot|)^{-m}$.

The proof of this lemma is given in [15] by using the same arguments of [33, Chapter V, Theorem 5]. The next three lemmas are from [10] where the first tells us that in most circumstances two convolutions are as good as one.

Lemma 3 *For $v_0, v_1 \in \mathbb{N}_0$ and $m > n$, we have*

$$\eta_{v_0,m} * \eta_{v_1,m} \approx \eta_{\min(v_0,v_1),m}$$

with the constant depending only on m and n.

For $v \in \mathbb{N}_0$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$, let $Q_{v,m}$ be the dyadic cube in $\mathbb{R}^n$, $Q_{v,m} = \{(x_1, ..., x_n) : m_i \leq 2^v x_i < m_i + 1, i = 1, 2, ..., n\}$. For the collection of all such cubes we use $\mathcal{Q} = \{Q_{v,m} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. For each cube Q, we denote by $x_{Q_{v,m}}$ the lower left-corner $2^{-v}m$ of $Q = Q_{v,m}$, its side length by $l(Q)$.

Lemma 4 *Let $v \in \mathbb{N}_0$ and $m > n$. Then for any $Q \in \mathcal{Q}$ with $l(Q) = 2^{-v}$, $y \in Q$ and $x \in \mathbb{R}^n$, we have*

$$\eta_{v,m} * \left(\frac{\chi_Q}{|Q|}\right)(x) \approx \eta_{v,m}(x-y)$$

with the constant depending only on m and n.

The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 5 *Let $0 < a < 1$, $\sigma \geq 0$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_k$ be a sequences of positive real numbers and denote $\delta_k = \sum_{j=1}^{\infty} |k-j|^\sigma a^{|k-j|}\varepsilon_j$. Then there exists constant $c > 0$ depending only on a and q such that*

$$\Big(\sum_{k=1}^{\infty} \delta_k^q\Big)^{1/q} \leq c\ \Big(\sum_{k=1}^{\infty} \varepsilon_k^q\Big)^{1/q}.$$

We will make use of the following statement, see [12], Lemma 3.3 for $w := 1$.

Lemma 6 *Let $p \in \mathcal{P}^{\log}$ and $w : \mathbb{R}^n \to \mathbb{R}^+$ be a weight function. Then for every $m > 0$ there exists $\gamma = e^{-2m/c_{\log}(1/p)} \in (0,1)$ such that*

$$\begin{aligned} &\Big(\frac{\gamma}{w(Q)}\int_Q |f(y)|\, w(y)dy\Big)^{p(x)} \\ \leq\ & \max\Big(1, (w(Q))^{1-\frac{p(x)}{p_Q^-}}\Big)\frac{1}{w(Q)}\int_Q |f(y)|^{p(y)}\, w(y)dy \\ & + \min\left(|Q|^m, 1\right)\Big(\frac{1}{w(Q)}\int_Q \left((e+|x|)^{-m} + (e+|y|)^{-m}\right) w(y)dy\Big), \end{aligned}$$

for every cube (or ball) $Q \subset \mathbb{R}^n$, $0 < w(Q) < \infty$, all $x \in Q \subset \mathbb{R}^n$ and all $f \in L^{p(\cdot)}(w) + L^\infty$ with $\left\|fw^{1/p(\cdot)}\right\|_{p(\cdot)} + \|f\|_\infty \leq 1$.

Notice that in the proof of this theorem we need only that

$$\int_Q |f(y)|^{p(y)} w(y)dy \le 1$$

and/or $\|f\|_\infty \le 1$. The proof of this lemma is postponed to the appendix.
The next two lemmas are the continuous version of Hardy-type inequality, where the second lemma for constant exponents is from [25].

Lemma 7 *Let $\alpha : \mathbb{R}^n \to \mathbb{R}, p \in \mathcal{P}(\mathbb{R}^n)$ and $q \in \mathcal{P}(\mathbb{R})$, with $1 \le q^+ < \infty$. Let $\{f_v\}_{v\in\mathbb{N}}$ be a sequence of measurable functions on $\mathbb{R}^n$. For all $x \in \mathbb{R}^n$, $v \in \mathbb{N}$ and all $\delta > 0$, let $g_v(x) = \sum_{k=1}^{\infty} 2^{-|k-v|\delta} f_k(x)$. Then there exists a positive constant c, independent of $\{f_v\}_{v\in\mathbb{N}}$ such that*

$$\Big\| \Big(t^{-\frac{1}{q(t)}} \|g_v\|_{p(\cdot)} \chi_{[2^{-v},2^{1-v}]} \Big)_v \Big\|_{\ell_{>}^{q(\cdot)}(L^{q(\cdot)})} \le c \Big\| \Big(t^{-\frac{1}{q(t)}} \|f_v\|_{p(\cdot)} \chi_{[2^{-v},2^{1-v}]} \Big)_v \Big\|_{\ell_{>}^{q(\cdot)}(L^{q(\cdot)})}.$$

Proof. By the scaling argument, we see that it suffices to consider when the right-hand side is less than or equal to 1. Observe that

$$\begin{aligned} &\sum_{v=1}^{\infty} \int_{2^{-v}}^{2^{1-v}} \Big\| \sum_{k=1}^{\infty} 2^{-|k-v|\delta} f_k \Big\|_{p(\cdot)}^{q(t)} \frac{dt}{t} \\ \le\ & \sum_{v=1}^{\infty} \int_0^2 \Big(\sum_{k=1}^{\infty} 2^{-|k-v|\delta} \|f_k\|_{p(\cdot)} \Big)^{q(t)} \chi_{[2^{-v},2^{1-v}]}(t) \frac{dt}{t}. \end{aligned} \tag{4}$$

We divide the second sum in two parts, $\sum_{k=1}^{v} \cdots$ and $\sum_{k=v+1}^{\infty} \cdots$. By Hölder's inequality,

$$\Big(\sum_{k=1}^{v} 2^{(k-v)\delta} \|f_k\|_{p(\cdot)} \Big)^{q(t)} \lesssim \sum_{k=1}^{v} 2^{(k-v)\delta/q^+} \|f_k\|_{p(\cdot)}^{q(t)}$$

for any $t \in [2^{-v}, 2^{1-v}]$ and any $v \in \mathbb{N}$. We have

$$\|f_k\|_{p(\cdot)}^{q(t)} = ((v-k+1)\log 2)^{q(t)} \Big[\frac{1}{(v-k+1)(\log 2)^2} \int_{2^{-v}}^{2^{1-k}} \|f_k\|_{p(\cdot)} \chi_{[2^{-k},2^{1-k}]}(\tau) \frac{d\tau}{\tau} \Big]^{q(t)}$$

for any $v \ge k$. Applying Lemma 6, we get

$$\begin{aligned} \|f_k\|_{p(\cdot)}^{q(t)} \ \lesssim\ & (v-k+1)^{q(t)-1} \int_0^2 \|f_k\|_{p(\cdot)}^{q(\tau)} \chi_{[2^{-k},2^{1-k}]}(\tau) \frac{d\tau}{\tau} \\ & + c\ (v-k+1)^{q(t)} 2^{-km} \end{aligned}$$

for any $t \in [2^{-v}, 2^{1-v}] \subset [2^{-v}, 2^{1-k}]$, where $c, m > 0$ are independent of k and t. By Lemma 5, (4), with $\sum_{k=1}^{v}$ in place of $\sum_{k=1}^{\infty}$, is bounded by

$$\begin{aligned} & c \sum_{v=1}^{\infty} \int_0^2 \sum_{k=1}^{v} \|f_k\|_{p(\cdot)}^{q(\tau)} \chi_{[2^{-k},2^{1-k}]}(\tau) 2^{(k-v)\delta/q^+} (v-k+1)^{q^+-1} \frac{d\tau}{\tau} + c \\ \lesssim\ & \sum_{k=1}^{\infty} \int_{2^{-k}}^{2^{1-k}} \|f_k\|_{p(\cdot)}^{q(\tau)} \frac{d\tau}{\tau} + c \lesssim 1. \end{aligned}$$

Obviously, (4) with $\sum_{k=v+1}^{\infty}$ in place of $\sum_{k=1}^{\infty}$, is

$$\sum_{v=1}^{\infty}\int_{0}^{2}\Big(\sum_{k=v+1}^{\infty}2^{(v-k)\delta}\left\|f_{k}\right\|_{p(\cdot)}\Big)^{q(t)}\chi_{[2^{-v},2^{1-v}]}(t)\frac{dt}{t}. \tag{5}$$

Again, Hölder's inequality gives

$$\Big(\sum_{k=v+1}^{\infty}2^{(v-k)\delta}\left\|f_{k}\right\|_{p(\cdot)}\Big)^{q(t)}\lesssim\sum_{k=v+1}^{\infty}2^{(v-k)\delta/q^{+}}\left\|f_{k}\right\|_{p(\cdot)}^{q(t)}.$$

We have

$$\begin{aligned}&\left\|f_{k}\right\|_{p(\cdot)}^{q(t)}\\ =\;&((k-v+1)\log 2)^{q(t)}\Big[\frac{1}{(k-v+1)\left(\log 2\right)^{2}}\int_{2^{-k}}^{2^{1-v}}\left\|f_{k}\right\|_{p(\cdot)}\chi_{[2^{-k},2^{1-k}]}(\tau)\frac{d\tau}{\tau}\Big]^{q(t)}\end{aligned}$$

for any $k\geq v+1$. Applying again Lemma 6, we get

$$\begin{aligned}\left\|f_{k}\right\|_{p(\cdot)}^{q(t)}\;\lesssim\;&(k-v+1)^{q(t)-1}\int_{2^{-k}}^{2^{1-v}}\left\|f_{k}\right\|_{p(\cdot)}^{q(\tau)}\chi_{[2^{-k},2^{1-k}]}(\tau)\frac{d\tau}{\tau}\\ &+c\,(k-v+1)^{q(t)}2^{-vm}\end{aligned}$$

for any $t\in[2^{-v},2^{1-v}]\subset[2^{-k},2^{1-v}]$, where $c,m>0$ are independent of k and t. Hence (5) is bounded by

$$\begin{aligned}&c\sum_{v=1}^{\infty}\int_{0}^{2}\sum_{k=v+1}^{\infty}\left\|f_{k}\right\|_{p(\cdot)}^{q(\tau)}\chi_{[2^{-k},2^{1-k}]}(\tau)2^{(v-k)\delta/q^{+}}(k-v+1)^{q^{+}-1}\frac{d\tau}{\tau}+c\\ \lesssim\;&\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{1-k}}\left\|f_{k}\right\|_{p(\cdot)}^{q(\tau)}\frac{d\tau}{\tau}+c\lesssim 1,\end{aligned}$$

by Lemma 5. The proof is complete. ■

Lemma 8 *Let $s>0$ and $q\in\mathcal{P}^{\log}(\mathbb{R})$ with $1\leq q^{-}\leq q^{+}<\infty$. Let $\{\varepsilon_{t}\}_{t}$ be a sequence of positive, measurable functions. Let*

$$\eta_{t}=t^{s}\int_{t}^{1}\tau^{-s}\varepsilon_{\tau}\frac{d\tau}{\tau}\quad\text{and}\quad\delta_{t}=t^{-s}\int_{0}^{t}\tau^{s}\varepsilon_{\tau}\frac{d\tau}{\tau}.$$

Then there exists constant $c>0$ depending only on s, q^{-}, $c_{\log}(q)$ and q^{+} such that

$$\left\|\eta_{t}\right\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})}+\left\|\delta_{t}\right\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})}\leq c\left\|\varepsilon_{t}\right\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})}.$$

Proof. We suppose that $\left\|\varepsilon_{t}\right\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})}\leq 1$. Notice that

$$\left\|\eta_{t}\right\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})}\approx\Big\|\Big(t^{-\frac{1}{q(t)}}\eta_{t}\chi_{[2^{-v},2^{1-v}]}\Big)_{v}\Big\|_{\ell_{>}^{q(\cdot)}(L^{q(\cdot)})}.$$

We see that

$$\int_{2^{-v}}^{1} \tau^{-s} \varepsilon_\tau \frac{d\tau}{\tau} = \sum_{i=0}^{v-1} \int_{2^{i-v}}^{2^{i-v+1}} \tau^{-s} \varepsilon_\tau \frac{d\tau}{\tau} = \sum_{j=1}^{v} \int_{2^{-j}}^{2^{1-j}} \tau^{-s} \varepsilon_\tau \frac{d\tau}{\tau}.$$

Let $\sigma > 0$, $\beta \in \mathbb{R}$ such that $q^+ < \sigma$ and $0 < \beta < s$. We have

$$\begin{aligned}
\Big(\sum_{j=1}^{v} \int_{2^{-j}}^{2^{1-j}} \tau^{-s} \varepsilon_\tau \frac{d\tau}{\tau}\Big)^{q(t)/\sigma} &\leq \sum_{j=1}^{v} \Big(\int_{2^{-j}}^{2^{1-j}} \tau^{-s} \varepsilon_\tau \frac{d\tau}{\tau}\Big)^{q(t)/\sigma} \\
&\leq \sum_{j=1}^{v} 2^{j(s-\beta)q(t)/\sigma} \Big(\int_{2^{-j}}^{2^{1-j}} \tau^{-\beta} \varepsilon_\tau \frac{d\tau}{\tau}\Big)^{q(t)/\sigma} \\
&= 2^{v(s-\beta)q(t)/\sigma} \sum_{j=1}^{v} 2^{(j-v)(s-\beta)q(t)/\sigma} \Big(\int_{2^{-j}}^{2^{1-j}} \tau^{-\beta} \varepsilon_\tau \frac{d\tau}{\tau}\Big)^{q(t)/\sigma}.
\end{aligned}$$

By Hölder's inequality, we estimate this expression by

$$2^{v(s-\beta)q(t)/\sigma} \Big(\sum_{j=1}^{v} 2^{(j-v)(s-\beta)q(t)/\sigma} \Big(\int_{2^{-j}}^{2^{1-j}} \tau^{-\beta} \varepsilon_\tau \frac{d\tau}{\tau}\Big)^{q(t)}\Big)^{1/\sigma}.$$

Observe that

$$\Big(\int_{2^{-j}}^{2^{1-j}} \tau^{-\beta} \varepsilon_\tau \frac{d\tau}{\tau}\Big)^{q(t)} \leq 2^{j\beta q(t)} \Big(\int_{2^{-j}}^{2^{1-j}} \varepsilon_\tau \frac{d\tau}{\tau}\Big)^{q(t)}.$$

By Lemma 6 we get

$$\begin{aligned}
&\Big(\frac{1}{(v-j+1)\log 2} \int_{2^{-v}}^{2^{1-j}} \varepsilon_\tau \chi_{[2^{-j},2^{1-j}]}(\tau) \frac{d\tau}{\tau}\Big)^{q(t)} \\
\lesssim\ & \frac{1}{v-j+1} \int_{2^{-v}}^{2^{1-j}} \varepsilon_\tau^{q(\tau)} \chi_{[2^{-j},2^{1-j}]}(\tau) \frac{d\tau}{\tau} + 2^{-jm} \\
\lesssim\ & \frac{2^{-j\beta q(t)}}{v-j+1} \int_{2^{-v}}^{2^{1-j}} \tau^{-\beta q(\tau)} \varepsilon_\tau^{q(\tau)} \chi_{[2^{-j},2^{1-j}]}(\tau) \frac{d\tau}{\tau} + 2^{-jm}
\end{aligned}$$

for any $v \geq j$ and any $t \in [2^{-v}, 2^{1-v}] \subset [2^{-v}, 2^{1-j}]$, where in the last inequality we use the fact that

$$\begin{aligned}
2^{j\beta q(t)} &\lesssim (1 + 2^j |t-\tau|)^{\beta c_{\log}(q)} 2^{j\beta q(\tau)} \\
&\lesssim 2^{j\beta q(\tau)}, \quad t \in [2^{-v}, 2^{1-v}],\ \tau \in [2^{-j}, 2^{1-j}],
\end{aligned}$$

by Lemma 1. Therefore,

$$\begin{aligned}
&\eta_t^{q(t)} \\
\lesssim\ & t^{\beta q(t)} \sum_{j=1}^{v} 2^{(j-v)(s-\beta)q^-/\sigma} (v-j+1)^{q^+-1} \int_{2^{-j}}^{2^{1-j}} \tau^{-\beta q(\tau)} \varepsilon_\tau^{q(\tau)} \frac{d\tau}{\tau} + h_v
\end{aligned}$$

for any $t \in [2^{-v}, 2^{1-v}]$, where

$$h_v = \sum_{j=1}^{v} 2^{(j-v)(s-\beta)q^-/\sigma} (v-j+1)^{q^+} 2^{j(\beta q^+ - m)}.$$

Observe that

$$\int_{2^{-v}}^{2^{1-v}} t^{\beta q(t)}\tau^{-\beta q(\tau)}\frac{dt}{t} \lesssim \int_{2^{-v}}^{2^{1-v}} \Big(\frac{t}{\tau}\Big)^{\beta q(t)}\frac{dt}{t} \le \int_0^{2\tau} \Big(\frac{t}{\tau}\Big)^{\beta q(t)}\frac{dt}{t} \lesssim 1.$$

Therefore,

$$\int_{2^{-v}}^{2^{1-v}} \eta_t^{q(t)}\frac{dt}{t} \lesssim \sum_{j=1}^{v} 2^{(j-v)(s-\beta)q^-/\sigma}(v-j+1)^{q^+-1}\int_{2^{-j}}^{2^{1-j}} \tau^{-\beta q(\tau)}\varepsilon_\tau^{q(\tau)}\frac{d\tau}{\tau} + h_v.$$

Applying Lemma 5 we get

$$\sum_{v=1}^{\infty}\int_{2^{-v}}^{2^{1-v}} \eta_t^{q(t)}\frac{dt}{t} \lesssim \sum_{j=1}^{\infty}\int_{2^{-j}}^{2^{1-j}} \varepsilon_\tau^{q(\tau)}\frac{d\tau}{\tau} + c \lesssim 1.$$

By taking m large enough such that $m > \beta q^+$. Now we prove that

$$\|\delta_t\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})} \le c\,\|\varepsilon_t\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})}\,.$$

We suppose that $\|\varepsilon_t\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})} \le 1$. We see that

$$\int_0^{2^{1-v}} \tau^s\varepsilon_\tau\frac{d\tau}{\tau} = \sum_{i=-\infty}^{-v}\int_{2^i}^{2^{i+1}} \tau^s\varepsilon_\tau\frac{d\tau}{\tau} = \sum_{j=v}^{\infty}\int_{2^{-j}}^{2^{1-j}} \tau^s\varepsilon_\tau\frac{d\tau}{\tau}.$$

Let $\sigma > 0$, $\beta \in \mathbb{R}$ such that $q^+ < \sigma$ and $0 < -\beta < \frac{s}{1+\frac{c_{\log}(q)}{q^-}}$. We have

$$\begin{aligned}
\Big(\sum_{j=v}^{\infty}\int_{2^{-j}}^{2^{1-j}} \tau^s\varepsilon_\tau\frac{d\tau}{\tau}\Big)^{q(t)/\sigma} &\le \sum_{j=v}^{\infty}\Big(\int_{2^{-j}}^{2^{1-j}} \tau^s\varepsilon_\tau\frac{d\tau}{\tau}\Big)^{q(t)/\sigma}\\
&\le \sum_{j=v}^{\infty} 2^{-j(\beta+s)q(t)/\sigma}\Big(\int_{2^{-j}}^{2^{1-j}} \tau^{-\beta}\varepsilon_\tau\frac{d\tau}{\tau}\Big)^{q(t)/\sigma}\\
&= 2^{-v(s+\beta)q(t)/\sigma}\sum_{j=v}^{\infty} 2^{(v-j)(\beta+s)q(t)/\sigma}\Big(\int_{2^{-j}}^{2^{1-j}} \tau^{-\beta}\varepsilon_\tau\frac{d\tau}{\tau}\Big)^{q(t)/\sigma}.
\end{aligned}$$

Again, by Hölder's inequality, we estimate this expression by

$$2^{-v(s+\beta)q(t)/\sigma}\Big(\sum_{j=v}^{\infty} 2^{(v-j)(s+\beta)q(t)/\sigma}\Big(\int_{2^{-j}}^{2^{1-j}} \tau^{-\beta}\varepsilon_\tau\frac{d\tau}{\tau}\Big)^{q(t)}\Big)^{1/\sigma}.$$

Applying again Lemmas 6 and 1 we get

$$\begin{aligned}
&2^{j\beta q(t)}\Big(\frac{1}{(j-v+1)\log 2}\int_{2^{-j}}^{2^{1-v}} \varepsilon_\tau\chi_{[2^{-j},2^{1-j}]}(\tau)\frac{d\tau}{\tau}\Big)^{q(t)}\\
\lesssim\ & \frac{2^{-(j-v)\beta c_{\log}(q)}}{j-v+1}\int_{2^{-j}}^{2^{1-v}} \tau^{-\beta q(\tau)}\varepsilon_\tau^{q(\tau)}\chi_{[2^{-j},2^{1-j}]}(\tau)\frac{d\tau}{\tau} + 2^{j\beta q(t)-vm},
\end{aligned}$$

where we used,

$$\begin{aligned} 2^{-j\beta q(\tau)} &\lesssim (1+2^j\left|t-\tau\right|)^{-\beta c_{\log}(q)}2^{-j\beta q(t)} \\ &\lesssim 2^{-(j-v)\beta c_{\log}(q)}2^{-j\beta q(t)}, \quad t\in[2^{-v},2^{1-v}],\ \tau\in[2^{-j},2^{1-j}], \end{aligned}$$

by Lemma 1. Therefore,

$$\begin{aligned} &\delta_t^{q(t)} \\ \lesssim\ & t^{\beta q(t)}\sum_{j=v}^{\infty}2^{(v-j)(s+\beta(1+\frac{c_{\log}(q)}{q(t)}))q(t)/\sigma}(j-v+1)^{q^+-1}\int_{2^{-j}}^{2^{1-j}}\tau^{-\beta q(\tau)}\varepsilon_\tau^{q(\tau)}\frac{d\tau}{\tau}+f_v \end{aligned}$$

for any $t\in[2^{-v},2^{1-v}]$, where

$$f_v=2^{-v(m+\beta q^+)}.$$

Observe that

$$\int_{2^{-v}}^{2^{1-v}}t^{\beta q(t)}\tau^{-\beta q(\tau)}\frac{dt}{t}\lesssim\int_{2^{-v}}^{2^{1-v}}\left(\frac{t}{\tau}\right)^{\beta q(t)}\frac{dt}{t}\leq\int_{\tau/2}^{\infty}\left(\frac{t}{\tau}\right)^{\beta q(t)}\frac{dt}{t}\lesssim 1$$

for any $\tau\in[2^{-j},2^{1-j}]$ and any $j\geq v$. Therefore,

$$\int_{2^{-v}}^{2^{1-v}}\delta_t^{q(t)}\frac{dt}{t}\lesssim\sum_{j=v}^{\infty}2^{(v-j)(s+\beta(1+\frac{c_{\log}(q)}{q(t)}))q(t)/\sigma}(j-v+1)^{q^+-1}\int_{2^{-j}}^{2^{1-j}}\tau^{-\beta q(\tau)}\varepsilon_\tau^{q(\tau)}\frac{d\tau}{\tau}+f_v.$$

By taking m large enough such that $m>-\beta q^+$. Applying again Lemma 5 we get

$$\sum_{v=1}^{\infty}\int_{2^{-v}}^{2^{1-v}}\delta_t^{q(t)}\frac{dt}{t}\lesssim\sum_{j=1}^{\infty}\int_{2^{-j}}^{2^{1-j}}\varepsilon_\tau^{q(\tau)}\frac{d\tau}{\tau}+c\lesssim 1.$$

The proof is completed by the scaling argument. ■

The following lemma is from [32, Lemma 1].

Lemma 9 *Let $\omega,\mu\in\mathcal{S}(\mathbb{R}^n)$ and $M\geq -1$, an integer such that $\int_{\mathbb{R}^n}x^\alpha\mu(x)dx=0$ for all $|\alpha|\leq M$. Then for any $N>0$, there is a constant $c_N>0$ so that*

$$\sup_{z\in\mathbb{R}^n}\left|t^{-n}\mu(t^{-1}\cdot)*\omega(z)\right|(1+|z|)^N\leq c_N\ t^{M+1}.$$

3 Variable Besov spaces

In this section we present the definition of Besov spaces of variable smoothness and integrability and we prove the basic properties in analogy to the Besov spaces with fixed exponents. Select a pair of Schwartz functions Φ and φ satisfy

$$\text{supp}\mathcal{F}\Phi\subset\{x\in\mathbb{R}^n:|x|<2\},\ \text{supp}\mathcal{F}\varphi\subset\{x\in\mathbb{R}^n:\frac{1}{2}<|x|<2\} \tag{6}$$

and

$$\mathcal{F}\Phi(\xi)+\int_0^1\mathcal{F}\varphi(t\xi)\frac{dt}{t}=1,\quad \xi\neq 0. \tag{7}$$

Such a resolution of unity can be constructed as follows. Let $\mu \in \mathcal{S}(\mathbb{R}^n)$ be such that $|\mathcal{F}\mu(\xi)| > 0$ for $\frac{1}{2} < |\xi| < 2$. There exists $\eta \in \mathcal{S}(\mathbb{R}^n)$ with supp$\mathcal{F}\eta \subset \{x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2\}$ such that

$$\int_0^\infty \mathcal{F}\mu(t\xi)\mathcal{F}\eta(t\xi)\frac{dt}{t} = 1, \quad \xi \in \mathbb{R}^n,$$

see [6], [21] and [22]. We set $\mathcal{F}\varphi = \mathcal{F}\mu\mathcal{F}\eta$ and

$$\mathcal{F}\Phi(\xi) = \begin{cases} \int_1^\infty \mathcal{F}\varphi(t\xi)\frac{dt}{t} & \text{if} \quad \xi \neq 0 \\ 1 & \text{if} \quad \xi = 0. \end{cases}$$

Then $\mathcal{F}\Phi \in \mathcal{S}(\mathbb{R}^n)$, and as $\mathcal{F}\eta$ is supported in $\{x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2\}$, we see that supp$\mathcal{F}\Phi \subset \{x \in \mathbb{R}^n : |x| < 2\}$.

Definition 1 *Let $\alpha : \mathbb{R}^n \to \mathbb{R}$, $p \in \mathcal{P}(\mathbb{R}^n)$ and $q \in \mathcal{P}(\mathbb{R})$. Let $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ be a resolution of unity and we put $\varphi_t = t^{-n}\varphi(\frac{\cdot}{t})$. The Besov space $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|\Phi * f\|_{p(\cdot)} + \left\| \left\| t^{-\alpha(\cdot)}(\varphi_t * f) \right\|_{p(\cdot)} \right\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})} < \infty. \tag{8}$$

When, $q := \infty$ the Besov space $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),\infty}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|\Phi * f\|_{p(\cdot)} + \sup_{t\in(0,1]} \left\| t^{-\alpha(\cdot)}(\varphi_t * f) \right\|_{p(\cdot)} < \infty.$$

One recognizes immediately that $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ is a normed space and if α, p and q are constants, then $\boldsymbol{B}^{\alpha}_{p,q}$ is the usual Besov spaces. For general literature on function spaces of variable smoothness and integrability we refer to [2-4, 10-11, 13-15, 18, 23-24, 27, 40-45].
Now, we are ready to show that the definition of these function spaces is independent of the chosen resolution of unity $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$. This justifies our omission of the subscript Φ and φ in the sequel.

Theorem 1 *Let $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$, $\{\mathcal{F}\Psi, \mathcal{F}\psi\}$ are two resolutions of unity, $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $q \in \mathcal{P}^{\log}(\mathbb{R})$ and $\alpha \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$. Then $\|f\|^{\Phi,\varphi}_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|f\|^{\Psi,\psi}_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$.*

Proof. It sufficient to show that there is $c > 0$ such that for all $f \in \mathcal{S}'(\mathbb{R}^n)$ we have $\|f\|^{\Phi,\varphi}_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c \|f\|^{\Psi,\psi}_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$. Interchanging the roles of (Ψ, ψ) and (Φ, φ) we obtain the desired result. We have

$$\mathcal{F}\Phi(\xi) = \mathcal{F}\Phi(\xi)\mathcal{F}\Psi(\xi) + \int_{1/4}^1 \mathcal{F}\Phi(\xi)\mathcal{F}\psi(\tau\xi)\frac{d\tau}{\tau}$$

and

$$\mathcal{F}\varphi(t\xi) = \int_{t/4}^{\min(1,4t)} \mathcal{F}\varphi(t\xi)\mathcal{F}\psi(\tau\xi)\frac{d\tau}{\tau}$$

for any $0 < t \leq 1$ and any $\xi \in \mathbb{R}^n$. Therefore,

$$\Phi * f = \Phi * \Psi * f + \int_{1/4}^1 \Phi * \psi_\tau * f\frac{d\tau}{\tau}$$

and

$$\varphi_t * f = \int_{t/4}^{\min(1,4t)} \varphi_t * \psi_\tau * f \frac{d\tau}{\tau}.$$

Since $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$:

$$\|\Phi * f\|_{p(\cdot)} \lesssim \|\Psi * f\|_{p(\cdot)} + \int_{1/4}^{1} \|\psi_\tau * f\|_{p(\cdot)} \frac{d\tau}{\tau} \lesssim \|f\|^{\Psi,\psi}_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$$

and

$$\left\| t^{-\alpha(\cdot)-1/q(t)} (\varphi_t * f) \right\|_{p(\cdot)} \lesssim \int_{t/4}^{\min(1,4t)} \left\| \tau^{-\alpha(\cdot)-1/q(\tau)} (\psi_\tau * f) \right\|_{p(\cdot)} d\tau,$$

where we used

$$t^{-1/q(t)} = \left(\frac{t}{\tau}\right)^{-1/q(t)} \tau^{-1/q(t)} \leq \tau^{-1/q(t)} \lesssim \tau^{-1/q(\tau)},$$

by Lemma 1. Hölder's inequality implies that

$$\left\| t^{-\alpha(\cdot)-1/q(t)} (\varphi_t * f) \right\|_{p(\cdot)} \lesssim \|f\|^{\Psi,\psi}_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \left\| \chi_{[t/4,\min(1,4t)]} \right\|_{q'(\cdot)} \lesssim \|f\|^{\Psi,\psi}_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Taking the $L^{q(\cdot)}((0,1])$-norm we obtain the desired estimate. Notice that the case $q := \infty$ can be easily solved. $\square$

Remark 1 *Let $\alpha : \mathbb{R}^n \to \mathbb{R}$, $p \in \mathcal{P}(\mathbb{R}^n)$ and $q \in \mathcal{P}(\mathbb{R})$, with $1 \leq q^+ < \infty$. Let $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ be a resolution of unity. We set*

$$\|f\|^{*}_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|\Phi * f\|_{p(\cdot)} + \left\| \left(t^{-\frac{1}{q(t)}} \left\| t^{-\alpha(\cdot)} (\varphi_t * f) \right\|_{p(\cdot)} \chi_{[2^{-v},2^{1-v}]} \right)_v \right\|_{\ell^{q(\cdot)}_{>}(L^{q(\cdot)})}.$$

Then

$$\|f\|_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|f\|^{*}_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{9}$$

We define for $a > 0$, $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, the Peetre maximal function

$$\varphi_t^{*,a} t^{-\alpha(\cdot)} f(x) = \sup_{y \in \mathbb{R}^n} \frac{t^{-\alpha(y)} |\varphi_t * f(y)|}{(1 + t^{-1}|x-y|)^a}, \qquad t > 0$$

and

$$\Phi^{*,a}(x) = \sup_{y \in \mathbb{R}^n} \frac{|\Phi * f(y)|}{(1 + |x-y|)^a}.$$

We now present a fundamental characterization of spaces under consideration.

Theorem 2 *Let $\alpha \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $q \in \mathcal{P}^{\log}(\mathbb{R})$ and $a > \frac{n}{p^-}$. Let $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ be a resolution of unity. Then*

$$\|f\|^{\blacktriangledown}_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|\Phi^{*,a} f\|_{p(\cdot)} + \left\| \left\| \varphi_t^{*,a} t^{-\alpha(\cdot)} f \right\|_{p(\cdot)} \right\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})} \tag{10}$$

is an equivalent norm in $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. It is easy to see that for any $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\|f\|^{\blacktriangledown}_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} < \infty$ and any $x \in \mathbb{R}^n$ we have

$$t^{-\alpha(x)} |\varphi_t * f(x)| \le \varphi_t^{*,a} t^{-\alpha(\cdot)} f(x).$$

This shows that the right-hand side in (8) is less than or equal (10).

We will prove that there is a constant $C > 0$ such that for every $f \in \boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$

$$\|f\|^{\blacktriangledown}_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \le C \|f\|_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{11}$$

By Lemmas 2 and 1 the estimates

$$\begin{aligned} t^{-\alpha(y)} |\varphi_t * f(y)| &\le C_1 \, t^{-\alpha(y)} \left(\eta_{t,wp^-} * |\varphi_t * f|^{p^-}(y) \right)^{1/p^-} \\ &\le C_2 \left(\eta_{t,(w-c_{\log}(\alpha))p^-} * (t^{-\alpha(\cdot)} |\varphi_t * f|)^{p^-}(y) \right)^{1/p^-} \end{aligned} \tag{12}$$

are true for any $y \in \mathbb{R}^n$ and any $w > n/p^-$, $t > 0$. Now divide both sides of (12) by $(1 + t^{-1}|x-y|)^a$, in the right-hand side we use the inequality

$$\left(1 + t^{-1}|x-y|\right)^{-a} \le \left(1 + t^{-1}|x-z|\right)^{-a} \left(1 + t^{-1}|y-z|\right)^{a}, \quad x, y, z \in \mathbb{R}^n,$$

in the left-hand side take the supremum over $y \in \mathbb{R}^n$ and get for all $f \in \boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ any $t > 0$ and any $w > \max(n/p^-, a + c_{\log}(\alpha))$

$$\varphi_t^{*,a} t^{-\alpha(\cdot)} f(x) \le C_2 \left(\eta_{t,ap^-} * (t^{-\alpha(\cdot)} |\varphi_t * f|^{p^-})(x) \right)^{1/p^-}$$

where $C_2 > 0$ is independent of x, t and f. Applying the $L^{p(\cdot)}$-norm and using the fact the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$, we get

$$\left\| \varphi_t^{*,a} t^{-\alpha(\cdot)-1/q(t)} f \right\|_{p(\cdot)} \lesssim \left\| t^{-\alpha(\cdot)-1/q(t)} \varphi_t * f \right\|_{p(\cdot)}, \quad t > 0.$$

Taking the $L^{q(\cdot)}((0,1])$-norm we obtain the desired estimate. The proof is complete. ∎

In order to formulate the main result of this section, let us consider $k_0, k \in \mathcal{S}(\mathbb{R}^n)$ and $S \ge -1$ an integer such that for an $\varepsilon > 0$

$$|\mathcal{F}k_0(\xi)| > 0 \quad \text{for} \quad |\xi| < 2\varepsilon \tag{13}$$

$$|\mathcal{F}k(\xi)| > 0 \quad \text{for} \quad \frac{\varepsilon}{2} < |\xi| < 2\varepsilon \tag{14}$$

and

$$\int_{\mathbb{R}^n} x^\alpha k(x) dx = 0 \quad \text{for any} \quad |\alpha| \le S. \tag{15}$$

Here (13) and (14) are Tauberian conditions, while (15) are moment conditions on k. We recall the notation

$$k_t(x) = t^{-n} k(t^{-1}x), \quad \text{for} \quad t > 0.$$

For any $a > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we denote

$$k_t^{*,a} t^{-\alpha(\cdot)} f(x) = \sup_{y \in \mathbb{R}^n} \frac{t^{-\alpha(y)} |k_t * f(y)|}{(1 + t^{-1}|x-y|)^a}, \quad j \in \mathbb{N}_0. \tag{16}$$

Usually $k_t * f$ is called local mean.

We are able now to state the main result of this section.

Theorem 3 *Let $\alpha \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q \in \mathcal{P}^{\log}(\mathbb{R})$. Let $a > \frac{n}{p^-}$ and $\alpha^+ < S+1$. Then*

$$\|f\|'_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|k_0^{*,a}f\|_{p(\cdot)} + \Big\|\big\|k_t^{*,a}t^{-\alpha(\cdot)}f\big\|_{p(\cdot)}\Big\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})} \tag{17}$$

and

$$\|f\|''_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|k_0 * f\|_{p(\cdot)} + \Big\|\big\|t^{-\alpha(\cdot)}(k_t * f)\big\|_{p(\cdot)}\Big\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})}, \tag{18}$$

are equivalent norms on $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. The idea of the proof is from V. S. Rychkov [32].

Step 1. Take any pair of functions φ_0 and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\begin{aligned} |\mathcal{F}\varphi_0(\xi)| > 0 \quad &\text{for} \quad |\xi| < 2\varepsilon \\ |\mathcal{F}\varphi(\xi)| > 0 \quad &\text{for} \quad \frac{\varepsilon}{2} < |\xi| < 2\varepsilon \end{aligned}$$

for some $\varepsilon > 0$. We will prove that there is a constant $c > 0$ such that for any $f \in \boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$

$$\|f\|'_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\,\|\varphi_0^{*,a}f\|_{p(\cdot)} + \Big\|\big\|\varphi_t^{*,a}t^{-\alpha(\cdot)}f\big\|_{p(\cdot)}\Big\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})}. \tag{19}$$

Let Λ, $\lambda \in \mathcal{S}(\mathbb{R}^n)$ so that

$$\operatorname{supp} \mathcal{F}\Lambda \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\varepsilon\}, \quad \operatorname{supp} \mathcal{F}\lambda \subset \{\xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon\} \tag{20}$$

$$\mathcal{F}\Lambda(\xi)\mathcal{F}\varphi_0(\xi) + \int_0^1 \mathcal{F}\lambda(\tau\xi)\mathcal{F}\varphi(\tau\xi)\frac{d\tau}{\tau} = 1, \quad \xi \in \mathbb{R}^n.$$

In particular, for any $f \in \boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ the identity is true

$$f = \Lambda * \varphi_0 * f + \int_0^1 \lambda_\tau * \varphi_\tau * f\frac{d\tau}{\tau}. \tag{21}$$

Hence we can write

$$k_t * f = k_t * \Lambda * \varphi_0 * f + \int_0^1 k_t * \lambda_\tau * \varphi_\tau * f\frac{d\tau}{\tau}, \quad 0 < t \leq 1.$$

We have

$$t^{-\alpha(y)}\,|k_t * \lambda_\tau * \varphi_\tau * f(y)| \leq t^{-\alpha(y)}\int_{\mathbb{R}^n} |k_t * \lambda_\tau(z)|\,|\varphi_\tau * f(y-z)|\,dz. \tag{22}$$

First let $t \leq \tau$. Writing for any $z \in \mathbb{R}^n$

$$k_t * \lambda_\tau(z) = \tau^{-n}k_{\frac{t}{\tau}} * \lambda(\frac{z}{\tau}),$$

we get by Lemma 9, that for any integer $S \geq -1$ and any $N > 0$ there is a constant $c > 0$ independent of t and τ

$$|k_t * \lambda_\tau(z)| \leq c \, \frac{\tau^{-n} \left(\frac{t}{\tau}\right)^{S+1}}{\left(1 + \tau^{-1} |z|\right)^N}, \quad z \in \mathbb{R}^n.$$

So the right-hand side of (22) can be estimated from above by

$$\begin{aligned} & c \, t^{-\alpha(y)} \tau^{-n} \left(\frac{t}{\tau}\right)^{S+1} \int_{\mathbb{R}^n} \left(1 + \tau^{-1} |z|\right)^{-N} |\varphi_\tau * f(y-z)| \, dz \\ = \; & c \left(\frac{t}{\tau}\right)^{S+1} t^{-\alpha(y)} \eta_{\tau,N} * |\varphi_\tau * f|(y). \end{aligned}$$

By Lemma 1 the estimates

$$\begin{aligned} t^{-\alpha(y)} \eta_{\tau,N} * |\varphi_\tau * f|(y) & \leq \left(\frac{\tau}{t}\right)^{\alpha^+} \eta_{\tau,N-c_{\log}(\alpha)} * (\tau^{-\alpha(y)} |\varphi_v * f|)(y) \\ & \leq \left(\frac{\tau}{t}\right)^{\alpha^+} \varphi_\tau^{*,a} \tau^{-\alpha(\cdot)} f(y) \left\| \eta_{\tau,N-a-c_{\log}(\alpha)} \right\|_1 \\ & \leq c \left(\frac{\tau}{t}\right)^{\alpha^+} \varphi_\tau^{*,a} \tau^{-\alpha(\cdot)} f(y), \end{aligned}$$

are true for any $N > n + a + c_{\log}(\alpha)$ and any $t \leq \tau$.
Let now $t \geq \tau$. Then, again by Lemma 9 we have for any $z \in \mathbb{R}^n$ and any $L > 0$

$$|k_t * \lambda_\tau(z)| = t^{-n} \left| k * \lambda_{\frac{\tau}{t}}\left(\frac{z}{t}\right) \right| \leq c \, \frac{t^{-n} \left(\frac{\tau}{t}\right)^{M+1}}{\left(1 + t^{-1} |z|\right)^L},$$

where $M \geq -1$ an integer can be taken arbitrarily large, since $D^\alpha \mathcal{F}\lambda(0) = 0$ for all α.
Therefore, for $t \geq \tau$, the right-hand side of (22) can be estimated from above by

$$\begin{aligned} & c \, t^{-\alpha(y)} t^{-n} \left(\frac{\tau}{t}\right)^{M+1} \int_{\mathbb{R}^n} \left(1 + t^{-1} |z|\right)^{-L} |\varphi_\tau * f(y-z)| \, dz \\ = \; & c \, t^{-\alpha(y)} t^{-n} \left(\frac{\tau}{t}\right)^{M+1} \eta_{t,L} * |\varphi_\tau * f|(y). \end{aligned}$$

We have for any $t \geq \tau$

$$\left(1 + t^{-1} |z|\right)^{-L} \leq \left(\frac{t}{\tau}\right)^L \left(1 + \tau^{-1} |z|\right)^{-L}.$$

Then, again, the right-hand side of (22) is dominated by

$$\begin{aligned} & c \, t^{-\alpha(y)} \left(\frac{\tau}{t}\right)^{M-L+1+n} \eta_{\tau,L} * |\varphi_\tau * f|(y) \\ \leq \; & c \left(\frac{\tau}{t}\right)^{M-L+1+n+\alpha^-} \eta_{\tau,L-c_{\log}(\alpha)} * (\tau^{-\alpha(\cdot)} |\varphi_\tau * f|)(y) \\ \leq \; & c \left(\frac{\tau}{t}\right)^{M-L+1+n+\alpha^-} \varphi_\tau^{*,a} \tau^{-\alpha(\cdot)} f(y) \left\| \eta_{\tau,L-a-c_{\log}(\alpha)} \right\|_1 \\ \leq \; & c \left(\frac{\tau}{t}\right)^{M-L+1+n+\alpha^-} \varphi_\tau^{*,a} \tau^{-\alpha(\cdot)} f(y), \end{aligned}$$

where in the first inequality we have used Lemma 1 (by taking $L > n + a + c_{\log}(\alpha)$). Taking $M > L - \alpha^- + a - n$ to estimate the last expression by

$$c\,\left(\frac{\tau}{t}\right)^{a+1} \varphi_\tau^{*,a}\tau^{-\alpha(\cdot)}f(y),$$

where $c > 0$ is independent of t, τ and f. Further, note that for all $x, y \in \mathbb{R}^n$ and all $t, \tau \in (0, 1]$

$$\begin{aligned}\varphi_\tau^{*,a}\tau^{-\alpha(\cdot)}f(y) &\leq \varphi_\tau^{*,a}\tau^{-\alpha(\cdot)}f(x)(1 + \tau^{-1}|x - y|)^a \\ &\leq \varphi_\tau^{*,a}\tau^{-\alpha(\cdot)}f(x)\max\left(1, \left(\frac{t}{\tau}\right)^a\right)(1 + t^{-1}|x - y|)^a.\end{aligned}$$

Hence

$$\sup_{y\in\mathbb{R}^n} \frac{t^{-\alpha(y)}|k_t * \lambda_\tau * \varphi_\tau * f(y)|}{(1 + t^{-1}|x - y|)^a} \leq C\,\varphi_\tau^{*,a}\tau^{-\alpha(\cdot)}f(x) \times \begin{cases} \left(\frac{t}{\tau}\right)^{S+1-\alpha^+} & \text{if} \quad t \leq \tau \\ \frac{\tau}{t} & \text{if} \quad t \geq \tau. \end{cases}$$

Using the fact that for any $z \in \mathbb{R}^n$, any $N > 0$ and any integer $S \geq -1$

$$|k_t * \Lambda(z)| \leq c\,\frac{t^{S+1}}{(1 + |z|)^N},$$

we obtain by the similar arguments that for any $t \in (0, 1]$

$$\sup_{y\in\mathbb{R}^n} \frac{t^{-\alpha(y)}|k_t * \Lambda * \varphi_0 * f(y)|}{(1 + t^{-1}|x - y|)^a} \leq C\,t^{S+1-\alpha^+}\varphi_0^{*,a}f(x).$$

Hence for all $f \in \boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, any $x \in \mathbb{R}^n$ and any $t \in (0, 1]$

$$k_t^{*,a}t^{-\alpha(\cdot)}f(x) \leq C\,t^{S+1-\alpha^+}\varphi_0^{*,a}f(x) + C\int_0^1 \min\left(\left(\frac{t}{\tau}\right)^{S+1-\alpha^+}, \frac{\tau}{t}\right)\varphi_\tau^{*,a}\tau^{-\alpha(\cdot)}f(x)\frac{d\tau}{\tau}. \tag{23}$$

Also we have for any $z \in \mathbb{R}^n$, any $N > 0$ and any integer $M \geq -1$

$$|k_0 * \lambda_\tau(z)| \leq c\,\frac{\tau^{M+1}}{(1 + |z|)^N}$$

and

$$|k_0 * \Lambda(z)| \leq c\,\frac{1}{(1 + |z|)^N}.$$

As before, we get for any $x \in \mathbb{R}^n$

$$k_0^{*,a}f(x) \leq C\,\varphi_0^{*,a}f(x) + C\int_0^1 \tau\varphi_\tau^{*,a}\tau^{-\alpha(\cdot)}f(x)\frac{d\tau}{\tau}. \tag{24}$$

In (23) and (24) taking the $L^{p(\cdot)}$-norm and then using Lemma 8, we get (19).

Step 2. We will prove in this step that there is a constant $c > 0$ such that for any $f \in \boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$

$$\|f\|'_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\,\|f\|''_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{25}$$

Analogously to (20), (21) find two functions Λ, $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\mathcal{F}\Lambda(t\xi)\mathcal{F}k_0(t\xi) + \int_0^1 \mathcal{F}\psi(\tau t\xi)\mathcal{F}k(\tau t\xi)\frac{d\tau}{\tau} = 1, \quad \xi \in \mathbb{R}^n.$$

and for all $f \in \boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $t \in (0,1]$

$$f = \Lambda_t * (k_0)_t * f + \int_0^t \psi_h * k_h * f\frac{dh}{h}.$$

Hence

$$k_t * f = \Lambda_t * (k_0)_t * k_t * f + \int_0^t k_t * \psi_h * k_h * f\frac{dh}{h}.$$

Writing for any $z \in \mathbb{R}^n$

$$k_t * \psi_h(z) = t^{-n}(k * \psi_{\frac{h}{t}})(\frac{z}{t}),$$

we get by Lemma 9, that for any integer $K \geq -1$ and any $M > 0$ there is a constant $c > 0$ independent of t and h

$$|k_t * \psi_h(z)| \leq c\,\frac{t^{-n}\left(\frac{h}{t}\right)^{K+1}}{\left(1+t^{-1}|z|\right)^M}, \quad z \in \mathbb{R}^n.$$

Analogous estimate

$$|\Lambda_t * (k_0)_t(z)| \leq c\,\frac{t^{-n}}{\left(1+t^{-1}|z|\right)^M}, \quad z \in \mathbb{R}^n,$$

is obvious. From this it follows that

$$\begin{aligned} t^{-\alpha(y)}|k_t * f(y)| &\leq c\,t^{-\alpha(y)}\eta_{t,M} * |k_t * f|(y) \\ &\quad + \int_0^t \left(\frac{h}{t}\right)^{K+1+\alpha^-} h^{-\alpha(y)}\eta_{t,M} * |k_h * f|(y)\frac{dh}{h}. \end{aligned}$$

Since

$$\left(1+t^{-1}|y-z|\right)^{-M} \leq \left(\frac{t}{h}\right)^M \left(1+h^{-1}|y-z|\right)^{-M}, \quad y,z \in \mathbb{R}^n$$

then by Lemma 1 and Hölder's inequality, we obtain

$$\begin{aligned} & t^{-\alpha(y)}|k_t * f(y)| \\ \leq\; & c\,t^{-\alpha(y)}\eta_{t,M} * |k_t * f|(y) + \int_0^t \left(\frac{h}{t}\right)^{K+1+\alpha^- - M} h^{-\alpha(y)}\eta_{h,M} * |k_h * f|(y)\frac{dh}{h} \\ \leq\; & c\,\left(\eta_{t,ap^-} * t^{-\alpha(\cdot)p^-}|k_t * f|^{p^-}(y)\right)^{1/p^-} \\ & + \int_0^t \left(\frac{h}{t}\right)^{K+1+\alpha^- - M+n} \left(\eta_{h,ap^-} * h^{-\alpha(\cdot)p^-}|k_h * f|^{p^-}(y)\right)^{1/p^-}\frac{dh}{h} \end{aligned}$$

by taking $M > a + n + c_{\log}(\alpha)$. Using the elementary estimates

$$\begin{aligned} \left(1+t^{-1}|x-y|\right)^{-a} &\leq \left(1+t^{-1}|x-z|\right)^{-a}\left(1+t^{-1}|y-z|\right)^{a} \\ &\leq \left(\frac{t}{h}\right)^{a}\left(1+h^{-1}|x-z|\right)^{-a}\left(1+h^{-1}|y-z|\right)^{a}, \end{aligned}$$

and again Hölder's inequality to get

$$\left(k_t^{*,a}t^{-\alpha(\cdot)}f(x)\right)^{p^-} \leq c\,\eta_{t,a} * t^{-\alpha(\cdot)p^-}|k_t * f|^{p^-}(x) + c\int_0^t \left(\frac{h}{t}\right)^N \eta_{h,ap^-} * h^{-\alpha(\cdot)p^-}|k_h * f|^{p^-}(x)\frac{dh}{h}, \tag{26}$$

where $N > 0$ can be still be taken arbitrarily large. Similarly we obtain

$$(k_0^{*,a}f(x))^{p^-} \leq c\eta_{1,ap^-} * |k_0 * f|(x) + \int_0^1 h^N \eta_{h,ap^-} * h^{-\alpha(\cdot)p^-}|k_h * f|^{p^-}(x)\frac{dh}{h}. \tag{27}$$

Observe that

$$\left\|\eta_{h,ap^-} * h^{-\alpha(\cdot)p^-}|k_h * f|^{p^-}\right\|_{p(\cdot)/p^-} \lesssim \left\|h^{-\alpha(\cdot)}k_h * f\right\|_{p(\cdot)}^{p^-},$$

$$\left\|\eta_{h,ap^-} * |k_0 * f|^{p^-}\right\|_{p(\cdot)/p^-} \lesssim \|k_0 * f\|_{p(\cdot)}^{p^-},$$

and

$$\left\|\eta_{t,ap^-} * t^{-\alpha(\cdot)p^-}|k_t * f|^{p^-}\right\|_{p(\cdot)/p^-} \lesssim \left\|t^{-\alpha(\cdot)}k_t * f\right\|_{p(\cdot)}^{p^-}$$

for any $h \in (0, t]$ and any $0 < t \leq 1$. Therefore (27), with power $1/p^-$, in $L^{p(\cdot)}$-norm is bounded by

$$\|k_0 * f\|_{p(\cdot)} + \int_0^1 h^N \left\|h^{-\alpha(\cdot)}k_h * f\right\|_{p(\cdot)} \frac{dh}{h} \lesssim \|f\|''_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

since, N can be still be taken arbitrarily large. In (26), taking the $L^{p(\cdot)/p^-}$-norm, using the above estimates, take the $1/p^-$ power and then the $L^{q(\cdot)}((0,1], \frac{dt}{t})$-norm, we get with the help of Lemma 8 that $\|f\|'_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\,\|f\|''_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$.

Step 3. We will prove in this step that for all $f \in \boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ the following estimates are true:

$$\|f\|'_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\,\|f\|_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\,\|f\|''_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Let $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ be a resolution of unity. The first inequality is proved by the chain of the estimates

$$\begin{aligned}\|f\|'_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} &\leq c\,\|\Phi^{*,a}f\|_{p(\cdot)} + \left\|\left\|\varphi_t^{*,a}t^{-\alpha(\cdot)}f\right\|_{p(\cdot)}\right\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})}\\ &\leq c\,\|\Phi * f\|_{p(\cdot)} + \left\|\left\|t^{-\alpha(\cdot)}\varphi_t * f\right\|_{p(\cdot)}\right\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})} \leq c\,\|f\|_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},\end{aligned}$$

where the first inequality is (19), see Step 1, the second inequality is (25) (with φ and φ_0 instead of k and k_0), see Step 2, and finally the third inequality is obvious. Now the second inequality can be obtained by the following chain

$$\begin{aligned}\|f\|_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} &\leq c\,\|\Phi^{*,a}f\|_{p(\cdot)} + \left\|\left\|\varphi_t^{*,a}t^{-\alpha(\cdot)}f\right\|_{p(\cdot)}\right\|_{L^{q(\cdot)}([0,1],\frac{dt}{t})}\\ &\leq c\,\|f\|'_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\,\|f\|''_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},\end{aligned}$$

where the first inequality is obvious, the second inequality is (19), see Step 1, with the roles of k_0 and k respectively φ_0 and φ interchanged, and finally the last inequality is (25), see Step 2. Hence the theorem is proved. $\square$

4 Embeddings

For the spaces $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ introduced above we want to show some embedding theorems. We say a quasi-Banach space A_1 is continuously embedded in another quasi-Banach space A_2, $A_1 \hookrightarrow A_2$, if $A_1 \subset A_2$ and there is a $c > 0$ such that $\|f\|_{A_2} \leq c\,\|f\|_{A_1}$ for all $f \in A_1$. We begin with the following elementary embeddings.

Theorem 4 *Let $\alpha \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q_0, q_1 \in \mathcal{P}^{\log}(\mathbb{R})$. If $(\alpha_0 - \alpha_1)^- > 0$, then*

$$\boldsymbol{B}^{\alpha_0(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow \boldsymbol{B}^{\alpha_1(\cdot)}_{p(\cdot),q_1(\cdot)}.$$

Proof. Let $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$, $\{\mathcal{F}\Psi, \mathcal{F}\psi\}$ are two resolutions of unity. We have

$$\varphi_t * f = \int_{t/4}^{\min(1,4t)} \varphi_t * \psi_\tau * f \frac{d\tau}{\tau}, \quad t \in (0,1].$$

Using the fact the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$, we get

$$\begin{aligned}
&\left\|t^{-\alpha_1(\cdot)}(\varphi_t * f)\right\|_{p(\cdot)} \\
\lesssim\ & t^{(\alpha_0-\alpha_1)^-} \int_{t/4}^{\min(1,4t)} \left\|\tau^{-\alpha_0(\cdot)}(\psi_\tau * f)\right\|_{p(\cdot)} \frac{d\tau}{\tau} \\
\lesssim\ & t^{(\alpha_0-\alpha_1)^-} \left\|\left\|\tau^{-\alpha_0(\cdot)}(\psi_\tau * f)\right\|_{p(\cdot)}\right\|_{L^{q_0(\cdot)}((0,1],\frac{d\tau}{\tau})} \|1\|_{L^{q_0'(\cdot)}([t/4,\min(1,4t)],\frac{d\tau}{\tau})} \\
\lesssim\ & t^{(\alpha_0-\alpha_1)^-} \left\|\left\|\tau^{-\alpha_0(\cdot)}(\psi_\tau * f)\right\|_{p(\cdot)}\right\|_{L^{q_0(\cdot)}((0,1],\frac{d\tau}{\tau})} \leq t^{(\alpha_0-\alpha_1)^-} \|f\|_{\boldsymbol{B}^{\alpha_0(\cdot)}_{p(\cdot),q_0(\cdot)}},
\end{aligned}$$

where we have used Hölder's inequality. Taking the $L^{q(\cdot)}((0,1],\frac{dt}{t})$-norm we obtain the desired estimate. ■

We next consider embeddings of Sobolev-type. It is well-known that

$$B^{\alpha_0}_{p_0,q} \hookrightarrow B^{\alpha_1}_{p_1,q},$$

if $\alpha_0 - n/p_0 = \alpha_1 - n/p_1$, where $0 < p_0 < p_1 \leq \infty$ and $0 < q \leq \infty$ (see e.g. [35, Theorem 2.7.1]). In the following theorem we generalize these embeddings to variable exponent case.

Theorem 5 *Let $\alpha_0, \alpha_1 \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, $p_0, p_1 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q \in \mathcal{P}^{\log}(\mathbb{R})$. If $\alpha_0 \geq \alpha_1$ and $\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}$, then*

$$\boldsymbol{B}^{\alpha_0(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow \boldsymbol{B}^{\alpha_1(\cdot)}_{p_1(\cdot),q(\cdot)}. \tag{28}$$

Proof. We will prove that

$$\|f\|_{\boldsymbol{B}^{\alpha_1(\cdot)}_{p_1(\cdot),q(\cdot)}} \lesssim \|f\|_{\boldsymbol{B}^{\alpha_0(\cdot)}_{p_0(\cdot),q(\cdot)}}$$

for any $f \in \boldsymbol{B}^{\alpha_0(\cdot)}_{p_0(\cdot),q(\cdot)}$. Let as prove that

$$\left\|t^{-\alpha_1(\cdot)-1}(\varphi_t * f)\right\|_{p_1(\cdot)} \lesssim \left\|t^{-\alpha_0(\cdot)-1}(\varphi_t * f)\right\|_{p_0(\cdot)} = \delta. \tag{29}$$

This is equivalent to

$$\left\|\delta^{-1}t^{-\alpha_1(\cdot)-1}(\varphi_t * f)\right\|_{p_1(\cdot)} \lesssim 1.$$

By Lemma 2 we have for any $m > n$, $d > 0$

$$|\varphi_t * f(x)| \leq c(\eta_{t,m} * |\varphi_t * f|^d(x))^{1/d}.$$

Hence

$$\begin{aligned} & \delta^{-1}t^{\frac{n}{p_1(x)}-\alpha_1(x)-1}|\varphi_t * f(x)| \\ \leq\ & c(\eta_{t,m-c_{\log}(\alpha_1)-c_{\log}(1/p_1)} * \delta^{-d}t^{-\alpha_1(\cdot)d+\frac{nd}{p_1(\cdot)}-d}|\varphi_t * f|^d(x))^{1/d} \\ \leq\ & c\left\|t^{n-\frac{n}{d}+\frac{n}{p_0(\cdot)}}\eta_{t,(m-c_{\log}(\alpha_1)-c_{\log}(1/p_1))/d}(x-\cdot)\right\|_{h(\cdot)}\left\|\delta^{-d}t^{-\alpha_0(\cdot)-1}(\varphi_t * f)\right\|_{p_0(\cdot)}. \end{aligned}$$

where $\frac{1}{d} = \frac{1}{p_0(\cdot)} + \frac{1}{h(\cdot)}$. The second norm on the right hand side is bounded by 1 due to the choice of δ. To show that the first norm is also bounded, we investigate the corresponding modular:

$$\begin{aligned} & \varrho_{h(\cdot)}(t^{n-\frac{n}{d}+\frac{n}{p_0(\cdot)}}\eta_{t,(m-c_{\log}(\alpha_1)-c_{\log}(1/p_1))/d}(x-\cdot)) \\ =\ & \int_{\mathbb{R}^n}\frac{t^{-n}}{(1+t^{-1}|x-z|)^{(m-c_{\log}(\alpha_1)-c_{\log}(1/p_1))h(z)/d}}dz < \infty \end{aligned}$$

for $m > 0$ large enough. Now

$$\begin{aligned} & \left|\delta^{-1}t^{-\alpha_1(x)-1}\varphi_t * f(x)\right|^{p_1(x)} \\ =\ & \left|\delta^{-1}t^{\frac{n}{p_1(x)}-\alpha_1(x)-1}\varphi_t * f(x)\right|^{p_1(x)-p_0(x)}\left|\delta^{-1}t^{-\alpha_0(x)-1}\varphi_t * f(x)\right|^{p_0(x)} \\ \lesssim\ & \left|\delta^{-1}t^{-\alpha_0(x)-1}\varphi_t * f(x)\right|^{p_0(x)}. \end{aligned}$$

Integrating this inequality over $\mathbb{R}^n$ and taking into account the definition of δ gives us the claim. The proof is complete by taking in (29) the $L^{q(\cdot)}((0,1],\frac{dt}{t})$-norm. ■

Let $\alpha \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$, $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q \in \mathcal{P}^{\log}(\mathbb{R})$. From (28), we obtain

$$\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \boldsymbol{B}^{\alpha(\cdot)+n/p^+-n/p(\cdot)}_{p^+,q(\cdot)} \hookrightarrow \boldsymbol{B}^{(\alpha+n/p^+-n/p)^-}_{p^+,q(\cdot)} \hookrightarrow \boldsymbol{B}^{(\alpha+n/p^+-n/p)^--\varepsilon}_{p^+,\infty} \hookrightarrow \mathcal{S}'(\mathbb{R}^n),$$

where $0 < \varepsilon < (\alpha + n/p^+ - n/p)^-$. Let $\alpha_0 \in \mathbb{R}$ be such that $\alpha_0 > (\alpha + n/p^- - n/p)^+$. We have

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \boldsymbol{B}^{\alpha_0}_{p^-,q^+} \hookrightarrow \boldsymbol{B}^{\alpha_0-n/p^-+n/p(\cdot)}_{p^-,q(\cdot)} \hookrightarrow \boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}.$$

Therefore,

Theorem 6 *Let $\alpha \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$, $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q \in \mathcal{P}^{\log}(\mathbb{R})$. Then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

5 Atomic decomposition

The idea of atomic decompositions leads back to M. Frazier and B. Jawerth in their series of papers [16], [17], see also [37]. The main goal of this section is to prove an atomic decomposition result for $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. Atoms are the building blocks for the atomic decomposition.

Definition 2 *Let $K \in \mathbb{N}_0, L+1 \in \mathbb{N}_0$ and let $\gamma > 1$. A K-times continuous differentiable function $a \in C^K(\mathbb{R}^n)$ is called $[K, L]$-atom centered at $Q_{v,m}$, $v \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, if*

$$\text{supp } a \subseteq \gamma Q_{v,m} \tag{30}$$

$$|D^\beta a(x)| \le 2^{v(|\beta|+1/2)}, \quad \textit{for} \quad 0 \le |\beta| \le K, x \in \mathbb{R}^n \tag{31}$$

and if

$$\int_{\mathbb{R}^n} x^\beta a(x) dx = 0, \quad \textit{for} \quad 0 \le |\beta| \le L \textit{ and } v \ge 1. \tag{32}$$

If the atom a located at $Q_{v,m}$, that means if it fulfills (30), then we will denote it by $a_{v,m}$. For $v = 0$ or $L = -1$ there are no moment conditions (32) required.

For proving the decomposition by atoms we need the following lemma, see Frazier & Jawerth [16, Lemma 3.3].

Lemma 10 *Let $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ be a resolution of unity and let $\varrho_{v,m}$ be an $[K, L]$-atom. If $j \in \mathbb{N}_0$ and $2^{-j} \le t \le 2^{1-j}$, then*

$$\left|\varphi_t * \varrho_{v,m}(x)\right| \le c\ 2^{(v-j)K+vn/2} \left(1 + 2^v \left|x - x_{Q_{v,m}}\right|\right)^{-M}$$

if $v \le j$ and

$$\left|\varphi_t * \varrho_{v,m}(x)\right| \le c\ 2^{(j-v)(L+n+1)+vn/2} \left(1 + 2^j \left|x - x_{Q_{v,m}}\right|\right)^{-M}$$

if $v \ge j$, where M is sufficiently large and $\varphi_t = t^{-n}\varphi(\frac{\cdot}{t})$. Moreover

$$\left|\Phi * \varrho_{v,m}(x)\right| \le c\ 2^{-v(L+n+1)+vn/2} \left(1 + \left|x - x_{Q_{v,m}}\right|\right)^{-M}.$$

Let $p \in \mathcal{P}(\mathbb{R}^n)$, $q \in \mathcal{P}(\mathbb{R})$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$. Then for all complex valued sequences $\lambda = \{\lambda_{v,m} \in \mathbb{C}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ we define

$$\boldsymbol{b}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} := \Big\{\lambda : \|\lambda\|_{\boldsymbol{b}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} < \infty\Big\},$$

where

$$\|\lambda\|_{\boldsymbol{b}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \Big\|\sum_{m \in \mathbb{Z}^n} \lambda_{0,m}\chi_{0,m}\Big\|_{p(\cdot)} + \Big\|\Big(\Big\|t^{-(\alpha(\cdot)+n/2)-\frac{1}{q(\cdot)}} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m}\chi_{v,m}\Big\|_{p(\cdot)} \chi_{[2^{-v},2^{1-v}]}\Big)_v\Big\|_{\ell^{q(\cdot)}_{>}(L^{q(\cdot)})}.$$

Here $\chi_{v,m}$ is the characteristic function of the cube $Q_{v,m}$.
Now we come to the atomic decomposition theorem.

Theorem 7 *Let $\alpha \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q \in \mathcal{P}^{\log}(\mathbb{R})$ with $1 \leq q^- \leq q^+ < \infty$. Let $K, L+1 \in \mathbb{N}_0$ such that*

$$K \geq ([\alpha^+]+1)^+, \tag{33}$$

and

$$L \geq \max(-1, [-\alpha^-]). \tag{34}$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, if and only if it can be represented as

$$f = \sum_{v=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{v,m} \varrho_{v,m}, \qquad \textit{converging in } \mathcal{S}'(\mathbb{R}^n), \tag{35}$$

where $\varrho_{v,m}$ are $[K,L]$-atoms and $\lambda = \{\lambda_{v,m} \in \mathbb{C}\}_{v\in\mathbb{N}_0, m\in\mathbb{Z}^n} \in \boldsymbol{b}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. Furthermore, $\inf \|\lambda\|_{\boldsymbol{b}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$, where the infimum is taken over admissible representations (35), is an equivalent norm in $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

The convergence in $\mathcal{S}'(\mathbb{R}^n)$ is postponed to the Appendix.
Proof. The proof follows the ideas in [16, Theorem 6] and [15, Theorem 4.3].

Step 1. Assume that $f \in \boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and let Φ and φ satisfy

$$\mathrm{supp}\mathcal{F}\Phi \subset \overline{B(0,2)} \text{ and } |\mathcal{F}\Phi(\xi)| \geq c \text{ if } |\xi| \leq \frac{5}{3} \tag{36}$$

and

$$\mathrm{supp}\mathcal{F}\varphi \subset \overline{B(0,2)} \backslash B(0,1/2) \text{ and } |\mathcal{F}\varphi(\xi)| \geq c \text{ if } \frac{3}{5} \leq |\xi| \leq \frac{5}{3}. \tag{37}$$

There exist functions $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (36) and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (37) such that for all $\xi \in \mathbb{R}^n$

$$f = \Psi * \Phi * f + \int_0^1 \psi_t * \varphi_t * f \frac{dt}{t},$$

see Section 3. Using the definition of the cubes $Q_{v,m}$ we obtain

$$f(x) = \sum_{m\in\mathbb{Z}^n} \int_{Q_{0,m}} \Phi(x-y)\Psi * f(y)dy + \sum_{v=1}^{\infty} \sum_{m\in\mathbb{Z}^n} \int_{2^{-v}}^{2^{1-v}} \int_{Q_{v,m}} \varphi_t(x-y)\psi_t * f(y)dy\frac{dt}{t}.$$

We define for every $v \geq 0$, $2^{-v} \leq t \leq 2^{-v+1}$ and all $m \in \mathbb{Z}^n$

$$\lambda_{v,m} = C_\varphi \Big(\int_{2^{-v}}^{2^{1-v}} \int_{Q_{v,m}} |\psi_t * f(y)|^2 \, dy \frac{dt}{t} \Big)^{1/2}, \tag{38}$$

where

$$C_\varphi = \max\{\sup_{|y|\leq c} |D^\alpha \varphi(y)| : |\alpha| \leq K\}, \quad c > 0.$$

Define also

$$\varrho_{v,m}(x) = \begin{cases} \frac{1}{\lambda_{v,m}} \int_{2^{-v}}^{2^{1-v}} \int_{Q_{v,m}} \varphi_t(x-y)\psi_t * f(y)dy\frac{dt}{t} & \text{if } \lambda_{v,m} \neq 0 \\ 0 & \text{if } \lambda_{v,m} = 0 \end{cases}. \tag{39}$$

Similarly we define for every $m \in \mathbb{Z}^n$ the numbers $\lambda_{0,m}$ and the functions $\varrho_{0,m}$ taking in (38) and (39) $v = 0$ and replacing ψ_t and φ by Ψ and Φ, respectively. Let us now check that such $\varrho_{v,m}$ are atoms in the sense of Definition 2. Note that the support and moment conditions are clear by (36) and (37), respectively. It thus remains to check (31) in Definition 2. We have

$$\begin{aligned}
&\left|D_x^\beta \varrho_{v,m}(x)\right| \\
\leq\ & \frac{1}{\lambda_{v,m}} \int_{2^{-v}}^{2^{1-v}} \Big(\int_{Q_{v,m}} \left|D_x^\beta \varphi_t(x-y)\right|^2 dy \Big)^{1/2} \Big(\int_{Q_{v,m}} \left|\psi_t * f(y)\right|^2 dy \Big)^{1/2} \frac{dt}{t} \\
\leq\ & \frac{1}{\lambda_{v,m}} \Big(\int_{2^{-v}}^{2^{1-v}} \int_{Q_{v,m}} \left|D_x^\beta \varphi_t(x-y)\right|^2 dy \frac{dt}{t} \Big)^{1/2} \Big(\int_{2^{-v}}^{2^{1-v}} \int_{Q_{v,m}} \left|\psi_t * f(y)\right|^2 dy \frac{dt}{t} \Big)^{1/2} \\
\leq\ & \frac{1}{C_\varphi} \Big(\int_{2^{-v}}^{2^{1-v}} t^{-2(n+|\beta|)} \int_{Q_{v,m}} \left|(D^\beta \varphi)(\frac{x-y}{t})\right|^2 dy \frac{dt}{t} \Big)^{1/2} \\
\leq\ & 2^{v(|\beta|+n/2)}.
\end{aligned}$$

The modifications for the terms with $v = 0$ are obvious.

Step 2. Next we show that there is a constant $c > 0$ such that $\|\lambda\|_{\mathbf{b}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c \|f\|_{\mathbf{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$. For that reason we exploit the equivalent norms given in Theorem 3 involving Peetre's maximal function. Let $v \geq 0$. Taking into account that $|x-y| \leq c\, 2^{-v}$ for $x, y \in Q_{v,m}$ we obtain

$$t^{\alpha(y)-\alpha(x)} \lesssim 2^{v(\alpha(x)-\alpha(y))} \leq \frac{c_{\log}(\alpha) v}{\log(e + 1/|x-y|)} \leq \frac{c_{\log}(\alpha) v}{\log(e + 2^{-v}/c)} \leq c, \quad 2^{-v} \leq t \leq 2^{-v+1},$$

if $v \geq [\log_2 c] + 2$. If $0 < v < [\log_2 c] + 2$, then $2^{v(\alpha(x)-\alpha(y))} \leq 2^{v(\alpha^- - \alpha^+)} \leq c$. Therefore,

$$t^{-\alpha(x)} \left|\psi_t * f(y)\right| \leq c\, t^{-\alpha(y)} \left|\psi_t * f(y)\right|$$

for any $x, y \in Q_{v,m}$ and any $v \in \mathbb{N}$. Hence,

$$\begin{aligned}
\sum_{m \in \mathbb{Z}^n} \lambda_{v,m} t^{-(\alpha(x)+n/2)} \chi_{v,m}(x) &\leq C_\varphi \sum_{m \in \mathbb{Z}^n} t^{-\alpha(x)} \sup_{2^{-v} \leq t \leq 2^{-v+1}} \sup_{y \in Q_{v,m}} \left|\psi_t * f(y)\right| \chi_{v,m}(x) \\
&\leq c \sum_{m \in \mathbb{Z}^n} \sup_{|y-x| \leq c\, 2^{-v}} \frac{t^{-\alpha(y)} \left|\psi_t * f(y)\right|}{(1 + t^{-1}|y-x|)^a} (1 + t^{-1}|y-x|)^a \chi_{v,m}(x) \\
&\leq c\, \psi_t^{*,a} t^{-\alpha(\cdot)} f(x) \sum_{m \in \mathbb{Z}^n} \chi_{v,m}(x) = c\, \psi_t^{*,a} t^{-\alpha(\cdot)} f(x),
\end{aligned}$$

where we have used $\sum_{m \in \mathbb{Z}^n} \chi_{v,m}(x) = 1$. Similarly we obtain

$$\sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \chi_{0,m}(x) \leq c\, \Psi^{*,a} f(x).$$

This estimates gives

$$\|\lambda\|_{\mathbf{b}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c \|f\|'_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c \|f\|_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

by Theorem 3 (with the equivalent norm (9)).

Step 3. Assume that f can be represented by (35), with K and L satisfying (33) and (34), respectively. We will show that $f \in \boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and that for some $c > 0$, $\|f\|_{\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c \|\lambda\|_{\boldsymbol{b}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$. We will use the equivalent norm given in (9). The arguments are very similar to those in [15]. We write

$$\begin{aligned} f &= \sum_{v=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{v,m}\varrho_{v,m} = \sum_{m\in\mathbb{Z}^n} \lambda_{0,m}\varrho_{0,m} + \sum_{v=1}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{v,m}\varrho_{v,m} \\ &= \sum_{m\in\mathbb{Z}^n} \lambda_{0,m}\varrho_{0,m} + \sum_{v=1}^{j} \cdots + \sum_{v=j+1}^{\infty} \cdots . \end{aligned}$$

From Lemmas 10 and 4, we have for any M sufficiently large, $t \in [2^{-j}, 2^{1-j}]$ and any $0 \leq v \leq j$

$$\begin{aligned} &\sum_{m\in\mathbb{Z}^n} t^{-\alpha(x)} |\lambda_{v,m}| \left|\varphi_t * \varrho_{v,m}(x)\right| \\ &\lesssim 2^{(v-j)(K-\alpha^+)} \sum_{m\in\mathbb{Z}^n} 2^{v(\alpha(x)-n/2)} |\lambda_{v,m}| \, \eta_{v,M}(x - x_{Q_{v,m}}) \\ &\lesssim 2^{(v-j)(K-\alpha^+)} \sum_{m\in\mathbb{Z}^n} 2^{v(\alpha(x)+n/2)} |\lambda_{v,m}| \, \eta_{v,M} * \chi_{v,m}(x). \end{aligned}$$

Lemma 1 gives $2^{v\alpha(\cdot)}\eta_{v,M} * \chi_{v,m} \lesssim \eta_{v,T} * 2^{v\alpha(\cdot)}\chi_{v,m}$, with $T = M - c_{\log}(\alpha)$ and since $K > \alpha^+$ we apply Lemma 7 to obtain

$$\begin{aligned} &\left\| \left(\left\| t^{-\frac{1}{q(\cdot)}} \sum_{v=1}^{j} 2^{(v-j)(K-\alpha^+)} \eta_{v,T} * \left(2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \, \chi_{v,m} \right) \right\|_{p(\cdot)} \chi_{[2^{-j},2^{1-j}]} \right)_j \right\|_{\ell^{q(\cdot)}_{>}(L^{q(\cdot)})} \\ \lesssim\ &\left\| \left(t^{-\frac{1}{q(\cdot)}} \sum_{v=0}^{j} 2^{(v-j)(K-\alpha^+)} \left\| 2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \, \chi_{v,m} \right\|_{p(\cdot)} \chi_{[2^{-j},2^{1-j}]} \right)_j \right\|_{\ell^{q(\cdot)}_{>}(L^{q(\cdot)})} \\ \lesssim\ &\left\| \left(t^{-\frac{1}{q(\cdot)}} \left\| 2^{j(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}| \, \chi_{j,m} \right\|_{p(\cdot)} \chi_{[2^{-j},2^{1-j}]} \right)_j \right\|_{\ell^{q(\cdot)}_{>}(L^{q(\cdot)})} \lesssim \|\lambda\|_{\boldsymbol{b}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} . \end{aligned}$$

For $v = 0$, we have

$$\begin{aligned} &\left\| \left(\left\| t^{-\frac{1}{q(\cdot)}} 2^{-j(K-\alpha^+)} \eta_{0,M} * \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{0,m}| \, \chi_{0,m} \right) \right\|_{p(\cdot)} \chi_{[2^{-j},2^{1-j}]} \right)_j \right\|_{\ell^{q(\cdot)}_{>}(L^{q(\cdot)})} \\ \lesssim\ &\left\| \sum_{m\in\mathbb{Z}^n} |\lambda_{0,m}| \, \chi_{0,m} \right\|_{p(\cdot)} \left\| \left(t^{-\frac{1}{q(\cdot)}} 2^{-j(K-\alpha^+)} \chi_{[2^{-j},2^{1-j}]} \right)_j \right\|_{\ell^{q(\cdot)}_{>}(L^{q(\cdot)})} \\ \lesssim\ &\|\lambda\|_{\boldsymbol{b}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} . \end{aligned}$$

Now from Lemma 10, we have for any M sufficiently large, $t \in [2^{-j}, 2^{1-j}]$ and $v \geq j$

$$\begin{aligned} &\sum_{m\in\mathbb{Z}^n} t^{-\alpha(x)} |\lambda_{v,m}| \left|\varphi_t * \varrho_{v,m}(x)\right| \\ &\lesssim 2^{(j-v)(L+1+n/2)} \sum_{m\in\mathbb{Z}^n} 2^{j(\alpha(x)-n/2)} |\lambda_{v,m}| \, \eta_{j,M}(x - x_{Q_{v,m}}) \\ &\lesssim 2^{(j-v)(L+1+n/2)} \sum_{m\in\mathbb{Z}^n} 2^{j(\alpha(x)-n/2)} |\lambda_{v,m}| \, \eta_{j,M} * \eta_{v,M}(x - x_{Q_{v,m}}), \end{aligned}$$

where the last inequality follows by Lemma 3, since $\eta_{j,M} = \eta_{\min(v,j),M}$. Again by Lemma 4, we have

$$\eta_{j,M} * \eta_{v,M}(x - x_{Q_{v,m}}) \lesssim 2^{vn} \eta_{j,M} * \eta_{v,M} * \chi_{v,m}(x).$$

Therefore, $\sum_{m\in\mathbb{Z}^n} t^{-\alpha(x)} |\lambda_{v,m}| \left|\varphi_t * \varrho_{v,m}(x)\right|$ is bounded by

$$\begin{aligned} & c\ 2^{(j-v)(L+1-n/2)} \sum_{m\in\mathbb{Z}^n} 2^{j(\alpha(x)+n/2)} |\lambda_{v,m}| \, \eta_{j,M} * \eta_{v,M} * \chi_{v,m}(x) \\ \lesssim\ & 2^{(j-v)(L+1+\alpha^-)} \eta_{j,T} * \eta_{v,T} * \Big[2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \, \chi_{v,m} \Big](x), \end{aligned}$$

by Lemmas 1 and 3, with $T = M - c_{\log}(\alpha)$. The convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$:

$$\Big\| \Big(t^{-\frac{1}{q(\cdot)}} \sum_{v=j}^{\infty} 2^{(j-v)(L+1+\alpha^-)} \Big\| \eta_{j,T} * \eta_{v,T} * 2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \, \chi_{v,m} \Big\|_{p(\cdot)} \chi_{[2^{-j},2^{1-j}]} \Big)_j \Big\|_{\ell_{>}^{q(\cdot)}(L^{q(\cdot)})}$$

is bounded by

$$\Big\| \Big(t^{-\frac{1}{q(\cdot)}} \sum_{v=j}^{\infty} 2^{(j-v)H} \Big\| 2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \, \chi_{v,m} \Big\|_{p(\cdot)} \chi_{[2^{-j},2^{1-j}]} \Big)_j \Big\|_{\ell_{>}^{q(\cdot)}(L^{q(\cdot)})},$$

where $H := L + 1 + \alpha^-$. Observing that $H > 0$, an application of Lemma 7 yields that the last expression is bounded by

$$c \Big\| \Big(t^{-\frac{1}{q(\cdot)}} \Big\| 2^{j(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}| \, \chi_{j,m} \Big\|_{p(\cdot)} \chi_{[2^{-j},2^{1-j}]} \Big)_j \Big\|_{\ell_{>}^{q(\cdot)}(L^{q(\cdot)})}^{1/r} \lesssim \|\lambda\|_{\boldsymbol{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

Clearly, $\sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \left|\Phi * \varrho_{v,m}(x)\right|$ is bounded by

$$c\ 2^{-v(L+1+\alpha^-)} \eta_{0,T} * \eta_{v,T} * \Big[2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \, \chi_{v,m} \Big](x).$$

Taking the $L^{p(\cdot)}$-norm and the fact the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$, we get

$$\begin{aligned} \big\| \Phi * f \big\|_{p(\cdot)} & \lesssim \sum_{v=0}^{\infty} 2^{-v(L+1+\alpha^-)} \Big\| 2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \, \chi_{v,m} \Big\|_{p(\cdot)} \\ & \lesssim \|\lambda\|_{\tilde{\boldsymbol{b}}_{p(\cdot),q(\cdot)}^{s(\cdot)}}. \end{aligned}$$

where

$$\|\lambda\|_{\tilde{\boldsymbol{b}}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} = \sup_{v\geq 0} \Big\| \Big\| t^{-(\alpha(\cdot)+n/2)v} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \, \chi_{v,m} \Big\|_{p(\cdot)} \Big\|_{L^{q(\cdot)}([2^{-v},2^{1-v}],\frac{dt}{t})}.$$

Indeed, by Hölder's inequality

$$\begin{aligned}
&\Big\| 2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\, \chi_{v,m} \Big\|_{p(\cdot)} \\
\lesssim\ & \frac{1}{\log 2} \int_{2^{-v}}^{2^{1-v}} \Big\| t^{-(\alpha(\cdot)+n/2)v} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\, \chi_{v,m} \Big\|_{p(\cdot)} \frac{dt}{t} \\
\lesssim\ & \Big\| \Big\| t^{-(\alpha(\cdot)+n/2)v} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\, \chi_{v,m} \Big\|_{p(\cdot)} \Big\|_{L^{q(\cdot)}([2^{-v},2^{1-v}],\frac{dt}{t})} \Big\| 1 \Big\|_{L^{q'(\cdot)}([2^{-v},2^{1-v}],\frac{dt}{t})}, \\
\lesssim\ & \|\lambda\|_{\tilde{\boldsymbol{b}}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},
\end{aligned}$$

where $q'(\cdot)$ is the conjugate exponent of $q(\cdot)$. Our estimate follows by the embedding $\boldsymbol{b}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \tilde{\boldsymbol{b}}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and hence the proof is complete. ■

Remark 2 *Let $\alpha : \mathbb{R} \to \mathbb{R}$, $p \in \mathcal{P}(\mathbb{R}^n)$ and $q \in \mathcal{P}(\mathbb{R})$. Let $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ be a resolution of unity and we put $\varphi_t = t^{-n}\varphi(\frac{\cdot}{t})$. The Besov space $\tilde{\boldsymbol{B}}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{\tilde{\boldsymbol{B}}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|\Phi * f\|_{p(\cdot)} + \Big\| \big\| t^{-\alpha(t)} \varphi_t * f \big\|_{p(\cdot)} \Big\|_{L^{q(\cdot)}((0,1],\frac{dt}{t})} < \infty.$$

Similar arguments above can be used to study these function spaces.

6 Appendix

Proof of Lemma 6. Let $p \in \mathcal{P}^{\log}$ with $1 \le p^- \le p^+ < \infty$ and $p_Q^- = \operatorname*{ess\text{-}inf}_{z\in Q} p(z)$. Define $q \in \mathcal{P}^{\log}(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$\frac{1}{q(x,y)} = \max\Big(\frac{1}{p(x)} - \frac{1}{p(y)}, 0\Big).$$

Then

$$\begin{aligned}
&\Big(\frac{\gamma}{w(Q)} \int_Q |f(y)|\, w(y)dy\Big)^{p(x)} \\
\le\ & \max\Big(1, (w(Q))^{1-\frac{p(x)}{p^-}}\Big) \frac{1}{w(Q)} \int_Q |f(y)|^{p(y)}\, w(y)dy + \frac{1}{w(Q)} \int_Q \gamma^{q(x,y)} w(y)dy
\end{aligned}$$

for every cube $Q \subset \mathbb{R}^n$, all $x \in Q$ and all $f \in L^{p(\cdot)}(w) + L^\infty$ with $\big\| f w^{1/p(\cdot)} \big\|_{p(\cdot)} + \|f\|_\infty \le 1$. Indeed, we split $f(y)$ into three parts

$$\begin{aligned}
f_1(y) &= f(y)\chi_{\{y:|f(y)|>1\}}(y) \\
f_2(y) &= f(y)\chi_{\{y:|f(y)|\le 1, p(y)\le p(x)\}}(y) \\
f_3(y) &= f(y)\chi_{\{y:|f(y)|\le 1, p(y)> p(x)\}}(y).
\end{aligned}$$

By convexity of $t \mapsto t^p$,

$$\begin{aligned}
\Big(\frac{\gamma}{w(Q)} \int_Q |f(y)|\, w(y)dy\Big)^{p(x)} &\le 3^{p^- -1} \sum_{i=1}^{3} \Big(\frac{\gamma}{w(Q)} \int_Q |f_i(y)|\, w(y)dy\Big)^{p(x)} \\
&= 3^{p^- -1} (I_1 + I_2 + I_3).
\end{aligned}$$

Estimation of I_1. By Jensen's inequality,

$$I_1 \leq \gamma^{p(x)}\Big(\frac{1}{w(Q)}\int_Q |f_1(y)|^{p_Q^-}\, w(y)dy\Big)^{\frac{p(x)}{p_Q^-}}.$$

Since $|f_1(y)| > 1$, we have $|f_1(y)|^{p_Q^-} \leq |f_1(y)|^{p(y)} \leq |f(y)|^{p(y)}$ and thus

$$\begin{aligned} I_1 &\leq \gamma^{p(x)}\Big(\frac{1}{w(Q)}\int_Q |f(y)|^{p(y)}\, w(y)dy\Big)^{\frac{p(x)}{p_Q^-}-1}\Big(\frac{1}{w(Q)}\int_Q |f(y)|^{p(y)}\, w(y)dy\Big) \\ &\leq (w(Q))^{1-\frac{p(x)}{p_Q^-}}\Big(\frac{1}{w(Q)}\int_Q |f(y)|^{p(y)}\, w(y)dy\Big), \end{aligned}$$

by the fact that $\int_Q |f(y)|^{p(y)}\, w(y)dy \leq 1$. If $\|f\|_\infty \leq 1$, then $f_1(y) = 0$ and $I = 0$.
Estimation of I_2. Again by Jensen's inequality,

$$I_2 \leq \gamma^{p(x)}\frac{1}{w(Q)}\int_Q |f_2(y)|^{p(x)}\, w(y)dy = J.$$

Since $|f_2(y)| \leq 1$ we have $|f_2(y)|^{p(x)} \leq |f_2(y)|^{p(y)}$ and thus

$$J \leq \frac{1}{w(Q)}\int_Q |f(y)|^{p(y)}\, w(y)dy.$$

Estimation of I_3. Again by Jensen's inequality,

$$\Big(\frac{\gamma}{w(Q)}\int_Q |f_3(y)|\, w(y)dy\Big)^{p(x)} \leq \frac{1}{w(Q)}\int_Q (\gamma\,|f(y)|)^{p(x)}\,\chi_{\{|f(y)|\leq 1, p(y)>p(x)\}}(y)w(y)dy.$$

Now, Young's inequality give that the last term is bounded by

$$\begin{aligned} &\frac{1}{w(Q)}\int_Q \Big(|f(y)|^{p(y)} + \gamma^{q(x,y)}\Big)\,\chi_{\{|f(y)|\leq 1, p(y)>p(x)\}}(y)w(y)dy \\ \leq\ &\frac{1}{w(Q)}\int_Q \Big(|f(y)|^{p(y)} + \gamma^{q(x,y)}\Big)\, w(y)dy. \end{aligned}$$

Now observe that

$$\frac{1}{q(x,y)} = \max(\frac{1}{p(x)} - \frac{1}{p(y)}, 0) \leq \frac{1}{s(x)} + \frac{1}{s(y)},$$

where $\frac{1}{s(\cdot)} = \left|\frac{1}{p(\cdot)} - \frac{1}{p_\infty}\right|$. We have

$$\gamma^{q(x,y)} = \gamma^{q(x,y)/2}\gamma^{q(x,y)/2} \leq \gamma^{q(x,y)/2}\left(\gamma^{s(x)/4} + \gamma^{s(y)/4}\right).$$

We suppose that $|Q| < 1$. Then

$$\frac{1}{q(x,y)} = \left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq \frac{c_{\log}(1/p)}{-\log|Q|}.$$

Hence, $\gamma^{q(x,y)/2} \leq |Q|^m$. If $|Q| \geq 1$, then we use $\gamma^{q(x,y)/2} \leq 1$ which follow from $\gamma < 1$. Now by [11, Proposition 4.1.8], see also [12, Lemma 3.3], we obtain the desired inequality.

The convergence of (35). Can be obtained as a by-product of the proof using the same method as in [15, Theorem 4.3]. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. By (30)-(31)-(32) and the Taylor expansion of φ up to order L with respect to the off-points $x_{Q_{v,m}}$, we obtain for fixed v

$$\begin{aligned} &\int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \lambda_{v,m}\varrho_{v,m}(y)\varphi(y)dy \\ =& \int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \lambda_{v,m}\varrho_{v,m}(y)\Big(\varphi(y) - \sum_{|\beta|\leq L}(y-x_{Q_{v,m}})^\beta \frac{\partial^\alpha\varphi(x_{Q_{v,m}})}{\beta!}\Big)dy. \end{aligned}$$

The last factor in the integral can be uniformly estimated from the above by

$$c\ 2^{-v(L+1)}(1+|y|^2)^{-M/2} \sup_{x\in\mathbb{R}^n}(1+|x|^2)^{M/2} \sum_{|\beta|\leq L+1} |\partial^\alpha\varphi(x)|\,,$$

where $M > 0$ is at our disposal. Let $0 < t < 1$ and $s(x) = \alpha(x) + \frac{n}{p(x)}(t-1)$ be such that $L+1 > -s(\cdot) > -\alpha(\cdot)$. Since $\varrho_{v,m}$ are $[K,L]$-atoms, then for every $S > 0$, we have $\left|\varrho_{v,m}(y)\right| \leq c2^{vn/2}\left(1+2^v\left|y-x_{Q_{v,m}}\right|\right)^{-S}$. Therefore,

$$\begin{aligned} &\Big|\int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \lambda_{v,m}\varrho_{v,m}(y)\varphi(y)dy\Big| \\ \leq&\ c\ 2^{-v(L+1)} \int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} 2^{vn/2}\left|\lambda_{v,m}\right| \frac{(1+|y|^2)^{-M/2}}{\left(1+2^v\left|y-x_{Q_{v,m}}\right|\right)^S}dy. \end{aligned}$$

Applying Lemma 4 to obtain

$$\sum_{m\in\mathbb{Z}^n}\left|\lambda_{v,m}\right|\left(1+2^v\left|y-x_{Q_{v,m}}\right|\right)^{-S} \lesssim \sum_{m\in\mathbb{Z}^n}\left|\lambda_{v,m}\right|\eta_{v,S} * \chi_{v,m}(y).$$

We split M into $R+S$. Since we have in addition the factor $(1+|y|^2)^{-S/2}$, Hölder's inequality and $(1+|y|^2)^{-R/2} \lesssim (1+|h|^2)^{-R/2}$ give that the term $\left|\int_{\mathbb{R}^n}\cdots dy\right|$ is bounded by

$$\begin{aligned} & c\ 2^{-v(L+1)} \sum_{h\in\mathbb{Z}^n}(1+|h|^2)^{-R/2}\Big\|\eta_{v,S} * \Big(\sum_{m\in\mathbb{Z}^n} 2^{vn/2}\left|\lambda_{v,m}\right|\chi_{v,m}\Big)\Big\|_{p(\cdot)/t} \\ \lesssim&\ c\ 2^{-v(L+1+s(x))} \sup_{j\geq 0}\Big\|2^{(s(\cdot)+n/2)j}\sum_{m\in\mathbb{Z}^n}\left|\lambda_{j,m}\right|\chi_{j,m}\Big\|_{p(\cdot)/t} \\ =&\ c\ 2^{-v(L+1+s(x))}\sup_{j\geq 0} g_j \end{aligned}$$

for some $x \in \mathbb{R}^n$ and by taking R large enough. By Hölder's inequality,

$$\begin{aligned} g_j &= \frac{1}{\log 2}\int_{2^{-j}}^{2^{1-j}} g_j\frac{d\tau}{\tau} \leq \frac{1}{\log 2}\int_{2^{-j}}^{2^{1-j}}\Big\|\tau^{-(s(\cdot)+n/2)j}\sum_{m\in\mathbb{Z}^n}\left|\lambda_{j,m}\right|\chi_{j,m}\Big\|_{p(\cdot)/t}\frac{d\tau}{\tau} \\ &\lesssim \Big\|\Big\|\tau^{-(s(\cdot)+n/2)j}\sum_{m\in\mathbb{Z}^n}\left|\lambda_{j,m}\right|\chi_{j,m}\Big\|_{p(\cdot)/t}\Big\|_{L^{q(\cdot)}([2^{-j},2^{1-j}],\frac{d\tau}{\tau})}\Big\|1\Big\|_{L^{q'(\cdot)}([2^{-j},2^{1-j}],\frac{d\tau}{\tau})}, \\ &\lesssim \|\lambda\|_{\tilde{\boldsymbol{b}}^{s(\cdot)}_{p(\cdot)/t,q(\cdot)}}\,, \end{aligned}$$

where

$$\|\lambda\|_{\tilde{\boldsymbol{b}}^{s(\cdot)}_{p(\cdot)/t,q(\cdot)}} = \sup_{j\geq 0} \left\| \left\| \tau^{-(s(\cdot)+n/2)j} \sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|\, \chi_{j,m} \right\|_{p(\cdot)/t} \right\|_{L^{q(\cdot)}([2^{-j},2^{1-j}],\frac{d\tau}{\tau})}.$$

The convergence of (35) is now clear by the fact that $L+1+s(x) > 0$ and the embeddings $\boldsymbol{b}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \boldsymbol{b}^{s(\cdot)}_{p(\cdot)/t,q(\cdot)} \hookrightarrow \tilde{\boldsymbol{b}}^{s(\cdot)}_{p(\cdot)/t,q(\cdot)}$. The proof is completed.

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