Back to the sampling theory: a practical and less redundant alternative to "Compressed sensing"

LEONID YAROSLAVSKY

Dept. of Physical Electronics, School of Electrical Engineering, Tel Aviv University, Tel-Aviv, 69978, Israel
**Corresponding author: yaro@eng.tau.ac.il*

Received XX Month XXXX; revised XX Month, XXXX; accepted XX Month XXXX; posted XX Month XXXX (Doc. ID XXXXX); published XX Month XXXX

Abstract
A method is suggested for restoration of images of N samples from their $M<N$ samples in the assumption that images can be replaced by their sparse approximations. The method represents a practical and less redundant alternative to "Compressed sensing". Results of experimental verification of the method are presented and some its limitations are discussed

OCIS codes: *(100.0100) Image processing; (100.2000) Digital image processing; (100.3020) Image reconstruction-restoration; (110.6980) Transforms*

1. Introduction

The very first step in digital image acquisition is discretization, i.e. obtaining a set of numbers, which is sufficient for image reconstruction with a given accuracy. Most frequently discretization is implemented as sampling of the image sensor signal. Before recently, image reconstruction from samples was performed by analog display devices. Because of this sampled image representation was very frequently redundant and, therefore, compressible. With an advent of digital computational imaging, an option of numerical reconstruction of sampled images emerged, which opened a possibility of non-redundant image sampling and numerical restoration with a required accuracy. In the paper we address the ways for implementation of this option.

The main issues in image sampling and numerical restoration are these: given a number M of image samples that can be taken and a number $N>M$ of nodes of a regular sampling grid for a desired image display,

- How to optimally arrange M samples over the image plane?
- How to reconstruct images of $N>M$ samples with the highest possible accuracy?

Two approaches exist to answering these questions: the "Compressive sensing" approach and sampling theory based approach.

The "Compressive sensing" approach suggests techniques of obtaining approximations to images by their "sparse" copies, i.e. by images that have only certain number of non-zero coefficients in domain of a certain "sparsifying" transform, such as Discrete wavelet, Discrete Fourier or Discrete Cosine transforms ([1] - [4]). It was mathematically proven ([5]) that if an image of N samples is known to have, in a domain of a certain transform, only K non-zero transform coefficients out of N, it can be precisely restored from its M random samples by means of **L1** norm minimization in the transform domain, provided the following relationship between the required number of samples N, the required number of measurements M and the number K of signal nonzero spectral coefficients holds:

$$M > 2 \cdot K \cdot \log(N/M)(1 + o(1)). \qquad (1)$$

In this formulation, the number N of required image samples is a free parameter. Obviously, applicability parameters of restored image, such as, for instance, image sharpness, do not depend on N and depend solely on the number K of non-zero spectral coefficients.

The "Compressed sensing" approach gained in recent years a considerable popularity as it promises image restoration from fewer number of samples than it is required to restore. However it has several drawbacks:

- As it follows from Eq. 1, $M>K$, i.e. the number M of required image samples for images with K nonzero spectral coefficients is considerably redundant with respect to the minimal required number, which is K, according to the discrete sampling theorem ([6]). For the range of sparsity K/N of real images from 10^{-1} to 10^{-3}, the required redundancy M/K reaches 5 to10 times ([7]).
- While "Compressive sampling" approach guaranties precise restoration of images with sparse spectrum, it does not say, what specific kind of sparse approximation it provides for real images with spectrum, which is not precisely sparse, and it does not provide numerical measures of sparse approximation error.
- Compressed sensing approach ignores aliasing sampling artifacts, which always emerge unless an appropriate antialiasing filtering is applied to signals at sampling.

2. Sampling theory based non-redundant image sampling and restoration

The sampling theory approach suggests a solution based on the Sampling Theorem, which tells that the minimal number of image samples per unit of image area required for image restoration with a given mean square error, is inverse to the area of image Fourier spectrum, which contains ($1\text{-}\varepsilon^2$) fraction of image energy, where ε^2 is the acceptable mean square error normalized to the image energy ([8]). The corresponding Discrete Sampling Theorem ([6], [8]) that assumes image numerical restoration tells that given K samples of an image, one can approximate this image by an image with K non-zero spectral coefficients in a domain of a certain transform with mean square approximation error equal to the energy of the rest $N\text{-}K$ transform coefficients. The approximation error is minimal if selected are K the most intensive transform coefficients. In case of Discrete Fourier and Cosine transforms, K samples can be taken in arbitrary positions. When N tends to infinity, Discrete Sampling Theorem converts to the classical Sampling Theorem.

Obviously that, in distinction from the "compressive sensing" approach, the discrete sampling theorem based approach provides non-redundant sampled representation of images. However its implementation requires specification, what particular spectral coefficients of the approximating image should be nonzero, which is not required in the "compressed sensing" approach. This, however, should not, in fact, be a problem in practical imager acquisition at least for three reasons:

- The relevant energy compacting transforms, such as DCT, DFT and wavelets, have a feature to compact image energy into few transform coefficients that form compact groups in transform domain rather than chaotically spread over it.

- For overwhelming number of real images it is known that appropriate transforms such as DCT compact image energy into the lower frequency part of spectral components. It is this property that is put in the base of transform coefficient zonal quantization tables in transform image coding such as JPEG.

- As soon as one believes that an image can be replaced by its copy with sparse spectrum, one usually knows, at least roughly, the region in spectral domain, where non-zero spectral coefficients are concentrated; otherwise this belief has no substance.

Therefore, one can, in addition to specifying the number N of desired images samples and the number M of samples to be taken make a natural assumption that image non-zero spectral components important for image reconstruction are concentrated within a certain, say, oval or rectangular shape that encompasses K=M spectral components with the lowest indices. With this assumption, one can reconstruct image sparse approximation defined by the selected spectral shape from a set of $K=M$ samples taken, in the case of sparsity of DCT or DFT spectra, in arbitrary positions.

For image restoration, there are two options:

- Direct inversion of the MxN transform matrix for computing M transform non-zero coefficients specified by the selected spectral shape from M samples with given indices, setting the rest $N\text{-}M$ transform coefficients to zero and applying inverse transform to the found zero-pad spectrum. Generally, matrix inversion is a very hard computational problem and no fast matrix inversion algorithms are known. In our specific case, a pruned transform matrix should be inverted. There exist pruned versions of fast transforms for computing subsets of transform coefficients of signals with all but some subset of samples being zeros ([9] [10]), which is inverse to what is required in given case. The question, whether these pruned algorithms can be adapted for computing subset of transform non-zero coefficients from subset of signal samples is open.

- An iterative Gershberg-Papoulis type algorithm, in which transform and inverse transform are performed alternatively at each iteration; in transform domain transform coefficients that are supposed to be zero are zeroed and then, in image domain, image samples in positions, where they were taken at sampling, are replaced by the corresponding available samples. Computational complexity of this method with using fast transform algorithms is $O(2N_{it}\cdot N\log N)$, where N_{it} is the number of iterations sufficient for achieving a required restoration accuracy.

For the sake of brevity we'll call this method of image sampling and restoration ArSBLR ("Arbitrary Sampling and Band-Limited Restoration")-method.

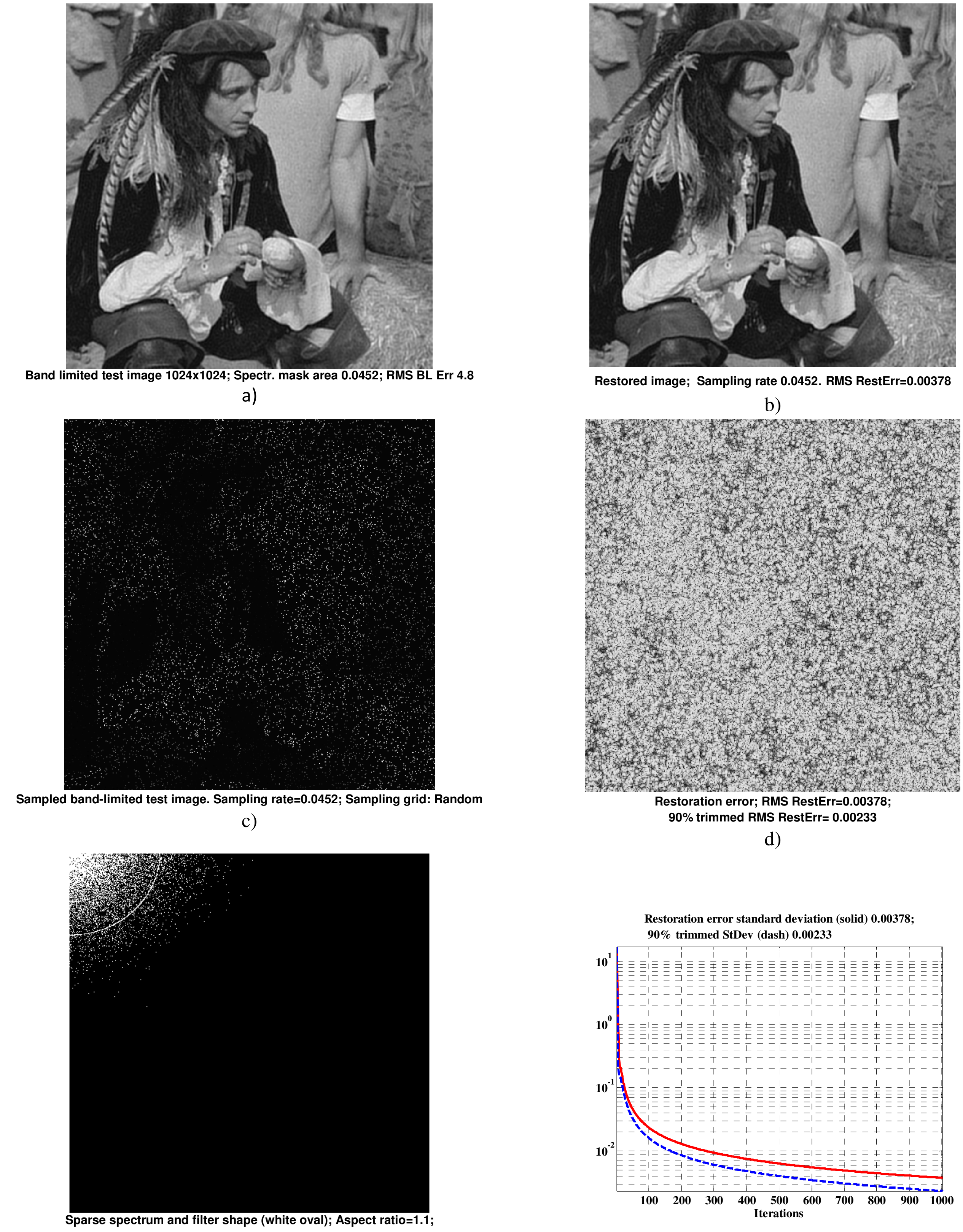
3. Experimental verification and practical considerations

The suggested ArSBLR-method of image sampling and restoration was experimentally verified on a large data base of test images. In the experiments, the described iterative Gershberg-Papoulis type algorithm was used and three types of sampling grid were tested: (i) uniform sampling grid, in which K image samples are uniformly distributed, with appropriate rounding up of their positions to nearest nodes of a dense square sampling grid of N samples; (ii) uniform sampling grid with jitter, in which each node of the grid is randomly placed, independently in each of two image coordinates, within primary uniform sampling intervals; and (iii) random sampling grid, in which positions of samples are random. As an image transform that compacts image spectrum, Discrete Cosine Transform was used. Two types of spectrum shapes of lower frequency part of image DCT spectra were tested in the experiments: rectangular and oval, with aspect ratio (ratio of maximal horizontal dimension to maximal vertical dimension) manually set on the base of visual evaluation of possible presence or absence of image spectrum anisotropy. For instance, if vertical edges prevailed in a particular image, the aspect ratio was set larger than one and when horizontal edges prevailed, it was set lower than one.

It was found in experiments that on the very first steps of the iteration restoration procedure restored image estimates are produced with a number of localized outliers, which very substantially slack the speed of convergence of the process. Introducing a simple outlier filtering on few first iteration steps allows overcoming this problem.

Figure 1 and Figure 2 illustrate the experiments' outcomes obtained for two images of the tested set: test images "Man" of 1024x1024 pixels and "Interferogram" of 256x256 pixels. The former is the image, which was used for demonstrating potentials of the "Compressed Sensing" approach in Ref. [11]. In order to make results of experiments comparable with those reported in [11], spectrum of the image was artificially sparsified by zeroing its all but 25000 most intensive DCT coefficients.

Shown in figures are: initial test image subjected to antialiasing pre-filtering according to the selected image spectrum shape, restored image, sparsely sampled image, restoration error (difference between initial test and restored images), image sparse DCT spectrum and graph of root mean square (RMS) restoration error vs iteration number. RMS error was evaluated in units of image gray level range (0-255). Image sparse spectra were evaluated by selecting image most intensive spectral component that reconstruct image with the same RMS error as regular JPEG compression does. Test image "Man" was sparsely sampled over random sampling grid, test image "Interferogram" was sampled uniformly.



Band limited test image 1024x1024; Spectr. mask area 0.0452; RMS BL Err 4.8

a)

Restored image; Sampling rate 0.0452. RMS RestErr=0.00378

b)

Sampled band-limited test image. Sampling rate=0.0452; Sampling grid: Random

c)

Restoration error; RMS RestErr=0.00378;
90% trimmed RMS RestErr= 0.00233

d)

Sparse spectrum and filter shape (white oval); Aspect ratio=1.1;
RMS Sprs Err=1.44 SpectrSparsity 0.0245. Spectr. mask area 0.0452; RMS BL Err 4.8

e)

f)

Figure 1. Results of experiments on sampling and restoration of test image "Man". Root mean square (RMS) of restoration error is given in units of image range 0-255.

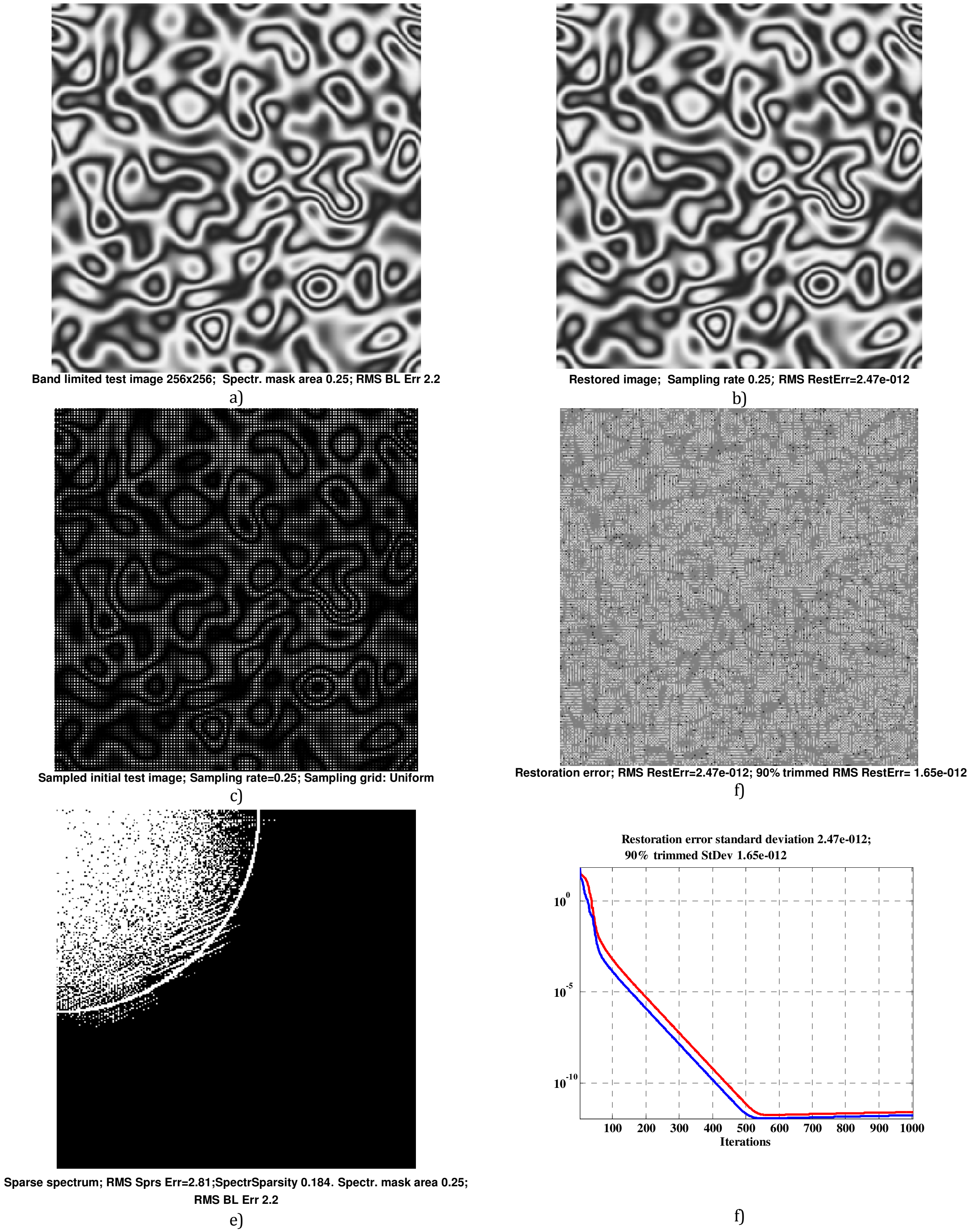


Figure 2. Results of experiments with test image “Interferogram” of 256x256 pixels.

As one can see in figures, test image “Man” is restored with high accuracy (RMS error < 0.004) from 0.0452x1024x1024=47396 samples. “Compressive sensing” approach required 96000 samples for the same image, i.e. more than two times more ([**11**]). Test image “Interferogram” was precisely (to the accuracy of Matlab computations) restored from 0.25x256x256=16384 samples. Note that RMS quantization error for images quantized to 256 gray levels is $1/\sqrt{12}$ =0.2887 in units of image gray level range (0-255). This level of accuracy was achieved in both cases after less than 20 iteration.

The presence of outliers is illustrated in graphs of RMS restoration error in **Figure 1**, f) and **Figure 2**, f), where two plots of RMS errors vs the number of iteration are presented: RMS error found for all errors (solid line) and trimmed RMS error found for 90% of lowest errors.

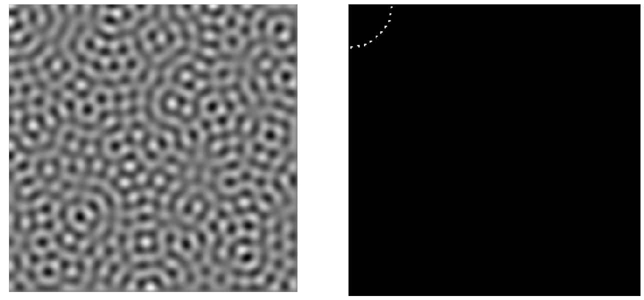
Several practical considerations can be added to conclude the discussion:

- When image anisotropy is clearly visible, one can apply anisotropic oval or rectangular band limitation; for instance in reconstruction of the test image “Man” an oval band limitation shape was used with aspect ratio (ratio of horizontal to vertical diameters) 1.1 in view of the prevalence in the image of vertical edges .

- The speed of convergence of the restoration error to zero depends of the type of the sampling grid used. It is the fastest when sampling grid is uniform; it is the slowest for random sampling grids; it is intermediate between above cases, when sampling grid is uniform with jitter. This has a simple and intuitive explanation. For the case of random sampling, restoration error converges to zero non-uniformly over the image area: considerably slower in places, where samples are happen to be more sparse, and faster in places, where samples are denser than on average.

- The method is somewhat redundant with respect to potential sparsity of image spectra. In Figure 1, e) and in Figure 2, e) shown white are image spectra non-zero spectral components that allow to reconstruct image with the same RMS error as that of image regular JPG compression. The ratio of the number of these coefficients to the number of image samples determines image spectrum sparsity at this level of image approximation accuracy. As one can see on these figures, within the selected spectrum shapes outlined by ovals there are a number of zero samples in image spectra. Experimental experience shows that ArSBLR-method requires 1.5-2 times excessive number of image samples with respect to the number of the image non-zero spectral coefficients. For instance, for the test image “Man” (Figure 1) the redundancy turns to be 1.84.

- There are some texture images, which contain very few periodic components such as an image shown in Figure 3, a), for which ArSBLR-method, with its assumption that image spectra most intensive components are concentrated in low frequency area in spectral domain, will require excessive number of image samples. The area bounded by image spectral components shown by white points in Figure 3, b) represents spectrum shape that would be selected by ArSBLR-method of sampling, and the number of spectral component within this area would be taken as the number of required samples. Obviously, this number substantially exceeds the number of non-zero spectral components of the image sufficient for its restoration. In such cases, for efficient application of the ArSBLR-method, more smart evaluation of signal spectrum shape using appropriate simple measurements of spatial parameters of the image texture is required.



a) b)

Figure 3. A texture image of extremely low sparsity (a) and its DCT spectrum (b, white dots)

4. Conclusion

The described ArSBLR-method of restoration of images of $\boldsymbol{N}$ samples from its arbitrarily taken $\boldsymbol{M}<\boldsymbol{N}$ samples represents a less redundant and practical alternative to “Compressed Sensing”. For the given transform, in which the image is supposed to have a sparse spectrum, and for the given number of image samples $\boldsymbol{M}$, it secures RMS image approximation error equal to the energy of $\boldsymbol{M}$-$\boldsymbol{N}$ transform coefficients set to zero and, if appropriately applied, the absence of aliasing errors. Image restoration from its samples can be easy implemented using fast transform algorithms in the iterative Gershberg-Papoulis procedure with computational complexity $\boldsymbol{O}(2\boldsymbol{N}_{it}\cdot\boldsymbol{N}\log\boldsymbol{N})$, where $\boldsymbol{N}_{it}$ is the number of iterations sufficient for achieving a required restoration accuracy.

References

[1] D. Donoho, “Compressed sensing,” IEEE Trans. Inform. Theory 52(4), 1289–1306 (2006).

[2] E. Candès, “Compressed sampling,” in Proc. Int. Congress of Math., Madrid, Spain (2006)

[3] E. Candès, J. Romberg, and T. Tao, “Stable signal recovery from incomplete and inaccurate measurements,” Commun. Pure Appl. Math. 59 (8), 1207 (2006).

[4] R. Willett, R. Marcia, and J. Nichols, “Compressed sensing for practical optical imaging systems: a tutorial,” Opt. Eng. 50(7), 072601 (July 06, 2011). doi:10.1117/1.3596602

[5] D. L. Donoho and J. Tanner, “Exponential bounds implying construction of compressed sensing matrices, error-correcting codes, and neighborly polytopes by random sampling,” IEEE Trans. Inf. Theory 56, 2002 (2010).

[6] L. P. Yaroslavsky et al., “Non-uniform sampling, image recovery from sparse data and the discrete sampling theorem,” J.Opt. Soc. Am. A 26, 566–575 (2009).

[7] L. P. Yaroslavsky; "Can compressed sensing **beat the Nyquist sampling rate?**" Opt. Eng., 54(7), 079701 (2015).

[8] L. Yaroslavsky, Theoretical Foundations of Digital Imaging Using Matlab, CRC Press, 2012

[9] L. Yaroslavsky, Digital Holography and Digital Image Processing, Kluwer Academic Publishers, Boston, 2004

[10] Y. Katiy, L. Yaroslavsky, Regular Matrix Methods for Synthesis of Fast Transforms: General Pruned and Integer-to-Integer Transforms, in Proc. Of International Workshop on Spectral Methods and Multirate Signal Processing, ed. By T. Saramäki, K. Egiazarioan & J. Astola, SMMSP’2001, Pula, Croatia, June 16-18, 2001, TICSP Series#13, ISSN 1456-2774, ISBN 952-15-0633-4

[11] E. Candès and M. Wakin, An introduction to compressive sampling. IEEE Signal Processing Magazine, 25(2), pp. 21 - 30, March 2008