

PERIODICITY OF CLUSTER TILTING OBJECTS

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ABSTRACT. Let \mathcal{T} be a locally finite triangulated category with an autoequivalence F such that the orbit category \mathcal{T}/F is triangulated. We show that if \mathcal{X} is an m -cluster tilting subcategory, then the image of \mathcal{X} in \mathcal{T}/F is an m -cluster tilting subcategory if and only if \mathcal{X} is F -periodic.

We show that for path-algebras of Dynkin quivers Δ one may study the periodic properties of n -cluster tilting objects in the n -cluster category $\mathcal{C}_n(k\Delta)$ to obtain information on periodicity of the preimage as n -cluster tilting subcategories of $\mathcal{D}^b(k\Delta)$.

Finally we classify the periodic properties of all 2-cluster tilting objects T of Dynkin quivers, in terms of symmetric properties of the quivers of the corresponding cluster tilted algebras $\text{End}_{\mathcal{C}_2}(T)^{op}$. This gives a complete overview of all 2-cluster tilting objects of all orbit categories of Dynkin diagrams.

1. INTRODUCTION

Cluster categories were introduced as a categorification of the combinatorics in cluster algebras, which were introduced in [12]. This was done for path algebras of Dynkin diagrams of type A by [11], and more general for finite-dimensional hereditary algebras by [5]. With this category in place, the authors of [5] were able to generalize the notion of tilting to the notion of cluster tilting. In the cluster category one has cluster tilting objects, which give rise to cluster tilted algebras. The notion of a cluster tilting object was however generalized again, see [13] and [15], to the notion of a cluster tilting subcategory of a triangulated category, and n -cluster tilting subcategory of a triangulated category.

Among the n -cluster tilting subcategories, the case $n = 2$ is of special interest, as results by [6] and [17] show that quotients of triangulated categories by a 2-cluster tilting subcategory produces an abelian category with enough projectives. However n -cluster tilting subcategories in general are also important in connection with higher dimensional Auslander-Reiten theory ([13],[15]).

In this paper we study how periodicity of n -cluster tilting subcategories under certain functors in a triangulated category gives rise to n -cluster tilting subcategories in orbit categories of the triangulated category. In particular we show that for representation-finite hereditary algebras H , the periodicity of an n -cluster tilting object of the m -cluster tilting category $\mathcal{C}_m(H)$ under the suspension functor and the Auslander-Reiten translate carries over to the preimage \mathcal{X} of the bounded derived category $\mathcal{D}^b(H)$. As there is a one-to-one correspondence between m -cluster tilting subcategories of $\mathcal{D}^b(H)$

and m -cluster tilting objects of $\mathcal{C}_m(H)$, this enables us to study the periodic properties within the cluster categories and still obtain the necessary information about periodicity in the bounded derived category $\mathcal{D}^b(H)$.

The mutation classes of quivers of cluster tilted algebras of representation-finite hereditary algebras have been described in [21], [9], and [2]. Based on the symmetry properties of the quivers of the corresponding cluster tilted algebras, we classify the periodic properties of all 2-cluster tilting objects of representation-finite hereditary algebras as an application of the method described above. This gives a complete overview of all 2-cluster tilting objects in all triangulated orbit categories of representation-finite hereditary algebras, with an easy way to determine all the indecomposable summands of the object. In regard to this, it is interesting to note two things. First, in many cases the combinatorics of type D is more involved than the combinatorics of type A. However this is not true for this case, the combinatorics of the proof for the 3-symmetric cases of type A is conceptually not any easier than the proofs of type D. Second, when considering cases of Dynkin type D_n , the parity of n affects the results in a substantial way.

Earlier results on the classification of orbit categories with a 2-cluster tilting object are given in [10], [7] and for a special case for type E in [18]. These results covers 2-cluster tilting objects in 2-Calabi Yau triangulated orbit categories, and the method used is to study orbit-categories of the 2-cluster category. As described above, our method can be applied regardless of the Calabi-Yau dimension of the category. Moreover, our method in itself provide a combinatorial means for finding explicit descriptions of any 2-cluster tilting object T in any orbit category in terms of its indecomposable summands, and hence also the quiver of the corresponding cluster tilted algebra. Furthermore, as noticed above, our method may be applied to any known m -cluster tilting object ($m > 2$) of a triangulated orbit category, to find other orbit categories with m -cluster tilting objects arising from the same preimage in the bounded derived category.

The paper is organized as follows: Section 2 contains the basic definitions from cluster tilting theory that will be used throughout the paper. In section 3 we show how periodicity of cluster tilting subcategories in a triangulated category \mathcal{T} determines cluster tilting subcategories in triangulated orbit categories of \mathcal{T} . Using properties of the bounded derived category for representation-finite hereditary algebras we show that some properties of periodic cluster tilting subcategories of cluster categories carry over to the bounded derived category. Based on this we describe our main method, which is applied in later sections.

Section 4 give a short overview of the quivers of cluster tilted algebras of Dynkin type A and D , and also states results connected to quivers of cluster tilted algebras, that will be used in sections 5 and 6. Then in section 5 we use the method described in section 3, and classify for all 2-cluster tilting subcategories of $\mathcal{D}^b(A_n)$ which functors they are periodic under. The classification is given in terms of symmetric properties of the quivers of the corresponding 2-cluster tilting subcategories in the cluster category $\mathcal{C}_2(A_n)$.

In section 6 we give the corresponding results for 2-cluster tilting subcategories of type D. Type E is studied in section 7. Finally we consider the Euclidean and wild cases in section 8.

2. BACKGROUND

For more background on the following definitions see for example [14].

Definition 1. Let \mathcal{C} be a category and \mathcal{T} a full subcategory of \mathcal{C} . Then \mathcal{T} is called a *functorially finite subcategory* if it is both covariantly finite and contravariantly finite. That is for each object $X \in \mathcal{C}$ there are objects $T, T' \in \mathcal{T}$ with morphisms and $f : X \rightarrow T'$ and $g : T \rightarrow X$ such that for any object $T'' \in \mathcal{T}$ there are epimorphisms $\text{Hom}_{\mathcal{C}}(T', T'') \xrightarrow{\text{Hom}_{\mathcal{C}}(f, T'')} \text{Hom}_{\mathcal{C}}(X, T'')$ and $\text{Hom}_{\mathcal{C}}(T'', T) \xrightarrow{\text{Hom}_{\mathcal{C}}(T'', g)} \text{Hom}_{\mathcal{C}}(T'', X)$.

Definition 2. A functorially finite subcategory \mathcal{T} is called an *r -cluster tilting subcategory* of \mathcal{C} if

$$\begin{aligned} \mathcal{T} &= \{X \in \mathcal{C} \mid \text{Ext}_{\mathcal{C}}^i(\mathcal{T}, X) = 0 \text{ for any } 0 < i < r\} \\ &= \{X \in \mathcal{C} \mid \text{Ext}_{\mathcal{C}}^i(X, \mathcal{T}) = 0 \text{ for any } 0 < i < r\} \end{aligned}$$

If $\mathcal{T} = \text{add } T$ for some object T , then T is called an *r -cluster tilting object*.

Definition 3. We call a triangulated category *locally finite* if for an indecomposable object X there is only a finite number of isomorphism classes of indecomposable objects Y such that $\text{Hom}_{\mathcal{T}}(X, Y) \neq 0$.

Note that definition 3 implies its own dual (see [22],[1]), i.e. for any indecomposable object Y there are only a finite number of isomorphism classes of indecomposable objects X such that $\text{Hom}_{\mathcal{T}}(X, Y) \neq 0$. Also note that any subcategory of a locally finite triangulated category is both contravariantly finite and covariantly finite.

Next we will define what we mean by an orbit category. One can define orbit categories of all additive categories (see [16]), however we will only use the definition for triangulated categories and therefore give the definition in this context. See for example also [5].

Definition 4. Let \mathcal{T} be a triangulated category with an autoequivalence $F : \mathcal{T} \rightarrow \mathcal{T}$. The orbit category $\mathcal{O}_F(\mathcal{T})$ has the same objects as \mathcal{T} . The morphisms $\text{Hom}_{\mathcal{O}}(X, Y)$ between two objects in $\mathcal{O}_F(\mathcal{T})$ are in bijection with the set $\bigoplus \text{Hom}_{\mathcal{T}}(X, F^n Y)$.

In later sections we will mostly consider orbit categories of the bounded derived category $\mathcal{D}^b(H)$ where H is a representation-finite hereditary algebra. We define the m -cluster category of H to be $\mathcal{C}_m(H) := \mathcal{D}^b(H)/\tau^{-}[m-1]$ for $m \geq 2$. In particular it is known ([16],[5]) that the m -cluster category of an hereditary algebra is triangulated, and $\pi_m : \mathcal{D}^b(H) \rightarrow \mathcal{C}_m(H)$ is a triangle functor. Note that in a triangulated category with only finitely many isomorphism classes of indecomposable objects, a cluster tilting subcategory always gives rise to a cluster tilting object. We will often use the term cluster tilting object about a cluster tiling subcategory in such a setting.

Furthermore, we note that the suspension functor in the bounded derived category of a representation-finite hereditary algebra $\mathcal{D}^b(H)$ will be denoted [1], whereas the suspension functor in a general triangulated category will be denoted by Σ . Throughout the paper τ always refers to the AR-translate of the category in question.

3. PERIODICITY DETERMINES CLUSTER TILTING SUBCATEGORIES

In this section we will study how periodicity under a functor F , will determine for an m -cluster tilting subcategory of $\mathcal{D}^b(H)$ in which orbit categories \mathcal{O}_F the image will again be an m -cluster tilting subcategory. The first three results we are able to state in a somewhat more general setting, that is, for a locally finite triangulated category \mathcal{T} in general, rather than for just $\mathcal{D}^b(H)$. For the remaining results however we will need some properties that are particular for $\mathcal{D}^b(H)$.

These results will be the foundation of the method applied throughout the rest of the paper.

Starting out is a lemma showing that the preimage of an m -cluster tilting subcategory of an orbit category \mathcal{O}_F is an m -cluster tilting subcategory of \mathcal{T} :

Lemma 5. *Let \mathcal{T} be a locally finite triangulated category, and F an autoequivalence such that the orbit category $\mathcal{O}_F := \mathcal{T}/F$ is triangulated and $\pi_F : \mathcal{T} \rightarrow \mathcal{O}_F$ is a triangle functor. If \mathcal{Y} is an m -cluster tilting subcategory of \mathcal{O}_F then $\mathcal{X} := \pi_F^{-1}(\mathcal{Y})$ is an m -cluster tilting subcategory of \mathcal{T} .*

Proof. Since \mathcal{T} is locally finite the subcategory \mathcal{X} is functorially finite. Also since the functor $\pi_F : \mathcal{T} \rightarrow \mathcal{O}_F$ is a faithful triangle functor we have the following inclusions:

$$\pi_F^{-1}(\mathcal{Y}) \subset \{Z \in \mathcal{T} \mid \text{Ext}_{\mathcal{T}}^i(\pi_F^{-1}(\mathcal{Y}), Z) = 0 \text{ for } 0 < i < m\}$$

$$\pi_F^{-1}(\mathcal{Y}) \subset \{Z \in \mathcal{T} \mid \text{Ext}_{\mathcal{T}}^i(Z, \pi_F^{-1}(\mathcal{Y})) = 0 \text{ for } 0 < i < m\}$$

We need to show that in fact we have equality, not just inclusion. Let $Z \in \mathcal{T}$ be an object such that $\text{Ext}_{\mathcal{T}}^i(\pi_F^{-1}(\mathcal{Y}), Z) = 0$ for $0 < i < m$, i.e. $\text{Hom}_{\mathcal{T}}(\pi_F^{-1}(\mathcal{Y}), \Sigma^i Z) = 0$ for $0 < i < m$. We want to show that $Z \in \pi_F^{-1}(\mathcal{Y})$, i.e. that $\pi_F(Z) \in \mathcal{Y}$.

Since \mathcal{Y} is an m -cluster tilting subcategory of \mathcal{O}_F we have that

$$\mathcal{Y} = \{? \in \mathcal{O}_F \mid \text{Ext}_{\mathcal{O}_F}^i(\mathcal{Y}, ?) = 0 \text{ for } 0 < i < m\}.$$

Let \bar{Y} be an object of \mathcal{Y} and let Y be its preimage in $\pi_F^{-1}(\mathcal{Y})$. Then we have the following:

$$\begin{aligned} \text{Ext}_{\mathcal{O}_F}^i(\bar{Y}, \pi_F(Z)) &= \text{Hom}_{\mathcal{O}_F}(\bar{Y}, \Sigma^i \pi_F(Z)) \\ &= \text{Hom}_{\mathcal{O}_F}(\pi_F(Y), \pi_F(\Sigma^i Z)) \\ &= \bigoplus \text{Hom}_{\mathcal{T}}(F^p(Y), \Sigma^i Z) = 0 \end{aligned}$$

for $0 < i < m$. Hence we conclude that $\pi_F(Z) \in \mathcal{Y}$. The last inclusion can be shown in a similar way. \square

The next result states a very important fact, if an m -cluster tilting category of a triangulated category \mathcal{T} is periodic under a functor F , then the image in the orbit category \mathcal{O}_F is an m -cluster tilting subcategory.

Lemma 6. *Let \mathcal{T} be a locally finite triangulated category and F an autoequivalence of \mathcal{T} such that $\mathcal{O}_F := \mathcal{T}/F$ is a triangulated category and $\pi_F : \mathcal{T} \rightarrow \mathcal{O}_F$ is a triangle functor. If \mathcal{X} is an F -periodic m -cluster tilting subcategory of \mathcal{T} , then $\mathcal{Y} := \pi_F(\mathcal{X})$ is an m -cluster tilting subcategory of \mathcal{O}_F .*

Proof. First let $\pi_F(X_1)$ and $\pi_F(X_2)$ be two objects of $\pi_F(\mathcal{X}) = \mathcal{Y}$. Then we have:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_F}(\pi_F(X_1), \Sigma^i \pi_F(X_2)) &= \mathrm{Hom}_{\mathcal{O}_F}(\pi_F(X_1), \pi_F(\Sigma^i X_2)) \\ &= \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X_1, \Sigma^i F^p(X_2)) = 0 \end{aligned}$$

for $0 < i < m$. Where the last expression is zero since $F^p(X_2) \in F^p \mathcal{X} = \mathcal{X}$. This gives the inclusions:

$$\begin{aligned} \pi_F(\mathcal{X}) &\subset \{\bar{Z} \in \mathcal{O}_F \mid \mathrm{Ext}_{\mathcal{O}_F}^i(\pi_F(\mathcal{X}), \bar{Z}) = 0 \text{ for } 0 < i < m\} \\ \pi_F(\mathcal{X}) &\subset \{\bar{Z} \in \mathcal{O}_F \mid \mathrm{Ext}_{\mathcal{O}_F}^i(\bar{Z}, \pi_F(\mathcal{X})) = 0 \text{ for } 0 < i < m\}. \end{aligned}$$

For the remaining inclusions, start with an object \bar{Z} of \mathcal{O}_F such that $\mathrm{Ext}_{\mathcal{O}_F}^i(\bar{Z}, \pi_F(\mathcal{X})) = 0$ for $0 < i < m$, i.e. $\mathrm{Hom}_{\mathcal{O}_F}(\bar{Z}, \Sigma^i \pi_F(\mathcal{X})) = 0$ for $0 < i < m$. Let X be any object of \mathcal{X} and $\bar{X} := \pi_F(X)$. Then we have

$$\begin{aligned} 0 &= \mathrm{Hom}_{\mathcal{O}_F}(\bar{Z}, \Sigma^i \bar{X}) = \mathrm{Hom}_{\mathcal{O}_F}(\pi_F(Z), \Sigma^i \pi_F(X)) \\ &= \mathrm{Hom}_{\mathcal{O}_F}(\pi_F(Z), \pi_F(\Sigma^i X)) \\ &\supseteq \mathrm{Hom}_{\mathcal{T}}(Z, \Sigma^i X) \end{aligned}$$

Therefore $Z \in \mathcal{X} = \pi^{-1}(\mathcal{Y})$ and $\bar{Z} = \pi_F(Z) \in \mathcal{Y}$. The last inclusion can be shown in a similar way. Functorial finiteness of \mathcal{Y} follows from \mathcal{T} being locally finite. \square

Theorem 7. *Let \mathcal{T} be a locally finite triangulated category and F an autoequivalence such that $\mathcal{O}_F := \mathcal{T}/F$ is triangulated and the projection functor $\pi_F : \mathcal{T} \rightarrow \mathcal{O}_F$ is a triangle functor. Then there is a bijection between the set of F -periodic m -cluster tilting subcategories of \mathcal{T} and m -cluster tilting subcategories of \mathcal{O}_F .*

Proof. This follows directly from lemma 5 and lemma 6. \square

For the rest of the section we will focus on the case when $\mathcal{T} = \mathcal{D}^b(H)$ where H is a representation-finite hereditary algebra. Note that $\mathcal{D}^b(H)$ is locally finite for any hereditary algebra H , therefore it follows immediately that all subcategories are both covariantly finite and contravariantly finite. Let \mathbb{S}_m denote the endofunctor $\tau[1-m]$ on $\mathcal{D}^b(H)$. It is known that any m -cluster tilting subcategory is \mathbb{S}_m periodic:

Lemma 8. [3] *Let \mathcal{T} be an m -cluster tilting subcategory of $\mathcal{D}^b(H)$ for a representation-finite hereditary algebra H and $m \geq 1$. Then \mathcal{T} is \mathbb{S}_m periodic, that is $\mathbb{S}_m \mathcal{T} = \mathcal{T}$.*

Proof. Let $T_1 \in \mathcal{T}$. Then for any $T_2 \in \mathcal{T}$ and for $m > 1$ we have that $\text{Hom}_{\mathcal{D}^b}(T_1, T_2[m-i]) = 0$ for $0 < i < m$. Using Serre duality we then obtain:

$$\begin{aligned} 0 &= \text{Hom}_{\mathcal{D}^b}(T_1, T_2[m-i]) \cong D \text{Hom}_{\mathcal{D}^b}(T_2[m-i], \mathbb{S}T_1) \\ &\cong D \text{Hom}_{\mathcal{D}^b}(T_2, (\mathbb{S}T_1[-m])[i]) \\ &\cong D \text{Hom}_{\mathcal{D}^b}(T_2, (\mathbb{S}_m T_1)[i]) \end{aligned}$$

Hence the object $\mathbb{S}T_1[m] = \mathbb{S}_m T_1$ is in $\text{add } \mathcal{T}$. Similarly one can show that the object $\mathbb{S}^{-1}T_1[m] = \mathbb{S}_m^{-1}T_1$ also belongs to \mathcal{T} . \square

This implies that we have a 1–1-correspondence between m -cluster tilting subcategories in $\mathcal{D}^b(H)$ and m -cluster tilting subcategories of the m -cluster category:

Corollary 9. *Let H be a representation-finite hereditary algebra. Then there is a 1-1 correspondence between m -cluster tilting subcategories of $\mathcal{D}^b(H)$ and m -cluster tilting subcategories of $\mathcal{C}_m(H) = \mathcal{D}^b(H)/\tau^- [m-1]$.*

Lemma 10. *Let \mathcal{T} be an n -cluster tilting subcategory of $\mathcal{C}_m(H)$, and let \mathcal{X} be the preimage of \mathcal{T} in $\mathcal{D}^b(H)$. If $F_{\mathcal{C}} : \mathcal{C}_m(H) \rightarrow \mathcal{C}_m(H)$ is a functor of the form $\tau^s [t]$, and \mathcal{T} is $F_{\mathcal{C}}$ -periodic then \mathcal{X} is periodic under the functor $F_{\mathcal{D}^b} : \mathcal{D}^b(H) \rightarrow \mathcal{D}^b(H)$ where $F_{\mathcal{D}^b} = \tau^s [t]$.*

Proof. It is known from [5] and [16] that the m -cluster category is triangulated and that $\pi_m : \mathcal{D}^b(H) \rightarrow \mathcal{C}_m(H)$ is a triangle functor for $m \geq 2$. Since π_m is a triangle functor it commutes with the suspension functors in $\mathcal{D}^b(H)$ and $\mathcal{C}_m(H)$ (both denoted [1] by abuse of notation).

From proposition 1.3 in [5] we also have that π_m commutes with the AR-translate in both categories.

Let X be any object in \mathcal{X} . We need to show that $F_{\mathcal{D}} X \in \mathcal{X}$. We therefore have that $F_{\mathcal{C}}(\pi_m(X)) = \pi_m(F_{\mathcal{D}}(X))$, and hence since $F_{\mathcal{C}}(\pi_m(X)) \in \mathcal{T}$ we have that $F_{\mathcal{D}}(X) \in \mathcal{X}$. \square

We will now describe the method that will be applied in sections 5, 6 and 7, where we will continue the focus on representation-finite hereditary algebras. As we see from theorem 7, whether or not an m -cluster tilting subcategory \mathcal{X} of $\mathcal{D}^b(H)$ can be pushed down to an m -cluster tilting subcategory in an orbit category $\mathcal{O}_F(H)$, depends on if \mathcal{X} is periodic under the functor F in $\mathcal{D}^b(H)$. Showing that a subcategory is F -periodic for a certain functor F will in the cases we will study require some counting, and is most easily carried out in a category with only finitely many isomorphism classes of indecomposable objects (i.e. in $\mathcal{C}_m(H)$ rather than in $\mathcal{D}^b(H)$). However, by corollary 9 and lemma 10, for an m -cluster tilting subcategory of $\mathcal{D}^b(H)$ with image $\pi_m(\mathcal{X})$ in the m -cluster category, we may study the periodicity of $\pi_m(\mathcal{X})$ under functors of the form $\tau^s [t]$ in the m -cluster category to obtain information about the periodicity of \mathcal{X} under $\tau^s [t]$ in $\mathcal{D}^b(H)$.

We will use this method to determine the period-properties under functors of the form $\tau^s [t]$ for all 2-cluster tilting subcategories of $\mathcal{D}^b(H)$ for Δ a Dynkin diagram. Note that in 2-cluster categories $\mathcal{C}_2(H)$, we have the isomorphism $\tau^- [1] \cong id$ as functors, and hence $\tau \cong [1]$. Hence we only need to check for periodicity under one of the functors τ^s or $[t]$. In sections 5, 6 and 7 we will study the periodicity of all 2-cluster tilting subcategories of the 2-cluster category of respectively type A, D and E under functors of the type τ^s . Finally we note that in these cases all 2-cluster tilting subcategories are also 2-cluster tilting objects, as the 2-cluster category in these cases have only finitely many isomorphism classes of indecomposable objects.

4. QUIVERS OF CLUSTER TILTED ALGEBRAS

For each cluster tilting object T in a cluster category $\mathcal{C}_2(kQ)$, with Q a Dynkin diagram, there is also a cluster tilted algebra defined as $\text{End}_{\mathcal{C}}(T)^{op}$. In subsequent sections we will investigate some properties of 2-cluster tilting objects of type A and D based on the shape of the quiver of the corresponding cluster tilted algebra. The possible quivers occurring as quivers of cluster tilted algebras of type A_n were classified by Vatne [21]. The first classification of quivers of cluster tilted algebras of type D_n was given in [9] also by Vatne and contained four main types. In [4], Oppermann, Bertani-Økland and Wrålsen were able to reduce this description to three main types of quivers. We will refer to the classification as it is presented in [4].

4.1. Cluster tilted algebras of type A. All cluster tilted algebras of type A are a connected subquiver of the quiver \mathcal{Q} below

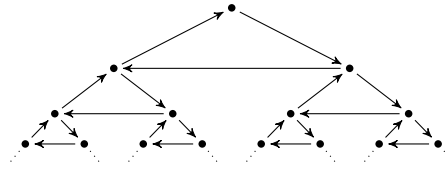


FIGURE 1. The quiver \mathcal{Q}

For type A_n , the cluster tilted algebras can be described as follows:

- all non-trivial cycles are oriented and have length 3.
- any vertex has at most four neighbours
- if a vertex has four neighbours then two of the arrows adjacent to the vertex are part of a 3-cycle, and the other two arrows also adjacent to the vertex are part of another 3-cycle.
- if a vertex has three neighbours then two of the adjacent arrows belong to a 3-cycle and the last arrow adjacent to the vertex does not belong to any 3-cycle.

The next result is not explicitly stated in [4], but follows directly from results in [4]. It shows that the position of a vertex in the quiver of a cluster tilted algebra is closely related to which τ -orbit the corresponding indecomposable summand lies in, in the AR-quiver of the cluster category. This will be very useful in section 5.

Theorem 11. *Let Q be a quiver of a 2-cluster tilting object T of $\mathcal{C}_2(A_n)$. Let i be a vertex of Q , such that deleting i from Q gives rise to two connected*

subquivers Q_1 and Q_2 , with $|Q_1| \leq |Q_2|$. Then the indecomposable summand T_i of T corresponding to vertex i lie in τ -orbit nr $|Q_1| + 1$, counted from the outermost τ -orbit of $\mathcal{C}_2(A_n)$.

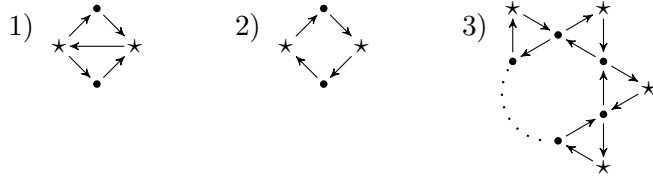
Proof. Suppose that T_i lies in the j outermost τ -orbit. Then by proposition 2.2 [4] this means that deleting the vertex i from Q one gives rise to connected subquivers Q_1^j and Q_2^j of size $n - j$ and $j - 1$. Hence we must have $j = i$. \square

Theorem 12. (lemma 3.2 in [4]) Any 2-cluster tilting object T in $\mathcal{C}_2(A_m)$ is induced by a tilting module over kA_m where A_m has linear orientation.

The next result is important as it gives us an easy method of determining the summands of all cluster tilting objects of $\mathcal{C}_2(A_n)$ having a certain quiver Q as the quiver of the cluster tilted algebra, given that we know the summands for one such cluster tilting object.

Theorem 13. [20] Let T and T' be two cluster tilting objects of type A_n . The cluster tilted algebras $\text{End}_{\mathcal{C}}(T)^{op}$ and $\text{End}_{\mathcal{C}}(T')^{op}$ are isomorphic if and only if $T = \tau^i T'$ for some integer i .

4.2. Cluster tilted algebras of type D. We will base the summary of type D on the description given in [4]. Given a quiver D_n then the quivers of the cluster tilted algebras of D_n are of the following forms:



Each star represents what is called a connecting vertex. At each connecting vertex there is attached a quiver from the mutation class of Dynkin type A_l , note that the case $l = 0$ can occur.

We now go on to distinguish between two different types of indecomposable objects in $\mathcal{C}_2(D_n)$. This will enable a more accurate description of the distribution of the indecomposable summands of each of the the three types of quivers of cluster tilted algebras. One should take note of the Ar-quiver of $\mathcal{D}^b(D_n)$, as shown in figure 2.

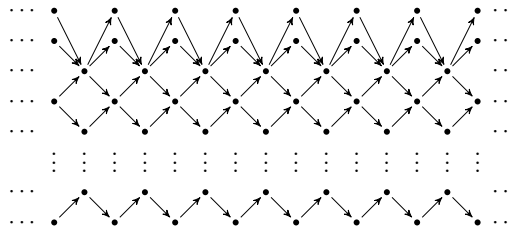


FIGURE 2. The AR-quiver of $\mathcal{D}^b(D_n)$.

Definition 14. *The objects lying in the two top τ -orbits of the AR-quiver of $\mathcal{D}^b(D_n)$ as illustrated above, are called α -objects. Objects in other τ -orbits are called β -objects. Also we define the flip ϕT_i of an α -object T_i to be the other α -object that is a summand of the middle-term in the same AR-triangle as T_i .*

The images of α objects, respectively β objects, in $\mathcal{C}_2(D_n)$ are also called α objects, respectively β objects.

From [4] we have the following theorem, giving us information about the distribution of α - objects for each type of quiver of type D .

Theorem 15. *(theorem 4.1 [4]) Let U be a 2-cluster tilting object of type D_n . Let Q be the quiver of the corresponding cluster tilted algebra. Then*

- *If Q is of type 1 listed above, then U has exactly two α -objects U_α and $\phi_\alpha(U_\alpha)$ as summands.*
- *If Q is of type 2 listed above, then U has exactly two α -objects U_1 and U_2 as summands, and $U_2 \neq \phi U_1$. Furthermore if the size of the subquivers at the connecting vertices are n_1 and n_2 then U_1 and U_2 are such that $U_1 = \tau^{n_1+1}U_2$ or $U_1 = \phi\tau^{n_1+1}U_2$ and $U_2 = \tau^{n_2+1}U_1$ or $U_2 = \phi\tau^{n_2+1}U_1$*
- *If Q is of type 3 then then U has more than two α -objects as summands. The α -objects are distributed within the AR-quiver depending on the size of the subquivers of type A at the various connecting vertices, as described for type 2.*

Consider a 2-cluster tilting object T of $\mathcal{C}_2(D_n)$, with quiver Q . Then it follows from the proof in [4] that if Q_l is a quiver in the mutation class of A_l that is attached at a connecting vertex of Q , then the indecomposable summands of T corresponding to the vertices in Q_l lie in a subcategory of $\mathcal{C}_2(D_n)$ that is equivalent to the category $\text{mod } kA_l$ where A_l has linear orientation. Subcategories of $\mathcal{C}_2(D_n)$ that are equivalent to $\text{mod } kA_n$ for some n with linear orientation on A_n will be called an A -triangle of size n . We will encounter such subcategories again in sections 5 and 6. Furthermore, if Δ is an A -triangle of size l in $\mathcal{C}_2(D_n)$, then we will denote the indecomposable object in $\mathcal{C}_2(D_n)$ corresponding to the projective-injective object of $\text{mod } kA_l$ by $\Pi(\Delta)$.

Lemma 16. [4] *Let T be 2-cluster tilting object in $\mathcal{C}_2(D_n)$, and let Δ be an A -triangle in $\mathcal{C}_2(D_n)$. Then all morphism between an indecomposable object in Δ and an indecomposable summand of T not in Δ must factor through the projective-injective object $\Pi(\Delta)$ of Δ .*

The last result of this section is the equivalent of theorem 13 for type D .

Theorem 17. [8] *Let T and T' be two cluster tilting objects of type D_n . Then the cluster tilted algebras $\text{End}_{\mathcal{C}}(T)^{op}$ and $\text{End}_{\mathcal{C}}(T')^{op}$ are isomorphic if and only if $T = \phi^j \tau^i T'$ for some integers i and j .*

5. 2-CLUSTER TILTING SUBCATEGORIES OF TYPE A

In this section we will apply the method described in the last part of section 3 to the cases where H is a path-algebra of type kA_n , where A_n is

some orientation of the Dynkin diagram A_n . As discussed in section 3, it is sufficient to study under which functors of the form τ^s the cluster tilting objects of the cluster category $\mathcal{C}_2(A_n)$ are periodic, where $s \in \mathbb{Z}$. We give a complete overview of the smallest possible positive value of s for each cluster tilting object of $\mathcal{C}_2(A_n)$. Furthermore, we show that the periodicity of a 2-cluster tilting object depends on symmetric properties of the quiver of the corresponding cluster tilted algebra.

First we need to recall some facts about the AR-quiver of the 2-cluster category of $\mathcal{C}_2(A_n)$. Recall that if n is even then all the τ -orbits in $\mathcal{C}_2(A_n)$ have $n + 3$ objects. If $n = 2l + 1$ is odd, then the innermost τ -orbit has $l + 2$ objects, and all the other τ -orbits have $n + 3$ -objects. Hence for any indecomposable object Z in $\mathcal{C}_2(A_n)$ we have $\tau^{(n+3)}Z = Z$ (for n even and odd). We first give a result narrowing down the possible values of s for which it is interesting to study periodicity of τ^s on the 2-cluster tilting subcategories.

Theorem 18. *Let T be a 2-cluster tilting object in $\mathcal{C}_2(A_n)$ such that $\tau^s T = T$ for some $0 < s < n + 3$. Then*

1. *either $s = \frac{n+3}{2}$, with n an odd number and T_i has exactly one summand in the innermost τ -orbit*
2. *or $s = \frac{n+3}{3}$ and n is divisible by 3.*

Proof. We will first assume that all indecomposable summands of T lie in τ -orbits in the AR-quiver of $\mathcal{C}_2(A_n)$ with $n + 3$ indecomposable objects. Since we have $\tau^{n+3}T = T$ and have assumed $\tau^s T = T$, we must have that s is a factor of $n + 3$, i.e. $n + 3 = sa$. Then $a = \frac{n+3}{s}$ must be a factor of n , since each τ -orbit containing one indecomposable summand of T must contain a multiple of a indecomposable summands of T , and T has n indecomposable summands. That is $(n + 3) = sa$ and $n = ab$ for some $b \in \mathbb{N}$. Hence $a \in \{1, 3\}$. However by assumption $s < n + 3$, so that $a > 1$, therefore we have $a = 3$ and the second part follows.

For the first part, assume that there is at least one indecomposable summand T_i of T that lie in a τ -orbit containing $\frac{n+3}{2}$ indecomposable objects (clearly this can only happen when n is odd). Then we have $\tau^{\frac{n+3}{2}}T_i = T_i$. Note that this is the minimal value of s such that $\tau^s T_i = T_i$. This is most easily seen in a figure. Drawing the Ext-support of T_i will reveal that it has extension to all other objects in the innermost τ -orbit. Hence the claim is proved. \square

We will study the first case of theorem 18 closer in subsection 5.1, and the second case in subsection 5.2.

5.1. 2-symmetric 2-cluster tilting objects of type A. In this subsection we will study 2-cluster tilting objects of $\mathcal{C}_2(A_n)$ which are covered by part 1 of theorem 18. It is clear from theorem 18 that if U is a 2-cluster tilting object of $\mathcal{C}_2(A_n)$ such that $\tau^{\frac{n+3}{2}}U = U$ in $\mathcal{C}_2(A_n)$, then U has one indecomposable summand lying in the innermost τ -orbit. Our aim is to describe the quiver Q of $\text{End}_{\mathcal{C}}(U)^{op}$ when U is such that $\tau^{\frac{n+3}{2}}U = U$. We start with an immediate consequence of U having an indecomposable summand in the innermost τ -orbit.

Theorem 19. *Let U be a basic 2-cluster tilting object of $\mathcal{C}_2(A_n)$ with a summand U_{l+1} in the innermost τ -orbit. Then the quiver of $\text{End}_{\mathcal{C}}(U)^{op}$ is of the form illustrated in figure 3, where Q_1 and Q_2 are quivers of cluster tilted algebras of type $A_{\frac{n+1}{2}}$, and Q_1 and Q_2 have exactly one vertex in common, namely the vertex corresponding to the summand U_{l+1} .*

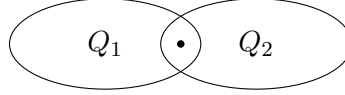


FIGURE 3. Quiver of type A_n with one summand in innermost τ -orbit

Proof. Direct from [4] we have that U may be considered as $U = U_1 \oplus U_{l+1} \oplus U_2$ where U_{l+1} lies in the innermost τ -orbit of $\mathcal{C}_2(A_n)$, U_1 and U_2 are 2-cluster tilting objects of type $A_{\frac{n-1}{2}}$. Since Q is connected $U_1 \oplus U_{l+1}$ and $U_{l+1} \oplus U_2$ is connected of type $A_{\frac{n+1}{2}}$. The distribution of summands is illustrated in figure 4. \square

Definition 20. *Let U be a 2-cluster tilting object of $\mathcal{C}_2(A_n)$ with quiver Q , where $n = 2l + 1$ and one summand U_{l+1} of U lie in the innermost τ -orbit of $\mathcal{C}_2(A_n)$. Then we call U a 2-symmetric 2-cluster tilting object if $Q_1 = Q_2$, (i.e. if the quiver Q is mirror-symmetric about the vertex corresponding to U_{l+1}).*

As we will go on to show, a 2-cluster tilting object T of $\mathcal{C}_2(A_n)$ is 2-symmetric if and only if $\tau^{(n+3)/2}T = T$. In order to show this we will first give a characterization of pairs of indecomposable summands that are closed under $\tau^{(n+3)/2}$.

Definition 21. *Let M and N be two indecomposable objects of $\mathcal{C}_2(A_n)$. We call M and N vertically aligned if there exists a sequence of indecomposable objects $M, M_1, M_2, \dots, M_t, N$ where $M \oplus M_1$ is the middle term of an*

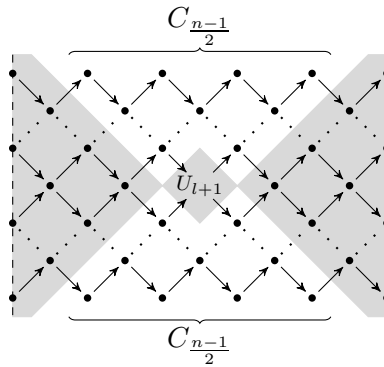


FIGURE 4. figure

AR-sequence, $M_i \oplus M_{i+1}$ is the middle term of an AR-sequence for each $i \in \{1, \dots, t\}$, and lastly $M_t \oplus N$ is the middle term of an AR-sequence.

Note that two vertically aligned indecomposable objects lying in the same τ -orbit of the AR-quiver, have to live in an τ -orbit with $n+3$ indecomposable objects.

Lemma 22. *Let $n = 2l + 1$, and let T_i and T_j be indecomposable vertically aligned objects lying in the τ -orbit of $\mathcal{C}^2(A_n)$. Then $\tau^{l+2}T_i = T_j$ and $\tau^{l+2}T_j = T_i$.*

Proof. Since T_i and T_j are vertically aligned, they can not be in the innermost τ -orbit. Without loss of generality we may assume that T_i and T_j lie in the part of the category corresponding to the module category of kA_n with linear orientation. This will make it possible to count how many objects lie between T_i and T_j , by first counting within the AR-quiver of kA_n and then adding the extra objects corresponding to the projectives shifted once. Let the row of the AR-quiver of kA_n with one object be row nr 1, and the row with n objects be row nr n , i.e. row nr 1 and row nr n are both contained in the outermost τ -orbit of the AR-quiver of $\mathcal{C}_2(A_n)$.

Assume now that T_i is located in row nr d of the AR-quiver of kA_n , where $0 < d < l + 1$. Moreover, assume that within this row of the AR-quiver of kA_n there are s objects to the right of T_i and $d - s - 1$ objects to the left of T_i (total of d objects).

Since T_i and T_j are assumed to be vertically aligned and in the same τ -orbit of the AR-quiver of $\mathcal{C}_2(A_n)$, the location of T_j correspond to being in row $n - d + 1$ in the AR-quiver of kA_n . Furthermore, row $n - d + 1$ has $n - d + 1 - d$ more objects than row nr d . Hence there is $\frac{1}{2}(n - 2d + 1) + (d - s - 1)$ objects to the left of T_j within the row $n - d + 1$ of the AR-quiver of kA_n , and there are $s + \frac{1}{2}(n - 2d + 1)$ objects to the right of T_j .

Starting from T_i and counting indecomposable objects along the τ -orbit towards T_j , there are $s + 1 + \frac{1}{2}(n - 2d + 1) + (d - s - 1) = l + 1$ objects between T_i and T_j . Hence $\tau^{l+2}T_i = T_j$. Counting along the τ -orbit from T_j towards T_i yields the same equations and hence $\tau^{l+2}T_j = T_i$. \square

Theorem 23. *Let $n = 2l + 1$ and let T_i and T_j be indecomposable objects lying in a tau-orbit of $\mathcal{C}_2(A_n)$ containing $n + 3$ objects. Then T_i and T_j are vertically aligned if and only if $\tau^{l+2}T_i = T_j$ and $\tau^{l+2}T_j = T_i$.*

Proof. One implication is shown in lemma 22. Assume that T_i and T_j are indecomposable objects such that $\tau^{\frac{n+3}{2}}T_i = T_j$ and $\tau^{\frac{n+3}{2}}T_j = T_i$. Since n is odd T_j and T_i must be vertically aligned. \square

Theorem 24. *Let U be a basic 2-cluster tilting object of $\mathcal{C}_2(A_n)$ where n is odd. Then $\tau^{\frac{n+3}{2}}U = U$ if and only if U is 2-symmetric.*

Proof. First assume that U is 2-symmetric. We now want to determine the distribution of the indecomposable summands of U within the AR-quiver of $\mathcal{C}_2(A_n)$. There is one summand U_{l+1} in the innermost τ -orbit, for which it is clear that $\tau^{\frac{n+3}{2}}U_{l+1} = U_{l+1}$.

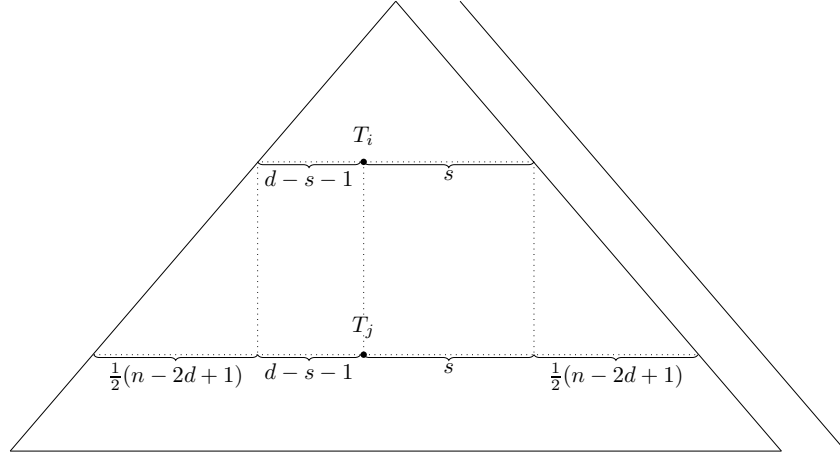


FIGURE 5. Schematic figure of $\mathcal{C}_2(A_n)$. Objects T_i and T_j lie in the d -outermost τ -orbit.

By the proof of theorem 19 we can write U on the form $U = U_1 \oplus U_{l+1} \oplus U_2$ where U_{l+1} lies in the innermost τ -orbit. Choose a basic tilting object T of $\text{mod } kA_{l+1}$ with the same quiver as $\text{End}_{\mathcal{C}}(U_1 \oplus U_{l+1})$. Let U_{l+1} be identified with the projective-injective object of $\text{mod } kA_{l+1}$, and the other indecomposables of T with the corresponding vertices in the subcategory of type A_{l+1} determined by U_{l+1} . By choosing the same tilting object T again we determine indecomposables in the AR-quiver that correspond to U_2 . The indecomposables of U_1 and U_2 are pairwise vertically aligned. By theorem 23 we are done.

Now assume that $\tau^{\frac{n+3}{2}}U = U$. By theorem 23 the indecomposable summands of U are pairwise vertically aligned. Since U has an odd number of indecomposable summands, there is one indecomposable summand U_{l+1} of U lying in the innermost τ -orbit. Hence the quiver Q of $\text{End}_{\mathcal{C}}(U)^{op}$ is of the shape described by figure 3, and U is a 2-symmetric 2-cluster tilting object in $\mathcal{C}_2(A_n)$. \square

Corollary 25. *Let T be a 2-symmetric 2-cluster tilting object of type A_n and \mathcal{X} be the preimage in the bounded derived category. Then $\tau^{\frac{n+3}{2}}\mathcal{X} = \mathcal{X}$.*

Proof. Follows from lemma 10. \square

5.2. 3-symmetric 2-cluster tilting objects of type A. We will now focus on the second case of theorem 18. We give a description of the quivers of the corresponding to the 2-cluster tilting objects \mathcal{T} such that $n = 3l$ and $\tau^{l+1}\mathcal{T} = \mathcal{T}$.

Definition 26. *Let T be a 2-cluster tilted object of type \mathbb{A}_n where $n = 3l$ and Q is the quiver of the corresponding cluster tilted algebra $\text{End}(T)^{op}$. Then we call T 3-symmetric if the quiver of $\text{End}(T)^{op}$ has a central 3-cycle with the same subquiver Q_l of type A_l attached at each vertex of the 3-cycle (Each subquiver of size A_l contains one of the vertices in the central 3-cycle).*

Theorem 27. *Let U be a 3-symmetric 2-cluster tilting object of $\mathcal{C}_2(A_n)$ where $n = 3l$. Then $\tau^{l+1}U = U$.*

Proof. Let U be a 3-symmetric 2-cluster tilting object of $\mathcal{C}_2(A_n)$ where $n = 3l$. Denote by Q the quiver of $\text{End}_{\mathcal{C}}(U)^{op}$ and by Q_l the subquivers of Q as indicated in figure 6. We will first define a set of indecomposable objects V of $\mathcal{C}_2(A_n)$. Then we will show that V is a 2-cluster tilting object with quiver Q (the same quiver as U), and $\tau^{l+1}V = V$. By Theorem 13 this completes the proof.

We now describe the set V of n indecomposable objects in $\mathcal{C}_2(A_n)$. Start by choosing a random object in the l outermost τ -orbit, and name this object V_l . Then choose $V_{2l} := \tau^{l+1}V_l$ and $V_{3l} := \tau^{l+1}V_{2l} = \tau^{2l+2}V_l$. The objects V_l, V_{2l} and V_{3l} determine 3 Abelian subcategories of type A_l (where A_l has linear orientation) in $\mathcal{C}_2(A_{3l})$. We denote the respective subcategories by Δ_1, Δ_2 and Δ_3 . An illustration of this situation is given in figure 7.

The projective-injective object $\Pi(\Delta)$ of Δ_i correspond to the object V_{il} . In Δ_1 we choose a tilting object $V_{1\dots l}$ with quiver Q_l . Note that V_l will be a summand of $V_{1\dots l}$ as it corresponds to the projective injective object. Then we choose the corresponding tilting objects $V_{l+1\dots 2l}$ in Δ_2 and $V_{2l+1\dots 3l}$ in Δ_3 , containing respectively V_{2l} and V_{3l} . We then define $V := V_{1\dots l} \oplus V_{l+1\dots 2l} \oplus V_{2l+1\dots 3l}$.

From figure 7 it is clear that the objects V_l, V_{2l} and V_{3l} are chosen in such a way that they are compatible in the same 2-cluster tilting object. There are no extensions between any objects in Δ_1 and Δ_2 , Δ_2 and Δ_3 and Δ_3 and Δ_1 due to the choice of V_l, V_{2l} and V_{3l} . Furthermore, there are no extensions between any indecomposable summands within Δ_1, Δ_2 or Δ_3 since $V_{1\dots l}, V_{l+1\dots 2l}$ and $V_{2l+1\dots 3l}$ are tilting objects in respective subcategories. Since V has exactly n indecomposable summands, we conclude that it is indeed a 2-cluster tilting object of $\mathcal{C}_2(A_n)$.

From the method used to choose $V_{1\dots l}, V_{l+1\dots 2l}$ and $V_{2l+1\dots 3l}$ we have that $\tau^{l+1}V = V$. Hence the theorem is proved. \square

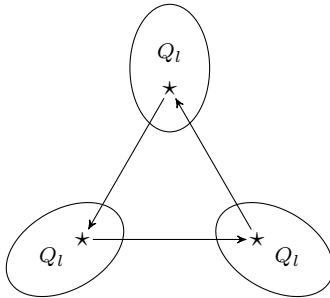


FIGURE 6. The general shape of the quiver of a 3-symmetric 2-cluster tilting object of type A_{3l} .

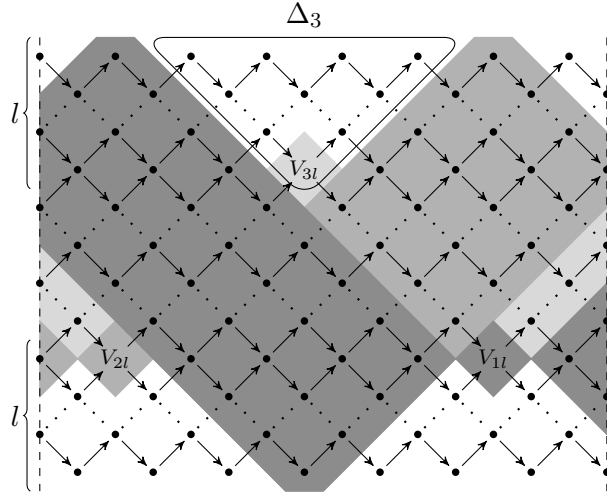


FIGURE 7. The indecomposables on a white background may be part of the same 2-cluster tilting object as V_l, V_{2l} and V_{3l} . The indecomposables with the same background color as V_{3l} or darker may not be part of a 2-cluster tilting object together with V_{3l} , and the same is also true for V_{2l} and V_l .

Lemma 28. *Let $n = 3l$ and U be a 2-cluster tilting object of $C_2(A_n)$ such that $\tau^{l+1}U = U$. Then all indecomposable summands of U lie in the l outermost τ -orbits of the AR-quiver of $C_2(A_n)$.*

Proof. Assume that U_i is an indecomposable summand of U , lying in τ -orbit nr i . Then for U_i we have that $\text{Ext}_{\mathcal{C}}(\tau^{-j}U_i, U_i) \neq 0$ for $j = 0, \dots, i$. If $i > l$ this makes it impossible to find any other indecomposable summand \overline{U}_i of U such that $\tau^{-l}U_i = \overline{U}_i$. This is illustrated in figure 8. \square

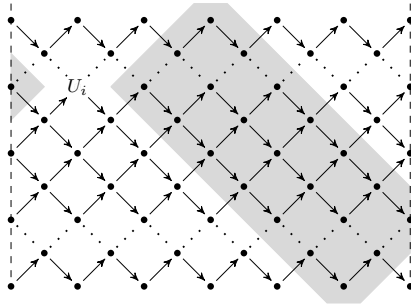


FIGURE 8. The Ext-support of an indecomposable object in $C_2(A_n)$.

Note that all 2-cluster tilting objects of type A_{3l} that do not contain a 3-cycle contain indecomposable objects that do not lie in the l -outermost τ -orbits.

We now go on to show the last implication for the 3-symmetric case, namely that if one has $\tau^{l+1}T = T$ for a 2-cluster tilting object T of $C_2(A_{3l})$, then T is 3-symmetric. As remarked before theorem 11, there is a close relation between the placement of a vertex in the quiver a cluster tilted algebra, and the placement of the corresponding indecomposable vertex in

the AR-quiver of the cluster category. Before proceeding with this last proof, we need to name a special type of vertices in the quiver, namely the maximal dividing vertices. Basically a maximal dividing vertex that divides the quiver in two subquivers of as equal size as possible. These will be the vertices that correspond to the indecomposables V_l , V_{2l} and V_{3l} as in the proof of theorem 27 and in figure 7.

Definition 29. *Let Q be the quiver of a 2-cluster tilting object of type A_n . We define a vertex M of Q to be a maximal dividing vertex if*

- *deleting M from Q yields to subquivers Q_1^M and Q_2^M that are each connected but not interconnected.*
- $\|Q_1^M\| - \|Q_2^M\| = \min \{ \|Q_1^i\| - \|Q_2^i\| \mid i = 1, \dots, n \}$

Theorem 30. *Let U be a 2-cluster tilting object of $\mathcal{C}_2(A_n)$ where $n = 3l$. If $\tau^{l+1}U = U$ then U is 3-symmetric.*

Proof. Let Q be the quiver of $\text{End}_{\mathcal{C}}(U)^{op}$. We aim to show that U must have at least one indecomposable in the l outermost τ -orbit.

Let i be a maximal dividing vertex of Q , dividing the quiver into two quivers Q_1^i and Q_2^i with respectively n_1^i and n_2^i vertices. Assume that $n_2^i \geq n_1^i$.

We start by studying how Q_2^i is connected to the vertex i in Q . There are two possibilities, either they are connected by a single arrow, or there is a 3-cycle in Q formed by the vertex i and two vertices of Q_2^i .

First assume that i and Q_2^i are connected by a single arrow in Q . Denote by $i+1$ the vertex in Q_2^i that is a neighbour of i in Q . Consider what happens if we instead divide the quiver Q at the vertex $i+1$, this would give two subquivers of size $n_2^i - 1$ and $n_1^i + 1$. Assume that vertex $i+1$ is also a maximal dividing vertex, and hence we obtain the same difference between subquivers by dividing at this vertex. Then $n_2^i - n_1^i = |(n_2^i - 1) - (n_1^i + 1)|$, thus $n_2^i = n_1^i + 1$. Given any 2-cluster tilting object T in $\mathcal{C}_2(A_{3l})$ with quiver Q , then by theorem 11 the indecomposable object T_i corresponding to vertex i lies in the innermost τ -orbit of the AR-quiver, contradicting theorem 28. Let us therefore assume that the vertex $i+1$ is not a maximal dividing vertex, giving rise to subquivers of size $n_2^i - 1$ and $n_1^i + 1$. Since $i+1$ is not a maximal dividing vertex we must have the inequality $n_2^i - n_1^i < |(n_2^i - 1) - (n_1^i + 1)|$, and so $n_2^i = n_1^i$. This implies that any indecomposable object T_i corresponding to the vertex i in a 2-cluster tilting object T with quiver Q must lie in the innermost τ -orbit in the AR-quiver by theorem 11. This contradicts lemma 28.

From the above considerations, it is clear that there is a 3-cycle in Q formed by the vertex i and two vertices of Q_2^i . Denote the other two vertices in the 3-cycle by $2i$ and $3i$. Dividing at vertex $2i$ gives rise to a subquiver Q_{2i} of size n_2^{2i} not containing i and $3i$, and similarly dividing at vertex $3i$ gives rise to a subquiver Q_{3i} of size n_3^{3i} not containing i or $2i$. Hence $n_1^i \geq \max\{n_2^{2i}, n_3^{3i}\}$ since i is a maximal dividing vertex. The situation at this point is illustrated in figure 9. Recall that the vertices $i, 2i$ and $3i$ are not counted as vertices in the quivers Q_2^i, Q_{2i} and Q_{3i} . Hence we have $n_1^i + n_2^{2i} + n_3^{3i} + 3 = n$. Combined with the last inequality we then have $3n_1^i + 3 \geq n$ so $n_1^i \geq l - 1$ (recall that $n=3l$). By theorem 11 and lemma 28 this means

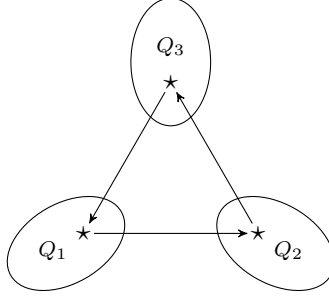


FIGURE 9. Illustration of proof of Theorem 30

that any indecomposable object U_i of U corresponding to vertex i must lie in the l^{th} outermost τ -orbit of the AR-quiver of $\mathcal{C}_2(A_{3l})$. However, by the assumption that $\tau^{l+1}U = U$, there are at least two more indecomposable objects of U lying in this τ -orbit, namely $\tau^{l+1}U_i$ and $\tau^{2l+2}U_i$. By the proof of lemma 28 this means that U is 3-symmetric. \square

Theorem 31. *Let T be a 3-symmetric 2-cluster tilting object of type A_n and \mathcal{X} be the preimage in the bounded derived category. Then $\tau^{l+1}\mathcal{X} = \mathcal{X}$.*

Proof. Follows from lemma 10. \square

5.3. Summary of results type A. We give a brief recount of the results obtained for type A.

Theorem 32. *Let T be a 2-cluster tilting object of $\mathcal{C}_2(A_n)$ such that $\tau^s T = T$ for some $0 < s < n + 3$. Then*

- $s = \frac{n+3}{2}$ for odd n or
- $s = \frac{n+3}{3}$ for n divisible by 3.

Furthermore

- $s = \frac{n+3}{2}$ for odd n if and only if T is 2-symmetric and
- $s = \frac{n+3}{3}$ for n divisible by 3 if and only if T is 3-symmetric.

Based on theorem 32 we give the following corollary 33, giving a complete overview of all triangulated orbit categories $\mathcal{O}_F(A_n)$ containing a 2-cluster tilting object.

Corollary 33. *Let $\mathcal{O}_F(A_n) = \mathcal{D}^b(A_n)/F$ for $n \geq 1$. Then $\mathcal{O}_F(A_n)$ has a 2-cluster tilting object if and only if F is of the following form:*

- $F = \tau^{t(n+3)}$ for all $n \geq 1$ and $t \in \mathbb{Z}$
- $F = \tau^{t(n+3)-1}[1]$ for all $n \geq 1$ and $t \in \mathbb{Z}$
- $F = \tau^{t(l+2)}$ for $n = 2l + 1$ and $t \in \mathbb{Z}$
- $F = \tau^{t(l+2)-1}[1]$ for $n = 2l + 1$ and $t \in \mathbb{Z}$
- $F = \tau^{t(l+1)}$ for $n = 3l$ and $t \in \mathbb{Z}$
- $F = \tau^{t(l+1)-1}[1]$ for $n = 3l$ and $t \in \mathbb{Z}$

.

Proof. There are two main types of AR-quivers to consider, cylindrical and Moebius-shaped. For each of these two main types we consider the three

different periodicities a 2-cluster tilting object of type A_n may have and apply lemma 6 and lemma 10. \square

6. 2-CLUSTER TILTING SUBCATEGORIES OF TYPE D

In this section the focus will be on Dynkin diagrams of type D and 2-cluster tilting subcategories T of these. We will first discuss the AR-quiver, in order to obtain some value of s such that $\tau^s T = T$. Recall from section 4, that quivers of cluster tilted algebras of type D are divided into three subcategories, depending on the number and distribution of α -objects. We will treat each subtype of quiver in its own subsection.

The AR-quiver of $\mathcal{D}^b(D_n)$ is illustrated in figure 2. It is important to note that for even values of n the suspension functor in $\mathcal{D}^b(D_n)$ sends an indecomposable object to an object in the same τ -orbit. However for odd values of n the suspension functor sends an α -object to an α -object in the other τ -orbit containing α -objects. For indecomposable β -objects the suspension functor sends the object to another object in the same τ -orbit, both for even and odd values of n .

This impacts the number of objects in each τ -orbit of the AR-quiver of $\mathcal{C}_2(D_n)$. For even values of n , all τ -orbits of the AR-quiver of $\mathcal{C}_2(D_n)$ contains n objects. However for odd values of n , it is well-known that the top two rows of objects, the α -objects, in the AR-quiver of $\mathcal{C}_2(D_n)$ form a Moebius-band. Hence there is only one τ -orbit of α -objects, containing a total of $2n$ objects. The τ -orbits containing β -objects contain n objects also for odd values of n .

From the above remarks on the length of τ -orbits in the AR-quiver of $\mathcal{C}_2(D_n)$ it follows that for a 2-cluster tilting object in $\mathcal{C}_2(D_n)$ we have $\tau^{2n} T = T$ for all n . For even values of n we have $\tau^n T = T$. However for any indecomposable β -object B we have $\tau^n B = B$ in $\mathcal{C}_2(D_n)$ for all values of n .

At this point one should recall that by theorem 17, we can only expect to classify 2-cluster tilting objects of type D_n up to flip of the α -objects. We will see that for 2-cluster tilting objects of type 1 this does not affect the results, however for type 2 and 3 this is of significance.

Before proceeding with the three subtypes of quivers of type D , we have a result limiting possible values that s may take such that $\tau^s T = T$:

Lemma 34. *If T is a 2-cluster tilting object of \mathcal{D}_n such that the quiver Q is of type 2 or 3, and $\tau^s T = T$, then s is an even number.*

Proof. All the three subtypes of quivers have at least two vertices corresponding to α -objects in the AR-quiver of $\mathcal{C}_2(D_n)$. Hence this can be seen directly from the Ext-support of an α -object, illustrated in figure 10. \square

6.1. 2-cluster tilting objects/subcategories of type D subtype 1.

We are now ready to consider the first subtype of cluster tilting objects of type D . Recall that a 2-cluster tilting object T giving rise to a cluster tilted algebra of type D_n of type 1 have exactly two indecomposable objects that are α -objects, T_1 and ϕT_1 . Knowing the exact number and placement of the α -objects give us the following result:

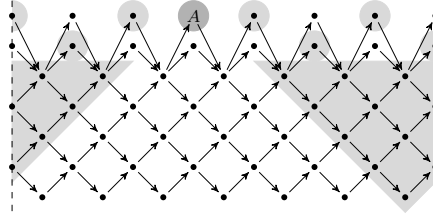


FIGURE 10. The indecomposables with a light grey background may not be part of the same 2-cluster tilting object as A .

Theorem 35. *Let T be a 2-cluster tilting object of $\mathcal{C}_2(D_n)$ giving rise to a cluster tilted algebra $\text{End}_{\mathcal{C}}(T)^{op}$ with quiver Q of type 1. Assume that $\tau^s T = T$ and s is the smallest integer such that this is true. Then $s = n$.*

Proof. First assume that n is an even number. Then there are two τ -orbits of α -objects, each containing n objects. Since T_1 and ϕT_1 do not lie in the same τ -orbit, and these are the only α -objects it is clear that n is the smallest value of s such that $\tau^s T_1 = T_1$. Now assume that n is an odd number. Then T_1 and ϕT_1 lie in the same τ -orbit containing $2n$ objects. Hence $\tau^n T_1 = \phi T_1$ and $\tau^n \phi T_1 = T_1$, and therefore n is the smallest value of s such that $\tau^s T = T$. \square

6.2. 2-cluster tilting objects/subcategories of type D subtype 2.

The second subtype of 2-cluster tilting objects of $\mathcal{C}_2(D_n)$ have two α -objects T_1 and T_2 , such that $T_2 \neq \phi T_1$. Furthermore from theorem 15 we know that if the size of the subquivers of type A attached at the connecting vertices are n_1 and n_2 then $T_1 = \tau^{n_1+1} T_2$ or $T_1 = \tau^{n_1+1} \phi T_2$, and $T_2 = \tau^{n_2+1} T_1$ or $T_2 = \tau^{n_2+1} \phi T_1$.

Definition 36. *A 2-cluster tilting object of subtype 2 is called 2-symmetric if Q has the same subquiver of type A attached at both connecting vertices.*

Theorem 37. *Let T be a 2-cluster tilting object of type D_n of subtype 2. Assume that s is the smallest number such that $\tau^s T = T$. Then*

1. *for even values of n we have $s = n/2$ if and only if T is 2-symmetric and $4|n$. Otherwise $s = n$.*
2. *for odd n we have $s = 2n$ for all T .*

Proof. Denote the α -objects by T_1 and T_2 .

First assume that T is 2-symmetric and that $4|n$. All τ -orbits have n objects, so $s \leq n$. The same subquiver of type $A_{n/2-1}$ is attached at both connecting vertices. We note that $n/2$ is an even number and thus theorem 15 implies that $\tau^{n/2} T_1 = T_2$ and $\tau^{n/2} T_2 = T_1$.

Denote by Δ_1 and Δ_2 the subcategories of type mod A_l attached at the connecting vertices of Q . By lemma 16 there is a morphism from Δ_1 to T_1 factoring through the projective injective object $\Pi(\Delta_1)$. Similarly there is a morphism from Δ_2 to T_2 factoring through $\Pi(\Delta_2)$. Hence for each pair of corresponding vertices T_{Δ_1} and T_{Δ_2} in Δ_1 and Δ_2 we have $\tau^{n/2} T_{\Delta_1} = T_{\Delta_2}$

and $\tau^{n/2}T_{\Delta_2} = T_{\Delta_1}$. Since it is the same subquiver of type A attached at both connecting vertices, we have $\tau^{n/2}T = T$.

Now assume that T is a 2-cluster tilting object of type D_n such that $\tau^{n/2}T = T$. This can only be the case if $2|n$. From lemma 34 it follows that $s = n/2$ is an even number, hence $4|n$.

Let Q_1 and Q_2 be the subquivers of type $A_{n/2-1}$ attached at the two connecting vertices. It is clear that Q_1 and Q_2 must be the same quiver in the mutation class of type A . Hence T is 2-symmetric.

If n is odd then the τ -orbit containing α -objects contains $2n$ objects. The only possibility of achieving a better value of s than $2n$, is that $\tau^n T_1 = T_2$ and $\tau^n T_2 = T_1$. This however would mean that $T_1 = \phi T_2$, a contradiction of the choice of T . \square

6.3. 2-cluster tilting objects/subcategories of type D subtype 3.

We recall from subsection 4.2 that cluster tilting objects of subtype 3 have at least three indecomposable α -objects. The distribution of these α -objects are described in theorem 15, they form the central cycle of the quiver of $\text{End}_{\mathcal{C}_2(D_n)}(T)^{op}$. We start off by studying how the number of α -objects is affected by T being closed under τ^s . Recall that if T is a 2-cluster tilting object of $\mathcal{C}_2(D_n)$ such that $\tau^s T = T$, then s is an even number by lemma 34.

Since we are interested in studying the cases $\tau^s T = T$ for $s < n$ for even n and $s < 2n$ for odd values of n , it is clear that if $\tau^s T = T$, then s divides n for even n and s divides $2n$ for odd n .

Lemma 38. *Let T be a 2-cluster tilting object of $\mathcal{C}_2(D_n)$ and $s = 2k$ an even number such that $\tau^s T = T$. Then the number of indecomposable α -objects a in T is divisible by $\frac{n}{s}$ for even n and is divisible by $\frac{2n}{s}$ for odd n .*

Proof. Assume that n is even. There are two τ -orbits of length n containing α -objects, each may contain multiple of $\frac{n}{s}$.

Now assume that n is an odd number. Then there is only one τ -orbit of α -objects of length $2n$. Clearly this τ -orbit contains a multiple of $\frac{2n}{s}$ indecomposables contained in T . \square

As for type A, there are quivers of type D which are particularly nice when one looks for symmetric properties. We now give a definition, describing the details of such quivers of type D.

Definition 39. *Let T be a 2-cluster tilting object of $\mathcal{C}_2(D_n)$ of type 3, with quiver as illustrated in figure 11, where a is the number of α -objects. If $l \leq a$ is the largest integer such that $a = l \cdot t$ and $Q_r = Q_{r+t} = Q_{r+2t} = \dots = Q_{r+(l-1)t}$ for each $r \in \{1, \dots, t\}$ then we call T l -symmetric. Denote the number of vertices in subquiver Q_i by n_i . If T is an l -symmetric cluster tilting object with quiver Q , then we define $p := \sum_{i=1}^t n_i + t$.*

For a 2-cluster tilting object of $\mathcal{C}_2(D_n)$ that is l -symmetric we can now see how l relates to n :

Lemma 40. *Let T be a 2-cluster tilting object of $\mathcal{C}_2(D_n)$ that is l -symmetric. Then we have that p divides n .*

Proof. Follows directly from definition 39. \square

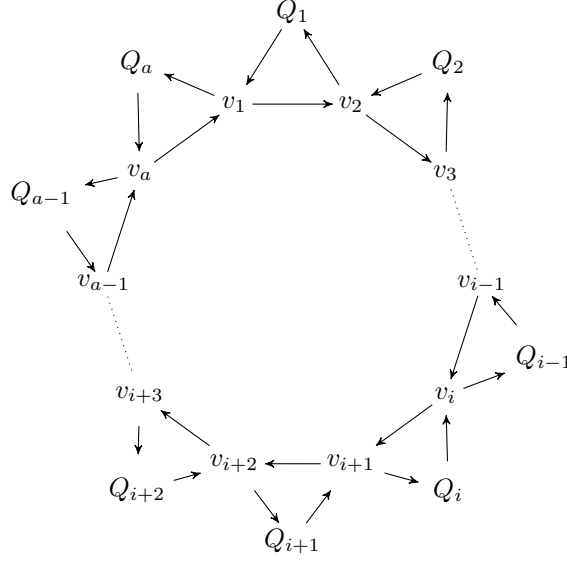


FIGURE 11. The quiver Q of $\text{End}_{\mathcal{C}_2}(T)^{op}$ of a 2-cluster tilting object T of $\mathcal{C}_2(D_n)$. Each Q_i represents the quiver of a cluster tilting object of type A.

Before proceeding with the main results of this section we need one more technical definition:

Definition 41. Let T be a 2-cluster tilting object, with quiver Q , such that $\tau^s T = T$. Denote the subquivers attached to the central cycle by Q_i for $i \in \{1, \dots, mt\}$. We then define an equivalence relation on the set of quivers Q_i by defining that $Q_i \sim_s Q_j$ if $\tau^{sw} T_{Q_i} = T_{Q_j}$ for some $w \in \mathbb{Z}$. We denote the equivalence class of a quiver Q_j by $[Q_i]$, where i is the smallest index of any quiver in the equivalence class. Denote by n_i the number of vertices in the quivers in the equivalence class $[Q_i]$.

This will help us distinguish quivers attached to the central cycle that are isomorphic as quivers, but where the corresponding summands are not closed under τ^s .

Theorem 42. Let T be a 2-cluster tiling object of $\mathcal{C}_2(D_n)$ with corresponding quiver of subtype 3. Assume that T is l -symmetric and that s is the smallest positive integer such that $\tau^s T = T$. Then $s = \begin{cases} n/l & \text{if } n/l \text{ is even} \\ 2n/l & \text{if } n/l \text{ is odd.} \end{cases}$

Proof. First assume that $\tilde{s} = \begin{cases} n/l & \text{if } n/l \text{ is even} \\ 2n/l & \text{if } n/l \text{ is odd.} \end{cases}$, hence \tilde{s} is an even number. Recall that T is l -symmetric, so we have $n = lp$. Hence if n/l is even we have $\tilde{s} = p$, and if n/l is odd we have $\tilde{s} = 2p$. We want to show that $\tau^{\tilde{s}} T = T$. By theorem 15 we know that $v_{i+1} = \tau^{n_i+1} v_i$ if $n_i + 1$ is even or $v_{i+1} = \phi \tau^{n_i+1} v_i$ if $n_i + 1$ is odd. Hence it follows in the case of n/l being even, from theorem 15, that $v_{i+t} = \tau^p v_i$, and $v_{mt+j} = \tau^p v_{(m+1)t+j}$ for each $j \in \{1, \dots, t\}$ and each $m \in \{1, \dots, l\}$. Furthermore, in the case of n/l being

odd, we have $v_{i+2t} = \tau^{2p}v_i$ and $v_{mt+j} = \tau^{2p}v_{(m+2)t+j}$ for each $j \in \{1, \dots, t\}$ and each $l \in \{1, \dots, l\}$. Hence the set of α -objects of T are closed under τ^s for both even and odd values of n/l .

Let T_{Q_i} be all indecomposable summands of T corresponding to all vertices in all the t occurrences of subquiver Q_i in Q . By lemma 16 it now follows that $\tau^s T_{Q_i} = T_{Q_i}$, hence we have $\tau^s T = T$. It follows that since s is minimal such that $\tau^s T = T$, we have $s|\tilde{s}$.

Now assume that $\tilde{l} = \begin{cases} n/s & \text{if } n \text{ is even} \\ 2n/s & \text{if } n \text{ is odd} \end{cases}$. Note that this means that

\tilde{l} is an integer, as $\tau^s T = T$ with s an even number and the length of τ -orbits is either n or $2n$. We want to show that the quiver of T satisfies all properties of being \tilde{l} -symmetric, with the exception that we do not require \tilde{l} being the maximal value such that this happens. That is, we want to show that the quiver of T has a central cycle corresponding to $a = \tilde{l} \cdot t$ number of α -objects with the following relations on the attached quivers: $Q_r = Q_{r+t} = Q_{r+2t} = \dots = Q_{r+(\tilde{l}-1)t}$ for each $r \in \{1, \dots, t\}$.

From lemma 38 it follows that the quiver Q of $\text{End}_{\mathcal{C}}(T)^{op}$ has the shape of an $\tilde{l}t$ -cycle with subquivers from the mutation class of type A attached at the connecting vertices. Let Q_i be a subquiver attached at a connecting vertex of Q . Since $\tau^s T = T$ the same quiver Q_i occurs at a total of \tilde{l} connecting vertices. Denote by n_i the number of vertices in quiver Q_i .

The total number of vertices in Q is n , so from the above considerations we get the following equation:

$$\tilde{l}t + \tilde{l}\sum_{i=1}^t n_i = \tilde{l}(t + \sum_{i=1}^t n_i) = n.$$

From which it follows that $t + \sum_{i=1}^t n_i = \begin{cases} s & \text{if } n \text{ is even} \\ s/2 & \text{if } n \text{ is odd} \end{cases}$.

We now want to show that the distribution of the quivers Q_i amongst the connecting vertices is as in the definition of \tilde{l} -symmetric. Vertices in the central $\tilde{l}t$ -cycle of Q will be denoted with the same notation as in definition 39.

Assume first that n is even, hence $\tilde{l} = n/s$ and $t + \sum_{i=1}^t n_i = s$. We start with the vertices v_1 and v_2 in the central $\tilde{l}t$ -cycle. Attached at these vertices is a quiver from the equivalence class $[Q_1]$. Next, at the vertices v_2 and v_3 there is a quiver attached from the equivalence class $[Q_2]$. Continuing like this, at vertices v_j and v_{j+1} there is a quiver attached from the equivalence class $[Q_j]$ for each $j \in \{1, \dots, t\}$. Therefore we have $\tau^s v_1 = v_{t+1}$, since s is even. By lemma 16 it follows that the quiver attached at vertices v_{t+1} and v_{t+2} is a quiver from the equivalence class $[Q_2]$. By iterating this process we find that $Q_r = Q_{r+t} = \dots = Q_{r+(\tilde{l}-1)t}$ for $r \in \{1, \dots, t\}$.

Now assume that n is odd, hence $\tilde{l} = 2n/s$ and $t + \sum_{i=1}^t n_i = s/2$. Notice that $n = \tilde{l}s/2$, so since n is odd both $s/2$ and \tilde{l} are odd integers. Again in this case we start with vertices v_1 and v_2 in the central $\tilde{l}t$ -cycle. Attached at these vertices is a quiver from the equivalence class $[Q_1]$. Next, at vertices v_2 and v_3 there is a quiver attached from the equivalence class $[Q_2]$. Continuing like this, at vertices v_j and v_{j+1} there is a quiver attached from the equivalence

class $[Q_j]$. Thus we find that $\phi\tau^{s/2}v_1 = v_{1+t}$. We now want to show that the quiver Q_x attached at the pair of vertices v_{1+t} and v_{2+t} is in the equivalence class of $[Q_1]$. Considering that $\phi = \tau^n$ for odd n we find that

$$\phi\tau^{s/2}v_1 = \tau^n\tau^{s/2}v_1 = \tau^{\tilde{l}s/2+s/2}v_1 = \tau^{s(\tilde{l}/2+1/2)}v_1 = \tau^{sw}v_1 = v_{t+1}$$

where $w = \tilde{l}/2 + 1/2$ is an integer since \tilde{l} is odd. Hence the quiver attached at vertices v_{t+1} and v_{t+2} is from the equivalence class $[Q_1]$. By iterating this process we find that $Q_r = Q_{r+t} = \dots = Q_{r+(\tilde{l}-1)t}$ for $r \in \{1, \dots, t\}$.

From this it follows that $\tilde{l}|l$, hence $l = \alpha\tilde{l}$. Also recall that

$$s|\tilde{s} = \begin{cases} n/l & \text{if } n/l \text{ is even} \\ 2n/l & \text{if } n/l \text{ is odd} \end{cases}.$$

We now again start by considering the case of n an even number. From the definition of \tilde{l} we then find that $s = n/\tilde{l} = \alpha n/l$. If the quotient n/l is even then since $s|\tilde{s}$ we have $\frac{\alpha n}{l} | \frac{n}{l}$ yielding $\alpha = 1$ and $n = sl$. If however the quotient n/l we find that since $s|\tilde{s}$ then $\frac{\alpha n}{l} | \frac{2n}{l}$ yielding $\alpha = 2$ and $2n = sl$.

Now consider the case of n being an odd number, in which case we have $\tilde{l} = 2n/s$. Then the quotient n/l can not be an even number, hence there is only one case to consider which is n/l an odd number. In that case we have as above that $s|\tilde{s}$ and hence $\frac{2\alpha n}{l} | \frac{2n}{l}$ yielding again $\alpha = 1$ and $2n = sl$ which concludes the proof. \square

6.4. Summary of result of type D_n . We give a brief recount of the result obtained for type D_n .

Theorem 43. *Let T be a 2-cluster tilting object of $\mathcal{C}_2(D_n)$ and s the smallest positive integer such that $\tau^s T = T$. Then*

- If T is of subtype 1 then $s = n$.
- If n is even and T is of subtype 2 then $s = n/2$ if and only if T is 2-symmetric of subtype 2 and $4|n$. For all other cases where n is even we have $s = n$.
- If n is odd and T is of subtype 2 then $s = 2n$ for all T .
- If T is of subtype 3 and l symmetric then $s = \begin{cases} n/l & \text{for } n/l \text{ even} \\ 2n/l & \text{for } n/l \text{ odd} \end{cases}$.

Based on theorem 43 we are able to give the following corollary, giving an overview of all triangulated orbit categories $\mathcal{O}_F(D_n)$ containing a 2-cluster tilting object.

Corollary 44. *Let $\mathcal{O}_F(D_n) = \mathcal{D}^b(D_n)/F$ be a triangulated orbit category with $n \geq 4$. Then $\mathcal{O}_F(D_n)$ has a 2-cluster tilting object if and only if F is one of the following functors:*

- τ^{tn} for odd $n \geq 5$ and $t \in \mathbb{Z}$
- $\tau^{tn-1}[1]$ for odd $n \geq 5$ and $t \in \mathbb{Z}$
- τ^{2t} for all $n \geq 4$ and $t \in \mathbb{Z}$.
- $\tau^{2t-1}[1]$ for odd $n \geq 5$ and $t \in \mathbb{Z}$.

Proof. For even values of n there is only one type of AR-quiver of \mathcal{O}_F to consider, which is cylindrical. However for odd n there are two possibilities, one is cylindrical and the other is cylindrical for β -objects with a Moebius-twist in the top two τ -orbits consisting of the α -objects.

From the last part of theorem 43 it follows that for both even and odd n the 2-cluster tilting object where the corresponding quiver of $\text{End}_{\mathcal{C}}(T)^{op}$ is an n -cycle is periodic under τ^2 . Since 2 is the smallest even natural number we get the last two parts of the corollary by applying lemma 6 and lemma 10. These cases also cover all categories having a 2-cluster tilting object due to the second part of theorem 43.

The last part to consider is the first part of theorem 43, which gives rise to the first two parts of this corollary using lemma 6 and lemma 10. It is only necessary to consider the cases for odd n as all cases of even n is considered in the third part of the corollary. □

7. 2-CLUSTER TILTING SUBCATEGORIES OF TYPE E

In this section we focus on the cases, E_6, E_7 and E_8 . We treat each case in its separate subsection. In each case we will use the properties of the AR-quiver of the 2-cluster category to study the periodic properties, as in the two previous sections.

7.1. Type E_6 . The AR-quiver of $\mathcal{C}_2(E_6)$ is illustrated in figure 12. It has a Moebius shape, the two outermost τ -orbits each contain 14 objects, whereas the two innermost τ -orbits contains only 7 objects each. Hence for any object X of $\mathcal{C}_2(E_6)$ we have that $\tau^{14}X = X$. From this information we deduce that other possible values of s such that $\tau^s\mathcal{T} = \mathcal{T}$ for a 2-cluster tilting subcategory of $\mathcal{C}_2(E_6)$ are $s = 2$ or $s = 7$.

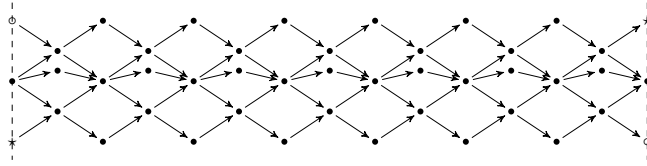
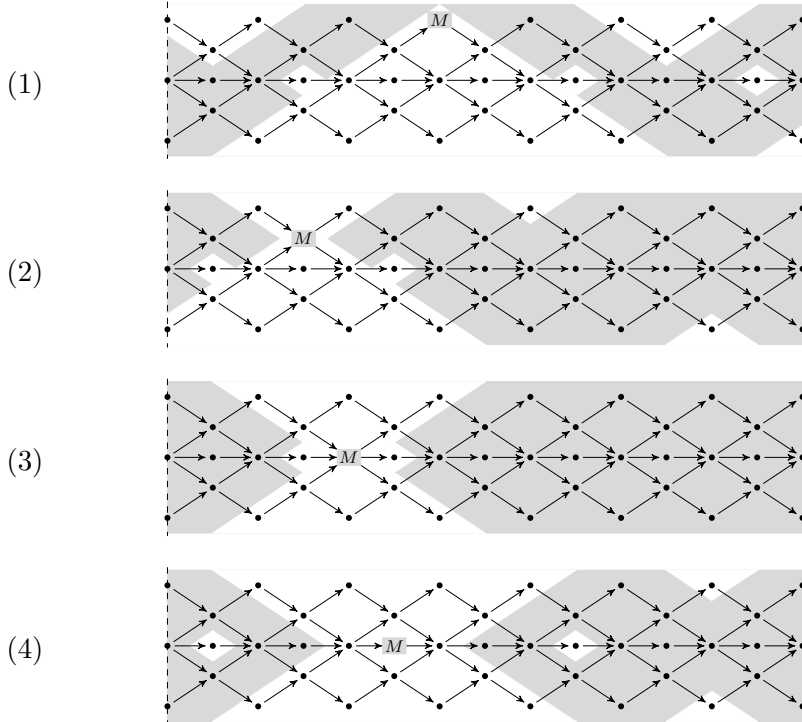


FIGURE 12. The AR-quiver of $\mathcal{C}_2(E_6)$, the objects of same shape on either side are identified.

Assume first that $s = 2$. If there is a summand of \mathcal{T} in any given τ -orbit, then this τ -orbit contains either 7 or 14 indecomposables from \mathcal{T} . This is not possible as \mathcal{T} only has 6 indecomposable summands up to isomorphism.

For the case $s = 7$, each τ -orbit of length 14 must contain an even number of summands from \mathcal{T} . If a τ -orbit has length 7 it may contain an odd number of indecomposable summands from \mathcal{T} . Below we have included an illustration for each τ -orbit of the AR-quiver of $\mathcal{C}_2(E_6)$. Each diagram shows in grey the indecomposable objects that can not be part of the same 2-cluster tilting object as M .



Based on these overviews for each τ -orbit, one may use combinations of them together with combinatorial arguments to get a complete classification of all 2-cluster tilting objects of $\mathcal{C}_2(E_6)$ that are periodic under τ^7 . An example of a calculation is given in subsection 7.3.

Before looking at the classification itself, we need a notation to keep track of the distribution of the indecomposables in the AR-quiver. For a 2-cluster tilting object T of $\mathcal{C}_2(E_6)$ that is closed under τ^7 , we will associate a vector (a_1, a_2, a_3, a_4) . The number a_1 will denote the number of indecomposables in the outermost τ -orbit. The value of a_1 will be denoted by 2 if there are one pair of indecomposables summands in the outermost τ -orbit, or $2 + 2$ if there are two pairs. The complete classification is listed in table 7.1.

7.2. Type E_7 . The AR-quiver of $\mathcal{C}_2(E_7)$ is illustrated in figure 13, it has the shape of a cylinder where each τ -orbit contains 10 objects. In order to achieve any relation of the type $\tau^s \mathcal{T} = \mathcal{T}$ for a 2-cluster tilting subcategory \mathcal{T} of $\mathcal{C}_2(E_7)$, we see that the only possibilities are $s = 2$ or $s = 5$.

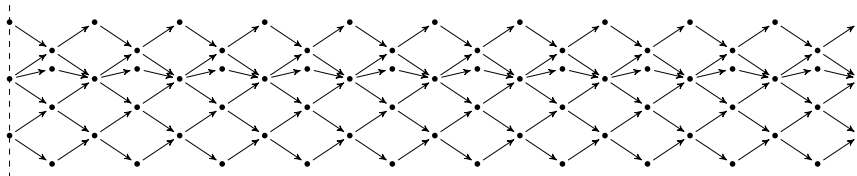


FIGURE 13. The AR-quiver of $\mathcal{C}_2(E_7)$.

Assume that $s = 2$. In that case each τ -orbit containing at least one indecomposable summand of \mathcal{T} contains a number of indecomposables from

| Distribution in AR-q. | $\tau^s T = T$ | Corresponding quivers of T |
|-----------------------|----------------|------------------------------|
| (2,2,1,1) | s=7 | |
| (2+2,0,1,1) | s=7 | |
| (2,2,2,0) | s=7 | |
| (2+2,0,2,0) | s=7 | |

TABLE 1. All quivers of 2-cluster tilting objects of $\mathcal{C}_2(E_6)$ where the corresponding 2-cluster tilting object is periodic under τ^7 . The first column gives the distribution of the indecomposable summands of T in the τ -orbits of the AR-quiver of $\mathcal{C}_2(E_6)$.

\mathcal{T} divisible by 5. However any 2-cluster tilting subcategory of $\mathcal{C}_2(E_7)$ has exactly 7 indecomposable objects up to isomorphism, hence this is an impossible scenario.

Now assume that $s = 5$. Then all τ -orbits containing at least one indecomposable summand of \mathcal{T} , contains an even number of summands of \mathcal{T} . This can not happen as 7 is odd. Hence we have the following result:

Theorem 45. *If \mathcal{T} is any 2-cluster tilting subcategory of $\mathcal{C}_2(E_7)$, then the smallest value of s such that $\tau^s \mathcal{T} = \mathcal{T}$ is $s = 10$.*

7.3. Type E_8 . The AR-quiver of $\mathcal{C}_2(E_8)$ has 8 τ -orbits, each with 16 indecomposable objects, as illustrated in figure 14. If T is a 2-cluster tilting object of $\mathcal{C}_2(E_8)$ such that $\tau^s T = T$ with $s < 16$, then $s \in \{2, 4, 8\}$. Below we have included for each τ -orbit, a figure where the grey area shows which indecomposables that can not be part of the same 2-cluster tilting object as M . By counting in each figure below, one may verify that it is not possible to have $\tau^2 T = T$ for a 2-cluster tilting object in $\mathcal{C}_2(E_8)$.

Furthermore, if T is a 2-cluster tilting object such that $\tau^4 T = T$, then the only τ -orbits that can contain indecomposable summands of T are the two bottom τ -orbits in figure 14.

Finally we study the case when $s = 8$. Again we turn to the 8 figures displaying the ext-support of an object of each τ -orbit. From these figures we see that if $\tau^8 T = T$ for a 2-cluster tilting object, then the indecomposable objects of T may only occur in the two bottom τ -orbits, or in the top τ -orbit.

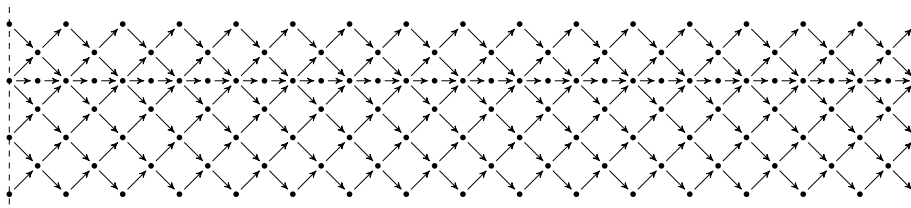
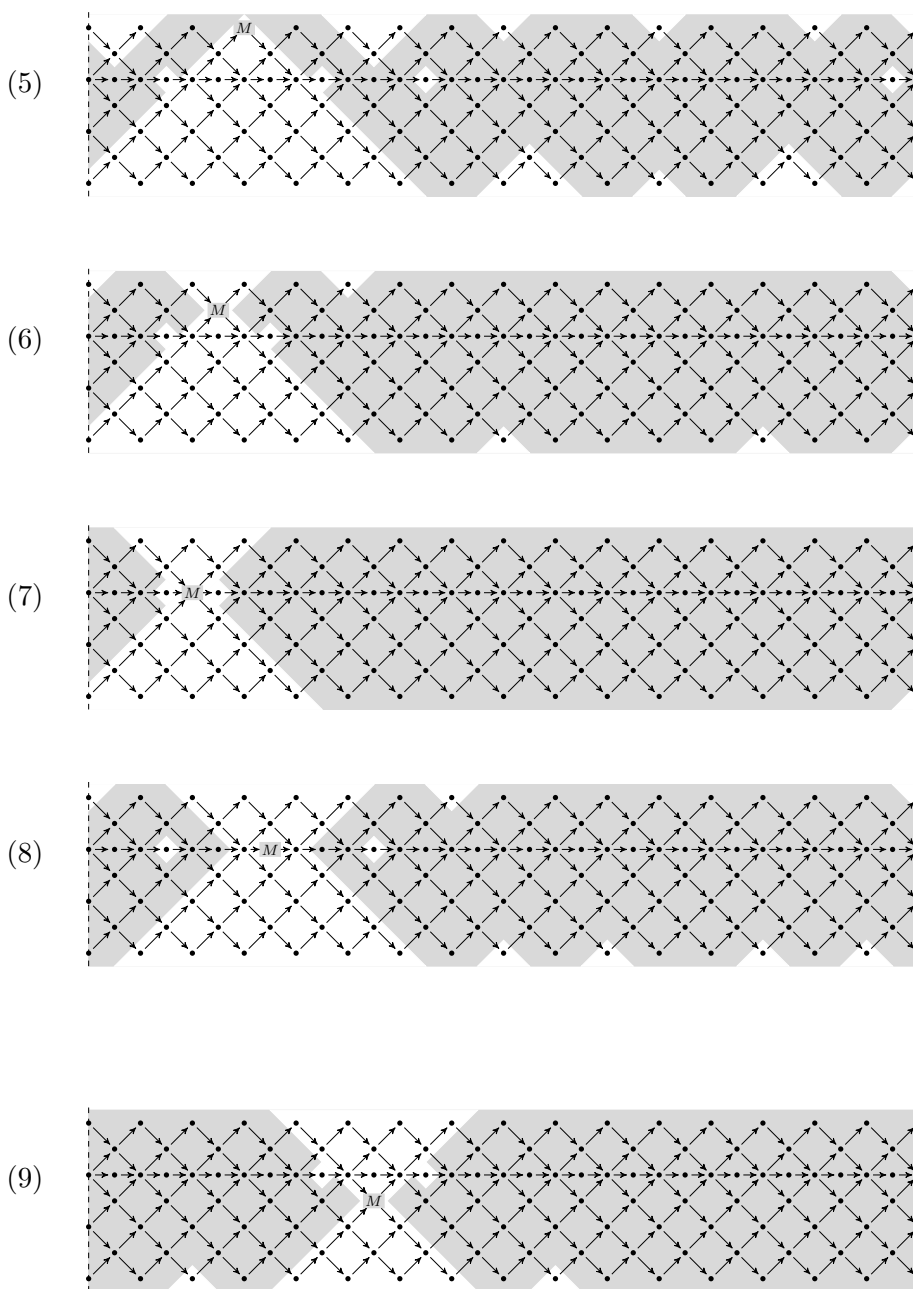
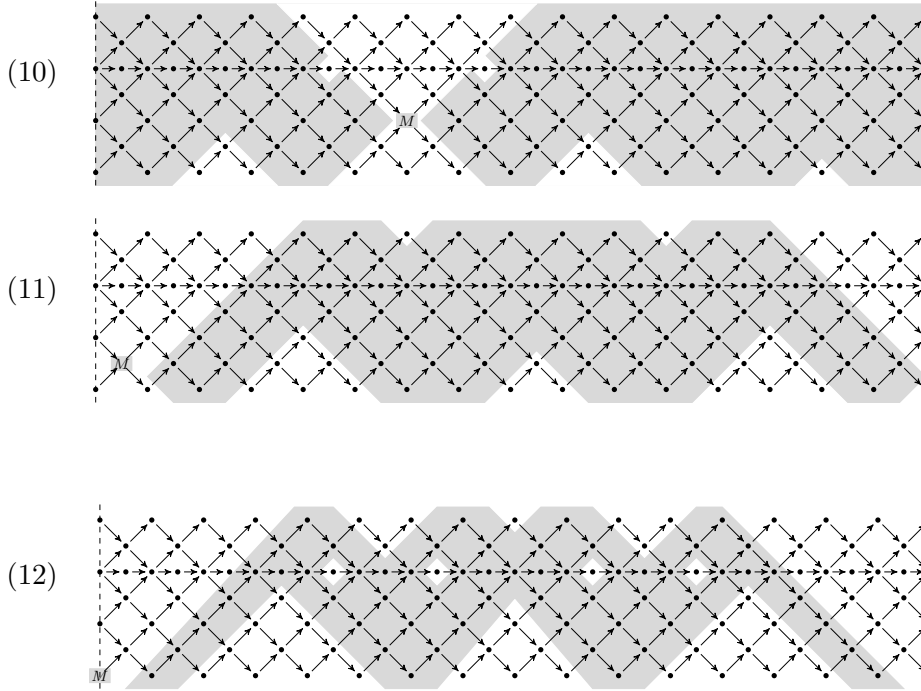


FIGURE 14. The AR-quiver of $C_2(E_8)$.





The number of possible distributions of the 8 indecomposable summands are now significantly reduced. The remaining possible distributions have all been checked, and the result is presented in table 7.3. Before proceeding to the table we present the necessary notation, as well as an example of the calculations.

For the rest of the subsection we will call the bottom τ -orbit of the AR-quiver nr 1, the second bottom τ -orbit nr 2 and the top τ -orbit nr 8. We will now introduce a short notation for the distribution of summands between these three τ -orbits. Denote the distribution of indecomposables by the vector (a, b, c) , where a is the indecomposables in τ -orbit nr 1, b the indecomposables in τ -orbit nr 2 and c the indecomposables in τ -orbit nr 8. If a 2-cluster tilting object T has 2 indecomposables closed under τ^8 in a τ -orbit, it will be denoted 2. If T has 4 indecomposables such that these are closed under τ^4 it will be denoted 4. If there are 4 indecomposables in a τ -orbit but these indecomposables are not closed under τ^4 but closed under τ^8 we will denote these objects by 2+2.

We now demonstrate an example of the calculations that have been done to achieve table 7.3, by considering the cases $(?, 2 + 2, ?)$ and $(?, 4, ?)$. First we consider the situation if there are two indecomposables of T placed in τ -orbit nr 2. This is pictured in figure 15.

From figure 15 we find/verify that there can maximally be 4 indecomposables of T in this τ -orbit. If there are 4 indecomposables in this τ -orbit then they are closed under τ^4 . This rules out the existence of cases of the form $(?, 2 + 2, ?)$.

Continuing under the assumption that there are 4 indecomposables in this τ -orbit that are closed under τ^4 , we have the situation pictured in figure 16.

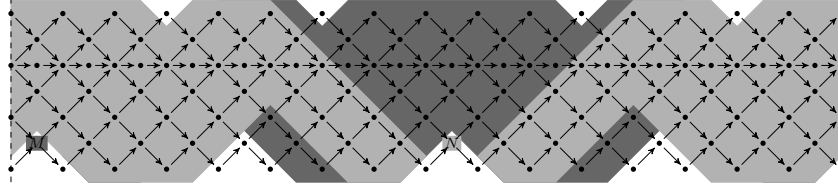


FIGURE 15. Indecomposables with a white background may be part of the same 2-cluster tilting object as M and N .

From this figure we find that the rest of the summands of T must be located in τ -orbit nr 8. The possible cases at this point are $(2 + 2, 4, 0)$ and $(4, 4, 0)$. Choosing first to focus on the case $(4, 4, 0)$, we see that this case will occur if one chooses the indecomposables in τ -orbit 8 with an arrow going into either M, N, O or R . The only other possibility is to choose the indecomposables in τ -orbit nr. 8 with an arrow from M, N, O or R . These two choices correspond to the quivers in the top row of table 7.3.

If one instead choose either the indecomposables in τ -orbit 8 with an arrow into M and N , and an arrow from O and R one obtains a cluster tilting object with the quiver listed in the second row of table 7.3. One may also choose the indecomposables with an arrow from M and N , and an arrow into O and R , but this quiver is isomorphic to the first, so there is really only one quiver occuring in this case.

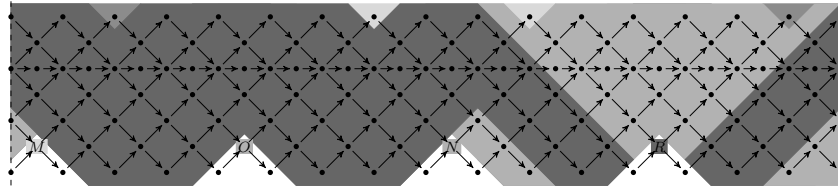


FIGURE 16. The indecomposable with white background are the ones that may be part of a 2-cluster tilting object together with M, N, O and P .

Table 7.3 lists all quivers of 2-cluster tilting objects of $\mathcal{C}_2(E_8)$ such that the corresponding cluster tilting objects is periodic under τ^8 or τ^4 .

7.4. Summary of results of type E_6, E_7 and E_8 . Based on the results regardin periodicity of 2-cluster tilting objects of $\mathcal{C}_2(E_6), \mathcal{C}_2(E_7)$ and $\mathcal{C}_2(E_8)$ we obtain the following corollary:

Corollary 46. *Let $\mathcal{O}_F(E_r) = \mathcal{D}^b(E_r)/F$ for $r \in \{6, 7, 8\}$. Then $\mathcal{O}_F(E_r)$ have a 2-cluster tilting object if and only if F is in the following list:*

- $F = \tau^{7t}$ for E_6 and $t \in \mathbb{Z}$
- $F = \tau^{7t-1}[1]$ for E_6 and $t \in \mathbb{Z}$
- $F = \tau^{10t}$ for E_7 and $t \in \mathbb{Z}$.
- $F = \tau^{4t}$ for E_8 and $t \in \mathbb{Z}$.

Proof. For the first two parts of the corollary we note that the AR-quiver of $\mathcal{O}_F(E_6)$ may be of two main types, cylindrical or of Moebius-shape. For

| Distribution in AR-q. | $\tau^s T = T$ | Corresponding quivers of T |
|-----------------------|----------------|------------------------------|
| (4,4,0) | s=4 | |
| (2+2,4,0) | s=8 | |
| (2,2,2+2) | s=8 | |
| (2+2,2,2) | s=8 | |
| (4,2,2) | s=8 | |
| (2+2,0,2+2) | s=8 | |

TABLE 2. All quivers of 2-cluster tilting objects of $\mathcal{C}_2(E_8)$ where the corresponding 2-cluster tilting object is periodic under τ^4 or τ^8 . The first column gives the distribution of the indecomposable summands of T in the AR-quiver of $\mathcal{C}_2(E_8)$.

the third and fourth parts recall that there is only the possibility of having a cylindrical AR-quiver with functors of the form $\tau^s[t]$ for orbit categories of type $\mathcal{O}_F(E_7)$ and $\mathcal{O}_F(E_8)$.

The results are then obtained by applying lemma 6 and lemma 10. \square

8. THE EUCLIDEAN AND WILD CASES

We have in the previous sections 5, 6 and 7 studied all the cases of representation finite hereditary cases. In this section we will focus on Euclidean and wild algebras, and argue that the phenomenon that we have observed for the representation-finite cases can never occur for cluster tilting objects of representation-infinite algebras.

First we discuss the Euclidean case. The AR-quiver of a path-algebra of some orientation of a Euclidean diagram contains three main components, a

preprojective component, regular components and a preinjective component. In the Euclidean cases the regular component consists of tubes.

The following result enables us to consider cluster tilting objects as tilting objects in a suitable module category.

Theorem 47. [5] *Let H be a tame hereditary algebra, and T a cluster tilting object of $\mathcal{C}_2(H)$. Then there is some algebra H' derived equivalent to H , such that T is a tilting object in $\text{mod } H'$.*

We may now use well-known theory about the placement of the indecomposable summands of tilting objects of path-algebras over a Euclidean diagram.

Lemma 48. [19] *(chapter 17) Let H be the path-algebra of a Euclidean diagram, and let T be a tilting object in $\text{mod } H$. Then T has at least two indecomposable non-isomorphic summand that do not lie in regular components.*

As a consequence of this lemma, every cluster tilting object T has an indecomposable summand T_p lying in the preprojective component or the preinjective component. Since there are infinitely many objects in either component it is impossible to find an integer s such that $\tau^s T = T$.

Now let us consider the case of a path algebra over a wild quiver. In this case all the regular components of the AR-quiver are of the type $\mathbb{Z}A_\infty$. It follows that in these cases there is no integer s such that $\tau^s T = T$ for any 2-cluster tilting object T .

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