ON THE CYCLE STRUCTURE OF THE PRODUCT OF RANDOM MAXIMAL CYCLES

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Abstract. The subject of this paper is the cycle structure of the random permutation σ of $[N]$, which is the product of k independent random cycles of maximal length N. We use the character-based Fourier transform to study the number of cycles of σ and also the distribution of the elements of the subset $[\ell]$ among the cycles of σ.

1. Introduction

Enumeration of permutations of a set $[N] = \{1, 2, \ldots, N\}$ according to the numbers of cycles of various lengths has a long and glorious history. The plentiful results are not infrequently cast in the probabilistic light, if the assumption is made that a permutation is chosen *uniformly at random* among all $N!$ permutations. The techniques wary widely, from bijective methods to multivariate generating functions to functional limit theorems, allowing to find solutions, exact or asymptotic, of rather delicate, enumerative-probabilistic, problems. More recently there has been a growing interest in the probabilities regarding distribution of the elements of a subset $S \subseteq [N]$ among the cycles of the random permutation. For instance, we can determine the probability that each of the entries in S will be in a different cycle, or that all entries of S will be in the same cycle, or that each cycle of p will contain at least one entry of S. See Lovász [12] for results of this kind.

The classic, and more recent, problems become much more difficult if instead of the uniformly random permutation, we consider a random permutation which is a *product* of random *maximal* cycles. That is, our sample space is now that of all ordered k-tuples $(p_1, p_2, \cdots, p_k)$, where all p_i are maximal cycles of length N. One can investigate the random permutation $\sigma := p_1 \cdots p_k$ under the assumption that $p_1, \ldots, p_k$ are maximal cycles, chosen uniformly at random, and independently of each other, from all $(N-1)!$ such cycles.

1.1. Motivation and recent results.

Among the sources of our inspiration are Zagier's formula for the distribution of the number of cycles in σ for $k = 2$, and the more recent results by Stanley [16] and Bernardi et al. [2],

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again for $k = 2$. For instance, in [2] a formula is proved for the probability that σ, the product of two maximal cycles, separates the *given disjoint* subsets of $[N]$, i.e. no two of those subsets are represented in the same cycle of σ. In particular, the probability that σ separates the entries $1, \dots, \ell$ is equal to $1/\ell!$ if $N - \ell$ is odd. In other words, in this aspect, the product of two independent maximal cycles behaves as the uniformly random permutation!

Beside their intrinsic interest, solutions of the mentioned problems may lead to surprising applications. In [3], Bóna and Flynn used a result of Stanley [16] concerning the special case $S = \{1, 2\}$ and $k = 2$ to prove an exact formula for the average number of block interchanges needed to sort a permutation, a problem motivated by genome sorting. Equally interesting are the methods that can be used, as they come from a wide array of areas in mathematics, such as character theory, multivariate Gaussian integration, bijective combinatorics and the summation techniques for hypergeometric sums.

1.2. **Overview.** In 1986 Harer and Zagier [8] discovered a remarkable formula for the bivariate generating function of the number of cycles in the product of a maximal cycle and the random, fixed-point free, involution of $[2n]$, thus solving a difficult problem of enumerating the chord diagrams by the genus of an associated surface. The proof was based on evaluation of the multidimensional Gaussian integrals. Soon after Jackson [9] and later Zagier [19] found alternative proofs that used characters of the symmetric group S_{2n}. Recently the second author [13] found a different, character-based proof. Its core is computing and marginally inverting the Fourier transform of the underlying probability measure on S_{2n}. In the present paper, we use the techniques in [13], see also an earlier paper by Chmutov and Pittel [5], to investigate the product of k maximal cycles in S_N. To make the discussion reasonably self-contained we will introduce the necessary definitions and facts from [13] in Section 2.

As far as our results go, we first prove an explicit formula for the probability distribution of the number of cycles in σ, the product of k random, independent, maximal cycles in S_N. Not surprisingly, the distribution is expressed through the Stirling numbers of first kind. The formula yields relatively simple corollaries for the probabilities that σ is the identity permutation, or that σ is a maximal cycle. Our analysis yields a well-known formula found by Zagier for the case $k = 2$; see Appendix by Zagier in Lando and Zvonkin [10]. We also obtain a bivariate generating function for the distribution of the number of cycles for the product of three cycles.

Then we turn to the following general question. Let $p_A(N, \ell; k)$ be the probability that the number of elements of $[\ell] = \{1, 2, \cdots, \ell\}$ in each cycle of σ comes from the set $A \subseteq \mathbb{Z}_{\geq 0}$. What can we say about $p_A(N, \ell; k)$?

To this end, for a general A, we first enumerate the admissible permutations by the cycle counts and then evaluate the sum of character values over all admissible permutations for irreducible representations labeled by one-hook Young diagrams. Then we consider the special case when $A=\mathbb{Z}_{>0}$, i.e. when each cycle of σ contains at least one element of $[\ell]$. Using the inverse Fourier transform, we find an alternating sum expression for this probability with $N-\ell+1$ binomial-type summands. For $k=2$, this sum reduces to two notably simpler expressions, that can be efficiently computed for moderate ℓ and moderate $N-\ell$ respectively.

Next we investigate the case of $A=\{0,\ell\}$, that is, when all elements of ℓ are in the same cycle of σ. This computation is longer than its counterpart in the previous case, and it leads to a general formula for $p_A(N,\ell;k)$ analogous to that for $A=\mathbb{Z}_{>0}$. Again, if $k=2$, then the formula shrinks to a pair of computationally efficient sums for moderate ℓ and moderate $N-\ell$ respectively. For $\ell=2$ and $\ell=3$, we recover the results obtained by Stanley [16].

Having experimented with Maple, we feel confident that the residual sums for $k=2$ in either of the two cases do not have a more compact presentation.

After this, we turn to our most general problem. We consider disjoint subsets $\mathcal{S}_1,\mathcal{S}_2,\cdots,\mathcal{S}_t$ of $[N]$ so that $|\mathcal{S}_j|=\ell_j$; define $\ell=\sum_j \ell_j$. Let $p(N,\vec{\ell};k)$ denote the probability that no cycle of σ contains elements from more than one $\mathcal{S}_j$, a property to which we refer by saying that σ *separates* the sets $\mathcal{S}_1,\mathcal{S}_2,\cdots,\mathcal{S}_t$. Bernardi et al. [2] found a striking formula for $p(N,\vec{\ell};2)$ that contained an alternating sum of $\ell-t+1$ terms. Remarkably, the factor $\prod_j \ell_j!$ aside, the rest of the formula depends on ℓ and t only. We show that the separation probability continues to have this latter property for all $k\geq 2$, and find an alternating sum formula with $N-\ell+t+1$ terms for this probability. So we find a computationally efficient formula if t and $N-\ell$ are both bounded as N grows. Then, for $k=2$, we are able to simplify this formula to one that is close in appearance, but is significantly different from the formula in [2]. Our formula still contains a sum of $\ell-t+1$ summands, but the signs are no longer alternating.

Finally, we consider the following question. Let us say that the elements of $[\ell]$ are blocked in a permutation s of $[N]$ if no two elements of $[\ell]$ are neighbors, *and* each element of $[\ell]$ has a neighbor from $[N]\setminus[\ell]$. Then, for a general $k\geq 2$, we find a *two-term* formula for the probability that σ blocks the elements of $[\ell]$.

2. Preliminaries

A key observation is that the set of all maximal cycles forms a *conjugacy class* in the symmetric group S_N, a class with particularly simple character values.

Let us start with the Fourier inversion formula for a general probability measure P on S_N:

$$P(s)=\frac{1}{N!}\sum_{\lambda\vdash N} f^\lambda \operatorname{tr}\big(\rho^\lambda(s^{-1})\hat{P}(\rho^\lambda)\big);\quad s\in S_N; \tag{1}$$

see Diaconis and Shahshahani [6] and Diaconis [7]. Here λ is a generic partition of the integer N, ρ^λ is the irreducible representation of S_N associated with λ, $f^\lambda = \dim(\rho^\lambda)$, and $\hat{P}(\rho^\lambda)$ is the $f^\lambda \times f^\lambda$ matrix-valued Fourier transform of $P(\cdot)$ evaluated at ρ^λ, $\hat{P}(\rho^\lambda) = \sum_{s\in S_N} \rho^\lambda(s)P(s)$. Let us evaluate the right-hand side of (1) for $P = P_\sigma$, the probability measure on S_N induced by $\sigma = \prod_{j=1}^k \sigma_j$, where σ_j is uniform on a conjugacy class C_j. As the σ_j are independent, we have that $P_\sigma(s) = \sum_{s_1,\dots,s_k} \prod_j P_{\sigma_j}(s_j)$, $(s_1 \cdots s_k = s)$, that is, P_σ is the convolution of $P_{\sigma_1}, \dots, P_{\sigma_k}$. So, by multiplicativity of the Fourier transform for convolutions, $\hat{P}_\sigma(\rho^\lambda) = \prod_j \hat{P}_{\sigma_j}(\rho^\lambda)$. Since each P_{σ_j} is supported by the single conjugacy class C_j, we have $\hat{P}_{\sigma_j}(\rho^\lambda) = \frac{\chi^\lambda(C_j)}{f^\lambda} I_{f^\lambda}$, I_{f^λ} being the $f^\lambda \times f^\lambda$ identity matrix, see [7]. So

$$\hat{P}_\sigma(\rho^\lambda) = \prod_{j=1}^k \hat{P}_{\sigma_j}(\rho^\lambda) = (f^\lambda)^{-k} \prod_{j=1}^k \chi^\lambda(C_j)\, I_{f^\lambda},$$

and (1) becomes

$$\begin{aligned} P_\sigma(s) &= \frac{1}{N!} \sum_\lambda (f^\lambda)^{-k+1} \left(\prod_{j=1}^k \chi^\lambda(C_j) \right) \operatorname{tr}\bigl(\rho^\lambda(s^{-1}) I_{f^\lambda}\bigr) \\ &= \frac{1}{N!} \sum_\lambda (f^\lambda)^{-k+1} \chi^\lambda(s) \prod_{j=1}^k \chi^\lambda(C_j). \end{aligned} \tag{2}$$

Note. For the special case $s = \mathrm{id}$, the identity (2) becomes

$$\mathrm{P}_\sigma(\mathrm{id}) = \frac{1}{N!} \sum_\lambda (f^\lambda)^{-k+2} \prod_{j=1}^k \chi^\lambda(C_j).$$

Since the left-hand side is just $\mathcal{N}(C_1, \dots, C_k)$, the number of ways to write the identity permutation as the product of elements of $C_1, \dots, C_k$, divided by $\prod_{j=1}^k |C_j|$, we obtain the S_N-version of Frobenius's identity

$$\mathcal{N}(C_1, \dots, C_k) = \frac{\prod_{j=1}^k |C_j|}{N!} \sum_\lambda (f^\lambda)^{-k+2} \prod_{j=1}^k \chi^\lambda(C_j). \tag{3}$$

We will use (2) for $C_j \equiv \mathcal{C}_N$, where $\mathcal{C}_N$ is the conjugacy class of all maximal cycles. By the Murnaghan-Nakayama rule, Sagan [14] (Lemma 4.10.2) or Stanley [15] (Section 7.17, Equation (7.75)), $\chi^\lambda(\mathcal{C}_N) = 0$ unless the diagram λ is a single hook λ^*, with one row of length λ_1 and one column of height λ^1, so $\lambda_1 + \lambda^1 = N + 1$. In that case

$$\chi^\lambda(\mathcal{C}_N) = (-1)^{\lambda^1 - 1}. \tag{4}$$

As for f^{λ^*}, the number of Standard Young Tableaux of shape λ^*, applying the hook length formula (or simply selecting the entries that go in the first column), we obtain

$$f^{\lambda^*} = \frac{N!}{N\prod_{r=1}^{\lambda_1-1} r \prod_{s=1}^{\lambda^1-1} s} = \binom{N-1}{\lambda_1-1}. \tag{5}$$

The equations (2), (4) and (5) imply

$$P_\sigma(s) = \frac{1}{N!}\sum_{\lambda^*}(-1)^{k(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1}\chi^{\lambda^*}(s). \tag{6}$$

By the Murnaghan-Nakayama rule, given a hook diagram λ^*, the value of $\chi^{\lambda^*}(s)$ depends on s only through $\vec{\nu} = \vec{\nu}(s) := \{\nu_r\}_{r\ge 1}$, where $\nu_r = \nu_r(s)$ is the total number of r-long cycles in the permutation s. It was proved in [13] that

$$\chi^{\lambda^*}(s) = (-1)^{\lambda^1+\nu}\,[\xi^{\lambda_1}]\,\frac{\xi}{1-\xi}\prod_{r\ge 1}(1-\xi^r)^{\nu_r}, \tag{7}$$

$\nu(s) := \sum_r \nu_r(s)$ being the total number of cycles of s. From (7) it follows that

$$\begin{aligned} &\sum_{s:\,\vec{\nu}(s)=\vec{\nu}} \chi^{\lambda^*}(s) = (-1)^N N!\,\mathcal{A}(N,\nu,\lambda_1),\\ &\mathcal{A}(N,\nu,\lambda_1) := \binom{N-1}{N-\lambda_1}\sum_{\ell\ge 1}(-1)^\ell\,\frac{s(\ell,\nu)}{\ell!}\binom{N-\lambda_1}{N-\ell}, \end{aligned} \tag{8}$$

where $s(\ell,\nu)$ is the signless, first-kind, Stirling number of permutations of $[\ell] = \{1,2,\cdots,\ell\}$ with ν cycles; see the proof of Theorem 2.1 and the equation (2.20) in [13]. The formulas (2), (7) and (8) are the basis of the proofs that follow.

3. Distribution of the number of cycles in σ

To stress dependence of σ on k, in this section we will write $\sigma^{(k)}$ instead of σ. Combining (8) and (6), and using $\lambda^1 + \lambda_1 = N + 1$, we obtain

$$\begin{aligned} \mathrm{P}(\nu(\sigma^{(k)}) = \nu) = (-1)^N &\sum_{\lambda_1=1}^{N}(-1)^{k(N-\lambda_1)}\binom{N-1}{N-\lambda_1}^{-k+2}\\ &\times\sum_{\ell\ge 1}(-1)^\ell\,\frac{s(\ell,\nu)}{\ell!}\binom{N-\lambda_1}{N-\ell}. \end{aligned} \tag{9}$$

For all $k \ge 2$, the formula (9) and $s(\ell,\nu) = 0$ for $\ell < \nu$ imply that

$$\mathrm{P}(\sigma^{(k)} = \mathrm{id}) = \mathrm{P}(\nu(\sigma) = N) = \frac{1}{N!}\sum_{r=0}^{N-1}(-1)^{kr}\binom{N-1}{r}^{-k+2}. \tag{10}$$

In particular, $\mathrm{P}(\sigma^{(2)} = \mathrm{id}) = \frac{1}{(N-1)!}$. This is an obvious result, since it simply says that if we multiply two maximal cycles, the probability that their product is the identity permutation is $1/(N-1)!$. Further, combining

(10) and the identity (14) for the alternating sum of binomial reciprocals in Suri et al. [17], we have a non-obvious answer

$$\mathrm{P}(\sigma^{(3)} = \mathrm{id}) = \frac{1 + (-1)^{N-1}}{(N-1)!(N+1)}. \tag{11}$$

That remarkable identity in [17] followed from the elementary, yet surprisingly powerful, formula

$$\binom{n}{r}^{-1} = (n+1) \int_0^1 t^r (1-t)^{n-r}\, dt. \tag{12}$$

Note that for even N, equation (11) returns zero probability, and that is how it should be, since the product of three even cycles is an odd permutation, and therefore, cannot be the identity. Furthermore, since $\sigma^{(k)} = \sigma^{(k-1)}\sigma_k$, $\sigma^{(k)}$ is the identity iff $\sigma^{(k-1)} = (\sigma_k)^{-1}$, which is a maximal cycle. As $(\sigma_k)^{-1}$ is uniform on the set of all $(N-1)!$ maximal cycles, *and* independent of $\sigma^{(k-1)}$, we see then that

$$\mathrm{P}(\sigma^{(k-1)} \text{ is a cycle}) = (N-1)!\mathrm{P}(\sigma^{(k)} = \mathrm{id}). \tag{13}$$

For $k = 2$ equations (11) and (13) imply that

$$\mathrm{P}(\sigma^{(2)} \text{ is a cycle}) = \frac{1 + (-1)^{N-1}}{N+1}. \tag{14}$$

The result for even N is obvious, since the product of two maximal cycles is an even permutation, and hence, it cannot be an N-cycle for even N. The result for odd N is equivalent to a well-known, not at all obvious, fact that there are $\frac{2(N-1)!}{N+1}$ ways to factor a given maximal cycle into a product of two maximal cycles; see for instance [4] and the references therein. In general, the equations (10), (13) imply

$$\mathrm{P}(\sigma^{(k)} \text{ is a cycle}) = \frac{1}{N} \sum_{r=0}^{N-1} (-1)^{(k+1)r} \binom{N-1}{r}^{-k+1}. \tag{15}$$

Further, it follows from (9) that for every real number x, we have

$$
\begin{aligned}
\mathrm{E}\big[x^{\nu(\sigma)}\big] &= (-1)^N \sum_{\lambda_1=1}^{N} (-1)^{k(N-\lambda_1)} \binom{N-1}{N-\lambda_1}^{-k+2} \\
&\quad \times \sum_{\ell\geq 1} \frac{(-1)^\ell}{\ell!} \binom{N-\lambda_1}{N-\ell} \sum_{\nu\geq 1} x^\nu s(\ell,\nu) \\
&= (-1)^N \sum_{\lambda_1=1}^{N} (-1)^{k(N-\lambda_1)} \binom{N-1}{N-\lambda_1}^{-k+2} \sum_{\ell\geq 1} \binom{N-\lambda_1}{N-\ell}\binom{-x}{\ell} \\
&= (-1)^N \sum_{\lambda_1=1}^{N} (-1)^{k(N-\lambda_1)} \binom{N-1}{N-\lambda_1}^{-k+2} \binom{N-\lambda_1-x}{N} \\
&= (-1)^N \sum_{r=0}^{N-1} (-1)^{kr} \binom{N-1}{r}^{-k+2} \binom{r-x}{N}. \qquad (16)
\end{aligned}
$$

For a positive integer x, the non-zero contributions to the sum come from $r < \min\{N, x\}$. So, for instance,

$$
\begin{aligned}
\mathrm{E}\big[2^{\nu(\sigma^{(k)})}\big] &= N + 1 + \frac{(-1)^k}{(N-1)^{k-2}}, \quad (N > 1), \\
\mathrm{E}\big[3^{\nu(\sigma^{(k)})}\big] &= 2(N+2)_2 - \frac{N+1}{(N-1)^{k-2}} + \binom{N-1}{2}^{-k+2}, \quad (N > 2).
\end{aligned}
$$

In particular, for $k = 2$ and $x > N$, equation (16) implies

$$
\begin{aligned}
\mathrm{E}\big[x^{\nu(\sigma^{(2)})}\big] &= (-1)^N \sum_{\lambda_1=1}^{N} \binom{N-\lambda_1-x}{N} \\
&= \sum_{\lambda_1=1}^{N} \binom{\lambda_1+x-1}{N} = \sum_{j=N}^{N+x-1} \binom{j}{N} - \sum_{j=N}^{x-1} \binom{j}{N} \\
&= \binom{N+x}{N+1} - \binom{x}{N+1} = \binom{N+x}{N+1} + (-1)^N \binom{N-x}{N+1}. \qquad (17)
\end{aligned}
$$

Of course, the identity (17) holds for all x. It is equivalent to Zagier's result, (see the Appendix by Zagier in Lando and Zvonkin [10]), stating that

$$
\mathrm{P}(\nu(\sigma^{(2)}) = \nu) = (1 + (-1)^{N-\nu})\,[x^\nu] \binom{N+x}{N+1}.
$$

For $k = 3$, the combination of (16) and the identity (12) allows us to show that

$$
\sum_{N\geq 1} \frac{y^N}{N} \mathrm{E}\big[x^{\nu(\sigma^{(3)}(N))}\big] = \int_0^1 \frac{(1-y(1-t))^{-x} - (1-y(1-t))^{x}}{1-yt(1-t)}\, dt; \qquad (18)
$$

here $\sigma^{(3)}(N)$ is the product of 3 random cycles of length N, and $|x| \le 1$, $|y| < 1$. The right-hand side is an odd function of x, which should be expected, since –regardless of the parity of N– the number of cycles in $\sigma^{(3)}(N)$ is odd. In particular, differentiating both sides at $x = 1$, we obtain: for $y \in [0, 1)$,

$$\begin{aligned}\sum_{N\ge 1}\frac{y^N}{N}\mathrm{P}(\sigma^{(3)}(N)\text{ is a cycle}) &= 2\int_0^1 \frac{\log\bigl(1-y(1-t)\bigr)^{-1}}{1-yt(1-t)}\,dt\\ &= 2\sum_{j>0}\frac{1}{j}\int_0^1\frac{\bigl(y(1-t)\bigr)^j}{1-yt(1-t)}\,dt = 2\sum_{j>0,h\ge 0}\frac{y^{j+h}}{j}\int_0^1(1-t)^{j+h}t^h\,dt\\ &= 2\sum_{j>0,h\ge 0}\frac{y^{j+h}}{j}(j+2h+1)^{-1}\binom{j+2h}{h}^{-1}.\end{aligned}$$

So

$$\mathrm{P}(\sigma^{(3)}(N)\text{ is a cycle}) = 2N\sum_{h<N}(N-h)^{-1}(N+h+1)^{-1}\binom{N+h}{h}^{-1}; \tag{19}$$

compare with the equation (15) for $k = 3$.

Let us prove (18). Since both sides of this equation are analytic for $|y| < 1$, it suffices to prove the identity for $|y| \le 1/3$. From (16), (12) and

$$(-1)^N\binom{r-x}{N} = [z^N](1-z)^{r-x},$$

we obtain

$$\begin{aligned}N^{-1}\mathrm{E}\bigl[x^{\nu(\sigma^{(3)}(N))}\bigr] &= [z^N]\sum_{r=0}^{N-1}(1-z)^{r-x}\int_0^1(1-t)^{N-1-r}t^r\,dt\\ &= [z^N](1-z)^{-x}\int_0^1(1-t)^{N-1}\sum_{r=0}^{N-1}\left(-\frac{(1-z)t}{1-t}\right)^r dt\\ &= [z^N](1-z)^{-x}\int_0^1\frac{(1-t)^N+(-1)^{N+1}\bigl((1-z)t\bigr)^N}{1-tz}\,dt.\end{aligned}$$

Next

$$y^N(1-t)^N[z^N]\,\frac{(1-z)^x}{1-tz} = [z^N]\,\frac{\bigl(1-(1-t)yz\bigr)^{-x}}{1-t(1-t)yz};$$

so

$$\begin{aligned}\int_0^1\sum_{N\ge 1}y^N(1-t)^N[z^N]\,\frac{(1-z)^{-x}}{1-tz}\,dt &= \int_0^1\sum_{N\ge 1}[z^N]\,\frac{\bigl(1-(1-t)yz\bigr)^{-x}}{1-t(1-t)yz}\,dt\\ &= \int_0^1\frac{\bigl(1-(1-t)y\bigr)^{-x}}{1-t(1-t)y}\,dt - 1.\end{aligned} \tag{20}$$

Further, by the Cauchy integral formula,

$$y^N[z^N]\frac{(1-z)^{-x}}{1-tz}\big((1-z)t\big)^N = \frac{1}{2\pi i}\oint_{|z|=2/3}\frac{(1-z)^{-x}}{z^{N+1}(1-tz)}\big(y(1-z)t\big)^N\,dz$$
$$= \frac{1}{2\pi i}\oint_{|z|=2/3}\frac{(1-z)^{-x}}{z(1-tz)}\left(\frac{y(1-z)t}{z}\right)^N\,dz.$$

On the circle $|z|=2/3$, we have $\left|\frac{y(1-z)t}{z}\right| \le 5|y|/2 \le 5/6$; so summing over $N \ge 1$,

$$\sum_{N\ge1}(-1)^{N+1}y^N[z^N]\frac{(1-z)^{-x}}{1-tz}\big((1-z)t\big)^N$$
$$=\frac{1}{2\pi i}\oint_{|z|=2/3}\frac{(1-z)^{-x}}{z(1-tz)}\frac{\frac{y(1-z)t}{z}}{1+\frac{y(1-z)t}{z}}\,dz$$
$$=\frac{1}{2\pi i}\oint_{|z|=2/3}\frac{(1-z)^{-x}}{z(1-tz)}\cdot\frac{y(1-z)t}{z+y(1-z)t}\,dz.$$

For $t>0$, in the circle $|z|\le 2/3$ the integrand has two poles, both simple, at $z=0$ and $z=-\frac{yt}{1-yt}$, with respective residues equal 1 and $-\frac{(1-yt)^x}{1-yt(1-t)}$. Thus

$$\sum_{N\ge1}(-1)^{N+1}y^N[z^N]\frac{(1-z)^{-x}}{1-tz}\big((1-z)t\big)^N = 1-\frac{(1-yt)^x}{1-yt(1-t)}.$$

Integrating for $t\in[0,1]$ and adding to (20), we obtain

$$\sum_{N\ge1}\frac{y^N}{N}\mathrm{E}\big[x^{\nu(\sigma^{(3)}(N))}\big] = \int_0^1\frac{(1-y(1-t))^{-x}-(1-yt)^x}{1-yt(1-t)}\,dt,$$

which is equivalent to (18), as $t(1-t)$ is symmetric with respect to $t=1/2$.

4. Probability that the occupancy numbers of the cycles of σ by the elements of $[\ell]$ belong to a given set

In the section title and elsewhere below σ is $\sigma^{(k)}$, the product of k random maximal cycles. Let $A\subseteq\mathbb{Z}_{\ge0}$ be given. Introduce $p_A(N,\ell;k)$, the probability that the number of elements of $[\ell]$ in each cycle of σ belongs to the set A.

The examples include: (1) $A_1=\mathbb{Z}_{>0}$; each cycle must contain at least one element of $[\ell]$; (2) $A_2=\{0,\ell\}$; one of the cycles of σ contains the whole set $[\ell]$; (3) $A_3=\{0,1\}$; each element of $[\ell]$ belongs to a distinct cycle of σ. The case of $k=2$, $\ell=2$ and $A=\{0,2\}$ or $A=\{0,1\}$ was solved by Stanley [16]. Very recently Bernardi et al. [2] solved the case $k=2$, $A=\{0,1\}$ for $\ell\ge2$. In fact they solved a general problem of separation probability for t disjoint sets $\mathcal{S}_1,\dots,\mathcal{S}_t$.

To evaluate $p_A(N,\ell;k)$, consider first $Q_A(\vec{\nu},\ell)$, the total number of permutations s of $[N]$, with $\vec{\nu}(s)=\{\nu_r(s)\}=\{\nu_r\}=\vec{\nu}$, such that the number of elements of $[\ell]$ in every cycle is an element of A. The reason we need $Q_A(\vec{\nu},\ell)$ is that the key formula (7) expresses $\chi^{\lambda*}(s)$ through the cycle counts $\nu_r(s)$, $r\ge 1$.

To evaluate $Q_A(\vec{\nu},\ell)$, introduce the non-negative integers $a_{r,j}$, $b_{r,j}$ that stand for the generic numbers of elements from $[\ell]$ and $[N]\setminus[\ell]$ and in the j-th cycle of length r, $(j\le \nu_r)$. For $\mathbf{a}$, $\mathbf{b}$ to be admissible we must have

$$a_{r,j}+b_{r,j}=r, \tag{21}$$
$$a_{r,j}\in A, \tag{22}$$
$$\sum_{r,j}a_{r,j}=\ell,\quad \sum_{r,j}b_{r,j}=N-\ell. \tag{23}$$

Therefore

$$\begin{aligned}
Q_A(\vec{\nu},\ell) &= (N-\ell)!\,\ell!\sum_{\substack{\mathbf{a},\,\mathbf{b}\text{ meet}\\(21),(22),(23)}}\prod_r\frac{((r-1)!)^{\nu_r}}{\nu_r!}\prod_{j\le r}\frac{1}{a_{r,j}!\,b_{r,j}!}\\
&= (N-\ell)!\,\ell!\,[w^\ell]\prod_r\frac{1}{r^{\nu_r}\nu_r!}\prod_{j\le\nu_r}\sum_{a_{r,j}\in A}\binom{r}{a_{r,j}}w^{a_{r,j}}\\
&= (N-\ell)!\,\ell!\,\,[w^\ell]\prod_r\frac{1}{\nu_r!}\left(\frac{\sum_{a\in A}\binom{r}{a}w^a}{r}\right)^{\nu_r}.
\end{aligned}\tag{24}$$

So, using (7) and $\nu=\sum_r\nu_r$, we conclude that

$$\sum_{s:\,\vec{\nu}(s)=\vec{\nu}}\chi^{\lambda^*}(s)=(-1)^{\lambda^1}(N-\ell)!\,\ell!$$
$$\times\,[\xi^{\lambda_1}w^\ell]\,\frac{\xi}{1-\xi}\prod_r\frac{1}{\nu_r!}\left(-\frac{(1-\xi^r)\big(\sum_{a\in A}\binom{r}{a}w^a\big)}{r}\right)^{\nu_r}.$$

Call a permutation s of $[N]$ admissible if the numbers of elements from $[\ell]$ in each cycle of s meet the constraint (22). The above identity implies

$$\begin{aligned}
\sum_{s\text{ admissible}}\chi^{\lambda^*}(s)&=(-1)^{\lambda^1}(N-\ell)!\,\ell!\\
\times\,[\xi^{\lambda_1}w^\ell]\,\frac{\xi}{1-\xi}&\sum_{\substack{\vec{\nu}:\\1\nu_1+2\nu_2+\cdots=N}}\prod_r\frac{1}{\nu_r!}\left(-\frac{(1-\xi^r)\big(\sum_{a\in A}\binom{r}{a}w^a\big)}{r}\right)^{\nu_r}.
\end{aligned}\tag{25}$$

The expression in the second line of (25) equals

$$
\begin{aligned}
&[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\sum_{\vec{\nu}\geq\mathbf{0}}\prod_r \frac{(x^r)^{\nu_r}}{\nu_r!}\left(-\frac{(1-\xi^r)\big(\sum_{a\in A}\binom{r}{a}w^a\big)}{r}\right)^{\nu_r}\\
=&[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\prod_r\sum_{\nu_r\geq 0}\frac{1}{\nu_r!}\left(-\frac{x^r(1-\xi^r)\big(\sum_{a\in A}\binom{r}{a}w^a\big)}{r}\right)^{\nu_r}\\
=&[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\prod_r \exp\left(-\frac{x^r(1-\xi^r)\big(\sum_{a\in A}\binom{r}{a}w^a\big)}{r}\right)\\
=&[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\exp\left(-\sum_{r\geq 1}\frac{x^r(1-\xi^r)\big(\sum_{a\in A}\binom{r}{a}w^a\big)}{r}\right). \qquad (26)
\end{aligned}
$$

4.1. **Probability that each cycle of σ contains at least one element of $[\ell]$.** In this case $A = A_1 = \mathbb{Z}_{>0}$. Therefore

$$
\sum_{a\in A}\binom{r}{a}w^a = (1+w)^r - 1.
$$

So, using (25), (26) and $\sum_{j\geq 1} z^j/j = -\log(1-z)$, $|z|<1$, we obtain

$$
\begin{aligned}
\sum_{s\text{ admissible}}\chi^{\lambda^*}(s) =&(-1)^{\lambda^1}(N-\ell)!\,\ell!\\
&\times [\xi^{\lambda_1} w^\ell x^N]\ \frac{\xi}{1-\xi}\ \frac{\big(1-x(1+w)\big)(1-\xi x)}{\big(1-\xi x(1+w)\big)(1-x)}. \qquad (27)
\end{aligned}
$$

Let us simplify this formula. Write

$$
\begin{aligned}
[w^\ell]\frac{1-x(1+w)}{1-\xi x(1+w)} =& \frac{\xi-1}{\xi}[w^\ell]\frac{1}{1-\xi x(1+w)}\\
=&\frac{1-\xi}{\xi^2 x}\,[w^\ell]\left(w-\frac{1-\xi x}{\xi x}\right)^{-1}\\
=&\frac{1-\xi}{\xi^2 x}\binom{-1}{\ell}(-1)^{-1-\ell}\left(\frac{1-\xi x}{\xi x}\right)^{-1-\ell}\\
=&-\frac{1-\xi}{\xi^2 x}\left(\frac{\xi x}{1-\xi x}\right)^{1+\ell}.
\end{aligned}
$$

Therefore

$$[\xi^{\lambda_1} w^\ell x^N]\ \frac{\xi}{1-\xi}\,\frac{\big(1-x(1+w)\big)(1-\xi x)}{\big(1-\xi x(1+w)\big)(1-x)} = -[\xi^{\lambda_1} x^N]\,(1-x)^{-1}\left(\frac{\xi x}{1-\xi x}\right)^\ell$$
$$= -[x^N]\,x^{\lambda_1}(1-x)^{-1}\cdot[y^{\lambda_1}]\left(\frac{y}{1-y}\right)^\ell$$
$$= -[y^{\lambda_1-k}](1-y)^{-\ell} = -\binom{\lambda_1-1}{\lambda_1-\ell},$$

where $\binom{a}{b}=0$ for $b<0$. So (27) becomes

$$\sum_{s \text{ admissible}} \chi^{\lambda^*}(s) = (-1)^{\lambda^1-1}(N-\ell)!\,\ell!\binom{\lambda_1-1}{\lambda_1-\ell}. \tag{28}$$

Combining (28) and (6) we conclude that

$$\begin{aligned} p_{A_1}(N,\ell;k) =& \frac{1}{N!}\sum_{\lambda^*}(-1)^{k(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1}\sum_{s \text{ admissible}}\chi^{\lambda^*}(s)\\ =&\binom{N}{\ell}^{-1}\sum_{\lambda_1=\ell}^{N}(-1)^{(k-1)(N-\lambda_1)}\binom{N-1}{N-\lambda_1}^{-k+1}\binom{\lambda_1-1}{\ell-1}. \end{aligned} \tag{29}$$

For $N\to\infty$, the dominant contribution to the right-hand side in (29) comes from $\lambda_1=\ell$ and $\lambda_1=N$, so that $p_{A_1}(N,\ell;k)=\ell/N+O(N^{-2\ell+1})$; the formula is useful for $\ell>1$. We remark that ℓ/N is the probability that every cycle of the uniformly random permutation of $[N]$ contains at least one element of $[\ell]$; see Lovász [12], Section 3, Exercise 6.

For $k=2$ we are able to replace the right-hand side of (29) with a sum of just $\ell+1$ terms, which will allow us to determine compact formulas for moderate values of ℓ. To do so we will need a certain binomial identity. Introduce

$$S_{n,a,b} = \sum_{r=a+b}^{n}(-1)^r\frac{\binom{r-a}{b}}{\binom{n}{r}}. \tag{30}$$

This function is relevant since (29) is equivalent to

$$p_{A_1}(N,\ell;2) = (-1)^{N-1}\binom{N}{\ell}^{-1}S_{N-1,0,\ell-1}. \tag{31}$$

Suri et al. [17] proved that

$$S_{n,0,0} = (1+(-1)^n)\,\frac{n+1}{n+2}. \tag{32}$$

The key element of their argument was an identity that we have previously mentioned and used, namely

$$\binom{n}{r}^{-1} = (n+1)\int_0^1 t^r(1-t)^{n-r}\,dt. \tag{33}$$

They used (33) to derive a sum-type formula, still with $n+1$ terms, for

$$\sum_{r=0}^{n}(-1)^r \frac{x^r}{\binom{n}{r}},$$

that yielded (32) via setting $x=1$. We also use (33) but avoid an intermediate sum with $n+1$ terms, and instead differentiate the resulting integral with respect to the parameter x. Here are the details. First define and evaluate $\mathcal{S}_{n,a,b}(x)$: for $a+b \le n$,

$$\begin{aligned}
\mathcal{S}_{n,a,b}(x) :=& \sum_{r=a+b}^{n} (-1)^r \frac{x^{r-a}}{\binom{n}{r}} \\
=&(n+1)\int_0^1 \left(\sum_{r=a+b}^{n} (-1)^r x^{r-a} t^r (1-t)^{n-r} \right) dt \\
=&(n+1)(-1)^{a+b} \int_0^1 t^{a+b}(1-t)^{n-a-b} x^b \sum_{r=a+b}^{n} \left(-\frac{xt}{1-t} \right)^{r-a-b} dt \\
=&(n+1)(-1)^{a+b} \int_0^1 \frac{x^b + (-1)^{n-a-b} x^{n-a+1} \left(\frac{t}{1-t}\right)^{n-a-b+1}}{1+\frac{xt}{1-t}} \\
&\times t^{a+b}(1-t)^{n-a-b}\, dt.
\end{aligned} \tag{34}$$

The connection between $\mathcal{S}(n,a,x)$ and $S(n,a,b)$ is: $S_{n,a,b} = \frac{1}{b!} \left.\frac{d^b \mathcal{S}_{n,a,b}(x)}{dx^b}\right|_{x=1}$.

To compute this derivative, we differentiate b times the right-hand side of (34) with respect to x by carrying the operation inside the integral and then setting $x=1$. So

(35)
$$\begin{aligned}
&\left.\frac{d^b \mathcal{S}(n,a,x)}{dx^b}\right|_{x=1} = (-1)^{a+b}(n+1) \\
&\times \int_0^1 \frac{\partial^b}{\partial x^b} \left.\frac{x^b + (-1)^{n-a-b} x^{n-a+1} \left(\frac{t}{1-t}\right)^{n-a-b+1}}{1+\frac{xt}{1-t}}\right|_{x=1} t^{a+b}(1-t)^{n-a-b}\, dt.
\end{aligned}$$

By $(uv)^{(b)} = \sum_j \binom{b}{j} u^{(j)} v^{(b-j)}$, the partial derivative at $x = 1$ is

$$\sum_{j=0}^{b} \binom{b}{j} \left((b)_j + (-1)^{n-a-b} \left(\tfrac{t}{1-t}\right)^{n-a-b+1} (n-a+1)_j \right)$$

$$\times (-1)^{b-j} \frac{(b-j)!}{\left(1+\frac{t}{1-t}\right)^{b-j+1}} \cdot \left(\frac{t}{1-t}\right)^{b-j}$$

$$= \sum_{j=0}^{b} (-1)^{b-j} \binom{b}{j} \left[b!\, t^{b-j}(1-t) + (-1)^{n-a-b}(b-j)!\,(n-a+1)_j \frac{t^{n-a+1-j}}{(1-t)^{n-a-b}} \right]$$

$$= b! \left[(1-t)^{b+1} + \sum_{j=0}^{b} (-1)^{n-a-j} \binom{n-a+1}{j} \frac{t^{n-a+1-j}}{(1-t)^{n-a-b}} \right].$$

Plugging the last expression into (35) and using (33) we obtain

$$S_{n,a,b} = \sum_{r=a+b}^{n} (-1)^r \frac{\binom{r-a}{b}}{\binom{n}{r}} = (n+1) \left[\frac{(-1)^{a+b}}{(n+2+b)\binom{n+b+1}{a+b}} + \sum_{j=0}^{b} (-1)^{n+b-j} \binom{n-a+1}{j} \frac{1}{n+2+b-j} \right]. \tag{36}$$

For large n, this formula is a significant improvement of the initial definition of $S_{n,a,b}$ if b remains moderately valued. Using yet another identity

$$\sum_{j=0}^{u} (-1)^j \binom{u}{j} \frac{1}{v+j+1} = \frac{1}{(u+v+1)\binom{u+v}{v}},$$

from Suri et al. [17], the equation (36) is easily transformed into

$$S_{n,a,b} = (-1)^{a+b}(n+1) \sum_{i=0}^{n-a-b} (-1)^i \binom{n-a+1}{i} \frac{1}{i+a+b+1}. \tag{37}$$

This alternative formula is efficient for the extreme case, when $n-a-b$ is moderately valued as n grows.

So, applying the formulas (36), (37) for $n = N-1$, $a = 0$ and $b = \ell - 1$, we obtain from (31) that

$$p_{A_1}(N, \ell; 2) = (-1)^{N-1} N \binom{N}{\ell}^{-1} \times \left[\frac{(-1)^{\ell-1}}{(N+\ell)\binom{N+\ell-1}{\ell-1}} + \sum_{j=0}^{\ell-1} (-1)^{N+\ell-j} \binom{N}{j} \frac{1}{N+\ell-j} \right]$$
$$= (-1)^{N+\ell} N \binom{N}{\ell}^{-1} \sum_{i=0}^{N-\ell} (-1)^i \binom{N}{i} \frac{1}{i+\ell}, \tag{38}$$

two expressions that can be efficiently computed for moderate ℓ and moderate $N-\ell$, respectively. In particular, using the first expression we obtain

$$p_{A_1}(N,1;2) = \begin{cases} \frac{2}{N+1} & \text{if } N \text{ is odd,} \\ 0 & \text{if } N \text{ is even.} \end{cases}$$

This is equivalent to the result already mentioned in Section 3, since $p_{A_1}(N,1;2)$ is indeed equal to the probability that σ is a maximal cycle.

4.2. **Probability that the elements $1,\dots,\ell$ are in the same cycle of σ.** This time $A = A_2 = \{0,\ell\}$, so that

$$\sum_{a\in A_2} \binom{r}{a} w^a = 1 + \binom{r}{\ell} w^\ell.$$

Therefore

$$Q_{A_2}(\vec{\nu},\ell) = (N-\ell)!\,\ell!\;[w^\ell]\prod_r \frac{1}{\nu_r!}\left(\tfrac{1+\binom{r}{\ell}w^\ell}{r}\right)^{\nu_r}. \tag{39}$$

So, using (7) and $\nu = \sum_r \nu_r$, we conclude that

$$\begin{aligned} \sum_{s \text{ admissible}} \chi^{\lambda^*}(s) &= (-1)^{\lambda^1}(N-\ell)!\,\ell! \\ &\times [\xi^{\lambda_1}w^\ell]\,\frac{\xi}{1-\xi}\sum_{\substack{\vec{\nu}:\\ 1\nu_1+2\nu_2+\dots=N}}\prod_r\frac{1}{\nu_r!}\left(-(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}\right)^{\nu_r}. \end{aligned} \tag{40}$$

Since $\sum_r r\nu_r = N$, the identity $\sum_r z^r/r = -\log(1-z)$, ($|z|<1$), implies that the second line expression in (40) equals

$$\begin{aligned} &[\xi^{\lambda_1}w^\ell x^N]\frac{\xi}{1-\xi}\sum_{\vec{\nu}\ge\mathbf{0}}\prod_r\frac{(x^r)^{\nu_r}}{\nu_r!}\left(-(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}\right)^{\nu_r} \\ &\quad= [\xi^{\lambda_1}w^\ell x^N]\frac{\xi}{1-\xi}\prod_r\sum_{\nu_r\ge 0}\frac{1}{\nu_r!}\left(-x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}\right)^{\nu_r} \\ &\quad= [\xi^{\lambda_1}w^\ell x^N]\frac{\xi}{1-\xi}\prod_r\exp\left(-x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}\right) \\ &\quad= [\xi^{\lambda_1}w^\ell x^N]\frac{\xi}{1-\xi}\,\exp\left(-\sum_{r\ge 1}x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}\right). \end{aligned}$$

Here, using $\sum_{b\ge a}\binom{b}{a}z^b=\frac{z^a}{(1-z)^{a+1}}$,

$$\begin{aligned}\sum_{r\ge 1}x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}&\\ &=-\log(1-x)+\log(1-x\xi)+\frac{w^\ell}{\ell}\sum_{r\ge 1}\binom{r-1}{\ell-1}\bigl(x^r-(x\xi)^r\bigr)\\ &=\log\frac{1-x\xi}{1-x}+\frac{w^\ell}{\ell}\left(\frac{x^\ell}{(1-x)^\ell}-\frac{(x\xi)^\ell}{(1-x\xi)^\ell}\right).\end{aligned}$$

Therefore

$$\begin{aligned}[w^\ell]\exp\left(-\sum_{r\ge 1}x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}\right)&\\ &=\frac{1-x}{1-x\xi}[w^\ell]\exp\left[-\frac{w^\ell}{\ell}\left(\frac{x^\ell}{(1-x)^\ell}-\frac{(x\xi)^\ell}{(1-x\xi)^\ell}\right)\right]\\ &=\frac{1}{\ell}\frac{1-x}{1-x\xi}\left(\frac{(x\xi)^\ell}{(1-x\xi)^\ell}-\frac{x^\ell}{(1-x)^\ell}\right).\end{aligned}$$

Therefore the expression in the second line of (40) is equal to

$$\begin{aligned}\frac{1}{\ell}[\xi^{\lambda_1}x^N]\frac{\xi}{1-\xi}\cdot\frac{1-x}{1-x\xi}\left(\frac{(x\xi)^\ell}{(1-x\xi)^\ell}-\frac{x^\ell}{(1-x)^\ell}\right)&\\ &=\frac{1}{\ell}[\xi^{\lambda_1}x^N]\left(\frac{1}{1-\xi}-\frac{1}{1-x\xi}\right)\left(\frac{(x\xi)^\ell}{(1-x\xi)^\ell}-\frac{x^\ell}{(1-x)^\ell}\right)\\ &=:\frac{1}{\ell}(T_1+T_2+T_3+T_4).\end{aligned}$$

Here

$$\begin{aligned}T_1&=[\xi^{\lambda_1}x^N]\frac{1}{1-\xi}\cdot\frac{(x\xi)^\ell}{(1-x\xi)^\ell}\\ &=[\xi^{\lambda_1}]\frac{\xi^N}{1-\xi}[y^N]\frac{y^\ell}{(1-y)^\ell}=1_{\{\lambda_1=N\}}\binom{N-1}{\ell-1};\end{aligned}\tag{41}$$

next

$$\begin{aligned}T_2&=-[\xi^{\lambda_1}x^N]\frac{1}{1-\xi}\cdot\frac{x^\ell}{(1-x)^\ell}\\ &=-[x^{N-\ell}]\frac{1}{(1-x)^\ell}=-\binom{N-1}{\ell-1};\end{aligned}\tag{42}$$

next

$$T_3 = -[\xi^{\lambda_1} x^N]\,\frac{(x\xi)^\ell}{(1-x\xi)^{\ell+1}} \tag{43}$$
$$= -1_{\{\lambda_1=N\}}\,[y^{N-\ell}]\,\frac{1}{(1-y)^{\ell+1}} = -1_{\{\lambda_1=N\}}\binom{N}{\ell};$$

and finally

$$T_4 = [\xi^{\lambda_1} x^N]\,\frac{1}{1-x\xi}\,\frac{x^\ell}{(1-x)^\ell}$$
$$= [x^N]\,\frac{x^{\lambda_1+\ell}}{(1-x)^\ell} = [x^{N-\lambda_1-\ell}]\,\frac{1}{(1-x)^\ell} \tag{44}$$
$$= 1_{\{\lambda_1<N\}}\binom{N-\lambda_1-1}{\ell-1}.$$

It follows from (41), (42), (43) and (44) that

$$\frac{1}{\ell}(T_1+T_2+T_3+T_4)$$
$$= -\frac{1}{\ell}\left\{1_{\{\lambda_1<N\}}\left[\binom{N-1}{\ell-1} - \binom{N-\lambda_1-1}{\ell-1}\right] + 1_{\{\lambda_1=N\}}\binom{N}{\ell}\right\}.$$

So (40) becomes

$$\sum_{s \text{ admissible}} \chi^{\lambda^*}(s) = (-1)^{\lambda^1-1}(N-\ell)!\,\ell! \tag{45}$$
$$\times \frac{1}{\ell}\left\{1_{\{\lambda_1<N\}}\left[\binom{N-1}{\ell-1} - \binom{N-\lambda_1-1}{\ell-1}\right] + 1_{\{\lambda_1=N\}}\binom{N}{\ell}\right\}.$$

Combining (45) and (6) we thus proved

$$p_{A_2}(N,\ell;k) = \frac{1}{N!}\sum_{\lambda_1=1}^{N}(-1)^{k(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1}\sum_{s \text{ admissible}}\chi^{\lambda^*}(s)$$
$$= \frac{1}{\ell}\binom{N}{\ell}^{-1}\sum_{\lambda_1}(-1)^{(k+1)(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1}$$
$$\times\left\{1_{\{\lambda_1<N\}}\left[\binom{N-1}{\ell-1} - \binom{N-\lambda_1-1}{\ell-1}\right] + 1_{\{\lambda_1=N\}}\binom{N}{\ell}\right\}.$$

For $k=2$, introducing $r = N-\lambda_1$, we equivalently have

$$p_{A_2}(N,\ell;2) = \frac{1}{\ell} + \frac{1}{\ell}\binom{N}{\ell}^{-1} \tag{46}$$
$$\times \sum_{r=1}^{N-1}(-1)^r\cdot\binom{N-1}{r}^{-1}\left[\binom{N-1}{\ell-1} - \binom{r-1}{\ell-1}\right].$$

By (30), the last sum is the linear combination of $S_{N-1,0,0}-1$ and $S_{N-1,1,\ell-1}$. According to (36) and (37), we have

$$\begin{aligned}
S_{N-1,0,0} =&\big[1+(-1)^{N-1}\big]\frac{N}{N+1},\\
S_{N-1,1,\ell-1} =&(-1)^{\ell}\left[\binom{N+\ell}{\ell}^{-1}+N\sum_{j=0}^{\ell-1}(-1)^{N-j}\binom{N-1}{j}\frac{1}{N+\ell-j}\right]\\
=&(-1)^{\ell}N\sum_{i=0}^{N-1-\ell}(-1)^{i}\binom{N-1}{i}\frac{1}{i+\ell+1}.
\end{aligned}$$

Plugging these expressions into (46), we obtain after simple algebra

$$\begin{aligned}
p_{A_2}(N,\ell;2) =&\frac{1}{\ell}+\left[\frac{1+(-1)^{N-1}}{N+1}-\frac{1}{N}\right]\\
&+\frac{(-1)^{\ell+1}}{\ell\binom{N}{\ell}}\left[\binom{N+\ell}{\ell}^{-1}+N\sum_{j=0}^{\ell-1}(-1)^{N-j}\binom{N-1}{j}\frac{1}{N+\ell-j}\right]\\
=&\frac{1}{\ell}-\frac{1}{(N+1)_2}\\
&+\frac{(-1)^{\ell+1}}{\ell\binom{N}{\ell}}\left[\binom{N+\ell}{\ell}^{-1}+N\sum_{j=0}^{\ell-2}(-1)^{N-j}\binom{N-1}{j}\frac{1}{N+\ell-j}\right] \qquad (47)\\
=&\frac{1}{\ell}-\frac{1}{(N+1)_2}\\
&+(-1)^{\ell+1}\binom{N-1}{\ell-1}^{-1}\sum_{i=0}^{N-\ell}(-1)^{i}\binom{N-1}{i}\frac{1}{i+\ell+1}. \qquad (48)
\end{aligned}$$

The equivalent formulas (47) and (48) are computationally efficient for moderate ℓ and moderate $N-\ell$ respectively. In particular, plugging $\ell=2,3$ into (47) and simplifying, we recover Stanley's results, [16].

5. The probability that σ separates the disjoint sets $\mathcal{S}_1,\dots,\mathcal{S}_t$

Let $\ell_j=|\mathcal{S}_j|$, $1\le j\le t$, $\ell=\sum_j\ell_j$. Introduce $p(N,\vec{\ell};k)$, the probability that the permutation σ separates the sets $\mathcal{S}_1,\dots,\mathcal{S}_t$, meaning that no cycle of σ contains a pair of elements from two distinct sets $\mathcal{S}_i$ and $\mathcal{S}_j$. Bernardi et al. [2] were able to derive a striking formula for $p(N,\vec{\ell};2)$:

(49)
$$p(N,\vec{\ell};2)=\frac{(N-\ell)!\prod_j\ell_j!}{(N+t)(N-1)!}\left[\frac{(-1)^{N+\ell}\binom{N-1}{t-2}}{\binom{N+\ell}{\ell-t}}+\sum_{j=0}^{\ell-t}\frac{(-1)^j\binom{\ell-t}{j}\binom{N+j+1}{\ell}}{\binom{N+t+j}{j}}\right],$$

which is a sum of $\ell - t + 2$ terms. Remarkably, $\prod_j \ell_j!$ aside, the rest of this expression does not depend on the individual ℓ_j. The equation (49) is very efficient for values of ℓ, t relatively small compared to N.

In this section we apply our approach to obtain an efficient formula for this probability when ℓ is close to N, implying in particular that, for $\ell = N$,

$$p(N, \vec{\ell}; 2) = \frac{\prod_j \ell_j!}{(N-1)!(N-t+1)}.$$

We will also obtain an equivalent version of (49).

Let $Q(\vec{\nu}, \vec{\ell})$ denote the total number of permutations of $[N]$ with cycle counts $\vec{\nu} = (\nu_1, \nu_2, \dots)$ that separate $\mathcal{S}_1, \dots, \mathcal{S}_t$. Each cycle of such a permutation either does not contain any element of $\cup_j \mathcal{S}_j$, or contains some of the elements of exactly one set $\mathcal{S}_j$. Since $|[N] \cup_j \mathcal{S}_j| = N - \ell$, Denoting $\prod_j w_j^{\ell_j} = \vec{w}^{\,\vec{\ell}}$, analogously to (24) we have

(50)

$$\begin{aligned}\frac{Q(\vec{\nu}, \vec{\ell})}{(N-\ell)! \prod_j \ell_j!} &= [y^{N-\ell} \vec{w}^{\,\vec{\ell}}] \left[\prod_r \frac{1}{\nu_r!} \left(\frac{y^r + \sum_{j=1}^t \sum_{a>0} \binom{r}{a} w_j^a y^{r-a}}{r} \right)^{\nu_r} \right] \\ &= [y^{N-\ell} \vec{w}^{\,\vec{\ell}}] \left[\prod_r \frac{1}{\nu_r!} \left(\frac{-(t-1)y^r + \sum_{j=1}^t (w_j + y)^r}{r} \right)^{\nu_r} \right].\end{aligned}$$

Using (7) and (50), we obtain

(51)

$$\begin{aligned}\sum_{s \text{ admissible}} \chi^{\lambda^*}(s) = (-1)^{\lambda^1} (N-\ell)! \prod_j \ell_j! \\ \times [\xi^{\lambda_1} y^{N-\ell} \vec{w}^{\,\vec{\ell}}] \, \frac{\xi}{1-\xi} \sum_{\vec{\nu}} \prod_r \frac{1}{\nu_r!} \left(- \frac{(1-\xi^r)\big(-(t-1)y^r + \sum_j (w_j + y)^r\big)}{r} \right)^{\nu_r},\end{aligned}$$

the sum being for $\vec{\nu} \geq \mathbf{0}$ with $\sum_r r\nu_r = N$. So the expression in the second line of (51) equals

$$\begin{aligned}
&[\xi^{\lambda_1} x^N y^{N-\ell} \vec{w}^{\,\vec{\ell}}] \frac{\xi}{1-\xi} \sum_{\vec{\nu} \geq \mathbf{0}} \prod_r \frac{(x^r)^{\nu_r}}{\nu_r!} \\
&\quad \times \left(- \frac{(1-\xi^r)\left[-(t-1)y^r + \sum_j (w_j + y)^r\right]}{r} \right)^{\nu_r} \\
&= [\xi^{\lambda_1} x^N y^{N-\ell} \vec{w}^{\,\vec{\ell}}] \frac{\xi}{1-\xi} \prod_r \exp\left(- \frac{x^r (1-\xi^r)\left[-(t-1)y^r + \sum_j (w_j+y)^r\right]}{r} \right) \\
&= [\xi^{\lambda_1} x^N y^{N-\ell}] \frac{\xi}{1-\xi} \left(\frac{1-\xi xy}{1-xy} \right)^{t-1} [\vec{w}^{\,\vec{\ell}}] \prod_j \frac{1 - x(w_j+y)}{1 - \xi x(w_j+y)} \\
&= [\xi^{\lambda_1} x^N y^{N-\ell}] \frac{\xi}{1-\xi} \left(\frac{1-\xi xy}{1-xy} \right)^{t-1} \left(\frac{1-xy}{1-\xi xy} \right)^{t} \cdot \prod_j [w_j^{\ell_j}] \frac{1 - \frac{x w_j}{1-xy}}{1 - \frac{\xi x w_j}{1-\xi xy}} \\
&= [\xi^{\lambda_1} x^N y^{N-\ell}] \frac{\xi}{1-\xi} \frac{1-xy}{1-\xi xy} \prod_j \left[\left(\frac{\xi x}{1-\xi xy} \right)^{\ell_j} - \frac{x}{1-xy} \left(\frac{\xi x}{1-\xi xy} \right)^{\ell_j - 1} \right] \\
&= [\xi^{\lambda_1} x^N y^{N-\ell}] \frac{\xi}{1-\xi} \frac{1-xy}{1-\xi xy} \left(\frac{\xi x}{1-\xi xy} \right)^{\ell - t} \left(\frac{(\xi-1)x}{(1-\xi xy)(1-xy)} \right)^{t} \\
&= (-1)^t \, [\xi^{\lambda_1 - 1 - (\ell - t)} x^{N-\ell} y^{N-\ell}] (1-\xi)^{t-1} (1-xy)^{-t+1} (1-\xi xy)^{-\ell-1} \\
&= (-1)^t K(N, \ell, t; \lambda_1 - 1).
\end{aligned}$$

Thus, ξ aside, we need to extract a coefficient of $(xy)^{N-\ell}$ from a power series of xy. So

$$\begin{aligned}
K(N,\ell,t;r) &:= [\xi^{r-\ell+t} z^{N-\ell}] \, (1-\xi)^{t-1} (1-z)^{-t+1} (1-\xi z)^{-\ell-1} \\
&= \sum_j (-1)^{r-\Delta-j} \binom{\ell+j}{j} \binom{t-1}{r-\Delta-j} \binom{N-\Delta-j-2}{t-2},
\end{aligned} \tag{52}$$

where we set $\Delta = \ell - t$. Obviously $K(N,\ell,t;r) = 0$ for $r < \ell - t$, and less obviously for $r \geq N$. Indeed

$$\begin{aligned}
&[z^{N-\ell}] \, (1-\xi)^{t-1} (1-z)^{-t+1} (1-\xi z)^{-\ell+1} \\
&\qquad = \sum_{j \leq N-\ell} (-1)^{N-\ell-j} \binom{-t+1}{N-\ell-j} [z^j] (1-\xi)^{t-1} (1-\xi z)^{-\ell-1},
\end{aligned} \tag{53}$$

and the $[z^j]$-factor is a polynomial of ξ of degree $t-1+j \leq t-1+N-\ell < r-\ell+t$ if $r \geq N$.

Combining this with equation (6), and $\lambda_1+\lambda^1=N+1$, we conclude that

$$
\begin{aligned}
p(N,\vec{\ell};k) &= \frac{1}{N!}\sum_{\lambda_1}(-1)^{k(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1}\sum_{s\text{ admissible}}\chi^{\lambda^*}(s)\\
&=\frac{(-1)^{\alpha_k(N,t)}\prod_j \ell_j!}{(N)_\ell}\sum_{r=0}^{N-1}(-1)^{(k+1)r}\binom{N-1}{r}^{-k+1}K(N,\ell,t;r),
\end{aligned}
\tag{54}
$$

where $\alpha_k(N,t)=t-1$ if k is odd, and $\alpha_k(N,t)=N+t$ if k is even.

The sum in the second line depends only on ℓ and t, rather than the individual $\ell_1,\dots,\ell_t$, and $K(N,\ell,t,r)$ is given by each of two lines in (52). In particular,

$$
K(N,N,t;r)=[\xi^{r-N+t}](1-\xi)^{t-1}=(-1)^{r-N+t}\binom{t-1}{r-N+t}.
$$

Let $\ell=\sum_j\ell_j=N$. Introducing $\beta_k(N)=N-1$ for k odd, $\beta_k(N)=0$ for k even, equation (54) becomes

$$
p(N,\vec{\ell};k)=\frac{(-1)^{\beta_k(N)}\prod_j\ell_j!}{(N)_\ell}\sum_{r=N-t}^{N-1}(-1)^{kr}\binom{N-1}{r}^{-k+1}\binom{t-1}{r-N+t},
$$

an alternating sum of t terms. For $t=N$, $p(N,\vec{\ell};k)=\mathrm{P}(\sigma=\mathrm{id})$; the resulting formula agrees with (10), since for k odd and N even the sum over $r\in[0,N-1]$ is zero.

From now on we focus on $k=2$, and general $\vec{\ell}$. In this case (54) becomes

$$
\begin{aligned}
p(N,\vec{\ell};2) &=\frac{(-1)^{N+t}\prod_j\ell_j!}{(N)_\ell}\sum_{r=\ell-t}^{N-1}(-1)^r\binom{N-1}{r}^{-1}K(N,\ell,t;r),\\
K(N,\ell,t;r) &:= [\xi^{r-\ell+t}z^{N-\ell}]\,(1-\xi)^{t-1}(1-z)^{-t+1}(1-\xi z)^{-\ell-1}.
\end{aligned}
\tag{55}
$$

In (55) we can extend the summation to $r\in[\ell-t,\infty)$, since $K(N,\ell,t;r)=0$ for $r\ge N$.

Let us evaluate the sum in (55) halfway, i.e. dropping $(1-z)^{-t+1}$ and postponing the extraction of the coefficient by $z^{N-\ell}$ till the next step. Using (33), *and* the observation above to replace $N-1$ with ∞, we reduce the halfway sum to

$$
\begin{aligned}
(56)\quad & N\sum_{r=\ell-t}^{\infty}(-1)^r[\xi^{r-\ell+t}]\,\frac{(1-\xi)^{t-1}}{(1-\xi z)^{\ell+1}}\int_0^1 u^r(1-u)^{N-1-r}\,du\\
&= N\int_0^1(1-u)^{N-1}\left(\sum_{r=\ell-t}^{\infty}\left(-\frac{u}{1-u}\right)^r\,[\xi^{r-\ell+t}]\frac{(1-\xi)^{t-1}}{(1-\xi z)^{\ell+1}}\right)du\\
&= N\int_0^1(1-u)^{N-1}\left(-\frac{u}{1-u}\right)^{\ell-t}\left(\sum_{r=\ell-t}^{\infty}[\xi^{r-\ell+t}]\frac{(1+\xi\frac{u}{1-u})^{t-1}}{(1+\xi z\frac{u}{1-u})^{\ell+1}}\right)du\\
&= N\int_0^1(1-u)^{N-1}\left(-\frac{u}{1-u}\right)^{\ell-t}\left(\sum_{r=\ell-t}^{\infty}[\xi^{r-\ell+t}]\frac{(1+\xi\frac{u}{1-u})^{t-1}}{(1+\xi z\frac{u}{1-u})^{\ell+1}}\right)du\\
&= N\int_0^1(1-u)^{N-1}\left(-\frac{u}{1-u}\right)^{\ell-t}\left.\frac{(1+\xi\frac{u}{1-u})^{t-1}}{(1+\xi z\frac{u}{1-u})^{\ell+1}}\right|_{\xi=1}du\\
&= (-1)^{\ell-t}N\int_0^1\frac{(1-u)^{N+1}u^{\ell-t}}{(1-u+zu)^{\ell+1}}\,du;
\end{aligned}
$$

(in the fifth line we used $\sum_{r\ge 0}[\xi^r]f(\xi)=f(1)$ for the series $f(\xi)=\sum_{r\ge 0}a_r\xi^r$). So (55) is transformed into

$$
\begin{aligned}
p(N,\vec{\ell};2) =&\frac{(-1)^{N+\ell}N\prod_j \ell_j!}{(N)_\ell}\\
&\times[z^{N-\ell}](1-z)^{-t+1}\int_0^1\frac{(1-u)^{N+1}u^{\ell-t}}{(1-u+zu)^{\ell+1}}\,du.
\end{aligned}
\tag{57}
$$

In particular, for $\ell=N$ this formula yields

$$
p(N,\vec{\ell};2)=\frac{N\prod_j \ell_j!}{N!}\int_0^1 u^{N-t}\,du=\frac{\prod_j \ell_j!}{(N-1)!(N-t+1)}. \tag{58}
$$

Notice that, for $\ell=N$, the separation probability for the uniformly random permutation of $[N]$ is

$$
\frac{\prod_j \ell_j!}{N!}.
$$

More generally,

$$
p(N,\vec{\ell};2)=\frac{N\prod_j \ell_j!}{(N)_\ell}\sum_{k\le N-\ell}(-1)^k\frac{\binom{t+k-2}{t-2}\binom{N-k}{\ell}}{(N-t+1)\binom{N-t}{k}}, \tag{59}
$$

an equation computationally efficient for moderate $N-\ell$, but progressively less useful for larger values of $N-\ell$.

We want to show that equation (57) can be transformed so that extraction of the coefficient of $z^{N-\ell}$ will lead to a sum with $\ell-t+2$ number of terms, close in appearance to the formula (49) by Bernardi et al.

Clearly it is the outside factor $(1-z)^{-t+1}$ that causes the number of summands in (59) grow indefinitely with N. To get rid of $(1-z)^{-t+1}$, we resort to repeated integration by parts of the integral, denote it $I(z)$, with each step producing the outside factor $1-z$. However the factor $u^{\ell-t}$ in the integrand of $I(z)$ would have made the integration process unwieldy; so we apply it instead to $K_1(z)$, where

$$K_\nu(z) := \int_0^1 \frac{(1-u)^{N+\nu}}{(1-u+zu)^{t+\nu}}\,du,$$

because

$$I(z) = \frac{(-1)^{\ell-t}}{(t+1)^{(\ell-t)}}\,\frac{d^{\ell-t}K_1(z)}{dz^{\ell-t}}. \tag{60}$$

One integration by parts leads to

$$\begin{aligned} K_1(z) =& \frac{1}{N+2} + \frac{(t+1)(1-z)}{N+2}\int_0^1 \frac{(1-u)^{N+2}}{(1-u+zu)^{t+2}}\,du \\ =& \frac{1}{N+2} + \frac{(t+1)(1-z)}{N+2}K_2(z). \end{aligned}$$

After $\ell-1$ integrations by parts, we get

$$K_1(z) = \sum_{j=1}^{\ell-1} \frac{(t+1)^{(j-1)}}{(N+2)^{(j)}}(1-z)^{j-1} + \frac{(t+1)^{(\ell-1)}}{(N+2)^{(\ell-1)}}\,(1-z)^{\ell-1}K_\ell(z).$$

So, using (60) and

$$\frac{d^{\ell-t}\big[(1-z)^{\ell-1}K_\ell\big]}{dz^{\ell-t}} = \sum_{\mu=0}^{\ell-t}(-1)^\mu\binom{\ell-t}{\mu}(\ell-1)_\mu(1-z)^{\ell-1-\mu}\frac{d^{\ell-t-\mu}K_\ell}{dz^{\ell-t-\mu}},$$

we obtain

$$\begin{aligned} \frac{(1-z)^{-t+1}I(z)}{\frac{(-1)^{\ell-t}}{(t+1)^{(\ell-t)}}} =& \ (-1)^{\ell-t}\sum_{j=1}^{\ell-1}\frac{(t+1)^{(j-1)}(j-1)_{\ell-t}}{(N+2)^{(j)}}(1-z)^{j-\ell} \\ &+ \frac{(t+1)^{(\ell-1)}}{(N+2)^{(\ell-1)}}\sum_{\mu=0}^{\ell-t}(-1)^\mu\binom{\ell-t}{\mu}(\ell-1)_\mu(1-z)^{\ell-t-\mu}\,\frac{d^{\ell-t-\mu}K_\ell(z)}{dz^{\ell-t-\mu}}. \end{aligned}$$

It remains to extract the coefficient of $[z^{N-\ell}]$ in the right-hand side expression. First,

$$[z^{N-\ell}](1-z)^{j-\ell} = (-1)^{N-\ell}\binom{j-\ell}{N-\ell}.$$

Next, for every $r \geq 0$,

$$[z^r]\frac{d^{\ell-t-\mu}K_\ell}{dz^{\ell-t-\mu}} = (-1)^{\ell-t-\mu}(t+\ell)^{(\ell-t-\mu)}[z^r]\int_0^1 \frac{(1-u)^{N+\ell}u^{\ell-t-\mu}}{(1-u+zu)^{2\ell-\mu}}\,du$$
$$= (-1)^{\ell-t-\mu}(t+\ell)^{(\ell-t-\mu)}\binom{-2\ell+\mu}{r}\int_0^1 (1-u)^{N-\ell+\mu-r}u^{\ell-t-\mu+r}\,du$$
$$= (-1)^{\ell-t-\mu}\frac{(t+\ell)^{(\ell-t-\mu)}\binom{-2\ell+\mu}{r}}{(N-t+1)\binom{N-t}{\ell-t-\mu+r}}.$$

So

$$\begin{aligned}
&[z^{N-\ell}]\left\{(1-z)^{\ell-t-\mu}\,\frac{d^{\ell-t-\mu}\,K_\ell}{dz^{\ell-t-\mu}}\right\}\\
&= \sum_{k\leq \ell-t-\mu}\left\{[z^k](1-z)^{\ell-t-\mu}\right\}\left\{[z^{N-\ell-k}]\,\frac{d^{\ell-t-\mu}\,K_\ell}{dz^{\ell-t-\mu}}\right\} \qquad (61)\\
&= \sum_{k\leq \ell-t-\mu}(-1)^k\binom{\ell-t-\mu}{k}(-1)^{\ell-t-\mu}\left.\frac{(t+\ell)^{(\ell-t-\mu)}\binom{-2\ell+\mu}{r}}{(N-t+1)\binom{N-t}{\ell-t-\mu+r}}\right|_{r=N-\ell-k}.
\end{aligned}$$

Collecting the pieces,

$$\begin{aligned}
&\frac{[z^{N-\ell}](1-z)^{-t+1}I(z)}{\frac{(-1)^{\ell-t}}{(t+1)^{(\ell-t)}}}\\
&= (-1)^{N-t}\sum_{j=1}^{\ell-1}\frac{(t+1)^{(j-1)}(j-1)_{\ell-t}}{(N+2)^{(j)}}\binom{j-\ell}{N-\ell}\\
&+ (-1)^{\ell-t}\frac{(t+1)^{(\ell-1)}}{(N+2)^{(\ell-1)}}\sum_{\mu=0}^{\ell-t}\binom{\ell-t}{\mu}(\ell-1)_\mu(t+\ell)^{(\ell-t-\mu)}\\
&\times \sum_{k\leq \ell-t-\mu}(-1)^k\binom{\ell-t-\mu}{k}\frac{\binom{-2\ell+\mu}{N-\ell-k}}{(N-t+1)\binom{N-t}{\mu+k}}.
\end{aligned}$$

So, since

$$\binom{-a}{b} = (-1)^b\binom{a+b-1}{a-1}, \quad \frac{(t+1)^{(\ell-1)}(t+\ell)^{(\ell-t-\mu)}}{(t+1)^{(\ell-t)}} = \frac{(2\ell-\mu-1)!}{\ell!},$$

equation (57) becomes

$$
\begin{aligned}
p(N,\vec{\ell};2) &= \frac{N\prod_j \ell_j!}{(N)_\ell} \\
&\times\Bigg[(-1)^{N+\ell}\sum_{j=1}^{\ell-1}\frac{(t+1)^{(j-1)}(j-1)_{\ell-t}}{(t+1)^{(\ell-t)}(N+2)^{(j)}}\binom{N-j-1}{\ell-j-1} \\
&+\frac{1}{\ell!(N+2)^{(\ell-1)}(N-t+1)} \\
&\times\sum_{\mu=0}^{\ell-t}\binom{\ell-t}{\mu}(\ell-1)_\mu\sum_{\nu=\mu}^{\ell-t}\binom{\ell-t-\mu}{\ell-t-\nu}\frac{(N+\ell-\nu-1)_{2\ell-\mu-1}}{\binom{N-t}{\nu}}\Bigg];
\end{aligned}
\tag{62}
$$

ν in the bottom sum comes from substitution $\nu = k+\mu$ in (61). Changing the order of summation, the double sum above equals

$$
\begin{aligned}
&\frac{(\ell-t)!}{(N-\ell)!}\sum_{\nu=0}^{\ell-t}\frac{(N+\ell-\nu-1)!}{(\ell-t-\nu)!}\frac{1}{\binom{N-t}{\nu}}\sum_{\mu=0}^{\nu}\binom{\ell-1}{\mu}\binom{N-\ell}{\nu-\mu} \\
&\qquad = \frac{(\ell-t)!}{(N-\ell)!}\sum_{\nu=0}^{\ell-t}\frac{(N+\ell-\nu-1)!}{(\ell-t-\nu)!}\,\frac{\binom{N-1}{\nu}}{\binom{N-t}{\nu}}.
\end{aligned}
\tag{63}
$$

Let $\Sigma(N,\ell,t)$ denote the top, ordinary, sum in (62). A Maple-aided symbolic computation of $\Sigma(N,\ell,t)$ for a few small values of ℓ, t makes it highly plausible that, in general,

$$
\Sigma(N,\ell,t) = \frac{(N-1)_{t-2}\,(\ell-t)!}{(t-2)!(N+t)^{(\ell-t+1)}}.
\tag{64}
$$

We confirmed this conjecture via the powerful Wilf-Zeilberger algorithm, see Nemes et al. [11], Wilf and Zeilberger [18]. Combining (63) and (64), we simplify the formula (62) to

$$
\begin{aligned}
p(N,\vec{\ell};2) &= \frac{(N-\ell)!\,\prod_j \ell_j!}{(N-1)!\,(N+t)}\Bigg[(-1)^{N+\ell}\frac{\binom{N-1}{t-2}}{\binom{N+\ell}{\ell-t}} \\
&+\frac{(N+t)(N+1)_{\ell+1}}{(N-t+1)(N+\ell)!\,(\ell)_t}\sum_{\nu=0}^{\ell-t}\frac{(N+\ell-\nu-1)!(N-1)_\nu}{(\ell-t-\nu)!(N-t)_\nu}\Bigg].
\end{aligned}
\tag{65}
$$

The outside factor and the first inside term are exactly those in (49) by Bernardi et al. The second inside term, a sum of $\ell-t+1$ terms, times $\frac{(N+t)(N+1)_{\ell+1}}{(N-t+1)(N+\ell)!\,(\ell)_t}$, is quite different in appearance from its counterpart in (49). For $\ell - t \le 5$, Maple confirms that the rational functions given by the sums are identical; we did not try to prove equality in general.

Here is the proof of (64) via the W-Z algorithm, as succinctly presented and illustrated by Andrews et al. [1]. Given $\Delta \ge 0$, introduce a function of

$t \geq 2$, defined by

$$S(t) = \sum_{j=1}^{t-1+\Delta} \frac{(t+1)^{(j-1)}(j-1)_\Delta}{(t+1)^{(\Delta)}(N+2)^{(j)}} \binom{N-j-1}{t+\Delta-j-1}.$$

The non-zero summands are those for $j \in [\Delta+1, t-1+\Delta]$. We can extend summation to $j \in [1, \infty)$, since the last binomial is zero for $j \geq t+\Delta$. We need to show that

$$S(t) = S^*(t) := \frac{(N-1)_{t-2}\Delta!}{(t-2)!(N+t)^{(\Delta+1)}}. \tag{66}$$

To do so, first we compute

$$\frac{S^*(t)}{S^*(t-1)} = \frac{\beta(t)}{\alpha(t)},$$
$$\alpha(t) := (t-2)(N+t+\Delta), \quad \beta(t) := (N-t+2)(N+t-1).$$

Next, let $F(t,j)$ stand for the j-term in the series $S(t)$. Introduce the "partner" sequence $G(t,j)$ (which again for each t is 0 for all but finitely many j) such that

$$G(t,j) - G(t,j-1) = \alpha(t)F(t,j) - \beta(t)F(t-1,j), \quad j \geq \Delta+1, \tag{67}$$

and $G(t,\Delta) = 0$.

The equation (66) will be proved if we demonstrate that $G(t,j) = 0$ for j large enough.

Computation by Maple shows that

$$G(t,\Delta+1) = -\frac{(\Delta+1)!(\Delta+2t-2)}{(N+2)^{(\Delta+1)}}\binom{N-\Delta-2}{t-3},$$
$$G(t,\Delta+2) = -\frac{(\Delta+2)!(\Delta+2t-2)(t+\Delta+1)}{(N+2)^{(\Delta+2)}}\binom{N-\Delta-3}{t-4},$$
$$G(t,\Delta+3) = -\frac{(\Delta+3)!(\Delta+2t-2)(t+\Delta+2)_2}{2(N+2)^{(\Delta+3)}}\binom{N-\Delta-4}{t-5}.$$

The evidence is unmistakable: it must be true that for all $u \geq 1$

$$G(t,\Delta+u) = -\frac{(\Delta+u)!(\Delta+2t-2)\binom{t+\Delta+u-1}{u-1}}{(N+2)^{(\Delta+u)}}\binom{N-\Delta-u-1}{t-u-2}. \tag{68}$$

Sure enough, the inductive step based on the recurrence (67) is easily carried out with a guided assistance of Maple. It remains to notice that the last binomial coefficient is zero for $u > t-2$.

6. Probability that σ blocks the elements of $[\ell]$

We say that the elements of $[\ell]$ are blocked in a permutation s of $[N]$ if in every cycle of s (1) no two elements of $[\ell]$ are neighbors, and (2) each element from $[\ell]$ has a neighbor from $[N] \setminus [\ell]$.

As customary, let us begin with $Q(\vec{\nu}, \ell)$, the total number of permutations with cycle counts $\vec{\nu}$ such that the elements of $[\ell]$ are blocked. To evaluate $Q(\vec{\nu},\ell)$, introduce the non-negative integers $a_{r,j}$, $b_{r,j}$ that stand for the generic numbers of elements from $[\ell]$ and $[N]\setminus[\ell]$ in the j-th cycle of length r, $(j\le \nu_r)$. Then

$$a_{r,j}+b_{r,j}=r, \tag{69}$$
$$b_{r,j}>0, \tag{70}$$
$$\sum_{r,\,j\le\nu_r} a_{r,j}=\ell,\qquad \sum_{r,\,j\le\nu_r} b_{r,j}=N-\ell. \tag{71}$$

For $a_{r,j}>0$, the number of admissible cycles with parameters $a_{r,j}$, $b_{r,j}$ is

$$c(a_{r,j},b_{r,j}):=(a_{r,j}-1)!\,b_{r,j}!\binom{b_{r,j}-1}{a_{r,j}-1}=(b_{r,j}-1)!a_{r,j}!\binom{b_{r,j}}{a_{r,j}}. \tag{72}$$

The last expression works for $a_{r,j}=0$ as well.

Indeed $(a_{r,j}-1)!$ is the total number of directed cycles formed by $a_{r,j}$ elements from $[\ell]$; $b_{r,j}!$ is the total number of ways to order, linearly, $b_{r,j}$ elements from $[N]\setminus\ell$, and $\binom{b_{r,j}-1}{a_{r,j}-1}$ is the total number of ways to break any such $b_{r,j}$-long sequence into $a_{r,j}$ blocks of positive lengths to be fitted between $a_{r,j}$ cyclically arranged elements from $[\ell]$, starting with the smallest element among them and moving in the cycle's direction, say.

Therefore

$$\begin{aligned} Q(\vec{\nu},\ell)&=(N-\ell)!\,\ell!\sum_{\substack{\mathbf{a},\,\mathbf{b}\text{ meet}\\ (69),(70),(71)}}\prod_{r\ge1}\frac{1}{\nu_r!}\prod_{j\le\nu_r}\frac{c(a_{r,j},b_{r,j})}{a_{r,j}!\,b_{r,j}!}\\ &=(N-\ell)!\,\ell!\,[w^\ell]\prod_{r\ge1}\frac{1}{\nu_r!}\left(\sum_{b>0,\,a+b=r}\frac{1}{b}\binom{b}{a}w^a\right)^{\nu_r}. \end{aligned} \tag{73}$$

Having found $Q(\vec{\nu},\ell)$, we turn to $p(N,\ell,k)$, the probability that σ blocks the elements of $[\ell]$. Using (7), the equality $\nu=\sum_r\nu_r$, and and (73), we obtain

$$\begin{aligned} \sum_{s:\,\vec{\nu}(s)=\vec{\nu}}\chi^{\lambda^*}(s)&=(-1)^{\lambda^1}(N-\ell)!\,\ell!\\ &\times[\xi^{\lambda_1}w^\ell]\,\frac{\xi}{1-\xi}\prod_r\frac{1}{\nu_r!}\left[-(1-\xi^r)\left(\sum_{\substack{b>0,\\ a+b=r}}\frac{1}{b}\binom{b}{a}w^a\right)\right]^{\nu_r}. \end{aligned}$$

Call a permutation s of $[N]$ admissible if it blocks the elements of $[\ell]$. The above identity implies

$$\sum_{s \text{ admissible}} \chi^{\lambda^*}(s) = (-1)^{\lambda^1}(N-\ell)!\,\ell! \tag{74}$$
$$\times [\xi^{\lambda_1} w^{\ell}]\,\frac{\xi}{1-\xi} \sum_{\substack{\vec{\nu}:\\ 1\nu_1+2\nu_2+\cdots=N}} \prod_r \frac{1}{\nu_r!}\left[-(1-\xi^r)\left(\sum_{\substack{b>0,\\ a+b=r}} \frac{1}{b}\binom{b}{a} w^a\right)\right]^{\nu_r}.$$

The expression in the second line of (74) equals

$$\begin{aligned}
&[\xi^{\lambda_1} w^{\ell} x^N]\frac{\xi}{1-\xi} \sum_{\vec{\nu}\ge \mathbf{0}} \prod_r \frac{(x^r)^{\nu_r}}{\nu_r!}\left[-(1-\xi^r)\left(\sum_{b>0,\, a+b=r} \frac{1}{b}\binom{b}{a} w^a\right)\right]^{\nu_r}\\
=\;&[\xi^{\lambda_1} w^{\ell} x^N]\frac{\xi}{1-\xi} \prod_r \sum_{\nu_r\ge 0} \frac{1}{\nu_r!}\left[-x^r(1-\xi^r)\left(\sum_{b>0,\, a+b=r} \frac{1}{b}\binom{b}{a} w^a\right)\right]^{\nu_r}\\
=\;&[\xi^{\lambda_1} w^{\ell} x^N]\frac{\xi}{1-\xi} \prod_r \exp\left[-x^r(1-\xi^r)\left(\sum_{b>0,\, a+b=r} \frac{1}{b}\binom{b}{a} w^a\right)\right]\\
=\;&[\xi^{\lambda_1} w^{\ell} x^N]\frac{\xi}{1-\xi} \exp\left[-\sum_{r\ge 1}[x^r-(x\xi)^r]\left(\sum_{b>0,\, a+b=r} \frac{1}{b}\binom{b}{a} w^a\right)\right].
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{r\ge 1} y^r \sum_{b>0,\, a+b=r} \frac{1}{b}\binom{b}{a} w^a &= \sum_{b>0} \frac{y^b}{b} \sum_a \binom{b}{a} (yw)^a\\
&= \sum_{b>0} \frac{y^b}{b}(1+yw)^b = \sum_{b>0} \frac{[y(1+yw)]^b}{b}\\
&= \log \frac{1}{1-y(1+yw)},
\end{aligned}$$

the bottom part (75) becomes

$$\begin{aligned}
&[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\exp\left(-\log\frac{1}{1-x(1+xw)}+\log\frac{1}{1-x\xi(1+x\xi w)}\right)\\
=&[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\frac{1-x(1+xw)}{1-x\xi(1+x\xi w)}\\
=&[\xi^{\lambda_1} x^N]\,\frac{\xi(1-x)}{(1-\xi)(1-x\xi)}\,[w^\ell]\frac{1-\frac{x^2}{1-x}w}{1-\frac{(x\xi)^2}{1-x\xi}w}\\
=&[\xi^{\lambda_1} x^N]\,\frac{\xi(1-x)}{(1-\xi)(1-x\xi)}\left[\left(\frac{(x\xi)^2}{1-x\xi}\right)^\ell-\frac{x^2}{1-x}\left(\frac{(x\xi)^2}{1-x\xi}\right)^{\ell-1}\right]\\
=&[\xi^{\lambda_1} x^N]\,\frac{\xi}{1-x\xi}\left(\frac{(x\xi)^2}{1-x\xi)}\right)^{\ell-1}\frac{x^2}{1-x\xi}\,(x\xi-1-\xi)\\
=&-[\xi^{\lambda_1} x^N]\left(\frac{x^{2\ell}\xi^{2\ell-1}}{(1-x\xi)^\ell}+\frac{x^{2\ell}\xi^{2\ell}}{(1-x\xi)^{\ell+1}}\right)\\
=&-[\xi^{\lambda_1-2\ell+1}x^{N-2\ell}](1-x\xi)^{-\ell}-[\xi^{\lambda_1-2\ell}x^{N-2\ell}](1-x\xi)^{-\ell-1}\\
=&-\binom{N-\ell-1}{\ell-1}1_{\{\lambda_1=N-1\}}-\binom{N-\ell}{\ell}1_{\{\lambda_1=N\}}.
\end{aligned}$$

So (74) simplifies, greatly, to

$$\sum_{s\text{ admissible}}\chi^{\lambda^*}(s)=(-1)^{\lambda^1-1}(N-\ell)!\,\ell!\times\left[\binom{N-\ell-1}{\ell-1}1_{\{\lambda_1=N-1\}}+\binom{N-\ell}{\ell}1_{\{\lambda_1=N\}}\right].\tag{75}$$

The rest is easy. By (6),

$$\begin{aligned}
p(N,\ell;k):=&\,\mathrm{P}(\sigma\text{ blocks elements of }[\ell])\\
=&\frac{1}{N!}\sum_{\lambda^*}(-1)^{k(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1}\sum_{s\text{ admissible}}\chi^{\lambda^*}(s).
\end{aligned}$$

Combining this with (75) we conclude that

$$p(N,\ell;k)=\frac{\binom{N-\ell}{\ell}}{\binom{N}{\ell}}+(-1)^{k+1}\frac{\binom{N-\ell-1}{\ell-1}}{(N-1)^{k-1}\binom{N}{\ell}}.\tag{76}$$

Note. The equation (76) shows that $\lim_{k\to\infty}p(N,\ell;k)=\binom{N-\ell}{\ell}/\binom{N}{\ell}$, the probability that the *uniformly* random permutation blocks $[\ell]$.

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