A new characterization of the invertibility of polynomial maps

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Abstract

In this paper we present an equivalent statement to the Jacobian conjecture. For a polynomial map F on an affine space of dimension n, we define recursively n finite sequences of polynomials. We give an equivalent condition to the invertibility of F as well as a formula for F^{-1} in terms of these finite sequences of polynomials. Some examples illustrate the effective aspects of our approach.

1 Introduction

The Jacobian Conjecture originated in the question raised by Keller in [8] on the invertibility of polynomial maps with Jacobian determinant equal to 1. The question is still open in spite of the efforts of many mathematicians. We recall in the sequel the precise statement of the Jacobian Conjecture, some reduction theorems and other results we shall use. We refer to [5] for a detailed account of the research on the Jacobian Conjecture and related topics.

Let K be a field and $K[X] = K[X_1, \ldots, X_n]$ the polynomial ring in the variables $X_1, \ldots, X_n$ over K. A *polynomial map* is a map $F = (F_1, \ldots, F_n) : K^n \to K^n$ of the form

$$(X_1, \ldots, X_n) \mapsto (F_1(X_1, \ldots, X_n), \ldots, F_n(X_1, \ldots, X_n)),$$

where $F_i \in K[X], 1 \leq i \leq n$. The polynomial map F is *invertible* if there exists a polynomial map $G = (G_1, \ldots, G_n) : K^n \to K^n$ such that $X_i = G_i(F_1, \ldots, F_n), 1 \leq i \leq n$. We shall call F a *Keller map* if the Jacobian matrix

$$J = \left(\frac{\partial F_i}{\partial X_j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$$

has determinant equal to 1. Clearly an invertible polynomial map F has a Jacobian matrix J with non zero determinant and may be transformed into a Keller map by composition with the linear automorphism with matrix $J(0)^{-1}$.

Jacobian Conjecture. *Let K be a field of characteristic zero. A Keller map $F : K^n \to K^n$ is invertible.*

In the sequel, K will always denote a field of characteristic 0. For $F = (F_1, \ldots, F_n) \in K[X]^n$, we define the *degree* of F as $\deg F = \max\{\deg F_i : 1 \leq i \leq n\}$. It is known that if F is a polynomial automorphism of K^n, then $\deg F^{-1} \leq (\deg F)^{n-1}$ (see [1] or [9]).

The Jacobian conjecture for quadratic maps was proved by Wang in [10]. We state now the reduction of the Jacobian conjecture to the case of maps of third degree (see [1], [11], [2] and [3]).

Proposition 1. *a) (Bass-Connell-Wright-Yagzhev) Given a Keller map* $F : K^n \to K^n$*, there exists a Keller map* $\widetilde{F} : K^N \to K^N$, $N \geq n$ *of the form* $\widetilde{F} = Id + H$*, where* $H(X)$ *is a cubic homogeneous map and having the following property: if* $\widetilde{F}$ *is invertible, then* F *is invertible too.*

b) (Drużkowski) The cubic part H *may be chosen of the form*

$$\left((\sum_{j=1}^{N} a_{1j} X_j)^3, \ldots, (\sum_{j=1}^{N} a_{Nj} X_j)^3 \right)$$

and with the matrix $A = (a_{ij})_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}}$ *satisfying* $A^2 = 0$.

Polynomial maps in the Drużkowski form are easier to handle than general cubic homogeneous polynomial maps. However we note the following result.

Proposition 2 ([6] Proposition 2.9)**.** *Let* $r \in \mathbb{N}$*. If the Jacobian Conjecture holds for all cubic homogeneous polynomial maps in* r *variables, then for all* $n \in \mathbb{N}$ *the Jacobian Conjecture holds for all polynomial maps of the form*

$$F = X + (AX)^3$$

with $A \in \mathrm{M}_n(K)$ *and* $\operatorname{rank} A \leq r$.

In [4] Drużkowski and Rusek give the following inversion formula for cubic homogeneous polynomial maps.

Theorem 3 ([4], Theorem 2.1)**.** *Let* $H : K^n \to K^n$ *be a cubic homogeneous polynomial map,* $F = Id - H$ *and let* $G = \sum_{j=0}^{\infty} G_j$*, where* $G_j : K^n \to K^n$ *is a homogeneous polynomial map of degree* j*, be the formal inverse of* F*. Then*

$$\begin{aligned} G_1 &= Id, \\ G_{2k+1} &= \textstyle\sum_{p+q+r=k-1} \varphi_H(G_{2p+1}, G_{2q+1}, G_{2r+1}), \ \forall k \geq 1, \\ G_{2k} &= 0, \ \forall k \geq 1, \end{aligned}$$

where φ_H *denotes the unique symmetric trilinear map such that* $\varphi_H(X, X, X) = H(X)$*.*

As a corollary, they obtain that, if for some natural number k, we have

$$G_{3^k+2} = \cdots = G_{3^{k+1}} = 0, \tag{1}$$

then F is a polynomial automorphism and $\deg F^{-1} \leq 3^k$. However, in [7], Gorni and Zampieri present an example of a polynomial automorphism of $\mathbb{C}^4$ for which condition (1) is not satisfied for any k (see example 7 below).

In this paper we present an algorithm providing a new characterization of the invertibility of polynomial maps. Given a polynomial map $F : K^n \to K^n$ of the form $F = Id + H$, where $H(X)$ has lower degree ≥ 2, we define recursively, for $1 \leq i \leq n$, a sequence P_k^i of polynomials in $K[X]$ with $P_0^i = X_i$ such that F is invertible if and only if the alternating sum $\sum_{j=0}^{m-1}(-1)^j P_j^i(X)$ satisfies a certain relation with P_m^i for all $i = 1, \ldots, n$, where m is an integer given explicitly and depending on the degrees of the components of H. When F is invertible, its inverse F^{-1} is given in terms of these alternating sums of polynomials. In the last section, we apply the algorithm to several examples of polynomial maps, including the one of Gorni and Zampieri.

2 A sufficient condition for invertibility

Let us consider a polynomial map $F : K^n \to K^n$. Given a polynomial $P(X_1, \ldots, X_n) \in K[X] = K[X_1, \ldots, X_n]$, we define the following sequence of polynomials in $K[X]$,

$$\begin{aligned} P_0(X_1, \ldots, X_n) &= P(X_1, \ldots, X_n), \\ P_1(X_1, \ldots, X_n) &= P_0(F_1, \ldots, F_n) - P_0(X_1, \ldots, X_n), \end{aligned}$$

and, assuming P_{k-1} is defined,

$$P_k(X_1, \ldots, X_n) = P_{k-1}(F_1, \ldots, F_n) - P_{k-1}(X_1, \ldots, X_n).$$

The following lemma is easy to prove.

Lemma 4. *For a positive integer m, we have*

$$P(X_1, \ldots, X_n) = \sum_{l=0}^{m-1}(-1)^l P_l(F_1, \ldots, F_n) + (-1)^m P_m(X_1, \ldots, X_n).$$

In particular, if we assume that for some integer m, $P_m(X_1, \ldots, X_n) = 0$, then

$$P(X_1, \ldots, X_n) = \sum_{l=0}^{m-1} (-1)^l P_l(F_1, \ldots, F_n).$$

Corollary 5. *Let $F : K^n \to K^n$ be a polynomial map. Let us consider the polynomial sequence (P_k^i) constructed with $P = X_i$, $i = 1, \ldots, n$. Let us assume that for all $i = 1, \ldots, n$, there exists an integer m_i such that $P_{m_i}^i = 0$. Then the inverse map G of F is given by*

$$G_i(Y_1, Y_2, \ldots, Y_n) = \sum_{l=0}^{m_i-1} (-1)^l P_l^i(Y_1, Y_2, \ldots, Y_n), \ 1 \leq i \leq n.$$

The condition $P_{m_i}^i = 0$, for some integer m_i for all $i = 1, \ldots, n$, is not necessary for the invertibility of F (see example 7). However we give in theorem 8 an equivalent condition to the invertibility of F using a finite number of terms of the polynomial sequences (P_k^i). The following lemma gives a precise description of the polynomials P_k^i.

Lemma 6. *Let $F : K^n \to K^n$ be a polynomial map of the form*

$$\begin{cases} F_1(X_1, \ldots, X_n) = X_1 + H_1(X_1, \ldots, X_n) \\ \qquad\vdots \\ F_n(X_1, \ldots, X_n) = X_n + H_n(X_1, \ldots, X_n), \end{cases}$$

where $H_i(X_1, \ldots, X_n)$ is a polynomial in $X_1, \ldots, X_n$ of degree D_i and lower degree d_i, with $d_i \geq 2$, for $i = 1, \ldots, n$. Let $d = \min d_i, D = \max D_i$. Then for the polynomial sequence (P_k^i) constructed with $P = X_i$ we have that P_k^i is a polynomial of degree $\leq D^{k-1} D_i$ and lower degree $\geq (k-1)(d-1) + d_i$.

In particular, if each H_i is a homogeneous polynomial of degree d, we have

$$P_k^i = \sum_{j=1}^{(d^k-1)/(d-1)-k+1} Q_{kj},$$

where Q_{kj} is a homogeneous polynomial in $X_1, \ldots, X_n$ of degree $(k + j - 1)(d-1) + 1$.

Proof. Let us consider, for a fixed i, the polynomial sequence

$$\begin{array}{rcl} P_0^i(X_1,\ldots,X_n) & = & X_i, \\ P_1^i(X_1,\ldots,X_n) & = & F_i(X_1,\ldots,X_n) - X_i = H_i(X_1,\ldots,X_n), \\ P_2^i(X_1,\ldots,X_n) & = & H_i(F_1,\ldots,F_n) - H_i(X_1,\ldots,X_n), \\ & \vdots & \end{array}$$

We write the Taylor series for the polynomial $H_i(F_1,\ldots,F_n) = H_i(X_1 + H_1,\ldots,X_n + H_n)$ and obtain

$$\begin{array}{rcl} P_2^i(X_1,\ldots,X_n) & = & H_i(F_1,\ldots,F_n) - H_i(X_1,\ldots,X_n) \\ & = & Q_{21}^i + Q_{22}^i + \ldots + Q_{2D_i}^i \end{array}$$

where

$$Q_{21}^i = \sum_{j=1}^n \frac{\partial H_i}{\partial X_j} H_j$$

$$Q_{22}^i = \frac{1}{2!} \sum_{1 \le j_1, j_2 \le n}^n \frac{\partial^2 H_i}{\partial X_{j_1} \partial X_{j_2}} H_{j_1} H_{j_2}$$

$$\vdots$$

$$Q_{2D_i}^i = \frac{1}{D_i!} \sum_{j_1, j_2, \ldots, j_{D_i}=1}^n \frac{\partial^{D_i} H_i}{\partial x_{j_1} \ldots \partial x_{j_{D_i}}} H_{j_1} \ldots H_{j_{D_i}}.$$

The polynomial P_2^i has lower degree equal to the lower degree of Q_{21}^i, which is $\ge d + d_i - 1$, and degree equal to the degree of $Q_{2D_i}^i$, which is $\le D \cdot D_i$. Let us prove by induction that P_k^i is a polynomial of degree $\le D^{k-1} D_i$ and lower degree $\ge (k-1)(d-1) + d_i$. We have already seen it for $k = 2$. Let us assume P_{k-1}^i is a polynomial of degree $\le D^{k-2} D_i$ and lower degree $\ge (k-2)(d-1) + d_i$. We want to prove the property for P_k^i. We have

$$P_k^i(X_1,\ldots,X_n) = P_{k-1}^i(F_1,\ldots,F_n) - P_{k-1}^i(X_1,\ldots,X_n)$$

If $Q(X_1,\ldots,X_n)$ is a polynomial of degree S and lower degree s,

$$Q(F_1,\ldots,F_n) - Q(X_1,\ldots,X_n) = \\ \textstyle\sum_{j=1}^n \dfrac{\partial Q}{\partial X_j} H_j + \ldots + \frac{1}{S!} \sum_{j_1,\ldots,j_S}^n \frac{\partial^S Q}{\partial x_{j_1} \ldots, \partial x_{j_S}} H_{j_1} \ldots H_{j_S}$$

is a polynomial of degree $\le S \cdot D$ and lower degree $\ge s-1+d$. Hence $P^i_{k-1}(F_1,\dots,F_n) - P^i_{k-1}(X_1,\dots,X_n)$ is a polynomial of degree $\le D^{k-1}D_i$ and lower degree $\ge (k-1)(d-1)+d_i$.

The homogeneous case is proved analogously using induction. □

Example 7. We shall consider the polynomial automorphism of $\mathbb{C}^4$ given in [7] to prove that the condition $P^i_{m_i}=0$, for some m_i, for all i, is not a necessary condition to the invertibility of F.

Let $p := X_1X_3 + X_2X_4$ and define F by

$$\begin{cases} F_1 &= X_1 + pX_4 \\ F_2 &= X_2 - pX_3 \\ F_3 &= X_3 + X_4^3 \\ F_4 &= X_4 \end{cases}$$

Clearly $P^4_1 = 0$ and $P^3_2 = 0$. But P^1_j and P^2_j are not zero for any j. In order to prove that $P^1_j \neq 0$, we shall prove by induction that the homogeneous summand of lowest degree Q^1_{j1} of P^1_j has the following form depending on the parity of j, for all $j \ge 2$.

$$\begin{aligned} Q^1_{2k,1} &= X_1X_4^{4k} \\ Q^1_{2k+1,1} &= X_1X_3X_4^{4k+1} + X_2X_4^{4k+2} \end{aligned}$$

By calculation we obtain $Q^1_{21} = X_1X_4^4, Q^1_{31} = X_1X_3X_4^5 + X_2X_4^6$. Now, $Q^1_{2k,1} = X_1X_4^{4k} \Rightarrow Q^1_{2k+1,1} = X_4^{4k}H_1 + 4kX_1X_4^{4k-1}H_4 = X_4^{4k}(X_1X_3 + X_2X_4)X_4 = X_1X_3X_4^{4k+1} + X_2X_4^{4k+2}$ and $Q^1_{2k+1,1} = X_1X_3X_4^{4k+1} + X_2X_4^{4k+2} \Rightarrow Q^1_{2k+2,1} = X_3X_4^{4k+1}H_1 + X_4^{4k+2}H_2 + X_1X_4^{4k+1}H_3 + ((4k+1)X_1X_3X_4^{4k} + (4k+2)X_2X_4^{4k+1})H_4 = X_3X_4^{4k+1}(X_1X_3 + X_2X_4)X_4 - X_4^{4k+2}(X_1X_3 + X_2X_4)X_3 + X_1X_4^{4k+4} = X_1X_4^{4(k+1)}$.

Analogously, in order to prove that $P^2_j \neq 0$, we shall prove by induction that the homogeneous summand of lowest degree Q^2_{j1} of P^2_j has the following form depending on the parity of j, for all $j \ge 2$.

$$\begin{aligned} Q^2_{2k,1} &= -2kX_1X_3X_4^{4k-1} - (2k-1)X_2X_4^{4k} \\ Q^2_{2k+1,1} &= -X_1X_3^2X_4^{4k} - X_2X_3X_4^{4k+1} - 2kX_1X_4^{4k+2} \end{aligned}$$

By calculation we obtain $Q^2_{21} = -2X_1X_3X_4^3 - X_2X_4^4, Q^1_{31} = -X_1X_3^2X_4^4 - X_2X_3X_4^5 - 2X_1X_4^6$. Now $Q^2_{2k,1} = -2kX_1X_3X_4^{4k-1} - (2k-1)X_2X_4^{4k} \Rightarrow Q^2_{2k+1,1} = -2kX_3X_4^{4k-1}H_1 - (2k-1)X_4^{4k}H_2 - 2kX_1X_4^{4k-1}H_3 = -X_1X_3^2X_4^{4k} - X_2X_3X_4^{4k+1} - 2kX_1X_4^{4k+2}$ and $Q^2_{2k+1,1} = -X_1X_3^2X_4^{4k} - X_2X_3X_4^{4k+1} - 2kX_1X_4^{4k+2}$

$\Rightarrow Q^2_{2k+2,1} = (-X_3^2X_4^{4k} - 2kX_4^{4k+2})H_1 - X_3X_4^{4k+1}H_2 - (2X_1X_3X_4^{4k} + X_2X_4^{4k+1})H_3 = -(2k+2)X_1X_3X_4^{4k+3} - (2k+1)X_2X_4^{4k+4}$.

3 An equivalent condition to invertibility

The following theorem gives an equivalent condition to the invertibility of F using a finite number of terms in the polynomial sequences (P_k^i).

Theorem 8. *Let $F : K^n \to K^n$ be a polynomial map of the form*

$$\begin{cases} F_1(X_1,\ldots,X_n) & = & X_1 + H_1(X_1,\ldots,X_n) \\ & \vdots & \\ F_n(X_1,\ldots,X_n) & = & X_n + H_n(X_1,\ldots,X_n), \end{cases}$$

where $H_i(X_1,\ldots,X_n)$ is a polynomial in $X_1,\ldots,X_n$ of degree D_i and lower degree d_i, with $d_i \geq 2$, for $i = 1,\ldots,n$. Let $d = \min d_i, D = \max D_i$. The following conditions are equivalent:

1) F is invertible.

2) For $i = 1,\ldots,n$ and every $m > \frac{D^{n-1}-d_i}{d-1} + 1$, we have

$$\sum_{j=0}^{m-1} (-1)^j P_j^i(X) = G_i(X) + R_m^i(X).$$

where $G_i(X)$ is a polynomial of degree $\leq D^{n-1}$, independent of m, and $R_m^i(X)$ is a polynomial satisfying $R_m^i(F) = (-1)^{m+1}P_m^i(X)$ (with lower degree $\geq (m-1)(d-1) + d_i > D^{n-1}$).

3) For $i = 1,\ldots,n$ and $m = \lfloor \frac{D^{n-1}-d_i}{d-1} + 1 \rfloor + 1$, we have

$$\sum_{j=0}^{m-1} (-1)^j P_j^i(X) = G_i(X) + R_m^i(X).$$

where $G_i(X)$ is a polynomial of degree $\leq D^{n-1}$, and $R_m^i(X)$ is a polynomial satisfying $R_m^i(F) = (-1)^{m+1}P_m^i(X)$.

Moreover the inverse G of F is given by

$$G_i(Y_1,\dots,Y_n)=\sum_{l=0}^{m-1}(-1)^l\tilde{P}_l^i(Y_1,\dots,Y_n),\ i=1,\dots,n,$$

where $\tilde{P}_l^i$ is the sum of homogeneous summands of P_l^i of degree $\le D^{n-1}$ and m is an integer $>\frac{D^{n-1}-d_i}{d-1}+1$.

Proof. 1) $\Rightarrow$ 2): If F is invertible, then $G=F^{-1}$ has degree $\le D^{n-1}$. Applying lemma 4, we obtain, for any positive integer m,

$$X_i=\sum_{l=0}^{m-1}(-1)^lP_l(F_1,\dots,F_n)+(-1)^mP_m(X_1,\dots,X_n).$$

Since $X_i=G_i(F_1,\dots,F_n)$, we obtain the following equality of polynomials in the variables $Y_1,\dots,Y_n$.

$$\begin{aligned}G_i(Y_1,\dots,Y_n) &= \textstyle\sum_{l=0}^{m-1}(-1)^lP_l(Y_1,\dots,Y_n)\\ &\quad +(-1)^mP_m(G_1(Y_1,\dots,Y_n),\dots,G_n(Y_1,\dots,Y_n)),\end{aligned}$$

which implies

$$\begin{aligned}\textstyle\sum_{l=0}^{m-1}(-1)^lP_l(Y_1,\dots,Y_n) &= G_i(Y_1,\dots,Y_n)\\ &\quad -(-1)^mP_m(G_1(Y_1,\dots,Y_n),\dots,G_n(Y_1,\dots,Y_n)),\end{aligned}$$

Hence, writing

$$R_m^i(Y_1,\dots,Y_n):=-(-1)^mP_m(G_1(Y_1,\dots,Y_n),\dots,G_n(Y_1,\dots,Y_n)),$$

we obtain 2). Now, G_i is a polynomial of degree at most D^{n-1} in $Y_1,\dots,Y_n$. For an integer m such that $m>\frac{D^{n-1}-d_i}{d-1}+1$, P_m is a polynomial in the variables $X_1,\dots,X_n$ of lower degree bigger than D^{n-1}, hence the lower degree of $P_m(G_1(Y_1,\dots,Y_n),\dots,G_n(Y_1,\dots,Y_n))$ in the variables $Y_1,\dots,Y_n$ is bigger than D^{n-1}. Therefore, the sum of homogeneous summands of degrees not bigger than D^{n-1} in the righthand side of the equality above is precisely $\sum_{l=0}^{m-1}(-1)^l\widetilde{P}_l^i(Y_1,Y_2,\dots,Y_n)$.

2) $\Rightarrow$ 3) is obvious.
3) $\Rightarrow$ 1): Let us assume that for $m = \lfloor \frac{D^{n-1}-d_i}{d-1} + 1 \rfloor + 1$, we have

$$\sum_{j=0}^{m-1} (-1)^j P_j^i(X) = G_i(X) + R_m^i(X),$$

where $G_i(X)$ is a polynomial of degree $\leq D^{n-1}$ and $R_m^i(X)$ is a polynomial satisfying $R_m^i(F) = (-1)^{m+1} P_m^i(X)$. By lemma 4, we have

$$X_i = \sum_{l=0}^{m-1} (-1)^l P_l^i(F_1, \ldots, F_n) + (-1)^m P_m^i(X_1, \ldots, X_n).$$

We obtain then

$$\begin{aligned} X_i &= G_i(F) + R_m^i(F) + (-1)^m P_m^i(X) \\ &= G_i(F) + (-1)^{m+1} P_m^i(X) + (-1)^m P_m^i(X) \\ &= G_i(F). \end{aligned}$$

Hence F is invertible with inverse $G = (G_1, \ldots, G_n)$. $\square$

4 Examples

4.1

We consider the following nonhomogeneous Keller map in dimension 2.

$$\begin{cases} F_1 = X_1 + (X_2 + X_1^3)^2 \\ F_2 = X_2 + X_1^3 \end{cases}$$

Let us write $H_1 := (X_2 + X_1^3)^2, H_2 := X_1^3$. With the notations in theorem 8, we have $d_1 = 2, d_2 = 3, d = 2, D = 6$ and we obtain

$$\sum_{i=0}^{5} (-1)^i P_i^1(X) = X_1 - X_2^2 + R_6^1(X),$$

where $R_6^1(X)$ is a polynomial of degree 6^5 and lower degree 9 satisfying $R_6^1(F) = -P_6^1(X)$, and

$$\sum_{i=0}^{4}(-1)^i P_i^2(X) = X_2 - X_1^3 + 3X_1^2X_2^2 - 3X_1X_2^4 + X_2^6 + R_5^2(X),$$

where $R_5^2(X)$ is a polynomial of degree $3 \cdot 6^3$ and lower degree 8 satisfying $R_5^2(F) = P_5^2(X)$. Hence the inverse of F is given by

$$\begin{cases} G_1 &= X_1 - X_2^2 \\ G_2 &= X_2 - X_1^3 + 3X_1^2X_2^2 - 3X_1X_2^4 + X_2^6 \end{cases}$$

4.2

We consider the following Keller map F in dimension 5.

$$\begin{cases} F_1 &= X_1 + a_1X_4^3 + a_2x_4^2X_5 + a_3X_4X_5^2 + a_4x_5^3 + \dfrac{a_2c_5X_2X_4^2}{c_2} + \dfrac{2a_3c_5X_2X_4X_5}{c_2} \\ & +\dfrac{3a_4c_5X_2X_5^2}{c_2} - \dfrac{a_2e_2X_3X_4^2}{c_2} - \dfrac{2a_3e_2X_3X_4X_5}{c_2} - \dfrac{3a_4e_2X_3X_5^2}{c_2} \\ & +\dfrac{a_3c_5^2X_2^2X_4}{c_2^2} + \dfrac{3a_4c_5^2X_2^2X_5}{c_2^2} - \dfrac{2a_3c_5e_2X_2X_3X_4}{c_2^2} - \dfrac{6a_4c_5e_2X_2X_3X_5}{c_2^2} \\ & +\dfrac{a_3e_2^2X_3^2X_4}{c_2^2} + \dfrac{3a_4e_2^2X_3^3X_5}{c_2^2} + \dfrac{a_4e_5^3X_2^3}{c_2^3} - \dfrac{3a_4c_5^2e_2X_2^2X_3}{c_2^3} \\ & +\dfrac{3a_4c_5e_2^2X_2X_3^2}{c_2^3} - \dfrac{a_4e_2^3X_3^3}{c_2^3} \\ F_2 &= X_2 + b_1X_4^3 \\ F_3 &= X_3 + c_5X_2X_4^2 + c_1X_4^3 + c_2X_4^4X_5 - e_2X_3X_4^2 \\ F_4 &= X_4 \\ F_5 &= X_5 + e_2X_4^2X_5 + \dfrac{c_5e_2X_2X_4^2}{c_2} - \dfrac{e_2^2X_3X_4^2}{c_2} - \dfrac{(b_1c_5 - c_1e_2)X_4^3}{c_2} \end{cases}$$

with parameters $a_1, a_2, a_3, a_4, b_1, c_1, c_2, c_5, e_2$. By applying the algorithm we obtain $P_2^i = 0$, for $i = 1, \ldots, 5$, hence F is a quasi-translation, i.e. $F^{-1} = 2Id - F$.

4.3

We consider the following Keller map F in dimension 6

$$\begin{cases} F_1 &=& X_1 + a_5e_1(X_1+X_2)^3/a_4 + a_4X_2X_4X_6 + a_5X_4X_5X_6 \\ F_2 &=& X_2 - a_5e_1(X_1+X_2)^3/a_4 \\ F_3 &=& X_3 + c_1X_1^3 + c_2(X_1+X_5)^3 + c_3(X_1+X_2)^3 + c_4(X_1+X_4)^3 + c_5X_6^3 \\ F_4 &=& X_4 + d_4X_2X_6^2 + a_5d_4X_5X_6^2/a_4 \\ F_5 &=& X_5 + e_1(X_1+X_2)^3 \\ F_6 &=& X_6 \end{cases}$$

with parameters $a_4, a_5, c_1, c_2, c_3, c_4, c_5, d_4, e_1$. Denoting $G = F^{-1}$ and taking variables $(Y_1, \ldots, Y_6)$ for G, we obtain $P_8^1 = 0$, $P_9^2 = 0$ and

$$\begin{aligned}
G_1 = -(&20a_5^4e_1Y_6^9d_4^3Y_2^3Y_5^3a_4^3 - 6a_5^4e_1Y_6^4Y_4Y_5^3d_4Y_2a_4^2 + a_5e_1a_4^6Y_6^9Y_2^6d_4^3 + 3Y_6^3a_5e_1Y_2^4d_4a_4^4 \\
&+3a_5e_1a_4^5Y_6^6Y_2^5d_4^2 - a_5e_1a_4^6Y_2^3Y_4^3Y_6^3 + a_4^5Y_2Y_4Y_6 - Y_6^3a_4^5d_4Y_2^2 + a_5^7e_1Y_6^9d_4^3Y_5^6 \\
&+a_5e_1a_4^3Y_1^3 + a_5e_1a_4^3Y_2^3 - Y_1a_4^4 - 3a_5^2e_1a_4^5Y_2^2Y_4^3Y_6^3Y_5 - 6a_5e_1Y_1Y_2^2Y_4Y_6a_4^4 \\
&+3a_5^5e_1Y_6^6Y_1d_4^2Y_5^4a_4 + 12a_5^4e_1Y_6^6Y_2^2d_4^2Y_5^3a_4^2 - 6a_5^4e_1Y_6^4Y_1Y_4d_4Y_5^3a_4^2 + 3a_5e_1a_4^3Y_1Y_2^2 \\
&-6a_5e_1a_4^5Y_6^4Y_1Y_2^3Y_4d_4 + 3Y_6^3a_5e_1Y_1^2Y_2^2d_4a_4^4 + 3a_5e_1a_4^5Y_1Y_2^2Y_4^2Y_6^2 - 30a_5^3e_1Y_6^7Y_4d_4^2Y_2^3Y_5^2a_4^4 \\
&+6Y_6^3a_5e_1Y_1Y_2^3d_4a_4^4 + 15a_5^3e_1Y_6^9d_4^3Y_2^4Y_5^2a_4^4 + 6a_5^2e_1a_4^5Y_6^9Y_2^5d_4^3Y_5 - 18a_5^2e_1Y_6^4Y_2^3Y_4d_4Y_5a_4^4 \\
&+12a_5^2e_1a_4^5Y_6^5Y_2^3Y_4^2d_4Y_5 - 15a_5^2e_1a_4^5Y_6^7Y_2^4Y_4d_4^2Y_5 + 12a_5^2e_1Y_6^6Y_1Y_2^3d_4^2Y_5a_4^4 - 18a_5^2e_1Y_6^4Y_1Y_2^2Y_4d_4Y_5a_4^4 \\
&-3a_5e_1a_4^6Y_6^7Y_2^5Y_4d_4^2 + 3a_5e_1a_4^3Y_1^2Y_2 + 18a_5^3e_1Y_6^5Y_4^2d_4Y_2^2Y_5^2a_4^4 + 12a_5^2e_1Y_6^6Y_2^4d_4^2Y_5a_4^4 \\
&+3a_5e_1a_4^5Y_6^6Y_1Y_2^4d_4^2 + 3a_5e_1a_4^6Y_6^5Y_2^4Y_4^2d_4 - 6a_5e_1Y_6^4a_4^5Y_2^4Y_4d_4 + a_5Y_4Y_5Y_6a_4^4 \\
&+15a_5^5e_1Y_6^9d_4^3Y_2^2Y_5^4a_4^2 + 6a_5^6e_1Y_6^9Y_2d_4^3Y_5^5a_4 - 18a_5^3e_1Y_6^4Y_2^2Y_4d_4Y_5^2a_4^3 + 18a_5^3e_1Y_6^6Y_1d_4^2Y_2^2Y_5^2a_4^3 \\
&-3a_5^6e_1Y_6^7Y_4Y_5^5d_4^2a_4 - 18a_5^3e_1Y_6^4Y_1Y_4d_4Y_2Y_5^2a_4^3 + 6a_5^3e_1Y_6^3Y_1d_4Y_5^2Y_2a_4^2 - 30a_5^4e_1Y_6^7Y_4d_4^2Y_2^2Y_5^3a_4^3 \\
&+12a_5^4e_1Y_6^5Y_2Y_4^2d_4Y_5^3a_4^3 + 12a_5^4e_1Y_6^6Y_1d_4^2Y_2Y_5^3a_4^2 + 3a_5^5e_1Y_6^6d_4^2Y_5^4Y_2a_4 + 3a_5^3e_1Y_6^3Y_1^2d_4Y_5^2a_4^2 \\
&+18a_5^3e_1Y_6^6Y_2^3d_4^2Y_5^2a_4^3 + 6a_5^2e_1Y_1Y_2Y_4^2Y_6^2Y_5a_4^4 - 3a_5e_1Y_2^3Y_4Y_6a_4^4 - 2Y_6^3Y_2a_5d_4Y_5a_4^4 \\
&+3a_5^5e_1Y_6^5Y_4^2Y_5^4d_4a_4^2 - 3a_5^3e_1Y_2Y_4^3Y_6^3Y_5^2a_4^4 - 3a_5e_1Y_1^2Y_2Y_4Y_6a_4^4 + 3a_5e_1a_4^5Y_2^3Y_4^2Y_6^2 \\
&+6a_5^2e_1Y_2^2Y_4^2Y_6^2Y_5a_4^4 - 3a_5^2e_1Y_4Y_5Y_6Y_2^2a_4^3 - a_5^2Y_6^3d_4Y_5^2a_4^3 - 3a_5^2e_1Y_1^2Y_4Y_5Y_6a_4^3 \\
&+3a_5^3e_1Y_4^2Y_5^2Y_6^2Y_2a_4^3 + 6Y_6^3a_5^2e_1Y_1^2Y_2d_4Y_5a_4^3 - 15a_5^5e_1Y_6^7Y_2Y_4d_4^2Y_5^4a_4^2 + 12Y_6^3a_5^2e_1Y_1Y_2^2d_4Y_5a_4^3 \\
&+6Y_6^3a_5^2e_1Y_2^3d_4Y_5a_4^3 + 3a_5^3e_1Y_6^3d_4Y_5^2Y_2^2a_4^2 + 3a_5^3e_1Y_1Y_4^2Y_5^2Y_6^2a_4^3 - 6a_5^2e_1Y_1Y_4Y_5Y_6Y_2a_4^3 \\
&-a_5^4e_1Y_4^3Y_5^3Y_6^3a_4^3)/a_4^4;
\end{aligned}$$

$$
\begin{aligned}
G_2 = {} & (20a_5^4e_1Y_6^9d_4^3Y_2^3Y_5^3a_4^3 - 6a_5^4e_1Y_6^4Y_4Y_5^3d_4Y_2a_4^2 + a_5e_1a_4^6Y_6^9Y_2^6d_4^3 + 3Y_6^3a_5e_1Y_2^4d_4a_4^4 \\
& +3a_5e_1a_4^5Y_6^6Y_2^5d_4^2 - a_5e_1a_4^6Y_2^3Y_4^3Y_6^3 + a_5^7e_1Y_6^9d_4^3Y_5^6 + a_5e_1a_4^3Y_1^3 \\
& +a_5e_1a_4^3Y_2^3 - 3a_5^2e_1a_4^5Y_2^2Y_4^3Y_6^3Y_5 - 6a_5e_1Y_1Y_2^2Y_4Y_6a_4^4 + 3a_5^5e_1Y_6^6Y_1d_4^2Y_5^4a_4 \\
& +12a_5^4e_1Y_6^6Y_2^2d_4^2Y_5^3a_4^2 - 6a_5^4e_1Y_6^4Y_1Y_4d_4Y_5^3a_4^2 + 3a_5e_1a_4^3Y_1Y_2^2 - 6a_5e_1a_4^5Y_6^4Y_1Y_2^3Y_4d_4 \\
& +3Y_6^3a_5e_1Y_1^2Y_2^2d_4a_4^4 + 3a_5e_1a_4^5Y_1Y_2^2Y_4^2Y_6^2 - 30a_5^3e_1Y_6^7Y_4d_4^2Y_2^3Y_5^2a_4^4 + 6Y_6^3a_5e_1Y_1Y_2^3d_4a_4^4 \\
& +15a_5^3e_1Y_6^9d_4^3Y_2^4Y_5^2a_4^4 + 6a_5^2e_1a_4^5Y_6^9Y_2^5d_4^3Y_5 - 18a_5^2e_1Y_6^4Y_2^3Y_4d_4Y_5a_4^4 + 12a_5^2e_1a_4^5Y_6^5Y_2^3Y_4^2d_4Y_5 \\
& -15a_5^2e_1a_4^5Y_6^7Y_2^4Y_4d_4^2Y_5 + 12a_5^2e_1Y_6^6Y_1Y_2^3d_4^2Y_5a_4^4 - 18a_5^2e_1Y_6^4Y_1Y_2^2Y_4d_4Y_5a_4^4 - 3a_5e_1a_4^6Y_6^7Y_2^5Y_4d_4^2 \\
& +3a_5e_1a_4^3Y_1^2Y_2 + 18a_5^3e_1Y_6^5Y_4^2d_4Y_2^2Y_5^2a_4^4 + 12a_5^2e_1Y_6^6Y_2^4d_4^2Y_5a_4^4 + 3a_5e_1a_4^5Y_6^6Y_1Y_2^4d_4^2 \\
& +3a_5e_1a_4^6Y_6^5Y_2^4Y_4^2d_4 - 6a_5e_1Y_6^4a_4^5Y_2^4Y_4d_4 + 15a_5^5e_1Y_6^9d_4^3Y_2^2Y_5^4a_4^2 + 6a_5^6e_1Y_6^9Y_2d_4^3Y_5^5a_4 \\
& -18a_5^3e_1Y_6^4Y_2^2Y_4d_4Y_5^2a_4^3 + 18a_5^3e_1Y_6^6Y_1d_4^2Y_2^2Y_5^2a_4^3 - 3a_5^6e_1Y_6^7Y_4Y_5^5d_4^2a_4 - 18a_5^3e_1Y_6^4Y_1Y_4d_4Y_2Y_5^2a_4^3 \\
& +6a_5^3e_1Y_6^3Y_1d_4Y_5^2Y_2a_4^2 - 30a_5^4e_1Y_6^7Y_4d_4^2Y_2^2Y_5^3a_4^3 + 12a_5^4e_1Y_6^5Y_2Y_4^2d_4Y_5^3a_4^3 + 12a_5^4e_1Y_6^6Y_1d_4^2Y_2Y_5^3a_4^2 \\
& +3a_5^5e_1Y_6^6d_4^2Y_5^4Y_2a_4 + 3a_5^3e_1Y_6^3Y_1^2d_4Y_5^2a_4^2 + 18a_5^3e_1Y_6^6Y_2^3d_4^2Y_5^2a_4^3 + 6a_5^2e_1Y_1Y_2Y_4^2Y_6^2Y_5a_4^4 \\
& -3a_5e_1Y_2^3Y_4Y_6a_4^4 + 3a_5^5e_1Y_6^5Y_4^2Y_5^4d_4a_4^2 - 3a_5^3e_1Y_2Y_4^3Y_6^3Y_5^2a_4^4 - 3a_5e_1Y_1^2Y_2Y_4Y_6a_4^4 \\
& +3a_5e_1a_4^5Y_2^3Y_4^2Y_6^2 + 6a_5^2e_1Y_2^2Y_4^2Y_6^2Y_5a_4^4 - 3a_5^2e_1Y_4Y_5Y_6Y_2^2a_4^3 - 3a_5^2e_1Y_1^2Y_4Y_5Y_6a_4^3 \\
& +3a_5^3e_1Y_4^2Y_5^2Y_6^2Y_2a_4^3 + 6Y_6^3a_5^2e_1Y_1^2Y_2d_4Y_5a_4^3 - 15a_5^5e_1Y_6^7Y_2Y_4d_4^2Y_5^4a_4^2 + 12Y_6^3a_5^2e_1Y_1Y_2^2d_4Y_5a_4^3 \\
& +6Y_6^3a_5^2e_1Y_2^3d_4Y_5a_4^3 + 3a_5^3e_1Y_6^3d_4Y_5^2Y_2^2a_4^2 + 3a_5^3e_1Y_1Y_4^2Y_5^2Y_6^2a_4^3 - 6a_5^2e_1Y_1Y_4Y_5Y_6Y_2a_4^3 \\
& -a_5^4e_1Y_4^3Y_5^3Y_6^3a_4^3 + Y_2a_4^4)/a_4^4.
\end{aligned}
$$

Now,

$$
\begin{aligned}
G_6 &= Y_6 \\
G_5 &= Y_5 - e_1(G_1 + G_2)^3 \\
G_4 &= Y_4 - d_4G_2G_6^2 - a_5d_4G_5G_6^2/a_4 \\
G_3 &= Y_3 - c_1G_1^3 - c_2(G_1 + G_5)^3 - c_3(G_1 + G_2)^3 - c_4(G_1 + G_4)^3 + c_5G_6^3.
\end{aligned}
$$

4.4

Let us consider again the polynomial automorphism of $\mathbb{C}^4$ given in example 7. We have $p := X_1X_3 + X_2X_4$ and F defined by

$$
\begin{cases}
F_1 = X_1 + pX_4 \\
F_2 = X_2 - pX_3 \\
F_3 = X_3 + X_4^3 \\
F_4 = X_4
\end{cases}
$$

We obtain

$$
\sum_{j=0}^{13} (-1)^j P_j^1(X) = X_1 - X_1X_3X_4 - X_2X_4^2 + X_1X_4^4 + R_{14}^1(X),
$$

where

$$
\begin{aligned}
R_{14}^1(X) \;=\;& -35X_1X_3X_4^{33} - 10X_1X_3X_4^{45} - X_1X_3X_4^{49} - 36X_1X_3X_4^{41} - 6X_1X_3X_4^{29} \\
& -56X_1X_3X_4^{37} - X_1X_4^{28} - 6X_2X_4^{30} - 15X_1X_4^{32} - 35X_2X_4^{34} \\
& -35X_1X_4^{36} - 56X_2X_4^{38} - 28X_1X_4^{40} - 36X_2X_4^{42} - 9X_1X_4^{44} \\
& -10X_2X_4^{46} - X_1X_4^{48} - X_2X_4^{50}
\end{aligned}
$$

satisfies $R_{14}^1(F) + P_{14}^1(X) = 0$. And

$$
\sum_{j=0}^{13}(-1)^j P_j^2(X) = X_2 - 2X_1X_3X_4^3 + X_2X_3X_4 - X_2X_4^4 + X_1X_4^6 + X_1X_3^2 + R_{14}^2(X),
$$

where

$$
\begin{aligned}
R_{14}^2(X) \;=\;& 14X_1X_3X_4^{27} + 6X_2X_3X_4^{29} + 282X_1X_3X_4^{31} + 6X_1X_3^2X_4^{28} + 35X_2X_3X_4^{33} \\
& +910X_1X_3X_4^{35} + 35X_1X_3^2X_4^{32} + 56X_2X_3X_4^{37} + 1064X_1X_3X_4^{39} + 56X_1X_3^2X_4^{36} \\
& +36X_2X_3X_4^{41} + 558X_1X_3X_4^{43} + 36X_1X_3^2X_4^{40} + 12X_1X_3X_4^{51} + 13X_2X_4^{28} \\
& +77X_1X_4^{30} + 267X_2X_4^{32} + 440X_1X_4^{34} + 875X_2X_4^{36} + 693X_1X_4^{38} \\
& +1036X_2X_4^{40} + 440X_1X_4^{42} + 549X_2X_4^{44} + 121X_1X_4^{46} + 133X_2X_4^{48} \\
& +12X_1X_4^{50} + 12X_2X_4^{52} + 10X_1X_3^2X_4^{44} + 134X_1X_3X_4^{47} + 10X_2X_3X_4^{45} \\
& +X_1X_3^2X_4^{48} + X_2X_3X_4^{49}
\end{aligned}
$$

satisfies $R_{14}^2(F) + P_{14}^2(X) = 0$. Hence $G = F^{-1}$ is given by

$$
\begin{cases}
G_1 = X_1 - X_1X_3X_4 - X_2X_4^2 + X_1X_4^4 \\
G_2 = X_2 - 2X_1X_3X_4^3 + X_2X_3X_4 - X_2X_4^4 + X_1X_4^6 + X_1X_3^2 \\
G_3 = X_3 - X_4^3 \\
G_4 = X_4
\end{cases}
$$

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