BRANCHING RULES FOR n-FOLD COVERING GROUPS OF SL_2 OVER A NON-ARCHIMEDEAN LOCAL FIELD

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ABSTRACT. Let $\widetilde{G^1}$ be the n-fold covering group of the special linear group of degree two, over a non-Archimedean local field. We determine the decomposition into irreducibles of the restriction of the principal series representations of $\widetilde{G^1}$ to a maximal compact subgroup of $\widetilde{G^1}$.

1. Introduction

In this paper, covering groups, also known in the literature as metaplectic groups, are central extensions of a simply connected simple and split algebraic group, over a non-Archimedean local field $\mathbb{F}$, by the group of the n-th roots of unity, μ_n. The problem of determining this class of groups was studied by Steinberg [22] and Moore [15] in 1968, and further completed by Matsumoto [12] in 1969 for simply connected Chevalley groups. Around the same time, Kubota independently constructed n-fold covering groups of SL_2 [10] and GL_2 [11], by means of presenting an explicit 2-cocycle. Kubota's cocycle is expressed in terms of the n-th Hilbert symbol.

Since then, there have been a number of studies of representations of this class of groups from different perspectives, among them being the work of H. Aritürk [1], D. A. Kazhdan and S. J. Patterson [9], C. Moen [14], D. Joyner [6, 7], G. Savin [20], M. Weissman and T. Howard [5], and P. J. McNamara [13].

In this paper, we consider the principal series representations of the n-fold covering group $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ of $\mathrm{SL}_2(\mathbb{F})$. The principal series representations of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ are those representations that are induced from the inverse image $\widetilde{B^1}$ of a Borel subgroup B^1 of $\mathrm{SL}_2(\mathbb{F})$. The construction of those representations of $\widetilde{B^1}$ that are trivial on the unipotent radical of $\widetilde{B^1}$ brings us to the study of the irreducible representations of the metaplectic torus $\widetilde{T^1}$, i.e., the inverse image of the split torus, T^1, of $\mathrm{SL}_2(\mathbb{F})$ in $\widetilde{\mathrm{SL}}_2(\mathbb{F})$.

An important feature of $\widetilde{T^1}$, which differentiates the nature of its representations from those of a linear torus, is that it is not abelian. However, it is a Heisenberg group and its irreducible representations are governed by the Stone-von Neumann theorem. The Stone-von Neumann theorem characterizes irreducible representations of Heisenberg groups, according to their central characters. Indeed, given a character of the centre of a Heisenberg group that satisfies some mild conditions, the Stone-von Neumann theorem provides a recipe to construct the corresponding, unique up to isomorphism, irreducible representation of the Heisenberg group. The construction involves induction from a maximal abelian subgroup of the Heisenberg group. We only consider those characters of the centre of $\widetilde{T^1}$ where μ_n acts by a fixed faithful character.

Once an irreducible representation ρ_χ, with central character χ, of $\widetilde{T^1}$ is obtained, the principal series representation π_χ of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ is $\mathrm{Ind}_{\widetilde{B^1}}^{\widetilde{\mathrm{SL}}_2(\mathbb{F})}\rho_\chi$, where ρ_χ is trivially extended on the unipotent radical subgroup of $\widetilde{B^1}$. These representations admit several open questions. The question we consider, and answer, in this paper is to decompose π_χ upon the restriction to the inverse image $\widetilde{K^1}$ of a maximal compact subgroup

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K^1 of $\mathrm{SL}_2(\mathbb{F})$. We refer to this decomposition as the K-type decomposition. We assume $n|q-1$, where q is the size of the residue field of $\mathbb{F}$, so that the central extension $\widetilde{\mathrm{SL}_2}(\mathbb{F})$ splits over K^1.

The study the decomposition of the restriction of representations to a particular subgroup is a common technique in representation theory. In the theory of real Lie groups, restriction to maximal compact subgroups retains a lot of information from the representation; in fact, such a restriction is a key step towards classifying irreducible unitary representations. In the case of reductive groups over p-adic fields, investigating the decomposition upon restriction to maximal compact subgroups reveals a finer structure of the representation, in the interests of recovering essential information about the original representation.

The K-type problem for reductive p-adic groups is visited and solved in certain cases, including the principal series representations of GL(3) [2, 3, 19], and SL(2) [16, 17], representations of GL(2) [4], and supercuspidal representations of SL(2) [18].

The main idea is to reduce the problem to calculating the dimensions of certain finite-dimensional Hecke algebras. The key calculation for determining the decomposition is the determination of certain double cosets that support intertwining operators for the restricted principal series representation (Proposition 3 and Proposition 4).

Our method is aligned with the one in [16] for the linear group $\mathrm{SL}_2(\mathbb{F})$; however, the technicalities in the covering case are much more involved than the linear case, and the results are fairly different. For instance, the K-type decomposition is no longer multiplicity-free (Corollary 2).

This paper is organized as follows. In Section 2, we present Kubota's construction of the covering group of $\mathrm{SL}_2(\mathbb{F})$, in Section 3 we overview the structure of this covering group and compute some subgroups of our interest. We compute the K-type decomposition for the principal series representations of $\widetilde{\mathrm{SL}_2}(\mathbb{F})$ in Section 4. This decomposition is completed by considering a similar problem for the n-fold covering group of $\mathrm{GL}_2(\mathbb{F})$ in Section 5. Our main result, Theorem 2, is stated in Section 6.

2. Notation and Background

Let $\mathbb{F}$ be a non-Archimedean local field with the ring of integers $\mathcal{O}$ and the maximal ideal $\mathfrak{p}$ of $\mathcal{O}$. Let $\kappa := \mathcal{O}/\mathfrak{p}$ be the residue field and $q = |\kappa|$ be its cardinality. Let $\mathcal{O}^\times$ denote the group of units in $\mathcal{O}$. We fix a unoformizing element ϖ of $\mathfrak{p}$. For every $x \in \mathbb{F}^\times$, the valuation of x is denoted by $\mathrm{val}(x)$, and $|x| = q^{-\mathrm{val}(x)}$. Let $n \geq 2$ be an integer such that $n|q-1$. Set $\underline{n} = n$ if n is odd, and $\underline{n} = \frac{n}{2}$ if n is even. We assume that $\mathbb{F}$ contains the group μ_n of n-th roots of unity.

Set $G = \mathrm{GL}_2(\mathbb{F})$, and $G^1 = \mathrm{SL}_2(\mathbb{F})$. Let B^1 (B) be the standard Borel subgroup of G^1 (G) and N^1 (N) be its unipotent radical, and let T^1 (T) be the standard torus in G^1 (G). Set $K^1 = \mathrm{SL}_2(\mathcal{O})$ $(K = \mathrm{GL}_2(\mathcal{O}))$ to be a maximal compact subgroup of G^1 (G). By the Iwasawa decomposition, we have $G^1 = T^1N^1K^1$ $(G = TNK)$. Our object of study is the central extension $\widetilde{G^1}$ of G^1 by μ_n,

$$0 \to \mu_n \xrightarrow{\mathsf{i}} \widetilde{G^1} \xrightarrow{\mathsf{p}} G^1 \to 0, \tag{1}$$

where i and p are natural injection and projection maps respectively. The group $\widetilde{G^1}$, which we call the n-fold covering group of G^1, is constructed explicitly by Kubota [10]. In order to describe Kubota's construction, we need knowledge of the n-th Hilbert symbol $(\ ,\ )_n : \mathbb{F}^\times \times \mathbb{F}^\times \to \mu_n$. Under our assumption on n, the n-th Hilbert symbol is given via $(a,b)_n = \overline{c}^{\frac{q-1}{n}}$, where $c = (-1)^{\mathrm{val}(a)\mathrm{val}(b)} \frac{a^{\mathrm{val}(b)}}{b^{\mathrm{val}(a)}}$, and $\overline{c}$ is the image of c in $\kappa^\times$. We benefit from the properties of the n-th Hilbert symbol, which can be found in [21, Ch XIV]. In particular, we benefit extensively from the following fact: $(a,b)_n = 1$ for all $a \in \mathbb{F}^\times$, if and only if $b \in {\mathbb{F}^\times}^n$.

Define the map $\beta : G^1 \times G^1 \to \mu_n$ by

$$\beta(\mathbf{g}_1,\mathbf{g}_2) = \left(\frac{X(\mathbf{g}_1\mathbf{g}_2)}{X(\mathbf{g}_1)}, \frac{X(\mathbf{g}_1\mathbf{g}_2)}{X(\mathbf{g}_2)}\right)_n, \text{ where } X\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{otherwise.}\end{cases} \tag{2}$$

In [10] Kubota proved that β is a non-trivial 2-cocycle in the continuous second cohomology group of G^1 with coefficients in μ_n; whence, $\widetilde{G^1} = G^1 \times \mu_n$ as a set, with the multiplication given via $(\mathbf{g}_1, \zeta_1)(\mathbf{g}_2, \zeta_2) = (\mathbf{g}_1\mathbf{g}_2, \beta(\mathbf{g}_1, \mathbf{g}_2)\zeta_1\zeta_2)$, for all $\mathbf{g}_1, \mathbf{g}_2 \in G^1$ and $\zeta_1, \zeta_2 \in \mu_n$.

In 1969, Kubota extends the map β to a 2-cocycle β' for $\widetilde{G}$ in [11], which defines the n-fold covering group $\widetilde{G} \cong \mathbb{F}^\times \ltimes \widetilde{G^1}$ of G. The covering group $\widetilde{G}$ fits into the exact sequence $0 \to \mu_n \xrightarrow{\mathsf{i}} \widetilde{G} \xrightarrow{\mathsf{p}} G \to 0$.

For all $t, s \in \mathbb{F}^\times$, set $\mathrm{dg}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right) \in T^1$, $\mathrm{dg}(t,s) = \left(\begin{smallmatrix} t & 0 \\ 0 & s\end{smallmatrix}\right) \in T$, and $\iota(t) = (\mathrm{dg}(t), 1) \in \widetilde{T^1}$. Set $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$, and $\widetilde{w} = (w, 1) \in \widetilde{G^1}$. Moreover, for matrices X and Y, with Y invertible, let $X^Y := Y^{-1}XY$ and ${}^YX := YXY^{-1}$ denote the conjugations of X by Y.

3. Structure Theory

For any subgroup H of G^1, the inverse image $\widetilde{H} := \mathsf{p}^{-1}(H)$ is a subgroup of $\widetilde{G^1}$. In particular, we are interested in the subgroups $\widetilde{T^1}$, $\widetilde{B^1}$, and $\widetilde{K^1}$ of $\widetilde{G^1}$. We say the central extension splits over the subgroup H of G^1, if there exists an isomorphism that yields $\mathsf{p}(H)^{-1} \cong H \times \mu_n$.

It is not difficult to see that $\widetilde{T^1}$ is not commutative, and hence, the central extension does not split over T^1 (and therefore neither over B^1). Additionally, it is easy to see that the commutator subgroup $[\widetilde{T^1}, \widetilde{T^1}] \cong \mu_{\underline{n}}$ is central in (1); which implies that $\widetilde{T^1}$ is a two-step nilpotent group, also known as a Heisenberg group. Clearly, $\mu_n \in Z(\widetilde{T^1})$, indeed, using the properties of the Hilbert symbol and some elementary calculation, one can show that $Z(\widetilde{T^1}) = \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^{\times \underline{n}}, \zeta \in \mu_n\}$.

Lemma 1. *The index of $Z(\widetilde{T^1})$ in $\widetilde{T^1}$ is $\underline{n}^2$.*

Proof. Note that $[\widetilde{T^1} : Z(\widetilde{T^1})] = [\mathbb{F}^\times : \mathbb{F}^{\times \underline{n}}]$, which because $\mathbb{F}^\times \cong \mathcal{O}^\times \times \mathbb{Z}$, is equal to $\underline{n}[\mathcal{O}^\times : \mathcal{O}^{\times \underline{n}}]$. Consider the homomorphism $\phi : \mathcal{O}^\times \to \mathcal{O}^{\times \underline{n}}$. Then $\ker(\phi) = \{x \in \mathcal{O}^\times \mid x^{\underline{n}} = 1\}$. Note that $f(x) = x^{\underline{n}} - 1 = 0$ has $(\underline{n}, q-1)$, which equals $\underline{n}$ under our assumption of $n | q-1$, solutions in the cyclic group $\kappa^\times$. By Hensel's lemma, any such root in $\kappa^\times$ lifts uniquely to a root in $\mathcal{O}^\times$. It follows that, $|\ker(\phi)| = \underline{n}$. Therefore, $[\mathcal{O}^\times : \mathcal{O}^{\times \underline{n}}] = |\ker(\phi)| = \underline{n}$, and the result follows. ☐

In order to construct principal series representations of $\widetilde{G^1}$ in Section 4, we need to construct irreducible representations of the Heisenberg group $\widetilde{T^1}$. To do so, we need to identify a maximal abelian subgroup of $\widetilde{T^1}$. Set $A^1 = C_{\widetilde{T^1}}(\widetilde{T^1} \cap \widetilde{K^1})$, to be the centralizer of $\widetilde{T^1} \cap \widetilde{K^1}$ in $\widetilde{T^1}$. It is not difficult to calculate that $A^1 = \{(\mathrm{dg}(a), \zeta) \mid a \in \mathbb{F}^\times, \underline{n} | \mathrm{val}(a), \zeta \in \mu_n\}$, and see that it is abelian. Observe that $\widetilde{T^1} \cap \widetilde{K^1} \subset A^1$ implies that A^1 is a maximal abelian subgroup. Note that $[\widetilde{T^1} : A^1] = [\mathbb{Z} : \underline{n}\mathbb{Z}] = \underline{n}$.

Let N^1 be the unipotent radical of B^1. It follows directly from the Kubota's formula for β that $\beta|_{N^1}$ is trivial, so $N^1 \times \{1\}$ is a subgroup of $\widetilde{G^1}$. We identify N^1 with $N^1 \times \{1\}$. Under this identification, we have the covering analogue of the Levi decomposition: $\widetilde{B^1} = \widetilde{T^1} \ltimes N^1$.

Next, we describe a family of compact open subgroups of $\widetilde{G^1}$. It is proven in [11] that

$$\widetilde{K^1} \to K^1 \times \mu_n, \quad (\mathbf{k}, \zeta) \mapsto (\mathbf{k}, s(\mathbf{k})\zeta), \text{ where } s\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = \begin{cases} (c,d)_n, & 0 < \mathrm{val}(c) < \infty \\ 1, & \text{otherwise.}\end{cases} \tag{3}$$

is an isomorphism. The image of K^1 in $\widetilde{K^1}$ under the isomorphism (3) is the subgroup $\widetilde{K}_0 := \{(\mathbf{k}, s(\mathbf{k})^{-1}) \mid \mathbf{k} \in K^1\}$ of $\widetilde{K^1}$. Consider the compact open congruent subgroups $K^1_j := \{\mathbf{g} \in K^1 \mid \mathbf{g} \equiv \mathrm{I}_2 \mod \mathfrak{p}^j\}$, for $j \geq 1$, of K^1.

Lemma 2. *The central extension* (1) *splits trivially over each of the subgroups K^1_j, $j \geq 1$, $T^1 \cap K^1$, and $B^1 \cap K^1$.*

Proof. Using the Hensel's lemma, it is easy to see that $1 + \mathfrak{p} \subset \mathcal{O}^{\times n}$. Then, it follows from (3) and properties of the n-th Hilbert symbol that, for all $i \geq 1$, $s|_{K^1_j}$ is trivial. On the other hand, it follows directly from (3) that $s|_{T^1 \cap K^1}$ and $s|_{B^1 \cap K^1}$ are trivial. ☐

We identify $K^1_j \cong K^1_j \times \{1\}$, $j \geq 1$, $B^1 \cap K^1 \cong (B^1 \cap K^1) \times \{1\}$ and $T^1 \cap K^1 \cong (T^1 \cap K^1) \times \{1\}$ as subgroups of $\widetilde{K^1}$.

In a similar way, we define the subgroups $\widetilde{T}$, $\widetilde{B}$ and $\widetilde{K}$ of $\widetilde{G}$ to be the inverse images of the standard torus, Borel, and the maximal compact $K = \mathrm{GL}(\mathcal{O})$ subgroups of G respectively. The central extension $\widetilde{G}$ does not split over T. Moreover, $\widetilde{T}$ is a Heisenberg group. It is not difficult to see that $Z(\widetilde{T}) = \{(\mathrm{dg}(s,t), \zeta) \mid s, t \in \mathbb{F}^{\times n}, \zeta \in \mu_n\}$, and $[\widetilde{T} : Z(\widetilde{T})] = n^4$. Moreover, set $A = C_{\widetilde{T}}(\widetilde{T} \cap \widetilde{K}) = \{(\mathrm{dg}(s,t), \zeta) \mid s, t \in \mathbb{F}^{\times}, n|\mathrm{val}(s), n|\mathrm{val}(t), \zeta \in \mu_n\}$. Then, A is a maximal abelian subgroup of $\widetilde{T}$ and $[\widetilde{T} : A] = n^2$. In addition, $\beta'|_N$ is trivial, where N is the unipotent radical of B. Hence, we can identify N with $N \times \{1\}$. Under this identification, we have the Levi decomposition: $\widetilde{B} = \widetilde{T} \ltimes N$. It is shown in [11] that the central extension $\widetilde{G}$ splits over K. For $j \geq 1$, let K_j denote the family of compact open congruent subgroups $\{\mathbf{g} \in K \mid \mathbf{g} \equiv \mathrm{I}_2 \mod \mathfrak{p}^j\}$ of K. Similar to Lemma 2, one can show that $\widetilde{G}$ splits over K_j, $T \cap K$ and $B \cap K$.

4. Branching Rules for $\widetilde{G^1}$

First, we present the construction of the principal series representations of $\widetilde{G^1}$ following [13]. Fix a faithful character $\epsilon : \mu_n \to \mathbb{C}^{\times}$. A representation of $\widetilde{G^1}$ is genuine if the central subgroup μ_n acts by ϵ. Such representations do not factor through representations of G^1. The construction of principal series representations of $\widetilde{G^1}$ is based on the essential fact that $\widetilde{T^1}$ is a Heisenberg subgroup, and hence its representations are governed by the Stone-von Neumann theorem, which we state here. See [13] for the proof.

Theorem 1 (Stone-von Neumann)**.** *Let H be a Heisenberg group with center $Z(H)$ such that $H/Z(H)$ is finite, and let χ be a character of $Z(H)$. Suppose that $\ker(\chi) \cap [H,H] = \{1\}$. Then there is a unique (up to isomorphism) irreducible representation π of H with central character χ. Let A be any maximal abelian subgroup of H and let χ_0 be any extension of χ to A. Then $\pi \cong \mathrm{Ind}_{\mathsf{A}}^{H} \chi_0$.*

Note that $[\widetilde{T^1} : Z(\widetilde{T^1})] = \underline{n}^2 < \infty$. Let χ be a genuine character of $Z(\widetilde{T^1})$, so that $\chi|_{\mu_n} = \epsilon$. Thus, $\ker(\chi) \cap [\widetilde{T^1}, \widetilde{T^1}]$ is trivial. Hence Theorem 1 applies: genuine irreducible smooth representations ρ of $\widetilde{T^1}$ are classified by genuine smooth characters of $Z(\widetilde{T^1})$. Moreover, $\dim(\rho) = [\widetilde{T^1} : \widetilde{A^1}] = \underline{n}$.

Let χ_0 be a fixed extension of χ to A^1; so that $(\rho, \mathrm{Ind}_{A^1}^{\widetilde{T^1}} \chi_0)$ is the unique smooth genuine irreducible representation of $\widetilde{T^1}$ with central character χ. Let us again write ρ for the genuine smooth irreducible representation of $\widetilde{T^1}$, with central character χ, extended trivially over N^1 to a representation of $\widetilde{B^1} = \widetilde{T^1} \ltimes N^1$. Then the genuine principal series representation of $\widetilde{G^1}$ associated to ρ is $\mathrm{Ind}_{\widetilde{B^1}}^{\widetilde{G^1}} \rho$, where Ind denotes the smooth (non-normalized) induction. In the rest of this section, we decompose $\mathrm{Res}_{\widetilde{K^1}}\mathrm{Ind}_{\widetilde{B^1}}^{\widetilde{G^1}}\rho$ into irreducible constituents. We drop the adjective "genuine" for simplicity.

Define the character

$$\vartheta : \mathbb{F}^\times \to \mu_n, \quad a \mapsto (\varpi, a)_n. \tag{4}$$

Observe that ϑ is ramified of degree one. Set $\vartheta_{\mathcal{O}^{\times 2}} := \vartheta|_{\mathcal{O}^{\times 2}}$. Observe that a typical element of A^1 can be written as $(\mathrm{dg}(a\varpi^{r\underline{n}}), \zeta)$, and a typical element of $\widetilde{T^1} \cap \widetilde{K^1}$ can be written as $(\mathrm{dg}(a), \zeta)$, where $a \in \mathcal{O}^\times$, $r \in \mathbb{Z}$, and $\zeta \in \mu_n$.

Lemma 3. *Let ρ be the unique irreducible representation of $\widetilde{T^1}$ with central character χ. Then $\mathrm{Res}_{A^1}\rho \cong \bigoplus_{i=0}^{\underline{n}-1} \chi_i$, where the χ_i are $\underline{n}$ distinct characters of A^1 defined by*

$$\chi_i \left(\mathrm{dg}(a\varpi^{\underline{n}r}), \zeta\right) = \chi_0 \left(\mathrm{dg}(a\varpi^{\underline{n}r}), \vartheta^{2i}(a)\zeta\right),$$

for all $a \in \mathcal{O}^\times$, $r \in \mathbb{Z}$, $\zeta \in \mu_n$, and $0 \leq i < \underline{n}$.

Proof. By Theorem 1, $\rho \cong \mathrm{Ind}_{A^1}^{\widetilde{T^1}}\chi_0$. By Mackey's theory, $\mathrm{Res}_{A^1}\mathrm{Ind}_{A^1}^{\widetilde{T^1}}\chi_0 = \bigoplus_{s\in S_{\underline{n}}} \mathrm{Ind}_{A^1\cap {}^sA^1}^{A^1}{\chi_0}^s$, where $S_{\underline{n}}$ is a complete set of coset representatives for $A^1\backslash\widetilde{T^1}/A^1$. It is not difficult to see that we can choose $S_{\underline{n}} = \{\left(\mathrm{dg}(\varpi^i), 1\right) \mid 0 \leq i < \underline{n}\}$. Since A^1 is stable under conjugation by $S_{\underline{n}}$, $\mathrm{Ind}_{A^1\cap {}^sA^1}^{A^1}{\chi_0}^s = {\chi_0}^s$. Let $(\mathrm{dg}(a\varpi^{r\underline{n}}), \zeta) \in A^1$, and $s = \left(\mathrm{dg}(\varpi^i), 1\right) \in S_{\underline{n}}$. Then

$$\begin{aligned} s^{-1}\left(\mathrm{dg}(a\varpi^{r\underline{n}}), \zeta\right) s &= \left(\mathrm{dg}(\varpi^{-i}), (\varpi^i, \varpi^i)_n\right)\left(\mathrm{dg}(a\varpi^{r\underline{n}}), \zeta\right)\left(\mathrm{dg}(\varpi^i), 1\right) \\ = \left(\mathrm{dg}(a\varpi^{r\underline{n}-i}), (a\varpi^{r\underline{n}}, \varpi^{-i})_n(\varpi^i, \varpi^i)_n\zeta\right)\left(\mathrm{dg}(\varpi^i), 1\right) &= \left(\mathrm{dg}(a\varpi^{r\underline{n}}), (\varpi^i, a\varpi^{r\underline{n}-i})_n(a\varpi^{r\underline{n}}, \varpi^{-i})_n(\varpi^i, \varpi^i)_n\zeta\right) \\ &= \left(\mathrm{dg}(a\varpi^{r\underline{n}}), (\varpi, a)_n^{2i}\zeta\right) = \left(\mathrm{dg}(a\varpi^{r\underline{n}}), \vartheta^{2i}(a)\zeta\right). \end{aligned}$$

Hence, ${\chi_0}^s\left((\mathrm{dg}(a\varpi^{r\underline{n}}), \zeta)\right) = \chi_0\left(\left(\mathrm{dg}(a\varpi^{r\underline{n}}), \vartheta^{2i}(a)\zeta\right)\right)$. Denote this character χ_i. To show that the χ_i, $0 \leq i < \underline{n}$, are distinct, it is enough to show that $\vartheta^{2i}|_{\mathcal{O}^\times} = 1$ if and only if $i = 0$. Observe that $\vartheta^{2i}(a) = \overline{a^{-1}}^{\frac{(q-1)2i}{n}}$, which is equal to 1 for all $a \in \mathcal{O}^\times$ if and only if $n|2i$. The result follows. ☐

The characters χ_i defined in Lemma 3 are clearly distinct when restricted to $\widetilde{T^1} \cap \widetilde{K^1}$ and, again writing χ_i for these restrictions,

$$\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\rho = \bigoplus_{i=0}^{\underline{n}-1}\chi_i. \tag{5}$$

Proposition 1. *Let χ_i, $0 \leq i < \underline{n}$, denote also the trivial extension of the characters in (5) to $\widetilde{B^1} \cap \widetilde{K^1}$. Then*

$$\mathrm{Res}_{\widetilde{K^1}}\mathrm{Ind}_{\widetilde{B^1}}^{\widetilde{G^1}}\rho \cong \bigoplus_{i=0}^{\underline{n}-1}\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i.$$

Proof. By Mackey's theorem, we have $\mathrm{Res}_{\widetilde{K^1}}\mathrm{Ind}_{\widetilde{B^1}}^{\widetilde{G^1}}\rho \cong \bigoplus_{x\in X}\mathrm{Ind}_{\widetilde{B^1}^{x^{-1}}\cap\widetilde{K^1}}^{\widetilde{K^1}}\mathrm{Res}_{\widetilde{B^1}^{x^{-1}}\cap\widetilde{K^1}}\rho^x$, where X is a complete set of double coset representatives of $\widetilde{K^1}$ and $\widetilde{B^1}$ in $\widetilde{G^1}$. The Iwasawa decomposition $\widetilde{K^1}\widetilde{B^1} = \widetilde{G^1}$ implies that $X = \{(\mathrm{I}_2, 1)\}$ and hence $\mathrm{Res}_{\widetilde{K^1}}\mathrm{Ind}_{\widetilde{B^1}}^{\widetilde{G^1}}\rho = \mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\mathrm{Res}_{\widetilde{B^1}\cap\widetilde{K^1}}\rho$. The result follows from (5). ☐

Hence, in order to calculate the K-types, it is enough to decompose each $\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i$, $0 \leq i < \underline{n}$, into irreducible representations. Note that the induction space $\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i$ is smooth and admissible. Fix

$i \in \{0, \cdots, \underline{n}-1\}$. The smoothness of $\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i$ implies that $\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i = \bigcup_{l\geq 1}\big(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i\big)^{K_l^1}$. Note that, by admissibility, $\big(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i\big)^{K_l^1}$ is finite-dimensional for every $l \geq 1$ and since K_l^1 is normal in $\widetilde{K^1}$, it is $\widetilde{K^1}$-invariant. Hence, to decompose $\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i$ into irreducible constituents, it is enough to decompose each $\big(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i\big)^{K_l^1}$ into irreducible constituents.

For any character γ of any subgroup D of $\widetilde{T^1}$, we say γ is primitive mod m if m is the smallest strictly positive integer for which $\mathrm{Res}_{D\cap K_m^1}\gamma = 1$. From now on, let $m \geq 1$ be a positive integer such that χ is primitive mod m. Because $1+\mathfrak{p} \subset \mathbb{F}^{\times n}$, $Z(\widetilde{T^1}) \cap K_m^1 = \widetilde{T^1} \cap K_m^1$, for all $m \geq 1$. Note that since $\chi_i|_{Z(\widetilde{T^1})} = \chi$, $\chi_i|_{\widetilde{T^1}\cap K_m^1} = \chi|_{Z(\widetilde{T^1})\cap K_m^1}$. Hence, χ is primitive mod m if and only if the χ_i for $0 \leq i < \underline{n}$ are primitive mod m. Set $\widetilde{B^1}_l := (\widetilde{B^1} \cap \widetilde{K^1})K_l^1$.

Lemma 4. *For every $0 \leq i < \underline{n}$,*

$$\big(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i\big)^{K_l^1} = \begin{cases} \{0\}, & 0 < l < m \\ \mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_i, & \textit{otherwise.} \end{cases}$$

Proof. Suppose $0 < l < m$, and that f is a vector in $\big(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i\big)^{K_l^1}$. Because $\chi_i\big|_{\widetilde{B^1}\cap K_l^1} \neq 1$ for $l < m$, we can choose $\mathtt{b} \in \widetilde{B^1} \cap K_l^1$ such that $\chi_i(\mathtt{b}) \neq 1$. Let $\mathtt{g} \in \widetilde{K^1}$. Note that K_l^1 is normal in $\widetilde{K^1}$ and hence $\mathtt{g}^{-1}\mathtt{b}\mathtt{g} \in K_l^1$. On the one hand, $f(\mathtt{bg}) = \chi_i(\mathtt{b})f(\mathtt{g})$; on the other hand, $f(\mathtt{bg}) = f(\mathtt{g}\mathtt{g}^{-1}\mathtt{b}\mathtt{g}) = \big(\mathtt{g}^{-1}\mathtt{b}\mathtt{g}\big)\cdot f(\mathtt{g}) = f(\mathtt{g})$, since f is fixed by K_l^1. It follows that $\chi_i(\mathtt{b})f(\mathtt{g}) = f(\mathtt{g})$. Our choice of $\mathtt{b}$ implies that $f(\mathtt{g}) = 0$ and because $\mathtt{g}$ is arbitrary, $f = 0$. However, if $l \geq m$ then $\chi_i|_{K_l^1} = 0$ and because K_l^1 is normal in $\widetilde{K^1}$, it is not difficult to see that every K_l^1-fixed vector f translates on the left by $\widetilde{B^1}_l$ and vice-versa. Hence the result follows. ☐

Lemma 4 tells us that, in order to decompose $(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i)^{K_l^1}$ into irreducible constituents, it is enough to decompose $\mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_i$. Hence, we are interested in counting the dimension of $\mathrm{Hom}_{\widetilde{K^1}}(\mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_i, \mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_i)$. By Frobenius reciprocity, this latter space is isomorphic to $\mathrm{Hom}_{\widetilde{B^1}_l}(\mathrm{Res}_{\widetilde{B^1}_l}\mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_i, \chi_i)$. It follows from Mackey's theory that

$$\mathrm{Res}_{\widetilde{B^1}_l}\mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_i \cong \bigoplus_{\mathtt{x}\in S}\mathrm{Ind}_{\widetilde{B^1}_l^{\mathtt{x}^{-1}}\cap\widetilde{B^1}_l}^{\widetilde{B^1}_l}\chi_i^{\mathtt{x}},$$

where S is a set of double coset representatives of $\widetilde{B^1}_l\backslash\widetilde{K^1}/\widetilde{B^1}_l$. The set S is a lift to the covering group $\widetilde{K^1}$ of a similar set of double coset representatives calculated in [16]. Using the latter set, and because $\mu_n \subset \widetilde{B^1}_l$, it is easy to see that

$$S = \{(\mathrm{I}_2, 1), \widetilde{w}, \widetilde{\mathrm{lt}}(x\varpi^r) \mid x \in \{1, \varepsilon\}, 1 \leq r < l\}, \tag{6}$$

where ε is a fixed non-square. For $0 \leq i, j < \underline{n}$, let $\mathcal{H}_{i,j}$ be the Hecke algebra

$$\mathcal{H}_{i,j} := \mathcal{H}(\widetilde{B^1}_l\backslash\widetilde{K^1}/\widetilde{B^1}_l,\ \chi_i, \chi_j) = \{f : \widetilde{K^1} \to \mathbb{C} \mid f(lgh) = \chi_i(l)f(g)\chi_j(h),\ l, h \in \widetilde{B^1}_l, g \in \widetilde{K^1}\}.$$

Proposition 2. *Let $0 \leq i, j < \underline{n}$. Then* $\dim \mathrm{Hom}_{\widetilde{K^1}}(\mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_i, \mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_j) = \dim \mathcal{H}_{i,j}$.

Proof. On the one hand, observe that $\mathrm{Hom}_{\widetilde{K^1}}(\mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_i, \mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_j) = \bigoplus_{\mathtt{x}\in S} \mathrm{Hom}_{\widetilde{B^1}_l}(\mathrm{Ind}_{\widetilde{B^1}_l^{\mathtt{x}^{-1}}\cap\widetilde{B^1}_l}^{\widetilde{B^1}_l}\chi_i^{\mathtt{x}}, \chi_j)$, which by Frobenius reciprocity is equal to $\bigoplus_{\mathtt{x}\in S}\mathrm{Hom}_{\widetilde{B^1}_l^{\mathtt{x}^{-1}}\cap\widetilde{B^1}_l}(\chi_i^{\mathtt{x}}, \chi_j)$. Let $S_{i,j}$ be the set of all $\mathtt{x}\in S$ such that $\chi_i(g) = \chi_j(h)$, whenever $h, g \in \widetilde{B^1}_l$ and $\mathtt{x}g\mathtt{x}^{-1} = h$. Then $\dim \mathrm{Hom}_{\widetilde{K^1}}(\mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_i, \mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_j) = |S_{i,j}|$. On the other hand, observe that for every $\mathtt{x}\in S$, there exists a function $f \in \mathcal{H}_{i,j}$ with support on the double coset represented by $\mathtt{x}$ if and only if $h = \mathtt{x}g\mathtt{x}^{-1}$ implies $\chi_i(g) = \chi_j(h)$ for all $h, g \in \widetilde{B^1}_l$. Moreover, the basis of $\mathcal{H}_{i,j}$ is parametrized by such double coset representatives. Hence, $\dim \mathcal{H}_{i,j} = |S_{i,j}|$. □

Hence, in order to decompose $(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i)^{K_l^1}$, we are interested in counting the dimension of $\mathcal{H}_{i,i}$. Set $(T^1\cap K^1)^2 := \{\mathrm{dg}(t^2) \mid t \in \mathcal{O}^\times\}$, $T_l^1 := \{\iota(t) \mid t \in \mathcal{O}^\times(1+\mathfrak{p}^l)\}$, and $(T_l^1)^2 := \{\iota(t^2) \mid t \in \mathcal{O}^\times(1+\mathfrak{p}^l)\}$. It is not difficult to see that T_l^1 and $(T_l^1)^2$ are subgroups of $(\widetilde{T^1}\cap\widetilde{K^1})K_l^1$.

Proposition 3. *Let $l \geq m$ and $0 \leq i < \underline{n}$. Then* $\dim\mathcal{H}_{i,i} = \begin{cases} 1 + 2(l-m), & \text{if } \chi_i|_{(T^1\cap K^1)^2} \neq 1; \\ 2l, & \text{otherwise.}\end{cases}$

Proof. Assume $l \geq m$. Note that $f(\mathtt{b}\mathtt{k}\mathtt{b}') = \chi_i(\mathtt{b})f(\mathtt{k})\chi_i(\mathtt{b}')$ for all $f \in \mathcal{H}_{i,i}$, $\mathtt{b}, \mathtt{b}' \in \widetilde{B^1}_l$ and $\mathtt{k} \in \widetilde{K^1}$. Hence, for every double coset representative $\mathtt{x}$ in (6), there exists a function $f \in \mathcal{H}_{i,i}$, with support on the double coset represented by $\mathtt{x}$ if and only if $\mathtt{b}\mathtt{x}\mathtt{b}' = \mathtt{x}$ implies that $\chi_i(\mathtt{b}\mathtt{b}') = 1$ for all $\mathtt{b}, \mathtt{b}' \in \widetilde{B^1}_l$. The set of such double cosets parameterizes a basis for $\mathcal{H}_{i,i}$. We now determine these double cosets. Let $\mathtt{b} = (\mathbf{b}, \zeta) = \left(\left(\begin{smallmatrix} t & s \\ 0 & t^{-1}\end{smallmatrix}\right), \zeta\right)$, and $\mathtt{b}' = (\mathbf{b}', \zeta') = \left(\left(\begin{smallmatrix} t' & s' \\ 0 & t'^{-1}\end{smallmatrix}\right), \zeta'\right)$, where $t, t' \in \mathcal{O}^\times(1+\mathfrak{p}^l)$, $s, s' \in \mathfrak{p}^l$ and $\zeta, \zeta' \in \mu_n$ denote arbitrary elements of $\widetilde{B^1}_l$.

- **The identity coset $\widetilde{B^1}_l$:** A function $f \in \mathcal{H}_{i,i}$ has support on $\widetilde{B^1}_l$ if and only if $f(\mathtt{b}) = \chi_i(\mathtt{b}), \forall \mathtt{b} \in \widetilde{B^1}_l$. So there is always a function with support on the identity coset, namely $f = \chi_i$.
- **The coset of $\widetilde{w}$:** For $\mathtt{b}$ and $\mathtt{b}'$ in $\widetilde{B^1}_l$, $\mathtt{b}\widetilde{w}\mathtt{b}' = \widetilde{w}$ implies, via a quick calculation, that $\mathbf{b} = \mathbf{b}' = \mathrm{dg}(t)$, for some $t \in \mathcal{O}^\times(1+\mathfrak{p}^l)$ and $\zeta' = \zeta^{-1}$. Therefore, $\chi_i(\mathtt{b}\mathtt{b}') = \chi_i\left((\mathrm{dg}(t), \zeta)(\mathrm{dg}(t), \zeta^{-1})\right) = \chi_i\left(\mathrm{dg}(t^2), (t,t)_n\right) = \chi_i\left(\mathrm{dg}(t^2), 1\right)$. So, $\mathcal{H}_{i,i}$ contains a function with support on this coset if and only if $\chi_i(\iota(t^2)) = 1$ for all $t \in \mathcal{O}^\times(1+\mathfrak{p}^l)$; that is if and only if $\chi_i|_{(T_l^1)^2} = 1$. Observe that for $0 \leq i < \underline{n}$, $\chi_i|_{(T_l^1)^2} = 1$, where $l \geq m$, if and only if $\chi_i|_{(T^1\cap K^1)^2} = 1$. Suppose $\chi_i|_{(T^1\cap K^1)^2} = 1$, for some $0 \leq i < \underline{n}$. We show that in this case, $m = 1$. Suppose $\alpha \in 1+\mathfrak{p}$, consider $f(X) = X^2 - \alpha$. Observe that $f(1) = 0 \mod \mathfrak{p}$, and $f'(1) = 2(1) \neq 0 \mod p$. By Hensel's lemma, $f(X)$ has a root in $\mathcal{O}$; that is $\alpha \in \mathcal{O}^{\times 2}$. Therefore $1+\mathfrak{p} \subset \mathcal{O}^{\times 2}$, which implies $\chi_i|_{\widetilde{T^1}\cap K_1^1} = 1$, so $m = 1$.
- **The coset of $\widetilde{\mathrm{lt}}(x\varpi^r)$:** For $\mathtt{b}$ and $\mathtt{b}'$ in $\widetilde{B^1}_l$, $\mathtt{b}\,\widetilde{\mathrm{lt}}(x\varpi^r)\mathtt{b}' = \widetilde{\mathrm{lt}}(x\varpi^r)$ implies that $tt' \in 1+\mathfrak{p}^r$ and $\zeta = \zeta'^{-1}$. Therefore, $\chi_i(\mathtt{b}\mathtt{b}') = \chi_i(\mathbf{b}\mathbf{b}', 1) = \chi_i(\left(\begin{smallmatrix} tt' & ts'+st'^{-1} \\ 0 & t^{-1}t'^{-1}\end{smallmatrix}\right), 1)$. Note that $\left(\begin{smallmatrix} tt' & ts'+st'^{-1} \\ 0 & t^{-1}t'^{-1}\end{smallmatrix}\right) \in \widetilde{B^1}\cap K_r^1$. Hence, $\chi_i(\mathtt{b}\mathtt{b}') = 1$ if and only if $\widetilde{B^1}\cap K_r^1 \subseteq \ker(\chi_i)$. The latter holds if and only if $r \geq m$, since χ_i is primitive mod m.

Now, let us summarize our result. There is always one function with support on the identity coset, and $2(l-m)$ functions on cosets represented by $\widetilde{\mathrm{lt}}(x\varpi^r)$, $x \in \{1, \varepsilon\}$, $m \leq r < l$. If $\chi_i|_{(T^1\cap K^1)^2} \neq 1$, no function in $\mathcal{H}_{i,i}$ has support on the double coset represented by $\widetilde{w}$, otherwise, there exists an additional function in $\mathcal{H}_{i,i}$ with support on the double coset represented by $\widetilde{w}$. □

Next two lemmas elaborate on the condition $\chi_i|_{(T^1\cap K^1)^2} = 1$ that appears in Proposition 3.

Lemma 5. *For each $0 \leq i < \underline{n}$, $\chi_i|_{(T^1\cap K^1)^2} = 1$ if and only if $\chi_0|_{(T^1\cap K^1)^2} = \epsilon\circ\vartheta_{\mathcal{O}^{\times 2}}^{-2i}$.*

Proof. Let $\iota(s) \in (T^1 \cap K^1)^2$, so $s \in \mathcal{O}^{\times 2}$. By Lemma 3, $\chi_i(\iota(s)) = \chi_0\left(\mathrm{dg}(s), \vartheta(s)^{2i}\right) = \chi_0(\iota(s))\epsilon(\vartheta(s)^{2i})$, which is equal to 1 if and only if $\chi_0|_{(T^1\cap K^1)^2} = \epsilon \circ \vartheta^{-2i}_{\mathcal{O}^{\times 2}}$. ☐

Lemma 6. *If $4 \nmid n$ then the characters $\vartheta^{-2i}_{\mathcal{O}^{\times 2}}$, $0 \le i < \underline{n}$ are distinct. Otherwise, the $\vartheta^{-2i}_{\mathcal{O}^{\times 2}}$, $0 \le i < \frac{n}{4}$, are distinct; for $\frac{n}{4} \le i < \frac{n}{2}$, $\vartheta^{-2i}_{\mathcal{O}^{\times 2}} = \vartheta^{-2(i-\frac{n}{4})}_{\mathcal{O}^{\times 2}}$.*

Proof. By definition of ϑ in (4), $\vartheta^{-2i}(s) = 1$ for all $s \in \mathcal{O}^{\times 2}$ if and only if $\overline{t^2}^{\frac{(q-1)2i}{n}} = 1$ for all $t \in \mathcal{O}^\times$, or equivalently when $n|4i$. Therefore, the equality holds only for $i = 0$ unless $4|n$, in which case the equality holds for both $i = 0$ and $i = \frac{n}{4}$. ☐

For $l > m$, let $\widetilde{W}_{i,l}$ denote the l-level representations $\widetilde{W}_{i,l} := (\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i)^{K^1_l}/(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i)^{K^1_{l-1}}$. Moreover, for $0 \le i < \underline{n}$, set $\widetilde{V}_i := \mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i$.

Corollary 1. *Assume $l \ge m$. We can decompose $\mathrm{Res}_{\widetilde{K^1}}\mathrm{Ind}_{\widetilde{B^1}}^{\widetilde{G^1}}\rho$ as follows:*

$$\mathrm{Res}_{\widetilde{K^1}}\mathrm{Ind}_{\widetilde{B^1}}^{\widetilde{G^1}}\rho \cong \bigoplus_{i=0}^{\underline{n}-1}\left(\widetilde{V}_i^{K^1_m} \oplus \bigoplus_{l>m}\left(\widetilde{W}^+_{i,l} \oplus \widetilde{W}^-_{i,l}\right)\right),$$

where $\widetilde{W}^+_{i,l} \oplus \widetilde{W}^-_{i,l} \cong \widetilde{W}_{i,l}$. All the pieces are irreducible, except when $m = 1$ and $\chi_0|_{(T^1\cap K^1)^2} = \epsilon \circ \vartheta^{-2i}_{\mathcal{O}^{\times 2}}$ for some $0 \le i < \underline{n}$, in which case, we are in one of the following situations:

1. *If $4 \nmid n$ then there is exactly one $0 \le i < \underline{n}$ for which $\widetilde{V}_i^{K^1_1}$ decomposes into two irreducible constituents. All other constituents are irreducible.*
2. *If $4|n$ then there are exactly two $0 \le i, k < \underline{n}$, $|i - k| = \frac{n}{4}$ for which $\widetilde{V}_i^{K^1_1}$ decomposes into two irreducible constituents. All other constituents are irreducible.*

Proof. It follows from Lemma 4 and Proposition 3 that for $l > m$, $\dim \mathrm{Hom}(\widetilde{W}_{i,l}, \widetilde{W}_{i,l}) = 2$. Hence, $\widetilde{W}_{i,l}$ decomposes into two inequivalent irreducible subrepresentations. Moreover,

$$\dim \mathrm{Hom}(\widetilde{V}_i^{K^1_m}, \widetilde{V}_i^{K^1_m}) = \begin{cases} 1, & \text{if } \chi_i\big|_{(T^1\cap K^1)^2} \ne 1 \\ 2, & \text{otherwise.}\end{cases} \tag{7}$$

By Lemma 5, $\chi_i\big|_{(T^1\cap K^1)^2} = 1$ is equivalent to $\chi_0|_{(T^1\cap K^1)^2} = \epsilon \circ \vartheta^{-2i}_{\mathcal{O}^{\times 2}}$, which also implies that $m = 1$. Hence, $\widetilde{V}_i^{K^1_m}$ is irreducible except when $m = 1$ and $\chi_0|_{(T^1\cap K^1)^2} = \epsilon \circ \vartheta^{-2i}_{\mathcal{O}^{\times 2}}$, where it decomposes into two irreducible constituents. If the latter is the case, by Lemma 6, there is exactly one $0 \le i < \underline{n}$ satisfying $\chi_0|_{(T^1\cap K^1)^2} = \epsilon \circ \vartheta^{-2i}_{\mathcal{O}^{\times 2}}$ if $4 \nmid n$, and there are exactly two $0 \le i < \underline{n}$ satisfying $\chi_0|_{(T^1\cap K^1)^2} = \epsilon \circ \vartheta^{-2i}_{\mathcal{O}^{\times 2}}$ if $4|n$. ☐

Next we determine the multiplicity of each constituent in the decomposition in Corollary 1. To do so, we count the dimension of $\mathrm{Hom}_{\widetilde{K^1}}\left(\mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_k, \mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}}\chi_i\right)$, which is equal to the dimension of the Hecke algebra $\mathcal{H}_{k,i} = \mathcal{H}(\widetilde{B^1}_l\backslash\widetilde{K^1}/\widetilde{B^1}_l, \chi_k, \chi_i)$.

Proposition 4. *Let $l \ge m$, $0 \le k, i < \underline{n}$, and $i \ne k$. Then*

$$\dim \mathcal{H}_{k,i} = \begin{cases} 2l - 1, & \textit{if } \chi_0|_{(T^1\cap K^1)^2} = \epsilon \circ \vartheta^{-(k+i)}_{\mathcal{O}^{\times 2}} \\ 2(l - m), & \textit{otherwise.}\end{cases}$$

Proof. Similar to the proof of Proposition 3, we determine which double cosets in $\widetilde{B^1}_l\backslash\widetilde{K^1}/\widetilde{B^1}_l$ support a function in $\mathcal{H}_{k,i}$. For every double coset representative $\mathtt{x}$ in Lemma (6), there exists a function $f \in \mathcal{H}_{k,i}$ with support on the double coset represented by $\mathtt{x}$ if and only if $\mathtt{b}\mathtt{x}\mathtt{b}' = \mathtt{x}$, $\mathtt{b},\mathtt{b}' \in \widetilde{B^1}_l$, implies that $\chi_k(\mathtt{b})\chi_i(\mathtt{b}') = 1$. Let $t, t' \in \mathcal{O}^\times(1+\mathfrak{p}^l)$, $s, s' \in \mathfrak{p}^l$ and $\zeta, \zeta' \in \mu_n$, so that $\mathtt{b} = (\mathbf{b}, \zeta) = \left(\left(\begin{smallmatrix} t & s \\ 0 & t^{-1}\end{smallmatrix}\right), \zeta\right)$ and $\mathtt{b}' = (\mathbf{b}', \zeta') = \left(\left(\begin{smallmatrix} t' & s' \\ 0 & t'^{-1}\end{smallmatrix}\right), \zeta'\right)$ are arbitrary elements of $\widetilde{B^1}_l$.

Because $\chi_k \neq \chi_i$, there is no function in $\mathcal{H}_{k,i}$ with support on the identity double coset.

For the double coset of $\widetilde{w}$, $\mathtt{b}\widetilde{w}\mathtt{b}' = \widetilde{w}$ implies that $\mathbf{b} = \mathbf{b}' = \mathrm{dg}(t)$, for some $t \in \mathcal{O}^\times(1+\mathfrak{p}^l)$ and $\zeta' = \zeta^{-1}$. Therefore, $\chi_k(\mathtt{b})\chi_i(\mathtt{b}') = \chi_k\left(\mathrm{dg}(t), \zeta\right)\chi_i\left(\mathrm{dg}(t), \zeta^{-1}\right)$ equals

$$\begin{aligned}\chi_0\left(\mathrm{dg}(t), \vartheta(t)^{2k}\zeta\right)\chi_0\left(\mathrm{dg}(t), \vartheta(t)^{2i}\zeta^{-1}\right) &= \chi_0\left(\mathrm{dg}(t^2), \vartheta(t)^{2(k+i)}\right) = \chi_0\left(\iota(t^2)(\mathrm{I}_2, \vartheta(t^2)^{k+i})\right) \\ &= \chi_0\left(\iota(t^2)\right)\epsilon\left(\vartheta(t^2)^{k+i}\right).\end{aligned}$$

Therefore, because $l \geq m$, $\chi_k(\mathtt{b})\chi_i(\mathtt{b}') = 1$ if and only if $\chi_0|_{(T^1\cap K^1)^2} = \epsilon \circ \vartheta_{\mathcal{O}^{\times 2}}^{-(k+i)}$. In this case, $m = 1$ and $\widetilde{w}$ supports a function in $\mathcal{H}_{k,i}$.

Finally, for the double cosets represented by $\widetilde{\mathrm{lt}}(x\varpi^r)$, $x \in \{1, \varepsilon\}$, $1 \leq r < l$, $\mathtt{b}\,\widetilde{\mathrm{lt}}(x\varpi^r)\mathtt{b}' = \widetilde{\mathrm{lt}}(x\varpi^r)$ implies that $\zeta' = \zeta^{-1}$, and $t + s\varpi^r = t'^{-1} \mod \mathfrak{p}^l$, or equivalently, $t = t'^{-1} \mod \mathfrak{p}^r$, and $t^{-1}\varpi^r = \varpi^r t'^{-1} \mod \mathfrak{p}^l$, or equivalently $t^{-1} = t'^{-1} \mod \mathfrak{p}^{l-r}$. Observe that, in general, $\chi_k(\mathtt{b})\chi_i(\mathtt{b}')$ is equal to

$$\begin{aligned}\chi_k\left(\mathrm{dg}(t), \zeta\right)\chi_i\left(\mathrm{dg}(t'), \zeta'\right) &= \chi_0\left(\mathrm{dg}(t), \vartheta(t)^{2k}\zeta\right)\chi_0\left(\mathrm{dg}(t'), \vartheta(t')^{2i}\zeta'\right) = \chi_0\left(\mathrm{dg}(tt'), \vartheta(t)^{2k}\vartheta(t')^{2i}\zeta\zeta'\right) \\ &= \chi_0\left(\iota(tt')\right)\epsilon\left(\vartheta(t)^{2k}\vartheta(t')^{2i}\zeta\zeta'\right).\end{aligned} \tag{8}$$

Note that ϑ is primitive mod one. Observe that $r \geq 1$ and $l - r \geq 1$. Therefore, $t = t'^{-1} \mod \mathfrak{p}$ and $t = t' \mod \mathfrak{p}$, which implies that $t = t' = \alpha \mod \mathfrak{p}$ where $\alpha \in \{\pm 1\}$. Hence, $\vartheta(t)^2 = \vartheta(t')^2 = 1$, and (8) simplifies to $\chi_0\left(\iota(tt')\right)\epsilon(\zeta\zeta')$. We are in one of the following situations:

- **Case 1:** Suppose $r \geq m$. Then we have $\zeta' = \zeta^{-1}$, and $t = t'^{-1} \mod \mathfrak{p}^m$; that is $tt' \in 1 + \mathfrak{p}^m$. Hence, $\chi_0\left(\iota(tt')\right)\epsilon(\zeta\zeta') = \chi_0(tt') = 1$, because χ_0 is primitive mod m. Therefore, in this case, there is always a function in $\mathcal{H}_{k,i}$ with support on these double cosets.
- **Case 2:** Suppose $r < m$. Then $\zeta' = \zeta^{-1}$, so $\chi_0\left(\iota(tt')\right)\epsilon(\zeta\zeta') = \chi_0(tt')$, which equals one if and only if $tt' \in 1 + \mathfrak{p}^m$, which is not the case in general. Hence, in this case, there is no function in $\mathcal{H}_{k,i}$ with support on these double cosets.

To summarize the result, the coset represented by $\widetilde{w}$ supports a function in $\mathcal{H}_{k,i}$ if and only if $\chi_0|_{(T^1\cap K^1)^2} = \epsilon \circ \vartheta_{\mathcal{O}^{\times 2}}^{-(k+i)}$. If $r \geq m$ then the cosets represented by $\mathrm{lt}(x\varpi^r)$ support a function in $\mathcal{H}_{k,i}$; otherwise, there is no function in $\mathcal{H}_{k,i}$ with support on these double cosets. □

Corollary 2. *In the decomposition of* $\mathrm{Res}_{\widetilde{K^1}}\mathrm{Ind}_{\widetilde{B^1}}^{\widetilde{G^1}}\rho$ *given in Corollary 1,*

1. *For each* $0 \leq i < \underline{n}$ *and* $l > m$, *there exists a way of decomposing* $\widetilde{W}_{i,l}$ *as* $\widetilde{W}_{i,l}^+ \oplus \widetilde{W}_{i,l}^-$ *such that for* $l > m$, $\widetilde{W}_{i,l}^+ \cong \widetilde{W}_{j,l}^+$ *and* $\widetilde{W}_{i,l}^- \cong \widetilde{W}_{j,l}^-$ *for all* $0 \leq i, j < \underline{n}$.
2. *For* $l = m$, $\{(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_i)^{K_m^1} \mid 0 \leq i < \underline{n}\}$ *consists of mutually inequivalent representations, except when* $m = 1$ *and* $\chi_0|_{(T^1\cap K^1)^2} = \epsilon \circ \vartheta_{\mathcal{O}^{\times 2}}^{-j}$, *for some* $0 \leq j < \underline{n}$, *where* $\widetilde{V}_i^{K_1^1} \cong \widetilde{V}_k^{K_1^1}$, *exactly when* $i + k \equiv j \mod \underline{n}$.

Proof. It follows from Proposition 4 that for $l > m$, $\dim \operatorname{Hom}_{\widetilde{K^1}}\left(\widetilde{W}_{i,l}, \widetilde{W}_{k,l}\right) = 2$, and when $i+k \equiv j \mod \underline{n}$

$$\dim \operatorname{Hom}_{\widetilde{K^1}}\left(\widetilde{V}_i^{K_m^1}, \widetilde{V}_k^{K_m^1}\right) = \begin{cases} 1, & \chi_0|_{(T^1\cap K^1)^2} = \epsilon \circ \vartheta_{\mathcal{O}^{\times 2}}^{-j} \\ 0, & \text{otherwise,} \end{cases}$$

and hence the result. □

In order to further investigate the irreducible spaces $\widetilde{W}_{i,l}^+$ and $\widetilde{W}_{i,l}^-$, we will show that $\widetilde{W}_{i,l}$, $0 \le i < \underline{n}$, is the restriction to $\widetilde{K^1}$ of an irreducible representation of the maximal compact subgroup $\widetilde{K}$ of the covering group $\widetilde{G}$ of $\mathrm{GL}_2(\mathbb{F})$.

5. Branching Rules for $\widetilde{G}$

We define the genuine principal series representations of $\widetilde{G}$ similarly by starting with a genuine smooth irreducible representation ρ' of $\widetilde{T}$ with the central character χ', which is constructed via the Stone-von Neumann theorem. Observe that $\dim \rho' = [\widetilde{T} : A] = n^2$. Then, after extending ρ' trivially over N, the genuine principal series representation π' of $\widetilde{G}$ is $\operatorname{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho'$. Applying a similar machinery as in Section 4, we obtain the K-type decomposition for $\operatorname{Res}_{\widetilde{K}} \pi'$. Since the argument in Section 4 goes through almost exactly, here we only overview the main steps and point out the differences. For detailed calculations, see [8].

Similar to Lemma 3, it follows that $\operatorname{Res}_A \rho' \cong \bigoplus_{i,j=0}^{n-1} \chi'_{i,j}$, where the $\chi'_{i,j}$ denote n^2 distinct characters of A, defined by $\chi'_{i,j}(\mathrm{dg}(a\varpi^{un}, b\varpi^{vn}), \zeta) = \chi'_0\left(\mathrm{dg}(a\varpi^{un}, b\varpi^{vn}), \vartheta(a)^{-j}\vartheta(b)^{-i}\zeta\right)$ where $a, b \in \mathcal{O}^\times$, $u, v \in \mathbb{Z}$ and $\zeta \in \mu_n$ and $\vartheta(a) = (\varpi, a)_n$ was defined in (4), and χ'_0 is a fixed extension of χ' to A. The $\chi'_{i,j}$ remain distinct when restricted to $\widetilde{T} \cap \widetilde{K}$, and again writing $\chi'_{i,j}$ for there restrictions, $\operatorname{Res}_{\widetilde{T}\cap\widetilde{K}} \rho' \cong \bigoplus_{i,j=0}^{n-1} \chi'_{i,j}$. Then similar to Proposition 1, we have $\operatorname{Res}_{\widetilde{K}}(\operatorname{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho') \cong \bigoplus_{i,j=0}^{n-1} \operatorname{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}} \chi'_{i,j}$. This latter isomorphism reduces the problem of decomposing the K-type to the one of decomposing each $\operatorname{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}} \chi'_{i,j}$, which, by smoothness, can be written as the union of its K_l, $l \ge 1$, fixed points.

Suppose χ' is primitive mod m. It follows that the $\chi'_{i,j}$ are also primitive mod m. Set $\widetilde{B}_l = (\widetilde{B} \cap \widetilde{K})K_l$. It can be seen that each level l representation $\left(\operatorname{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}} \chi'_{i,j}\right)^{K_l} = \operatorname{Ind}_{\widetilde{B}_l}^{\widetilde{K}} \chi'_{i,j}$ if $l \ge m$, and is zero if $l < m$. Similar to Proposition 2, one can see that $\dim \operatorname{Hom}_{\widetilde{K}}(\operatorname{Ind}_{\widetilde{B}_l}^{\widetilde{K}} \chi'_{i,j}, \operatorname{Ind}_{\widetilde{B}_l}^{\widetilde{K}} \chi'_{i,j}) = \dim \mathcal{H}'_{i,j}(\widetilde{B}_l \backslash \widetilde{K} / \widetilde{B}_l, \chi'_{i,j}, \chi'_{i,j})$. We count the dimension of $\mathcal{H}'_{i,j}$ using a method similar to the one we used in Proposition 3. To do so, we need to calculate a set of double coset representatives of $\widetilde{B}_l$ in $\widetilde{K}$.

Lemma 7. *A complete set of double coset representatives of $\widetilde{B}_l$ in $\widetilde{K}$ is given by $\{(\mathrm{I}_2, 1), \widetilde{w}, \widetilde{\mathrm{lt}}(\varpi^r) \mid 1 \le r < l\}$.*

Proof. Note that this set is a subset of the set S in (6). Observe that under the isomorphism

$$\mathbb{F}^\times \ltimes \widetilde{G^1} \cong \widetilde{G}, \quad (y, (\mathbf{g}, \zeta)) \mapsto (\mathrm{dg}(1, y)\mathbf{g}, \zeta)\,, \tag{9}$$

$\mathcal{O}^\times \times \widetilde{K^1}$ maps to $\widetilde{K}$ and $\mathcal{O}^\times \times \widetilde{B^1}_l$ maps to $\widetilde{B}_l$. For every $\mathtt{k}' \in \widetilde{K}$, let $(y, \mathtt{k})$ be the inverse image of $\mathtt{k}'$ under the isomorphism (9), and let $\mathtt{b}_1, \mathtt{b}_2 \in \widetilde{B^1}_l$ be such that $\mathtt{b}_1\mathtt{x}\mathtt{b}_2 = \mathtt{k}$, for some $\mathtt{x} \in S$. Let $\mathtt{b}'_1$ and $\mathtt{b}'_2$ be the image of $(y, \mathtt{b}_1)$ and $(y, \mathtt{b}_2)$ under (9) respectively. It follows from the multiplication of $\mathbb{F}^\times \ltimes \widetilde{G^1}$ and the isomorphism map (9), that $\mathtt{b}'_1\mathtt{x}\mathtt{b}'_2 = \mathtt{k}'$. Thus, $\widetilde{K} = \bigcup_{\mathtt{x}\in S} \widetilde{B}_l \mathtt{x} \widetilde{B}_l$. A short calculation shows that

$$\left(\mathrm{dg}(\varepsilon^{-1}, 1), 1\right) \widetilde{\mathrm{lt}}(\varpi^r) \left(\mathrm{dg}(\varepsilon, 1), 1\right) = \left(\mathrm{lt}(\varepsilon\varpi^r), (\varpi^r, \varepsilon)_n(\varepsilon, \varpi^r)_n\right) = \widetilde{\mathrm{lt}}(\varepsilon\varpi^r),$$

where ε is a fixed non-square and $1 \leq r < l$. It is not difficult to see that other cosets of S remain distinct in $\widetilde{K}$. ☐

The following proposition can be proved similar to Proposition 3.

Proposition 5. *Let $l \geq m$. Then* $\dim \mathcal{H}'_{i,j} = \begin{cases} 1 + (l-m), & \text{if } \chi'_{i,j}|_{T\cap K} \neq 1; \\ 2 + (l-m), & \text{otherwise.} \end{cases}$

Lemma 8. *For $0 \leq i,j < n$, $\chi'_{i,j}|_{T\cap K} = 1$ if and only if $\chi'_{0,0}|_{T\cap K} = \epsilon \circ \vartheta^{j-i}$.*

Proof. Note that $\chi'_{i,j}(\iota(a)) = \chi'_{0,0}\left(\mathrm{dg}(a), \vartheta^{i-j}(a)\right)$, which is equal to 1 if and only if $\chi'_{0,0}|_{T\cap K} = \epsilon \circ \vartheta^{j-i}$. ☐

For $l > m$, let $\widetilde{W'}_{i,j,l}$ denote the l-level quotient representation $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}} \chi'_{i,j})^{K_l} / (\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}} \chi'_{i,j})^{K_{l-1}}$. The K-type decomposition $\mathrm{Res}_{\widetilde{K}}(\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho')$ is given in the following Corollary.

Corollary 3. *We can decompose $\mathrm{Res}_{\widetilde{K}}(\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho')$ as follows:*

$$\mathrm{Res}_{\widetilde{K}}(\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho') \simeq \bigoplus_{i,j=0}^{n-1} \left((\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}} \chi'_{i,j})^{K_m} \oplus \bigoplus_{l>m} \widetilde{W'}_{i,j,l} \right). \tag{10}$$

If $\chi'_{0,0}|_{T\cap K} \neq \vartheta^k|_{\mathcal{O}^\times}$, for all $0 \leq k < n$, then all the pieces are irreducible. Otherwise, there are exactly n pairs (i,j), $0 \leq i,j < n$, such that $j - i \equiv k \mod n$, and $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}} \chi'_{i,j})^{K_m}$ decomposes into two irreducible constituents. The rest of the constituents are irreducible.

Proof. It follows from Proposition 5 that for $0 \leq i,j < n$, $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}} \chi'_{i,j})^{K_m}$ is irreducible if $\chi'_{0,0}|_{T\cap K} \neq \vartheta|_{\mathcal{O}^\times}^{j-i}$, and decomposes into two inequivalent constituents otherwise. Moreover, for $l > m$, the quotients $\widetilde{W'}_{i,j,l}$ are irreducible. Note that the map $(i,j) \to j - i \mod n$ has a kernel of size n. Hence, if there exists a pair such that $\chi'_{0,0}|_{T\cap K} = \vartheta|_{\mathcal{O}^\times}^{j-i}$, then there are exactly n distinct such pairs. ☐

5.1. **Restriction of $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho'$ to $\widetilde{K^1}$.** Fix a genuine irreducible representation ρ of $\widetilde{T^1}$ with central character χ, where χ is primitive mod m. Let $\widetilde{W}_{k,l}$, $\widetilde{W}^+_{k,l}$, and $\widetilde{W}^-_{k,l}$ be the representations of $\widetilde{K^1}$ that appear in the K-type decomposition of $\mathrm{Res}_{\widetilde{K^1}} \mathrm{Ind}_{\widetilde{B^1}}^{\widetilde{G^1}} \rho$ in Corollary 1. In this section, we show that, for each $0 \leq k < \underline{n}$, $\widetilde{W}_{k,l} \cong \mathrm{Res}_{\widetilde{K^1}} W'$, where W' is some irreducible representation of $\widetilde{K}$. We deduce that $\widetilde{W}^+_{k,l}$ and $\widetilde{W}^-_{k,l}$ have the same dimension.

Let ρ' be a genuine irreducible representation of $\widetilde{T}$ with central character χ', such that depth of χ' is equal to depth of χ, and that ρ appears in $\mathrm{Res}_{\widetilde{T^1}} \rho'$. Let $\chi'_{i,j}$, $0 \leq i,j < n$ be all possible extensions of χ' to A. To find W', we consider the restriction of the principal series representation $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho'$ to $\widetilde{K^1}$. Because the structure of the $\widetilde{T^1}$ depends on the parity of n, we consider the cases for even and odd n separately.

5.1.1. *n odd.* Recall that for odd n we have $Z(\widetilde{T^1}) = \{(\mathrm{dg}(a), \zeta) \mid a \in \mathbb{F}^{\times n}, \zeta \in \mu_n\}$, $A^1 = \{(\mathrm{dg}(a), \zeta) \mid a \in \mathbb{F}^\times, \zeta \in \mu_n,\ n|\mathrm{val}(a)\}$, $\widetilde{T^1} \cap \widetilde{K^1} = \{(\mathrm{dg}(a), \zeta) \mid a \in \mathcal{O}^\times, \zeta \in \mu_n\}$, $Z(\widetilde{T}) = \{(\mathrm{dg}(a,b), \zeta) \mid a, b \in \mathbb{F}^{\times n}, \zeta \in \mu_n\}$, $A = \{(\mathrm{dg}(a,b), \zeta) \mid a, b \in \mathbb{F}^\times, \zeta \in \mu_n, n|\mathrm{val}(a), n|\mathrm{val}(b)\}$, $\widetilde{T} \cap \widetilde{K} = \{(\mathrm{dg}(a,b), \zeta) \mid a, b \in \mathcal{O}^\times, \zeta \in \mu_n\}$. Observe that $Z(\widetilde{T}) \cap \widetilde{T^1} = Z(\widetilde{T^1})$ and $A \cap \widetilde{T^1} = A^1$.

We compute $\mathrm{Res}_{\widetilde{K^1}} \mathrm{Res}_{\widetilde{K}} \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho'$, where the decomposition of $\mathrm{Res}_{\widetilde{K}} \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho'$ is given in Corollary 3. The assumption ρ appears in $\mathrm{Res}_{\widetilde{T^1}} \rho'$ implies $\chi'|_{Z(\widetilde{T^1})} = \chi$. We further assume that the choice of χ_0 is such that $\mathrm{Res}_A \chi'_0 = \chi_0$. In order to study the restriction of each piece in (10), we need to restrict the characters $\chi'_{i,j}$ to $\widetilde{T^1} \cap \widetilde{K^1}$.

Lemma 9. *Assume n is odd. For $0 \le i, j < n$, let k be the integer in $\{0, \cdots, n-1\}$ such that $k \equiv \frac{i-j}{2} \mod n$. Then $\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}} \chi'_{i,j} = \chi_k$.*

Proof. Let $(\mathrm{dg}(u), \zeta) \in \widetilde{T^1} \cap \widetilde{K^1}$. Then $\chi'_{i,j}(\mathrm{dg}(u), \zeta) = \chi'_0\left(\mathrm{dg}(u), \vartheta(u)^{i-j}\zeta\right)$, which by Lemma 3, and because $\chi'_0|_{A^1} = \chi_0$, is equal to $\chi_0\left(\mathrm{dg}(u), \vartheta(u)^{2k}\zeta\right) = \chi_k(\mathrm{dg}(u), \zeta)$. □

The cardinality of the kernel of the map $(i, j) \to k \mod n$, in Lemma 9, is n; that is for each k, there are exactly n distinct characters $\chi'_{i,j}$ of $\widetilde{T} \cap \widetilde{K}$ that restrict to χ_k on $\widetilde{T^1} \cap \widetilde{K^1}$.

Lemma 10. *Assume n is odd. Let i, j and k be in $\{0, \cdots, n-1\}$, such that $\chi'_{i,j}|_{\widetilde{T^1}\cap\widetilde{K^1}} = \chi_k$. Then, for all $l \ge m$, $\mathrm{Res}_{\widetilde{K^1}}\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}} \chi'_{i,j}\right)^{K_l} \cong \left(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}} \chi_k\right)^{K^1_l}$.*

Proof. It is enough to show that $\mathrm{Res}_{\widetilde{K^1}} \mathrm{Ind}_{\widetilde{B}_l}^{\widetilde{K}} \chi'_{i,j} \cong \mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}} \chi_k$. Note that $\widetilde{K^1}\backslash\widetilde{K}/\widetilde{B}_l$ is trivial and $\widetilde{B}_l \cap \widetilde{K^1} = \widetilde{B^1}_l$. So by Mackey's theory, we have $\mathrm{Res}_{\widetilde{K^1}} \mathrm{Ind}_{\widetilde{B}_l}^{\widetilde{K}} \chi'_{i,j} \cong \mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}} \mathrm{Res}_{\widetilde{B^1}_l} \chi'_{i,j}$, which is equal to $\mathrm{Ind}_{\widetilde{B^1}_l}^{\widetilde{K^1}} \chi_k$ by choice of i, j and k. □

5.1.2. *n even.* Recall that for even n, $Z(\widetilde{T^1}) = \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^{\times n/2}, \zeta \in \mu_n\}$, $A^1 = \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^\times, \frac{n}{2}|\mathrm{val}(t), \zeta \in \mu_n\}$, $\widetilde{T^1} \cap \widetilde{K^1} = \{(\mathrm{dg}(t), \zeta) \mid t \in \mathcal{O}^\times, \zeta \in \mu_n\}$, $Z(\widetilde{T}) = \{(\mathrm{dg}(t, s), \zeta) \mid t, s \in \mathbb{F}^{\times n}, \zeta \in \mu_n\}$, $A = \{(\mathrm{dg}(t, s), \zeta) \mid t, s \in \mathbb{F}^\times, n|\mathrm{val}(t), n|\mathrm{val}(s), \zeta \in \mu_n\}$, $\widetilde{T} \cap \widetilde{K} = \{(\mathrm{dg}(t, s), \zeta) \mid t, s \in \mathcal{O}^\times, \zeta \in \mu_n\}$, and therefore, $Z(\widetilde{T}) \cap \widetilde{T^1} = \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^{\times n}, \zeta \in \mu_n\}$, and $A \cap \widetilde{T^1} = \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^\times, n|\mathrm{val}(t), \zeta \in \mu_n\}$.

Unlike the case for odd n, the centre $Z(\widetilde{T})$ and the maximal abelian subgroup A of $\widetilde{T}$ do not restrict to those of $\widetilde{T^1}$ upon restriction to $\widetilde{T^1}$. Observe that $[Z(\widetilde{T^1}) : Z(\widetilde{T}) \cap \widetilde{T^1}] = 4$, $[A^1 : A \cap \widetilde{T^1}] = 2$. This mismatch makes the computation of $\mathrm{Res}_{\widetilde{K^1}} \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho'$ more delicate. Indeed, our assumption that ρ appears in $\mathrm{Res}_{\widetilde{T^1}} \rho'$ does not imply that ρ' is ρ isotypic, upon restriction to $\widetilde{T^1}$. We show that ρ is one of the four distinct irreducible representations of $\widetilde{T^1}$ that appear in $\mathrm{Res}_{\widetilde{T^1}} \rho'$.

Set $\underline{\chi} := \mathrm{Res}_{Z(\widetilde{T})\cap\widetilde{T^1}} \chi'$. Note that $|\underline{n}\mathbb{Z}/n\mathbb{Z}| = |\mathcal{O}^{\times \underline{n}}/\mathcal{O}^{\times n}| = 2$. We denote the coset representatives of the former by $\{e, o\}$. Let L denote the set of coset representatives for $Z(\widetilde{T^1})/(Z(\widetilde{T}) \cap \widetilde{T^1})$, so $|L| = 4$. The representation $\mathrm{Ind}_{Z(\widetilde{T})\cap\widetilde{T^1}}^{Z(\widetilde{T^1})} \underline{\chi}$ decomposes into 4 distinct characters ${}_\ell\chi$:

$$\mathrm{Ind}_{Z(\widetilde{T})\cap\widetilde{T^1}}^{Z(\widetilde{T^1})} \underline{\chi} = \bigoplus_{\ell \in L} {}_\ell\chi. \tag{11}$$

We denote the irreducible genuine representation of $\widetilde{T^1}$ with central character ${}_\ell\chi$ by ρ_ℓ.

Proposition 6. *Assume n is even. Let ${}_\ell\chi$, $\ell \in L$ be as in (11). Then $\mathrm{Res}_{\widetilde{T^1}} \rho' = \bigoplus_{\ell \in L} \left[(\rho_\ell)^{\oplus n/2}\right]$, where ρ_ℓ are mutually inequivalent and $\rho \cong \rho_\ell$ for some $\ell \in L$.*

Proof. Note that $X = \{\left(\mathrm{dg}(1, \varpi^j), 1\right) \mid 0 \le j < n\}$ is a system of coset representatives for $\widetilde{T^1}\backslash\widetilde{T}/A$, and that A is stable under conjugation by $\mathtt{x} \in X$. Moreover, it is not difficult to see that for $\mathtt{x} = \left(\mathrm{dg}(1, \varpi^j), 1\right)$,

${\chi'_0}^{\mathtt{x}} = \chi'_{0,j}$. Therefore, by Mackey's theory,

$$\mathrm{Res}_{\widetilde{T^1}}\rho' = \bigoplus_{\mathtt{x}\in X}\left(\mathrm{Ind}_{(\widetilde{T^1}\cap A^{\mathtt{x}})}^{\widetilde{T^1}}{\chi'_0}^{\mathtt{x}}\right) = \bigoplus_{j=0}^{n-1}\mathrm{Ind}_{A^1}^{\widetilde{T^1}}\left(\mathrm{Ind}_{\widetilde{T^1}\cap A}^{A^1}\chi'_{0,j}\right).$$

Observe that $[A^1 : \widetilde{T^1}\cap A] = 2$, with coset representatives $\{e, o\}$. Therefore, for every $0 \le j < n$, $\mathrm{Ind}_{\widetilde{T^1}\cap A}^{A^1}\chi'_{0,j}$ is a 2-dimensional representation of the abelian group A^1 and hence decomposes into direct sum of two characters: ${}_e\chi'_j \oplus {}_o\chi'_j$.

Next, we show that the elements of the set $\{{}_e\chi'_j, {}_o\chi'_j \mid 0 \le j < n\}$ are distinct. Note that for $0 \le j < n$, $\mathrm{Res}_{\widetilde{T^1}\cap A}\mathrm{Ind}_{\widetilde{T^1}\cap A}^{A^1}\chi'_{0,j} \cong \chi'_{0,j} \oplus \chi'_{0,j}$. Suppose $0 \le i, j < n$, by Frobenius reciprocity

$$\mathrm{Hom}_{A^1}\left(\mathrm{Ind}_{\widetilde{T^1}\cap A}^{A^1}\chi'_{0,j}, \mathrm{Ind}_{\widetilde{T^1}\cap A}^{A^1}\chi'_{0,i}\right) = \mathrm{Hom}_{\widetilde{T^1}\cap A}\left(\mathrm{Res}_{\widetilde{T^1}\cap A}\mathrm{Ind}_{\widetilde{T^1}\cap A}^{A^1}\chi'_{0,j}, \chi'_{0,i}\right) = \mathrm{Hom}_{\widetilde{T^1}\cap A}\left(\chi'_{0,j}\oplus\chi'_{0,j}, \chi'_{0,i}\right).$$

We can easily see that $\chi'_{0,j}$ and $\chi'_{0,i}$ coincide on $\widetilde{T^1}\cap A$ if and only if $i = j$. Whence,

$$\dim \mathrm{Hom}_{A^1}\left(\mathrm{Ind}_{\widetilde{T^1}\cap A}^{A^1}\chi'_{0,j}, \mathrm{Ind}_{\widetilde{T^1}\cap A}^{A^1}\chi'_{0,i}\right) = \begin{cases} 2, & i = j \\ 0, & \text{otherwise.}\end{cases}$$

Therefore, the elements of $\{{}_e\chi'_j, {}_o\chi'_j \mid 0 \le j < n\}$ are $2n$ distinct characters of A, which because $[A : Z(\widetilde{T^1})] = n/2$, implies that they restrict to, at least 4, distinct characters upon restriction to $Z(\widetilde{T^1})$. Moreover, because ρ appears in $\mathrm{Res}_{\widetilde{T^1}}\rho'$, at least one of these 4 central characters is χ. Observe that, for $0 \le j < n$, and $\alpha \in \{e, o\}$, $\mathrm{Res}_{Z(\widetilde{T})\cap\widetilde{T^1}}\ {}_\alpha\chi'_j = \underline{\chi}$.

Consider $\mathrm{Ind}_{Z(\widetilde{T})\cap\widetilde{T^1}}^{A^1}\underline{\chi} = \mathrm{Ind}_{Z(\widetilde{T^1})}^{A^1}\mathrm{Ind}_{Z(\widetilde{T})\cap\widetilde{T^1}}^{Z(\widetilde{T^1})}\underline{\chi} = \mathrm{Ind}_{Z(\widetilde{T^1})}^{A^1}\bigoplus_{\ell\in L}{}_\ell\chi = \bigoplus_{\ell\in L, 0\le k<n/2}{}_\ell\chi_k$. Observe that, the ${}_\ell\chi_k$, are $2n$ distinct characters that restrict to $\underline{\chi}$ on $Z(\widetilde{T})\cap\widetilde{T^1}$, and exhaust every such character. Hence, the sets $\{{}_e\chi'_{0,j}, {}_o\chi'_{0,j} \mid 0 \le j < n\}$ and $\{{}_\ell\chi_k \mid \ell \in L, 0 \le k < n/2\}$ are equal. In particular, $\mathrm{Res}_{\widetilde{T^1}}\rho' \cong \mathrm{Ind}_{A^1}^{\widetilde{T^1}}\bigoplus_{0\le j<n}{}_e\chi'_j \oplus {}_o\chi'_j = \mathrm{Ind}_{A^1}^{\widetilde{T^1}}\left(\bigoplus_{\ell\in L, 0\le k<n/2}{}_\ell\chi_k\right) \cong \bigoplus_{\ell\in L}\rho_\ell^{\oplus\frac{n}{2}}$. The last equality is because the ${}_\ell\chi_k$ extend ${}_\ell\chi$. Moreover, ρ_ℓ are mutually inequivalent because ${}_\ell\chi$ are mutually inequivalent. Finally, because $\chi = {}_\ell\chi$ for some $\ell \in L$, $\rho \cong \rho_\ell$ for some $\ell \in L$. □

We compute $\mathrm{Res}_{\widetilde{K^1}}\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho'$. First, we need to study $\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\chi'_{i,j}$.

Lemma 11. *Let* $\chi'_{i,j}$, $0 \le i, j < n$ *be as in* (10). *Then* $\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\chi'_{i,j}(\mathrm{dg}(t), \zeta) = \chi'_{0,0}\left(\mathrm{dg}(t), \vartheta(t)^{i-j}\zeta\right)$, *for all* $(\mathrm{dg}(t), \zeta) \in \widetilde{T^1}\cap\widetilde{K^1}$.

Proof. Let $(\mathrm{dg}(t)\zeta) \in \widetilde{T^1}\cap\widetilde{K^1}$. Then

$$\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\chi'_{i,j}(\mathrm{dg}(t), \zeta) = \chi'_{0,0}\left(\mathrm{dg}(t), \vartheta(t)^{-j}\vartheta(t)^{i}\zeta\right) = \chi'_{0,0}\left(\mathrm{dg}(t), \vartheta(t)^{i-j}\zeta\right).$$

□

Therefore, $\{\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\chi'_{i,j} \mid 0 \le i, j < n\}$ consists of n distinct characters of $\widetilde{T^1}\cap\widetilde{K^1}$. In the next lemma and proposition, we realize these characters as characters of $\widetilde{T^1}\cap\widetilde{K^1}$ that come from central characters ${}_\ell\chi$, $\ell \in L$, of $Z(\widetilde{T^1})$.

Lemma 12. *Each* $\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\chi'_{i,j}$ *appears exactly twice in* $\bigoplus_{\ell\in L, 0\le k<\frac{n}{2}}\ {}_\ell\chi_k$.

Proof. Note that $\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\mathrm{Ind}_{Z(\widetilde{T})\cap\widetilde{T^1}}^{A^1}\underline{\chi} = \mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\left(\bigoplus_{\ell\in L,0\leq k<\frac{n}{2}}\ell\chi_k\right)$. Consider

$$\mathrm{Hom}_{\widetilde{T^1}\cap\widetilde{K^1}}\left(\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\chi'_{i,j}, \mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\mathrm{Ind}_{Z(\widetilde{T})\cap\widetilde{T^1}}^{A^1}\underline{\chi}\right). \tag{12}$$

Observe that $\widetilde{T^1}\cap\widetilde{K^1}\backslash A^1/Z(\widetilde{T})\cap\widetilde{T^1}\cong \underline{n}\mathbb{Z}/n\mathbb{Z}$. So, by Mackey's theory and Frobenius reciprocity (12) is

$$\mathrm{Hom}_{\widetilde{T^1}\cap\widetilde{K^1}}\left(\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\chi'_{i,j}, \left(\mathrm{Ind}_{Z(\widetilde{T})\cap\widetilde{K^1}}^{\widetilde{T^1}\cap\widetilde{K^1}}\underline{\chi}\right)^{\oplus 2}\right)\cong \mathrm{Hom}_{Z(\widetilde{T})\cap\widetilde{K^1}}\left(\mathrm{Res}_{Z(\widetilde{T})\cap\widetilde{K^1}}\chi'_{i,j}, \mathrm{Res}_{Z(\widetilde{T})\cap\widetilde{K^1}}\underline{\chi}^{\oplus 2}\right).$$

Because $\mathrm{Res}_{Z(\widetilde{T^1})}\chi' = \underline{\chi}$, for all $0\leq i,j<n$, $\mathrm{Res}_{Z(\widetilde{T})\cap\widetilde{K^1}}\chi'_{i,j} = \mathrm{Res}_{Z(\widetilde{T})\cap\widetilde{K^1}}\underline{\chi}$, and hence, (12) is 2-dimensional, which shows that $\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\chi'_{i,j}$ appears exactly twice in $\bigoplus_{\ell\in L,0\leq k<\underline{n}}\ell\chi_k$. □

Note that the map $(i,j)\to i-j \mod n$, which appears in Lemma 11, has a kernel of size n. Therefore, it is easy to see that $\{\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\chi'_{0,j}\mid 0\leq j<n\}$ consists of n distinct characters of $\widetilde{T^1}\cap\widetilde{K^1}$, each appearing exactly twice in $\bigoplus_{\ell\in L,0\leq k<\underline{n}}\ell\chi_k$ by Lemma 12. By a simple counting argument, we deduce that for every $0\leq k<\underline{n}$ and $\ell\in L$, there exists a $0\leq j<n$, such that $\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\chi'_{0,j} = {}_\ell\chi_k$. Similar to Lemma 10, we see that, for n even, if $0\leq j<n$, $0\leq k<\underline{n}$ and $\ell\in L$ are such that $\mathrm{Res}_{\widetilde{T^1}\cap\widetilde{K^1}}\chi'_{0,j} = {}_\ell\chi_k$, then, for all $l\geq m$, $\left(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}{}_\ell\chi_k\right)^{K_l^1}\cong\mathrm{Res}_{\widetilde{K^1}}\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi'_{0,j}\right)^{K_l}$.

The following proposition sums up the result in this section.

Proposition 7. *Let ρ and ρ' be irreducible representations of $\widetilde{T^1}$ and $\widetilde{T}$ with central characters χ and χ', primitive mod m, respectively, such that ρ appears in $\mathrm{Res}_{\widetilde{T^1}}\rho'$. For $l>m$, $0\leq k<\underline{n}$, $0\leq i,j<n$, let $\widetilde{W}_{k,l} = \widetilde{W}_{k,l}^- \oplus\widetilde{W}_{k,l}^+$ and $\widetilde{W'}_{i,j,l}$ be the quotient spaces that appear in the decompositions in Corollary 1 and Corollary 3 respectively. Then, for each $0\leq k<\underline{n}$, $l>m$, $\widetilde{W}_{k,l} = \mathrm{Res}_{\widetilde{K^1}}\widetilde{W'}_{i,j,l}$, for some $0\leq i,j<n$.*

Proof. If n is odd, it follows from Lemma 10 that for a given k and l there exists $0\leq i,j<n$ such that $\left(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_k\right)^{K_l^1}\cong\mathrm{Res}_{\widetilde{K^1}}\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi'_{i,j}\right)^{K_l}$. Without loss of generality, we can assume $i=0$. For n even, it follows from Proposition 6 that $\chi = {}_\ell\chi$ for some $\ell\in L$, where ${}_\ell\chi$ are defined in (11). It is a consequence of Lemma 12 that, for a given k and l there exists $0\leq j<n$ such that $\left(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}{}_\ell\chi_k\right)^{K_l^1}\cong \mathrm{Res}_{\widetilde{K^1}}\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi'_{0,j}\right)^{K_l}$. Consider $\widetilde{W}'_{0,j,l} = (\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi'_{0,j})^{K_l}/(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi'_{0,j})^{K_{l-1}}$. Observe that

$$\begin{aligned}\mathrm{Res}_{\widetilde{K^1}}\widetilde{W}'_{0,j,l} &= \mathrm{Res}_{\widetilde{K^1}}\left[(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi'_{0,j})^{K_l}\right]/\mathrm{Res}_{\widetilde{K^1}}\left[(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi'_{0,j})^{K_{l-1}}\right]\\ &= \left(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_k\right)^{K_l}/\left(\mathrm{Ind}_{\widetilde{B^1}\cap\widetilde{K^1}}^{\widetilde{K^1}}\chi_k\right)^{K_{l-1}} = \widetilde{W}_{k,l}^-\oplus\widetilde{W}_{k,l}^+.\end{aligned}$$

□

Corollary 4. *The inequivalent irreducible representations $\widetilde{W}_{k,l}^-$ and $\widetilde{W}_{k,l}^+$, $0\leq k<\underline{n}$, $l>m$, that appear in the* K*-type decomposition $\mathrm{Res}_{\widetilde{K^1}}\mathrm{Ind}_{\widetilde{B^1}}^{\widetilde{G^1}}\rho$ in Corollary 1 are of the same dimension.*

Proof. By Proposition 7, for any $0\leq k<\underline{n}$, $l>m$, $\widetilde{W}_{k,l} = \widetilde{W}_{k,l}^-\oplus\widetilde{W}_{k,l}^+$, is restriction of some irreducible representation $\widetilde{W}'_{i,j}$ of $\widetilde{K}$, for some $0\leq i,j<n$. Hence, there exists an element of $\widetilde{K}\setminus\widetilde{K^1}$ that maps $\widetilde{W}_{k,l}^-$ to $\widetilde{W}_{k,l}^+$ bijectively. □

6. Main Result

Finally, we put all of our results together to make the main result of this paper.

Theorem 2. *Let ρ be a genuine irreducible representation of $\widetilde{T^1}$ with central character χ, primitive mod m, and let χ_k, $0 \le k < \underline{n}$, be all the possible extensions of χ to A^1. Then*

$$\mathrm{Res}_{\widetilde{K^1}} \mathrm{Ind}_{\widetilde{B^1}}^{\widetilde{G^1}} \rho \cong \bigoplus_{k=0}^{\underline{n}-1} \left(\widetilde{V}_k^{K_m^1} \right) \oplus \bigoplus_{l>m} \left(\widetilde{W}_{0,l}^{+} \oplus \widetilde{W}_{0,l}^{-} \right)^{\oplus \underline{n}},$$

where $\widetilde{W}_{0,l}^{+}$ and $\widetilde{W}_{0,l}^{-}$ are two inequivalent irreducible representations of $\widetilde{K^1}$ with the same dimension, and $\left(\widetilde{W}_{0,l}^{+} \oplus \widetilde{W}_{0,l}^{-} \right) \cong (\mathrm{Ind}_{\widetilde{B^1} \cap \widetilde{K^1}}^{\widetilde{K^1}} \chi_0)^{K_l^1} / (\mathrm{Ind}_{\widetilde{B^1} \cap \widetilde{K^1}}^{\widetilde{K^1}} \chi_0)^{K_{l-1}^1}$, and $\widetilde{V}_k = \mathrm{Ind}_{\widetilde{B^1} \cap \widetilde{K^1}}^{\widetilde{K^1}} \chi_k$.

We consider $T^1 \cap K^1$ as a subgroup of $\widetilde{T^1}$. The m-level representations $\widetilde{V}_k^{K_m^1}$, where $0 \le k < \underline{n}$, are irreducible and mutually inequivalent, except when $m = 1$, and for some $0 \le k < \underline{n}$, $\chi_k|_{(T^1 \cap K^1)}$ is a quadratic character. In this case, up to relabelling, we can assume that $\chi_0|_{(T^1 \cap K^1)}$ is a quadratic character, and we are in one of the following situations:

1. *If $4 \nmid n$ then $\widetilde{V}_k^{K_1^1}$ is reducible if and only if $k = 0$, in which case it decomposes into two irreducible constituents. Moreover, $\widetilde{V}_i^{K_1^1} \cong \widetilde{V}_k^{K_1^1}$, exactly when $i + k = \underline{n}$.*
2. *If $4|n$ then $\widetilde{V}_k^{K_1^1}$ is reducible if and only if $k = 0$ or $k = \frac{n}{4}$. In which case, it decomposes into two irreducible constituents Moreover, $\widetilde{V}_i^{K_1^1} \cong \widetilde{V}_k^{K_1^1}$, exactly when $i + k = \underline{n}$.*

Proof. The decomposition and irreducibility results follow from Corollary 1. The multiplicity results are shown in Corollary 2, and the fact that $\widetilde{W}_{0,l}^{+}$ and $\widetilde{W}_{0,l}^{-}$ have the same degree follows from Corollary 4. ☐

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