GEOMETRIC INVARIANT THEORY FOR GRADED UNIPOTENT GROUPS AND APPLICATIONS

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Abstract. Let U be a graded unipotent group over the complex numbers, in the sense that it has an extension $\hat{U}$ by the multiplicative group such that the action of the multiplicative group by conjugation on the Lie algebra of U has all its weights strictly positive. Given any action of U on a projective variety X extending to an action of $\hat{U}$ which is linear with respect to an ample line bundle on X, then provided that one is willing to replace the line bundle with a tensor power and to twist the linearisation of the action of $\hat{U}$ by a suitable (rational) character, and provided an additional condition is satisfied which is the analogue of the condition in classical GIT that there should be no strictly semistable points for the action, we show that the $\hat{U}$-invariants form a finitely generated graded algebra; moreover the natural morphism from the semistable subset of X to the enveloping quotient is surjective and expresses the enveloping quotient as a geometric quotient of the semistable subset. Applying this result with X replaced by its product with the projective line gives us a projective variety which is a geometric quotient by $\hat{U}$ of an invariant open subset of the product of X with the affine line and contains as an open subset a geometric quotient of a U-invariant open subset of X by the action of U. Furthermore these open subsets of X and its product with the affine line can be described using criteria similar to the Hilbert-Mumford criteria in classical GIT.

Mumford's geometric invariant theory (GIT) allows us to construct and study quotients of algebraic varieties by linear actions of reductive groups [28, 30]. When a complex reductive group G acts linearly (with respect to an ample line bundle L) on a complex projective variety X, the associated GIT quotient $X/\!/G$ is the projective variety $\mathrm{Proj}(\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^G)$ associated to the ring of invariants $\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^G$, which is a finitely generated graded complex algebra. Geometrically the variety $X/\!/G$ can be described as the image of a surjective morphism from an open subset X^{ss} of X, consisting of the semistable points for the action, or as X^{ss} modulo the equivalence relation $\sim$ such that if $x, y \in X^{ss}$ then $x \sim y$ if and only if the closures of the G-orbits of x and y meet in X^{ss}. The stable points for the action form a subset X^s of X^{ss} which has a geometric quotient X^s/G which is an open subset of $X/\!/G$. Moreover the subsets X^s and X^{ss} can be described using the Hilbert–Mumford criteria for (semi)stability. The GIT quotient $X/\!/G$ and its open subset X^s/G can also be described in terms of symplectic geometry and a moment map [20, 29].

In suitable situations GIT can be generalised to allow us to construct GIT-like quotients for linear actions of non-reductive groups [11, 12, 13, 15, 16, 22, 39]. However there is an immediate difficulty in extending GIT to non-reductive group actions, since now the ring of invariants

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is not necessarily finitely generated as a graded algebra, and when it is not finitely generated there is no associated projective variety.

Every affine algebraic group H has a unipotent radical $U \trianglelefteq H$ such that $R = H/U$ is reductive (and over $\mathbb{C}$ we have a semi-direct product decomposition $H \cong R \ltimes U$), and understanding GIT-theoretic questions about the action – such as whether invariants are finitely generated – often follows from understanding the action of the unipotent group U. In some cases the U-invariants happen to be finitely generated. For example, if U is the unipotent radical of a parabolic subgroup P of a complex reductive group G and an action of U on a complex projective variety X, which is linear with respect to an ample line bundle L, extends to a linear action of G, then the ring of invariants $\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^U$ is finitely generated [17, 23]. In this case the 'enveloping quotient' $X \wr\wr U$ (in the sense of [11] but using the notation of [3]) is the projective variety $\mathrm{Proj}(\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^U)$ associated to the ring of invariants, and it contains as an open subset a geometric quotient X^s/U where X^s is a U-invariant open subset of X. However there is still no analogue for $X \wr\wr U$ of the geometric description of $X//G$ when G is reductive as X^{ss} modulo an equivalence relation, since the natural morphism from X^{ss} to $X \wr\wr U$ is not in general surjective, although there are alternative geometric descriptions [23].

In this paper we consider a more general situation. Instead of taking U to be the unipotent radical of a parabolic subgroup of a complex reductive group G which acts linearly on X, we assume that U is a unipotent group over $\mathbb{C}$ with an extension $\hat{U} = U \rtimes \mathbb{C}^*$ by $\mathbb{C}^*$ such that the action of $\mathbb{C}^*$ by conjugation on the Lie algebra of U has all its weights strictly positive; we call such a U a graded unipotent group. (The unipotent radical of a parabolic subgroup of a complex reductive group G always has such an extension contained in the parabolic subgroup). We are interested in linear actions of U on projective varieties X which extend to linear actions of $\hat{U}$. Given any action of U on a projective variety X extending to an action of $\hat{U}$ which is linear with respect to an ample line bundle on X, then *provided* that we are willing to replace the line bundle with a tensor power and to twist the linearisation of the action of $\hat{U}$ by a suitable (rational) character of $\hat{U}$, and provided an additional condition is satisfied which is the analogue of the condition in classical GIT that there should be no strictly semistable points for the action (that is, 'semistability coincides with stability'), we find that the $\hat{U}$-invariants form a finitely generated algebra; moreover the natural morphism $\phi : X^{ss,\hat{U}} \to X \wr\wr \hat{U}$ is surjective and indeed expresses $X \wr\wr \hat{U}$ as a geometric quotient of $X^{ss,\hat{U}}$, so that ϕ satisfies $\phi(x) = \phi(y)$ if and only if the $\hat{U}$-orbits of x and y coincide in $X^{ss,\hat{U}}$. Applying this result with X replaced by $X \times \mathbb{P}^1$ gives us a projective variety $(X \times \mathbb{P}^1) \wr\wr \hat{U}$ which is a geometric quotient by $\hat{U}$ of a $\hat{U}$-invariant open subset of $X \times \mathbb{C}$ and contains as an open subset a geometric quotient of a U-invariant open subset $X^{\hat{s},U}$ of X by U. Furthermore the subsets $X^{s,\hat{U}} = X^{ss,\hat{U}}$ and $X^{\hat{s},U}$ of X can be described using Hilbert–Mumford-like criteria.

This situation arises even for the Nagata counterexamples to Hilbert's 14th problem, which provide examples of linear actions of unipotent groups U on projective space such that the corresponding U-invariants are not finitely generated. In these cases the linear action extends to a linear action of an extension $\hat{U} = U \rtimes \mathbb{C}^*$ by $\mathbb{C}^*$ such that the action of $\mathbb{C}^*$ by conjugation on the Lie algebra of U has all its weights strictly positive. Thus when the condition that semistability coincides with stability is satisfied, we obtain open subsets $X^{s,\hat{U}} = X^{ss,\hat{U}}$ and $X^{\hat{s},U}$ of X, which are determined by analogues of the Hilbert–Mumford criteria, with geometric quotients $X^{s,\hat{U}}/\hat{U}$ and $X^{\hat{s},U}/U$, such that $X^{s,\hat{U}}/\hat{U}$ is projective and $X^{\hat{s},U}/U$ is quasi-projective with a projective completion in which the complement of $X^{\hat{s},U}/U$ is $X^{s,\hat{U}}/\hat{U}$.

A related situation is studied in [4], where it is assumed that the linear action of the graded unipotent group U extends to a linear action of a general linear group $GL(n)$. Here U and $\hat{U}$ are embedded in $GL(n)$ as subgroups 'generated along the first row' in the sense that there are integers $1 = \omega_1 < \omega_2 \leq \omega_3 \leq \cdots \leq \omega_n$ and polynomials $\mathfrak{p}_{i,j}(\alpha_1, \ldots, \alpha_n)$ in $\alpha_1, \ldots, \alpha_n$ with complex coefficients for $1 < i < j \leq n$ such that

$$\hat{U} = \left\{ \left(\begin{array}{ccccc} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_n \\ 0 & \alpha_1^{\omega_2} & p_{2,3}(\alpha) & \ldots & p_{2,n}(\alpha) \\ 0 & 0 & \alpha_1^{\omega_3} & \ldots & p_{3,n}(\alpha) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \alpha_1^{\omega_n} \end{array} \right) : \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^* \times \mathbb{C}^{n-1} \right\} \tag{1}$$

and U is the unipotent radical of $\hat{U}$, defined by $\alpha_1 = 1$. The main results of [4] also involve the subgroup $\tilde{U}$ of $SL(n)$ which is the intersection of $SL(n)$ with the product $\hat{U}Z(GL(n))$ of $\hat{H}$ with the central one-parameter subgroup $Z(GL(n)) \cong \mathbb{C}^*$ of $GL(n)$. Like $\hat{U}$, the subgroup $\tilde{U}$ of $GL(n)$ is a semi-direct product $\tilde{U} = U \rtimes \mathbb{C}^*$ where $\mathbb{C}^*$ acts on the Lie algebra of U with all weights strictly positive. When $GL(n)$ acts linearly on a projective variety X with respect to an ample line bundle L on X, and the linearisation of the action of $\tilde{U}$ on X is twisted by a suitable rational character χ (which is 'well adapted' to the action in the sense of [4]), then it is shown in [4] Theorem 1.1 that the corresponding algebra of $\tilde{U}$-invariants is finitely generated, and the projective variety $X /\!\!/ \tilde{U}$ associated to this algebra of invariants is a categorical quotient of an open subset $X^{ss,\tilde{U}}$ of X by $\tilde{U}$ and contains as an open subset a geometric quotient of an open subset $X^{s,\tilde{U}}$ of X. Applying a similar argument after replacing X with $X \times \mathbb{P}^1$ provides a projective variety $(X \times \mathbb{P}^1) /\!\!/ \tilde{U}$ which is a categorical quotient by $\tilde{U}$ of a $\tilde{U}$-invariant open subset of $X \times \mathbb{C}$ and contains as an open subset a geometric quotient of a U-invariant open subset $X^{\hat{s},U}$ of X by U.

The results of this paper are more general than those of [4] in that the linear action of the unipotent group U is only required to extent to a linear action of $\hat{U}$ rather than a general linear group in which U and $\hat{U}$ are embedded in a very special way. On the other hand in [4] the additional condition that 'semistability coincides with stability' is not required. We will address the removal of this additional condition in future work, using a partial desingularisation construction analogous to that of [21].

Let $\chi : \hat{U} \to \mathbb{C}^*$ be a character of $\hat{U}$ with kernel containing U; we will identify such characters χ with integers so that the integer 1 corresponds to the character which fits into the exact sequence $U \to \hat{U} \to \mathbb{C}^*$. Suppose that $\omega_{\min} < \omega_{\min+1} < \cdots < \omega_{\max}$ are the weights with which the one-parameter subgroup $\mathbb{C}^* \leq \hat{U}$ acts on the fibres of the tautological line bundle $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*)}(-1)$ over points of the connected components of the fixed point set $\mathbb{P}((H^0(X,L)^*)^{\mathbb{C}^*}$ for the action of $\mathbb{C}^*$ on $\mathbb{P}((H^0(X,L)^*)$; when L is very ample X embeds in $\mathbb{P}((H^0(X,L)^*)$ and the line bundle L extends to the dual $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*)}(1)$ of the tautological line bundle $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*)}(-1)$. We will assume that there exist at least two distinct such weights since otherwise the action of

U on X is trivial. Let c be a positive integer such that

$$\frac{\chi}{c} = \omega_{\min} + \epsilon$$

where $\epsilon > 0$ is sufficiently small; we will call rational characters χ/c with this property *well adapted* to the linear action of $\hat{U}$, and we will call the linearisation well adapted if $\omega_{\min} < 0 \leq \omega_{\min} + \epsilon$ for sufficiently small $\epsilon > 0$. The linearisation of the action of $\hat{U}$ on X with respect to the ample line bundle $L^{\otimes c}$ can be twisted by the character χ so that the weights ω_j are replaced with $\omega_j c - \chi$; let $L_\chi^{\otimes c}$ denote this twisted linearisation. Let $X^{s,\mathbb{C}^*}_{\min+}$ denote the stable subset of X for the linear action of $\mathbb{C}^*$ with respect to the linearisation $L_\chi^{\otimes c}$; by the theory of variation of (classical) GIT [10, 38], if L is very ample then $X^{s,\mathbb{C}^*}_{\min+}$ is the stable set for the action of $\mathbb{C}^*$ with respect to any rational character χ/c such that $\omega_{\min} < \chi/c < \omega_{\min+1}$. We set

$$X^{s,\hat{U}}_{\min+} = X \setminus \hat{U}(X \setminus X^{s,\mathbb{C}^*}_{\min+}) = \bigcap_{u \in U} u X^{s,\mathbb{C}^*}_{\min+}$$

to be the complement of the $\hat{U}$-sweep (or equivalently the U-sweep) of the complement of $X^{s,\mathbb{C}^*}_{\min+}$.

The main theorem of this paper concerns a linear action of $\hat{U}$ on a projective variety X which is well adapted in the sense above and satisfies an additional condition to which we will refer as the condition that 'semistability coincides with stability'. More precisely, any element ξ of the Lie algebra of U defines a derivation $\delta_\xi : H^0(X,L) \to H^0(X,L)$, and we require that whenever U' is a subgroup of U and ξ belongs to the Lie algebra of U but not the Lie algebra of U', then the weight space with weight $-\omega_{\min}$ for the action of $\mathbb{C}^*$ on $H^0(X,L)$ is contained in the image $\delta_\xi(H^0(X,L)^{U'})$ of $H^0(X,L)^{U'}$ under the derivation δ_ξ.

Theorem 0.1. *Let U be a unipotent group over $\mathbb{C}$ and let $\hat{U} = U \rtimes \mathbb{C}^*$ be a semidirect product of U by $\mathbb{C}^*$ where the conjugation action of $\mathbb{C}^*$ on U is such that all the weights of the induced $\mathbb{C}^*$-action on the Lie algebra of U are strictly positive. Suppose that $\hat{U}$ acts linearly on a projective variety X with respect to an ample line bundle L, and that $\chi : \hat{U} \to \mathbb{C}^*$ is a character of $\hat{U}$ with kernel containing U and c is a positive integer such that the rational character χ/c is well adapted for the linear action of $\hat{U}$. Suppose also that the linear action of $\hat{U}$ on X satisfies the condition that 'semistability coincides with stability' as above. Then the algebra of invariants $\oplus_{m=0}^\infty H^0(X, L_{m\chi}^{\otimes cm})^{\hat{U}}$ is finitely generated for any well-adapted rational character χ/c of $\hat{U}$. Moreover the enveloping quotient $X /\!/ \hat{U}$ is the projective variety associated to this algebra of invariants and is a geometric quotient of the open subset $X^{s,\hat{U}}_{\min+}$ of X by $\hat{U}$.*

Applying this result after replacing X with $X \times \mathbb{P}^1$ we obtain geometric information about the action of the unipotent group U on X:

Corollary 0.2. *In the situation above let $\hat{U}$ act diagonally on $X \times \mathbb{P}^1$ where the action on $\mathbb{P}^1$ is via*

$$\hat{u} \cdot [x : y] = [\chi_1(\hat{u})x : y]$$

where $\chi_1 : \hat{U} \to \mathbb{C}^*$ is the character of $\hat{U}$ with kernel U which fits into the extension $\{1\} \to U \to \hat{U} \to \mathbb{C}^* \to \{1\}$, and linearise this action using the tensor product of L_χ with $O_{\mathbb{P}^1}(M)$ for *suitable $M \geq 1$. Then $(X \times \mathbb{P}^1) /\!/ \hat{U}$ is a projective variety which is a geometric quotient by $\hat{U}$ of a $\hat{U}$-invariant open subset of $X \times \mathbb{C}$ and contains as an open subset a geometric quotient of a U-invariant open subset $X^{\hat{s},U}$ of X by U.*

Remark 0.3. We can also deduce that the algebra $A = \oplus_{m=0}^{\infty} H^0(X \times \mathbb{P}^1, L_{m\chi}^{\otimes cm} \otimes \mathcal{O}_{\mathbb{P}^1}(M))^{\hat{U}}$ of $\hat{U}$-invariants on $X \times \mathbb{P}^1$ is finitely generated for a well-adapted rational character χ/c of $\hat{U}$ when c is a sufficiently divisible positive integer. This graded algebra A can be identified with the subalgebra of the algebra of U-invariants $\oplus_{m=0}^{\infty} H^0(X, L^{\otimes cm})^U$ on X generated by the U-invariants in $\oplus_{m=0}^{\infty} H^0(X, L^{\otimes cm})^U$ which are weight vectors with non-positive weights for the action of $\mathbb{C}^* \leq \hat{U}$ after twisting by the well-adapted rational character χ/c. The sections σ of L which are weight vectors with weight $-\omega_{min}$ are all U-invariant, and after twisting by χ/c these are the only weight vectors in $H^0(X, L)$ which have non-positive (in fact strictly negative) weights. If we localise the U-invariants at any such σ then we get a finitely generated algebra of invariants $\mathcal{O}(X_\sigma)^U$, since this algebra can be identified with the localisation of A at σ.

This theorem has another immediate corollary:

Corollary 0.4. *Let $H \cong R \ltimes U$ be a complex linear algebraic group with unipotent radical U and $R \cong H/U$ reductive, and suppose that R contains a central subgroup isomorphic to $\mathbb{C}^*$ which acts by conjugation on the Lie algebra of U with all weights strictly positive. Let $\hat{U}$ be the subgroup of H which is the semidirect product of U and this one-parameter subgroup $\mathbb{C}^*$ of R. Suppose that H acts linearly on a projective variety X with respect to an ample line bundle L, and that $\chi : H \to \mathbb{C}^*$ is a character of H, that c is a sufficiently divisible positive integer such that the restriction to $\hat{U}$ of the rational character χ/c is well adapted for the linear action of $\hat{U}$ on X, and that the linear action of $\hat{U}$ on X satisfies the condition that 'semistability coincides with stability' as above. Then the algebra of H-invariants $\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^H$ is finitely generated, and the projective variety $X /\!/ H$ associated to this algebra of invariants is a categorical quotient of an open subset $X^{ss,H}$ of X by H, and the canonical H-invariant morphism $\phi : X^{ss,H} \to X /\!/ H$ is surjective with $\phi(x) = \phi(y)$ if and only if the closures of the H-orbits of x and y meet in $X^{ss,H}$.*

Example: Consider the weighted projective plane $\mathbb{P}(1,1,2)$ which is $\mathbb{C}^3 \setminus \{0\}$ modulo the action of $\mathbb{C}^*$ with weights $1, 1, 2$. The automorphism group of $\mathbb{P}(1,1,2)$ is

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) \cong R \ltimes U$$

with $R \cong GL(2)$ reductive and $U \cong (\mathbb{C}^+)^3$ unipotent; here $(\lambda, \mu, \nu) \in (\mathbb{C}^+)^3 \cong U$ acts on the weighted projective plane $\mathbb{P}(1,1,2)$ as $[x, y, z] \mapsto [x, y, z + \lambda x^2 + \mu xy + \nu y^2]$. The central one-parameter subgroup $\mathbb{C}^*$ of $R \cong GL(2)$ acts on $Lie(U)$ with all positive weights, and the associated extension $\hat{U} = U \rtimes \mathbb{C}^*$ can be identified with a subgroup of $\mathrm{Aut}(\mathbb{P}(1,1,2))$. Thus Corollary 0.4 applies to every linear action of $\mathrm{Aut}(\mathbb{P}(1,1,2))$ on a projective variety X with respect to an ample line bundle L after twisting by a well adapted rational character.

The weighted projective plane $\mathbb{P}(1,1,2)$ is a simple example of a toric variety; in fact as we shall see in §4 below, the automorphism group of any complete simplicial toric variety satisfies the conditions of Corollary 0.4.

Remark 0.5. Popov [31] has shown that if H is any non-reductive group then there is an affine variety Y on which H acts such that the algebra of invariants $\mathcal{O}(Y)^H$ is not finitely generated.

Our main motivation for considering linear actions of groups of the form $\hat{U}$ in this article and in [4] came from the study of jet differentials. The groups $\mathbb{G}_k$ of k-jets of holomorphic reparametrizations of $(\mathbb{C}, 0)$ (and more generally the groups $\mathbb{G}_{k,p}$ of k-jets of holomorphic reparametrizations of $(\mathbb{C}^p, 0)$ for $p \geq 1$) play an important role in the strategy of Demailly, Siu and others [1, 5, 7, 8, 9, 14, 24, 26, 34, 35, 36] towards the Green-Griffiths conjecture on entire holomorphic curves in hypersurfaces of large degree in projective spaces. Here $\mathbb{G}_k$ is a non-reductive complex linear algebraic group which is a semi-direct product $\mathbb{G}_k = \mathbb{U}_k \rtimes \mathbb{C}^*$ of its unipotent radical $\mathbb{U}_k$ by $\mathbb{C}^*$ acting with weights $1, 2, 3, \ldots, k$ on the Lie algebra of $\mathbb{U}_k$, while if $p > 1$ then $\mathbb{G}_{k,p} = \mathbb{U}_{k,p} \rtimes GL(p; \mathbb{C})$ where all the weights of the central one-parameter subgroup $\mathbb{C}^*$ of $GL(p; \mathbb{C})$ on the Lie algebra of the unipotent radical $\mathbb{U}_{k,p}$ of $\mathbb{G}_{k,p}$ are strictly positive. So the results above apply to linear actions of the reparametrization group $\mathbb{G}_k$ and its generalizations $\mathbb{G}_{k,p}$ for $p \geq 1$). In particular the reparametrization group $\mathbb{G}_k$ acts fibrewise in a natural way on the Semple jet bundle $J_k(T^*X) \to X$ over a complex manifold X of dimension n with fibre

$$J_{k,x} \cong \bigoplus_{j=1}^{k} \mathrm{Sym}^j(\mathbb{C}^n)$$

at x consisting of the k-jets of holomorphic curves at x. There is an induced action of $\mathbb{G}_k$ on the polynomial ring $\mathcal{O}(J_{k,x})$, which can be identified with the algebra $\oplus_{m=0}^{\infty} H^0(\mathbb{P}(J_{k,x}), \mathcal{O}_{\mathbb{P}(J_{k,x})}(1)^{\otimes m})$ of sections of powers of the hyperplane line bundle on the associated projective space $\mathbb{P}(J_{k,x})$, and the bundle $E_k \to X$ of Demailly-Semple invariant jet differentials of order k has fibre at x given by $(E_k)_x = \mathcal{O}(J_{k,x})^{\mathbb{U}_k}$.

The layout of the paper is as follows. §1 reviews the results of [11] and [3] on non-reductive GIT, and §2 considers the case when $\dim(U) = 1$ and proves Theorem 0.1 in this case. §3 uses these results to prove Theorem 0.1 and Corollaries 0.2 and 0.4. In §4 we observe that Corollary 0.4 applies to the automorphism groups of all complete simplicial toric varieties, while §5 discusses applications to Demailly-Semple jet differentials and their generalisations to maps $\mathbb{C}^p \to X$.

1. Classical and non-reductive geometric invariant theory

Let X be a complex quasi-projective variety and let G be a complex reductive group acting on X. To apply (classical) geometric invariant theory (GIT) we require a linearisation of the action; that is, a line bundle L on X and a lift $\mathcal{L}$ of the action of G to L.

Remark 1.1. Usually L is assumed to be ample, and it makes no difference for classical GIT if we replace L with $L^{\otimes k}$ for any integer $k > 0$, so then we lose little generality in supposing that for some projective embedding $X \subseteq \mathbb{P}^n$ the action of G on X extends to an action on $\mathbb{P}^n$ given by a representation

$$\rho : G \to GL(n+1),$$

and taking for L the hyperplane line bundle on $\mathbb{P}^n$.

Definition 1.2. Let X be a quasi-projective complex variety with an action of a complex reductive group G and linearisation $\mathcal{L}$ with respect to a line bundle L on X. Then $y \in X$ is *semistable* for this linear action if there exists some $m > 0$ and $f \in H^0(X, L^{\otimes m})^G$ not vanishing at y such that the open subset

$$X_f := \{x \in X \mid f(x) \neq 0\}$$

is affine, and y is *stable* if also the action of G on X_f is closed with all stabilisers finite.

Remark 1.3. This definition comes from [28], although in [28] the terminology 'properly stable' is used instead of stable. When X is projective and L is ample and $f \in H^0(X, L^{\otimes m})^G$ for $m > 0$, then X_f is affine if and only if f is nonzero. The reason for introducing the requirement that X_f must be affine in Definition 1.2 above is to ensure that X^{ss} has a quasi-projective categorical quotient $X^{ss} \to X/\!/G$, which restricts to a geometric quotient $X^s \to X^s/G$ (see [28] Theorem 1.10).

From now on in this section we will assume that X is projective and L is ample. We have an induced action of G on the homogeneous coordinate ring

$$\hat{O}_L(X) = \bigoplus_{k \geq 0} H^0(X, L^{\otimes k})$$

of X. The subring $\hat{O}_L(X)^G$ consisting of the elements of $\hat{O}_L(X)$ left invariant by G is a finitely generated graded complex algebra because G is reductive, and the GIT quotient $X/\!/G$ is the projective variety $\mathrm{Proj}(\hat{O}_L(X)^G)$. The subsets X^{ss} and X^s of X are characterised by the following properties (see [28, Chapter 2] or [30]).

Proposition 1.4. *(Hilbert-Mumford criteria) (i) A point $x \in X$ is semistable (respectively stable) for the action of G on X if and only if for every $g \in G$ the point gx is semistable (respectively stable) for the action of a fixed maximal torus of G.*
(ii) A point $x \in X$ with homogeneous coordinates $[x_0 : \ldots : x_n]$ in some coordinate system on $\mathbb{P}^n$ is semistable (respectively stable) for the action of a maximal torus of G acting diagonally on $\mathbb{P}^n$ with weights $\alpha_0, \ldots, \alpha_n$ if and only if the convex hull

$$\mathrm{Conv}\{\alpha_i : x_i \neq 0\}$$

contains 0 *(respectively contains* 0 *in its interior).*

Now let H be any affine algebraic group, with unipotent radical U, acting linearly on a complex projective variety X with respect to an ample line bundle L. Then the ring of invariants

$$\hat{O}_L(X)^H = \bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^H$$

is not necessarily finitely generated as a graded complex algebra, so that $\mathrm{Proj}(\hat{O}_L(X)^H)$ is not well-defined as a projective variety, although $\mathrm{Proj}(\hat{O}_L(X)^H)$ does make sense as a scheme, and the inclusion of $\hat{O}_L(X)^H$ in $\hat{O}_L(X)$ gives us a rational map of schemes q from X to $\mathrm{Proj}(\hat{O}_L(X)^H)$, whose image is a constructible subset of $\mathrm{Proj}(\hat{O}_L(X)^H)$ (that is, a finite union of locally closed subschemes). The action on X of the unipotent radical U of H is studied in [11] following earlier work [12, 13, 15, 16, 39].

Definition 1.5. (See [11] §4). Let $I = \bigcup_{m>0} H^0(X, L^{\otimes m})^U$ and for $f \in I$ let X_f be the U-invariant affine open subset of X where f does not vanish, with $\mathcal{O}(X_f)$ its coordinate ring. A point $x \in X$ is called *naively semistable* if there exists some $f \in I$ which does not vanish at x, and the set of naively semistable points is denoted $X^{nss} = \bigcup_{f \in I} X_f$. The *finitely generated semistable set* of X is $X^{ss,fg} = \bigcup_{f \in I^{fg}} X_f$ where

$$I^{fg} = \{ f \in I \mid \mathcal{O}(X_f)^U \text{ is finitely generated } \}.$$

The set of *naively stable* points of X is $X^{ns} = \bigcup_{f \in I^{ns}} X_f$ where

$$I^{ns} = \{ f \in I^{fg} \mid q : X_f \longrightarrow \operatorname{Spec}(\mathcal{O}(X_f)^U) \text{ is a geometric quotient}\},$$

and the set of *locally trivial stable* points is $X^{lts} = \bigcup_{f \in I^{lts}} X_f$ where

$$I^{lts} \;=\; \{ f \in I^{fg} \mid q : X_f \longrightarrow \operatorname{Spec}(\mathcal{O}(X_f)^U) \text{ is a locally trivial geometric quotient}\}.$$

The *enveloped quotient* of $X^{ss,fg}$ is $q : X^{ss,fg} \to q(X^{ss,fg})$, where $q : X^{ss,fg} \to \operatorname{Proj}(\hat{\mathcal{O}}_L(X)^U)$ is the natural morphism of schemes and $q(X^{ss,fg})$ is a dense constructible subset of the *enveloping quotient*

$$X /\!\!/ U = \bigcup_{f \in I^{ss,fg}} \operatorname{Spec}(\mathcal{O}(X_f)^U)$$

of $X^{ss,fg}$.

Remark 1.6. Because of Theorem 1.13 below, we also call a point $x \in X$ *stable* for the linear U-action if $x \in X^{lts}$ and *semistable* if $x \in X^{ss,fg}$. We write X^s (or $X^{s,U}$) for X^{lts}, and we write X^{ss} (or $X^{ss,U}$) for $X^{ss,fg}$ (cf. [11] 5.3.7).

Remark 1.7. $q(X^{ss})$ is not necessarily a subvariety of $X /\!\!/ U$ (see for example [11] §6).

Proposition 1.8. *If $\hat{\mathcal{O}}_L(X)^U$ is finitely generated then $X /\!\!/ U$ is the projective variety* $\operatorname{Proj}(\hat{\mathcal{O}}_L(X)^U)$.

Remark 1.9. In [11] 4.2.9 and 4.2.10 it is claimed that the enveloping quotient $X /\!\!/ U$ is a quasi-projective variety with an ample line bundle $L_H \to X /\!\!/ U$ which pulls back to a positive tensor power of L under the natural map $q : X^{ss} \to X /\!\!/ U$. The argument given there fails in general since the morphisms $X_f \to \operatorname{Spec}(\mathcal{O}(X_f)^U)$ for $f \in I^{ss,fg}$ are not necessarily surjective. However it is still true that the enveloping quotient $X /\!\!/ U$ has quasi-projective open subvarieties ('inner enveloping quotients' $X /\!\!/\!\!\!/ H$) which contain the enveloped quotient $q(X^{ss})$ and have ample line bundles pulling back to positive tensor powers of L under the natural map $q : X^{ss} \to X /\!\!/ U$ (see [3] for details).

Now let G be a complex reductive group with the unipotent group U as a closed subgroup, and let $G \times_U X$ denote the quotient of $G \times X$ by the free action of U defined by $u(g,x) = (gu^{-1}, ux)$ for $u \in U$, which is a quasi-projective variety by [32] Theorem 4.19. Then there is an induced G-action on $G \times_U X$ given by left multiplication of G on itself. In cases where the action of U on X extends to an action of G there is an isomorphism of G-varieties

$$G \times_U X \cong (G/U) \times X \tag{2}$$

given by $[g,x] \mapsto (gU, gx)$. If U acts linearly on X with respect to a very ample line bundle L and linearisation $\mathcal{L}$ inducing a U-equivariant embedding of X in $\mathbb{P}^n$, and if G is a subgroup of

$\mathrm{SL}(n+1;\mathbb{C})$, then we get a very ample G-linearisation (by abuse of notation also denoted by $\mathcal{L}$) on $G \times_U X$ using the inclusions

$$G \times_U X \hookrightarrow G \times_U \mathbb{P}^n \cong (G/U) \times \mathbb{P}^n,$$

and the trivial bundle on the quasi-affine variety G/U. We can choose a G-equivariant embedding of G/U in an affine space $\mathbb{A}^m$ with a linear G-action to get a G-equivariant embedding of $G \times_U X$ in $\mathbb{A}^m \times \mathbb{P}^n$ and thus in $\mathbb{P}^m \times \mathbb{P}^n$ embedded in $\mathbb{P}^{nm+m+n}$, and the G-invariants on $G \times_U X$ are given by

$$(3) \qquad \bigoplus_{m \geq 0} H^0(G \times_U X, L^{\otimes m})^G \cong \bigoplus_{m \geq 0} H^0(X, L^{\otimes m})^U = \hat{O}_L(X)^U.$$

Definition 1.10. (See [11] §5). The sets of *Mumford stable points* and *Mumford semistable points* in X are $X^{ms} = i^{-1}((G \times_U X)^s)$ and $X^{mss} = i^{-1}((G \times_U X)^{ss})$ where $i : X \to G \times_U X$ is the inclusion given by $x \mapsto [e, x]$ for e the identity element of G. Here $(G \times_U X)^s$ and $(G \times_U X)^{ss}$ are defined as in Definition 1.2 for the induced linear action of G on the quasi-projective variety $G \times_U X$. (In fact X^{ms} and X^{mss} are equal and are independent of the choice of G: see Theorem 1.13 below). A *finite separating set of invariants* for the linear action of U on X is a collection of invariant sections $\{f_1, \ldots, f_n\}$ of positive tensor powers of L such that, if x, y are any two points of X then $f(x) = f(y)$ for all invariant sections f of $L^{\otimes k}$ and all $k > 0$ if and only if

$$f_i(x) = f_i(y) \qquad \forall i = 1, \ldots, n.$$

If G is any reductive group containing U, a finite separating set S of invariant sections of positive tensor powers of L is a *finite fully separating set of invariants* for the linear U-action on X if

(i) for every $x \in X^{ms}$ there exists $f \in S$ with associated G-invariant F over $G \times_U X$ (under the isomorphism (3)) such that $x \in (G \times_U X)_F$ and $(G \times_U X)_F$ is affine; and

(ii) for every $x \in X^{ss}$ there exists $f \in S$ such that $x \in X_f$ and S is a generating set for $O(X_f)^U$.

(This definition is in fact independent of the choice of G: see [11] Remark 5.2.3).

Definition 1.11. (See [11] §5). Let X be a quasi-projective variety with a linear U-action with respect to an ample line bundle L on X, and let G be a complex reductive group containing U as a closed subgroup. A G-equivariant projective completion $\overline{G \times_U X}$ of $G \times_U X$, together with a G-linearisation with respect to a line bundle L which restricts to the given U-linearisation on X, is a *reductive envelope* of the linear U-action on X if every U-invariant f in some finite fully separating set of invariants S for the U-action on X extends to a G-invariant section of a tensor power of L over $\overline{G \times_U X}$. If moreover there exists such an S for which every $f \in S$ extends to a G-invariant section F over $\overline{G \times_U X}$ such that $(\overline{G \times_U X})_F$ is affine, then $(\overline{G \times_U X}, L')$ is a *fine reductive envelope*, and if L is ample (in which case $(\overline{G \times_U X})_F$ is always affine) it is an *ample reductive envelope*. If every $f \in S$ extends to a G-invariant F over $\overline{G \times_U X}$ which vanishes on each codimension 1 component of the boundary of $G \times_U X$ in $\overline{G \times_U X}$, then a reductive envelope for the linear U-action on X is called a *strong* reductive envelope.

Definition 1.12. (See [11] §5 and [22] §3). Let X be a projective variety with a linear U-action and a reductive envelope $\overline{G \times_U X}$. The set of *completely stable points* of X with respect to the reductive envelope is

$$X^{\overline{s}} = (j \circ i)^{-1}(\overline{G \times_U X}^{s})$$

and the set of *completely semistable points* is

$$X^{\overline{ss}} = (j \circ i)^{-1}(\overline{G \times_U X}^{ss}),$$

where $i : X \hookrightarrow G \times_U X$ and $j : G \times_U X \hookrightarrow \overline{G \times_U X}$ are the inclusions, and $\overline{G \times_U X}^{s}$ and $\overline{G \times_U X}^{ss}$ are the stable and semistable sets for the linear G-action on $\overline{G \times_U X}$. In addition we set

$$X^{\overline{nss}} = (j \circ i)^{-1}(\overline{G \times_U X}^{nss})$$

where $y \in \overline{G \times_U X}$ belongs to $\overline{G \times_U X}^{nss}$ (and is said to be *naively semistable* for the linear action of G) if there exists some $m > 0$ and $f \in H^0(X, L^{\otimes m})^G$ not vanishing at y; then $X^{\overline{nss}} = X^{\overline{ss}}$ when the reductive envelope is ample, but not in general otherwise (cf. Remark 1.3).

Theorem 1.13. ([11] 5.3.1 and 5.3.5). *Let X be a normal projective variety with a linear U-action, for U a connected unipotent group, and let $(\overline{G \times_U X}, L)$ be any fine reductive envelope. Then*

$$X^{\overline{s}} \subseteq X^{s} = X^{ms} = X^{mss} \subseteq X^{ns} \subseteq X^{ss} \subseteq X^{\overline{ss}} = X^{nss} \subseteq X^{\overline{nss}}.$$

The stable sets $X^{\overline{s}}$, $X^{s} = X^{ms} = X^{mss}$ and X^{ns} admit quasi-projective geometric quotients, given by restrictions of the quotient map $q = \pi \circ j \circ i$ where

$$\pi : (\overline{G \times_U X})^{ss} \to \overline{G \times_U X}/\!/G$$

is the classical GIT quotient map for the reductive envelope and i, j are as in Definition 1.12. The quotient map q restricted to the open subvariety X^{ss} is an enveloped quotient with $q : X^{ss} \to X /\!\!\wr U$ an enveloping quotient, and there is an open subvariety $X/\!\wr U$ of $\overline{G \times_U X}/\!/G$ which is an inner enveloping quotient of X by the linear action of U. Moreover there is an ample line bundle L_U on $X/\!\wr U$ which pulls back to a tensor power $L^{\otimes k}$ of the line bundle L for some $k > 0$ and extends to an ample line bundle on $\overline{G \times_U X}/\!/G$.

If furthermore $\overline{G \times_U X}$ is normal and provides a fine strong reductive envelope for the linear U-action on X, then $X^{\overline{s}} = X^{s}$ and $X^{ss} = X^{nss}$.

Thus we have a diagram of quasi-projective varieties

$$\begin{array}{ccccccccc}
X^{\overline{s}} & \subseteq & X^{s} & \subseteq & X^{ns} & \subseteq & X^{ss} & \subseteq & X^{\overline{ss}} = X^{nss} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^{\overline{s}}/U & \subseteq & X^{s}/U & \subseteq & X^{ns}/U & \subseteq & X/\!\wr U & \subseteq & \overline{G \times_U X}/\!/G
\end{array}$$

where all the inclusions are open and all the vertical morphisms are restrictions of the GIT quotient map $\pi : (\overline{G \times_U X})^{ss} \to \overline{G \times_U X}/\!/G$, and each except the last is a restriction of the map of schemes $q : X^{nss} \to \mathrm{Proj}(\hat{\mathcal{O}}_L(X)^U))$ associated to the inclusion $\hat{\mathcal{O}}_L(X)^U \subseteq \hat{\mathcal{O}}_L(X)$. Note here that $X /\!\!\wr U$ is not always projective, and (even if the ring of invariants $\hat{\mathcal{O}}_L(X)^U$ is finitely generated and $X/\!\wr U = \mathrm{Proj}(\hat{\mathcal{O}}_L(X)^U)$ is projective) the morphism $X^{ss} \to X/\!\wr U$ is not in general surjective.

There always exists an ample, and hence fine, but not necessarily strong, reductive envelope for any linear U-action on a projective variety X, at least if we replace the line bundle L with a suitable positive tensor power of itself, by [11] Proposition 5.2.8. By Theorem 1.13 above a choice of fine reductive envelope $\overline{G \times_U X}$ provides a projective completion

$$\overline{X/\!\wr U} = \overline{G \times_U X}/\!/G$$

of the inner enveloping quotient $X/\!\wr\, U$. This projective completion in general depends on the choice of reductive envelope, but when $\hat{O}_L(X)^U$ is finitely generated then $X /\!\!\wr\, U = \mathrm{Proj}(\hat{O}_L(X)^U)$ is itself projective, and if $(\overline{G \times_U X}, L)$ is a fine reductive envelope with respect to a finite fully separating set of invariants S containing generators of $\hat{O}_L(X)^U$ then $X /\!\!\wr\, U = \overline{G \times_U X} /\!/ G$ (note such a $(\overline{G \times_U X}, L)$ always exists when $\hat{O}_L(X)^U$ is finitely generated).

[11] also gives a geometric criteria (Theorem 1.14 below) for the graded algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^U$ to be finitely generated. A slight modification of this geometric criteria will be used in the proof of our results.

Theorem 1.14. ([11] 5.3.19). *Let X be a nonsingular complex projective variety on which U acts linearly with respect to an ample line bundle L. Let $\overline{G \times_U X}$ be a nonsingular G-equivariant completion of $G \times_U X$, with a G-linearisation $\mathcal{L}'$ of the G-action on a line bundle L' which extends the given linearisation $\mathcal{L}$. Let $D_1, \dots, D_r$ be the codimension 1 components of the boundary of $G \times_U X$ in $\overline{G \times_U X}$, and let $\mathcal{L}'_N$ be the induced G-linearisation on $L'_N = L'[N \sum_{j=1}^r D_j]$ when N is such that ND_j is Cartier for $1 \leq j \leq r$. Then the algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^U$ is finitely generated if there exists N_0 such that, for all $N > N_0$ for which L'_N is defined, L'_N is ample and every codimension 1 component D_j in the boundary of $G \times_U X$ in $\overline{G \times_U X}$ is unstable for the G-action with the linearization $\mathcal{L}'_N$.*

Remark 1.15. Note that there is an error in the proof of [11] Theorem 5.3.18, which should include as an additional hypothesis that the algebra

$$A = \bigoplus_{k \geq 0} H^0(\overline{G \times_U X}, (L'_N)^{\otimes k})$$

is finitely generated, although its corollary [11] 5.3.19 is correct since there L'_N is assumed ample for N large enough.

We also have the following result, which allows us to study the geometry of $X /\!\!\wr\, U$ when $G \times_U X$ has a nonsingular G-equivariant completion $\overline{G \times_U X}$.

Proposition 1.16. ([11] 5.3.10). *Let $\overline{G \times_U X}$ be a nonsingular G-equivariant completion of $G \times_U X$, with a G-linearisation $\mathcal{L}'$ of the G-action on a line bundle L' which extends the given linearisation $\mathcal{L}$. Let $D_1, \dots, D_r$ be the codimension 1 components of the boundary of $G \times_U X$ in $\overline{G \times_U X}$, and let $\mathcal{L}'_N$ be the induced G-linearisation on $L'[N \sum_{j=1}^r D_j]$ when N is such that ND_j is Cartier for $1 \leq j \leq r$. Given a finite fully separating set S of invariants on X, then $(\overline{G \times_U X}, L'_N)$ is a strong reductive envelope with respect to S for suitable sufficiently large N.*

If moreover $(\overline{G \times_U X}, L')$ is an ample (or more generally a fine) reductive envelope with respect to S then $(\overline{G \times_H X}, L'_N)$ is a fine strong reductive envelope with respect to S. In this situation Theorem 1.13 applies, and $X^{\bar{s}} = X^s$ and $X^{\overline{ss}} = X^{ss}$.

These results can be generalised to allow us to study H-invariants for linear algebraic groups H which are neither unipotent nor reductive [3, 4]. Over $\mathbb{C}$ any linear algebraic group H is a semi-direct product $H = H_u \rtimes R$ where $H_u \subset H$ is the unipotent radical of H (its maximal unipotent normal subgroup) and $R \simeq H_r = H/H_u$ is a reductive subgroup of H. When H acts linearly on a projective variety X with respect to an ample line bundle L, the naively semistable and (finitely generated) semistable sets X^{nss} and $X^{ss} = X^{ss,fg}$, enveloped and enveloping quotients and inner enveloping quotients

$$q : X^{ss} \to q(X^{ss}) \subseteq X/\!\wr\, H \subseteq X /\!\!\wr H$$

are defined in [3] as for the unipotent case in Definition 1.5 and Remark 1.9. However the definition of the stable set X^s combines the unipotent and reductive cases as follows.

Definition 1.17. Let H be a linear algebraic group acting on an irreducible variety X and $L \to X$ a linearisation for the action. The *stable locus* is the open subset

$$X^{\mathrm{s}} = \bigcup_{f \in I^{\mathrm{s}}} X_f$$

of X^{ss}, where $I^{\mathrm{s}} \subseteq \bigcup_{r>0} H^0(X, L^{\otimes r})^H$ is the subset of H-invariant sections satisfying the following conditions:

(1) the open set X_f is affine;
(2) the action of H on X_f is closed with all stabilisers finite groups; and
(3) the restriction of the H_u-enveloping quotient map

$$q_{H_u} : X_f \to \mathrm{Spec}((S^{H_u})_{(f)})$$

is a principal H_u-bundle for the action of H_u on X_f.

If it is necessary to indicate the group H we will write $X^{s,H}$ and $X^{ss,H}$ for X^s and X^{ss}.

Remark 1.18. This definition of stability extends the definition of stability in [11] for unipotent groups, and in the case where H is reductive, then H_u is trivial and the definition reduces to Mumford's notion of properly stable points in [28]. Note that

(i) if R is a reductive subgroup of H then it follows straight from the definition that $X^{s,R} \subseteq X^{s,H}$;

(ii) if N is a normal subgroup of H such that the canonical projection $H_u \to H_u/N_u$ splits, and if W is an H-invariant open subvariety of $X^{s,N}$ with a geometric quotient W/N which is an H/N-invariant open subvariety of $X^{s,N}/N \subseteq X/\!\wr\, N$, where $X/\!\wr\, N$ is an inner enveloping quotient of X by N such that a tensor power $L^{\otimes m}$ of L induces a very ample line bundle on $X/\!\wr\, N$ and hence an embedding of $X/\!\wr\, N$ in the corresponding projective space with closure $\overline{X/\!\wr\, N}$, and if $W/N \subseteq (\overline{X/\!\wr\, N})^{s,H/N}$, then $W \subseteq X^{s,H}$.

The following result which we will need is proved in [3] Cor 3.1.20.

Proposition 1.19. *Suppose H is a linear algebraic group, X an irreducible H-variety and $L \to X$ a linearisation. If the enveloping quotient $X /\!\!\wr H$ is quasi-compact and complete, then for suitably divisible integers $r > 0$ the algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L^{\otimes kr})^H$ is finitely generated and the enveloping quotient $X /\!\!\wr H$ is the associated projective variety; moreover the line bundle $L^{\otimes r}$ induces an ample line bundle $L_{[H]}^{\otimes r}$ on $X /\!\!\wr H$ such that the natural structure map*

$$\bigoplus_{k \geq 0} H^0(X, L^{\otimes kr})^H \to \bigoplus_{k \geq 0} H^0(X /\!\!\wr H, L_{[H]}^{\otimes kr})^H$$

is an isomorphism.

If a linear algebraic group H is a subgroup of a reductive group G then there is an induced right action of R on G/H_u which commutes with the left action of G. Similarly if H acts on a projective variety X then there is an induced action of $G \times R$ on $G \times_{H_u} X$ with an induced $G \times R$-linearisation. The same is true if we replace the requirement that H is a subgroup of G with the existence of a group homomorphism $H \to G$ whose restriction to H_u is injective.

Definition 1.20. *A group homomorphism $H \to G$ from a linear algebraic group H to a reductive group G will be called H_u-faithful if its restriction to the unipotent radical H_u of H is injective.*

As noted in [4] the proof of [11] Theorem 5.1.18 gives us

Theorem 1.21. *Let X be a nonsingular complex projective variety acted on by a linear algebraic group $H = H_u \rtimes R$ where H_u is the unipotent radical of H and let L be a very ample linearisation of the H action defining an embedding $X \subseteq \mathbb{P}^n$. Let $H \to G$ be an H_u-faithful homomorphism into a reductive subgroup G of* $\mathrm{SL}(n+1;\mathbb{C})$ *with respect to an ample line bundle L. Let L' be a $G \times R$-linearisation over a nonsingular projective completion $\overline{G \times_{H_u} X}$ of $G \times_{H_u} X$ extending the $G \times R$ linearisation over $G \times_{H_u} X$ induced by L. Let $D_1, \ldots, D_r$ be the codimension one components of the boundary of $G \times_{H_u} X$ in $\overline{G \times_{H_u} X}$, and suppose for all sufficiently divisible N that $L'_N = L'[N \sum_{j=1}^r D_j]$ is an ample line bundle on $\overline{G \times_{H_u} X}$. Then the algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^H$ is finitely generated if and only if for all sufficiently divisible N any $G \times R$-invariant section of a positive tensor power of L'_N vanishes on every codimension one component D_j.*

Remark 1.22. The proof of Theorem 1.21 tells us that when the hypotheses hold and the algebra of invariants $\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^H$ is finitely generated then the enveloping quotient

$$X \wr\!\wr H = \mathrm{Proj}(\oplus_{k\geq 0} H^0(X, L^{\otimes k})^H) \simeq \overline{G \times_{H_u} X} /\!/_{L'_N} (G \times R) \tag{4}$$

for sufficiently divisible N.

In general even when the algebra of invariants $\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^H$ on X is finitely generated and (4) is true, the morphism $X \to X \wr\!\wr H$ is not surjective and in order to study the geometry of $X \wr\!\wr H$ by identifying it with $\overline{G \times_{H_u} X} /\!/_{L'_N} (G \times R)$ we need information about the boundary $\overline{G \times_{H_u} X} \setminus G \times_{H_u} X$ of $\overline{G \times_{H_u} X}$. If, however, we are lucky enough to find a $G \times R$-equivariant projective completion $\overline{G \times_{H_u} X}$ with a linearisation L such that for sufficiently divisible N the line bundle L'_N is ample and the boundary $\overline{G \times_{H_u} X} \setminus G \times_{H_u} X$ is unstable for L'_N, then we have a situation which is almost as well behaved as for reductive group actions on projective varieties with ample linearisations, as follows.

Definition 1.23. *Let $X^{\overline{ss}} = X \cap \overline{G \times_{H_u} X}^{ss, G\times R}$ and $X^{\overline{s}} = X \cap \overline{G \times_{H_u} X}^{s, G \times R}$ where X is embedded in $G \times_{H_u} X$ in the obvious way as $x \mapsto [1, x]$.*

Theorem 1.24. ([4] Thm 2.9). *Let X be a complex projective variety acted on by a linear algebraic group $H = H_u \rtimes R$ where H_u is the unipotent radical of H and let L be a very ample linearisation of the H action defining an embedding $X \subseteq \mathbb{P}^n$. Let $H \to G$ be an H_u-faithful homomorphism into a reductive subgroup G of* $\mathrm{SL}(n+1;\mathbb{C})$ *with respect to an ample line bundle L. Let L' be a $G \times R$-linearisation over a projective completion $\overline{G \times_{H_u} X}$ of $G \times_{H_u} X$*

extending the $G \times R$ linearisation over $G \times_{H_u} X$ induced by L. Let $D_1, \dots, D_r$ be the codimension 1 components of the boundary of $G \times_{H_u} X$ in $\overline{G \times_{H_u} X}$, and suppose that $L'_N = L'[N \sum_{j=1}^r D_j]$ is an ample line bundle on $\overline{G \times_{H_u} X}$ for all sufficiently divisible N. If for all sufficiently divisible N any $G \times R$-invariant section of a positive tensor power of L'_N vanishes on the boundary of $G \times_{H_u} X$ in $\overline{G \times_{H_u} X}$, then

(1) *the algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^H$ is finitely generated;*
(2) *the enveloping quotient $X \wr\wr H \cong \overline{G \times_{H_u} X} /\!/_{L'_N} (G \times R) \cong \mathrm{Proj}(\oplus_{k \geq 0} H^0(X, L^{\otimes k})^H)$ for sufficiently divisible N;*
(3) *$\overline{G \times_{H_u} X}^{ss, G \times R, L'_N} \subseteq G \times_{H_u} X$ and therefore the morphism*

$$\phi : X^{\overline{ss}} \to X \wr\wr H$$

is surjective and $X \wr\wr H$ is a categorical quotient of $X^{\overline{ss}}$;
(4) *if $x, y \in X^{\overline{ss}}$ then $\phi(x) = \phi(y)$ if and only if the closures of the H-orbits of x and y meet in $X^{\overline{ss}}$;*
(5) *ϕ restricts to a geometric quotient $X^{\overline{s}} \to X^{\overline{s}}/H \subseteq X \wr\wr H$.*

Remark 1.25. As in Proposition 1.16 we have $X^{\bar{s}} = X^s$ and $X^{ss} = X^{\overline{ss}}$ here. This is a consequence of the following generalisation of Theorem 1.13.

Theorem 1.26. ([11] 5.3.1 and 5.3.5). *Let X be a normal projective variety acted on by a linear algebraic group $H = H_u \rtimes R$ where H_u is the unipotent radical of H and let L be a very ample linearisation of the H action defining an embedding $X \subseteq \mathbb{P}^n$. Let $H \to G$ be an H_u-faithful homomorphism into a reductive subgroup G of $\mathrm{SL}(n+1; \mathbb{C})$ with respect to an ample line bundle L. Let $(\overline{G \times_U X}, L)$ be any fine $G \times R$-equivariant reductive envelope. Then*

$$X^{\overline{s}} \subseteq X^s \subseteq X^{ss} \subseteq X^{\overline{ss}} = X^{nss}.$$

The stable sets $X^{\overline{s}}$ and X^s admit quasi-projective geometric quotients by the action of H. The quotient map q restricted to the open subvariety X^{ss} is an enveloped quotient with $q : X^{ss} \to X \wr\wr H$ an enveloping quotient. There is an open subvariety $X/\!\wr H$ of $\overline{G \times_U X} /\!/ (G \times R)$ which is an inner enveloping quotient of X by the linear action of H. Moreover there is an ample line bundle L_U on $X/\!\wr H$ which pulls back to a tensor power $L^{\otimes k}$ of the line bundle L for some $k > 0$ and extends to an ample line bundle on $\overline{G \times_U X} /\!/ (G \times R)$.

If furthermore $\overline{G \times_U X}$ is normal and provides a fine strong $G \times R$-equivariant reductive envelope, then $X^{\overline{s}} = X^s$ and $X^{ss} = X^{nss}$.

2. Actions of $\mathbb{C}^+ \rtimes \mathbb{C}^*$

We will prove Theorem 0.1 by induction on the dimension of U. In this section we will study the case when $\dim(U) = 1$ so that $U \cong \mathbb{C}^+$.

Definition 2.1. Let X be a complex projective variety equipped with a linear action (with respect to an ample line bundle L) of a semi-direct product $\hat{U} = \mathbb{C}^* \ltimes \mathbb{C}^+$, where the weight of the induced $\mathbb{C}^*$ action on the Lie algebra of $U = \mathbb{C}^+$ is strictly positive. Let ξ be a non-zero element of the Lie algebra of $U = \mathbb{C}^+$ and let $\delta_\xi : H^0(X, L) \to H^0(X, L)$ be the corresponding derivation. We say that *semistability coincides with stability for the linear action of* $\hat{U}$ if the weight space with minimum weight $-\omega_{\min}$ for the action of $\mathbb{C}^*$ on $H^0(X, L)$ is contained in the image $\delta_\xi(H^0(X, L))$ of $H^0(X, L)$ under the derivation δ_ξ.

Definition 2.2. Let $\chi : \hat{U} \to \mathbb{C}^*$ be a character of the semi-direct product $\hat{U} = \mathbb{C}^* \ltimes \mathbb{C}^+$ acting linearly on X as above. Suppose that $\omega_{\min} < \omega_{\min+1} < \cdots < \omega_{\max}$ are the weights with which the one-parameter subgroup $\mathbb{C}^* \leq \hat{U}$ acts on the fibres of the line bundle $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*)}(1)$ over points of the connected components of the fixed point set $\mathbb{P}((H^0(X, L)^*)^{\mathbb{C}^*}$ for the action of $\mathbb{C}^*$ on $\mathbb{P}((H^0(X, L)^*)$; when L is very ample X embeds in $\mathbb{P}((H^0(X, L)^*)$ and the line bundle L extends to $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*)}(1)$. Let c be a positive integer such that

$$\frac{\chi}{c} = \omega_{\min} + \epsilon$$

where $\epsilon > 0$ is a sufficiently small rational number; we will call rational characters χ/c with this property *well adapted* to the linear action of $\hat{U}$, and we will call the linearisation well adapted if the trivial character 0 is well adapted. The linearisation of the action of $\hat{U}$ on X with respect to the ample line bundle $L^{\otimes c}$ can be twisted by the character χ so that the weights ω_j are replaced with $\omega_j c - \chi$; let $L_\chi^{\otimes c}$ denote this twisted linearisation. Note that the unipotent group $U = \mathbb{C}^+$ is contained in the kernel of χ and so the restriction of the linearisation to the action of U is unaffected by this twisting.

Let $X^{s,\mathbb{C}^*}_{\min+}$ denote the stable subset of X for the linear action of $\mathbb{C}^*$ with respect to the linearisation $L_\chi^{\otimes c}$ for any well adapted rational character χ/c. Let

$$X^{s,\hat{U}}_{\min+} = X \setminus \hat{U}(X \setminus X^{s,\mathbb{C}^*}_{\min+}) = \bigcap_{u \in U} u X^{s,\mathbb{C}^*}_{\min+}$$

be the complement of the $\hat{U}$-sweep (or equivalently the U-sweep) of the complement of $X^{s,\mathbb{C}^*}_{\min+}$, and let $X^{s,\hat{U}}_{L_\chi^{\otimes c}}$ and $X^{ss,\hat{U}}_{L_\chi^{\otimes c}}$ denote the stable and semistable subsets for the action of $\hat{U}$ on X with respect to the linearisation $L_\chi^{\otimes c}$. Let $X /\!\!/_{L_\chi^{\otimes c}} \hat{U}$ be the corresponding enveloping quotient. Let $X^{s,U}$ denote the stable subset for the action of U on X with respect to the linearisation L.

The aim of this section is to prove the following theorem, which we will use for our inductive proof of Theorem 0.1.

Theorem 2.3. *Let X be a complex projective variety equipped with a linear action (with respect to an ample line bundle L) of a semi-direct product* $\hat{U} = \mathbb{C}^* \ltimes \mathbb{C}^+$*, where the weight of the induced* $\mathbb{C}^*$ *action on the Lie algebra of* $U = \mathbb{C}^+$ *is strictly positive. Suppose that the linear action of* $\hat{U}$ *on X satisfies the condition that 'semistability coincides with stability' as above. If* $\chi : \hat{U} \to \mathbb{C}^*$ *is a character of* $\hat{U}$ *and c is a sufficiently divisible positive integer such that the rational character* χ/c *is well adapted for the linear action of* $\hat{U}$ *with respect to L, then after twisting this linear action by* χ/c *we have*

(1) *the* $\hat{U}$*-invariant open subset* $X^{s,\hat{U}}_{\min+}$ *of X has a geometric quotient* $\pi : X^{s,\hat{U}}_{\min+} \to X^{s,\hat{U}}_{\min+}/\hat{U}$ *by the action of* $\hat{U}$*;*

(2) *this geometric quotient* $X^{s,\hat{U}}_{\min+}/\hat{U}$ *is a projective variety and the tensor power* $L^{\otimes c}$ *of L descends to a very ample line bundle* $L_{(c,\hat{U})}$ *on* $X^{s,\hat{U}}_{\min+}/\hat{U}$*;*

(3) $X^{s,\hat{U}}_{L_\chi^{\otimes c}} = X^{ss,\hat{U}}_{L_\chi^{\otimes c}} = X^{s,\hat{U}}_{\min+}$;

(4) *the geometric quotient* $X^{s,\hat{U}}_{\min+}/\hat{U}$ *is the enveloping quotient* $X \wr\!\wr_{L_\chi^{\otimes c}} \hat{U}$;

(5) *the algebra of invariants* $\bigoplus_{k\geq 0} H^0(X, L_{k\chi}^{\otimes ck})^{\hat{U}}$ *is finitely generated and the enveloping quotient* $X \wr\!\wr_{L_\chi^{\otimes c}} \hat{U} \cong \mathrm{Proj}(\oplus_{k\geq 0} H^0(X, L_{k\chi}^{\otimes ck})^{\hat{U}})$ *is the associated projective variety;*

(6) *the tensor power* $L^{\otimes c}$ *of* L *induces a very ample line bundle on an inner enveloping quotient* $X /\!\wr U$ *for the action of* U *on* X *with a* $\mathbb{C}^*$*-equivariant embedding*

$$X /\!\wr U \to \mathbb{P}((H^0(X, L^{\otimes c})^U)^*)$$

as a quasi-projective subvariety, containing the geometric quotient $X^{s,U}/U$ *as an open subvariety, with closure* $\overline{X /\!\wr U}$ *in* $\mathbb{P}((H^0(X, L^{\otimes c})^U)^*)$;

(7) $X^{s,\hat{U}}_{\min+}$ *is a* U*-invariant open subset of* $X^{s,U}$ *and has a geometric quotient* $X^{s,\hat{U}}_{\min+}/U$ *which is a* $\mathbb{C}^*$*-invariant open subset of* $X^{s,U}/U$ *and coincides with both the stable and semistable sets* $(\overline{X /\!\wr U})^{s,\mathbb{C}^*} = (\overline{X /\!\wr U})^{ss,\mathbb{C}^*}$ *for the* $\mathbb{C}^*$ *action with respect to the linearisation on* $\mathcal{O}_{\mathbb{P}((H^0(X,L^{\otimes c})^U)^*)}(1)$ *induced by* $L_\chi^{\otimes c}$*, so that the associated GIT quotient of* $\overline{X /\!\wr U}$ *by* $\mathbb{C}^*$ *is given by*

$$\overline{X /\!\wr U}/\!/\mathbb{C}^* \cong (X^{s,\hat{U}}_{\min+}/U)/\mathbb{C}^* \cong X^{s,\hat{U}}_{\min+}/\hat{U} = X \wr\!\wr_{L_\chi^{\otimes c}} \hat{U}.$$

In order to prove Theorem 2.3, we will first prove the theorem in the case where $X = \mathbb{P}(V)$ and $L = \mathcal{O}_{\mathbb{P}(V)}(1)$, for a finite dimensional $\hat{U}$-representation V. This is done by explicit calculation: we construct a strong ample reductive envelope for the twisted linearisation $\mathcal{O}(c)_{(\chi)}$, and use the Hilbert-Mumford criteria to compute stability and semistability for this reductive envelope. Through the choice of twist, stability and semistability for the reductive envelope will turn out to be equivalent conditions. We can then deduce that the stable and finitely generated semistable loci are equal for the linear action of $\hat{U}$ with this linearisation, and that the associated enveloping quotient is projective. An explicit description of the stable locus is used to prove that the enveloping quotient map $\mathbb{P}(V)^{ss,\hat{U}} \to \mathbb{P}(V) \wr\!\wr \hat{U}$ is a geometric quotient for the $\hat{U}$-action on $\mathbb{P}(V)^{ss,\hat{U}}$. Theorem 2.3 will then follow by embedding X into a projective space and using the fact that stability behaves well under closed immersions.

2.1. The Case $(X, L) = (\mathbb{P}(V), \mathcal{O}(1))$.

Let V be a finite-dimensional representation of

$$\hat{U} = \hat{U}^{[\ell]} = U \rtimes \mathbb{C}^*$$

where $U = \mathbb{C}^+$ and $\mathbb{C}^*$ acts on $\mathrm{Lie}(U)$ with weight $\ell \geq 1$, and let $X = \mathbb{P}(V)$, with $\hat{U} \curvearrowright L = \mathcal{O}(1) \to \mathbb{P}(V)$ the canonical linearisation. As usual we write points in $\mathbb{P}(V)$ as equivalence classes $[v]$ of nonzero vectors $v \in V$ under the scaling action of $\mathbb{C}^*$ on V.

Definition 2.4. Let $V_{\min}$ be the $\mathbb{C}^*$-weight space in V of minimal weight $\omega_{\min}$, and let $\mathbb{P}(V)^0_{\hat{U},\mathcal{O}(1)}$ be the open subset of points flowing to $\mathbb{P}(V_{\min})$ under the action of $t \in \mathbb{C}^*$, as $t \to 0$.

Remark 2.5. In this situation the condition that semistability coincides with stability given in Definition 2.1 is equivalent to saying that $V_{\min}$ does not contain any fixed points for the $\mathbb{C}^+$-action on V; moreover $X^{s,\hat{U}}_{\min+} = \mathbb{P}(V)^0_{\hat{U},\mathcal{O}(1)} \setminus (U \cdot \mathbb{P}(V_{\min}))$.

We wish to prove the following proposition.

Proposition 2.6. *If $V_{\min}$ does not contain any fixed points for the $\mathbb{C}^+$-action on V, and the linearisation is twisted by a well adapted rational character χ/c, then*

(1) *there are equalities $\mathbb{P}(V)^{s,\hat{U}} = \mathbb{P}(V)^{ss,\hat{U}} = \mathbb{P}(V)^0_{\hat{U},O(1)} \setminus (U \cdot \mathbb{P}(V_{\min}))$;*
(2) *the enveloping quotients $\mathbb{P}(V)/\!\!/\hat{U}$ and $\mathbb{P}(V)/\!\!/U$ are projective varieties, and for suitably divisible integers $r > 0$ the algebras of invariants $\bigoplus_{k\geq 0} H^0(X, L^{\otimes kr})^{\hat{U}}$ and $\bigoplus_{k\geq 0} H^0(X, L^{\otimes kr})^U$ are finitely generated; and*
(3) *the enveloping quotient map $\mathbb{P}(V)^{ss,\hat{U}} \to \mathbb{P}(V)/\!\!/\hat{U}$ is a geometric quotient for the $\hat{U}$-action on $\mathbb{P}(V)^{ss,\hat{U}}$.*

In order to study the linear action of $\hat{U} = \hat{U}^{[\ell]}$ on $O(1) \to \mathbb{P}(V)$ we shall use the following trick. Consider the surjective homomorphism

$$\eta_\ell : \hat{U}^{[2\ell]} \to \hat{U}^{[\ell]}, \quad (u;t) \mapsto (u;t^2).$$

We can pull back the linear action of $\hat{U} = \hat{U}^{[\ell]}$ to a linear action of $\hat{U}^{[2\ell]}$ via η_ℓ. The (finitely generated) semistable loci for the linear actions of $\hat{U}^{[\ell]}$ and $\hat{U}^{[2\ell]}$ then coincide, and the same is true for the enveloping quotients. Moreover the stable loci $X^{s(\hat{U}^{[2\ell]},O(c)_{(\chi^{[2]})})}$ and $X^{s(\hat{U}^{[\ell]},O(1)^{(\chi)})}$ coincide.

Lemma 2.7. $X^{s(\hat{U}^{[2\ell]},O(c)_{(\chi^{[2]})})} = X^{s(\hat{U}^{[\ell]},O(1)^{(\chi)})}$.

Proof. The algebras of invariants for the linear actions $\hat{U}^{[2\ell]} \curvearrowright O(c)_{(\chi^{[2]})} \to \mathbb{P}(V)$ and $\hat{U}^{[\ell]} \curvearrowright O(c)_{(\chi)}$ are equal. Let f be an invariant section with X_f affine. Because $\eta_\ell : \hat{U}^{[2\ell]} \to \hat{U}^{[\ell]}$ has finite kernel, the action of $\hat{U}^{[\ell]}$ on X_f is closed with all stabilisers finite if and only if the same is true for the action of $\hat{U}^{[2\ell]}$, and because η_ℓ restricts to identify the unipotent radicals of $\hat{U}^{[\ell]}$ and $\hat{U}^{[2\ell]}$, the natural morphism $X_f \to \operatorname{Spec}(O(X_f)^{\hat{U}^{[\ell]}}) = \operatorname{Spec}(O(X_f)^{\hat{U}^{[2\ell]}})$ is a principal $(\hat{U}^{[\ell]})_u$-bundle if and only if it is a principal $(\hat{U}^{[2\ell]})_u$-bundle. Thus, by Definition 1.17, the stable loci are equal. □

In order to prove Proposition 2.6 we may therefore work with the linear action $\hat{U}^{[2\ell]} \curvearrowright O(c)_{(\chi^{[2]})} \to \mathbb{P}(V)$, without loss of generality.

Now the $\hat{U}^{[2\ell]}$-representation V defined by η_ℓ admits a decomposition

$$V \overset{\hat{U}^{[2\ell]}}{\cong} \bigoplus_{i=1}^{q} \mathbb{C}^{(a_i)} \otimes \operatorname{Sym}^{l_i}\mathbb{C}^2, \tag{5}$$

of $\hat{U}^{[2\ell]}$-modules, where

- $\mathbb{C}^{(a_i)}$ is the one dimensional representation of $\hat{U}^{[2\ell]}$ defined by the character $\hat{U}^{[2\ell]} \to \mathbb{C}^*$ of weight $a_i \in \mathbb{Z}$;
- $\operatorname{Sym}^{l_i}\mathbb{C}^2$ is the standard irreducible representation of $G := \operatorname{SL}(2,\mathbb{C})$ of highest weight $l_i \geq 0$, upon which $\hat{U}^{[2\ell]}$ acts via the surjective homomorphism

$$\rho_\ell : \hat{U}^{[2\ell]} \to \hat{U}^{[2]}, \quad (u;t) \mapsto (u;t^\ell)$$

and the identification of $\hat{U}^{[2]}$ with the Borel subgroup $B \subseteq G$ of upper triangular matrices given by

$$\hat{U}^{[2]} = \mathbb{C}^+ \rtimes \mathbb{C}^* \to B, \quad (u;t) \mapsto \left(\begin{smallmatrix} t & tu \\ 0 & t^{-1} \end{smallmatrix}\right);$$

and

- because the action of $\hat{U}^{[2\ell]}$ factors through $\eta_\ell : \hat{U}^{[2\ell]} \to \hat{U}^{[\ell]}$, we have $a_i \equiv \ell\, l_i \mod 2$ for each $i = 1, \ldots, q$.

We wish to produce a strong reductive envelope for the linear action $\hat{U}^{[2\ell]} \curvearrowright \mathcal{O}(c)_{(\chi^{[2]})} \to \mathbb{P}(V)$ twisted by the pullback of χ/c. Observe that $\rho_\ell : \hat{U}^{[2\ell]} \to \hat{U}^{[2]} \cong B \subseteq G$ restricts to give the standard inclusion of the unipotent radical $U = \mathbb{C}^+ = (\hat{U}^{[2\ell]})_u$ of $\hat{U}^{[2\ell]}$ inside G as the subgroup of strictly upper triangular matrices, so ρ_ℓ is an $(\hat{U}^{[2\ell]})_u$-faithful homomorphism, in the sense of Definition 1.20. The linear action of $(\hat{U}^{[2\ell]})_u$ on V extends to a linear action of G by demanding that G act on $\mathrm{Sym}^{l_i}\mathbb{C}^2$ in the usual manner and trivially on $\mathbb{C}^{(a_i)}$, for each i.

There is therefore an isomorphism of $G \times (\hat{U}^{[2\ell]})_r$-spaces (where $(\hat{U}^{[2\ell]})_r = \hat{U}^{[2\ell]}/(\hat{U}^{[2\ell]})_u$)

$$G \times_U \mathbb{P}(V) \cong (G/U) \times \mathbb{P}(V)$$

which lifts to an isomorphism of linearisations. A straightforward examination of the corresponding $G \times (\hat{U}^{[2\ell]})_r$-actions and linearisations on $(G/U) \times \mathbb{P}(V)$ yields

Lemma 2.8. *Let $\mathcal{P}$ denote the $G \times (\hat{U}^{[2\ell]})_r$-linearisation on $\mathcal{O}(1)$ induced by the action $\underset{G}{\cdot}$ of G on V and the following action of $(\hat{U}^{[2\ell]})_r = \mathbb{C}^*$:*

$$(6) \qquad t \cdot v = \textstyle\sum_i (t^{a_i} z_i) \otimes s_i, \quad v \in V,\ t \in \mathbb{C}^*, \quad v = \sum_i z_i \otimes s_i \in \bigoplus_{i=1}^q \mathbb{C}^{(a_i)} \otimes \mathrm{Sym}^{l_i}\mathbb{C}^2 \text{ via (5).}$$

Then the linearisation of $G \times (\hat{U}^{[2\ell]})_r$ on $(G/U) \times (\mathcal{O}(c)_{(\chi^{[2]})}) \to (G/U) \times \mathbb{P}(V)$ is equal to the product of the twisted linearisation $(\mathcal{P}_{(\chi^{[2\ell]})})^{\otimes c}$ with the $G \times (\hat{U}^{[2\ell]})_r$-action on G/U given by left multiplication by G and right multiplication by $(\hat{U}^{[2\ell]})_r = \mathbb{C}^*$.

The homomorphism ρ_ℓ embeds $U = (\hat{U}^{[2\ell]})_u$ into $G = \mathrm{SL}(2;\mathbb{C})$ as a Grosshans subgroup, since there is an isomorphism $G/U \cong \mathbb{C}^2 \setminus \{0\}$ given by considering the orbit of $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathbb{C}^2$ under the defining representation of G. The inclusion $\mathbb{C}^2 \setminus \{0\} \hookrightarrow \mathbb{C}^2$ defines a nonsingular affine completion $\overline{G/H_u}^{\mathrm{aff}}$ which contains G/U with codimension 2 complement. We may therefore construct a strong ample reductive envelope as a $G \times \hat{U}^{[2\ell]}$-equivariant nonsingular projective completion of G/U by regarding elements of $\mathbb{C}^3$ as column vectors, and adding a hyperplane at infinity to $\overline{G/U}^{\mathrm{aff}} = \mathbb{C}^2$: if $\mathbb{P}^2 = \{[v_0 : v_1 : v_2] \mid 0 \neq (v_0, v_1, v_2)^{\mathrm{t}} \in \mathbb{C}^3\}$ with the hyperplane at infinity defined by $v_0 = 0$ then the action of $G \times (\hat{U}^{[2\ell]})_r = G \times \mathbb{C}^*$ on $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}^3)$ is the one defined by the representation given in block form

$$(g,t) \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g\left(\begin{smallmatrix} t^{-\ell} & 0 \\ 0 & t^{-\ell} \end{smallmatrix}\right) \end{array}\right) \in \mathrm{GL}(3;\mathbb{C}), \quad g \in G,\ t \in \mathbb{C}^*,$$

where $\mathrm{GL}(3;\mathbb{C})$ acts on $\mathbb{C}^3$ by left multiplication. For any integer $N > 0$, this representation canonically defines a $G \times (\hat{U}^{[2\ell]})_r$-linearisation on $\mathcal{O}_{\mathbb{P}^2}(N) \to \mathbb{P}^2$ which restricts to the canonical linearisation on $\mathcal{O}_{G/U} \to G/U$.

Let $\beta : G \times_U \mathbb{P}(V) \cong (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}(V) \hookrightarrow \mathbb{P}^2 \times \mathbb{P}(V)$ be the induced open immersion and for $N > 0$ let

$$\mathcal{P}'_N := \mathcal{O}_{\mathbb{P}^2}(N) \boxtimes \mathcal{P}^{(\chi^{[2]})} \to \mathbb{P}^2 \times \mathbb{P}(V)$$

equipped with its natural rational $G \times (\hat{U}^{[2\ell]})_r$-linearisation, where the $G \times (\hat{U}^{[2\ell]})_r$-linearisation $\mathcal{P}_{(\chi^{[2]})} \to \mathbb{P}(V)$ is defined as in Lemma 2.8. Then the triple

$$(\mathbb{P}^2 \times \mathbb{P}(V), \beta, \mathcal{P}'_N)$$

defines a strong ample reductive envelope for $\hat{U}^{[2\ell]} \curvearrowright \mathcal{O}(c)_{(\chi^{[2]})} \to \mathbb{P}(V)$, when $N > 0$ is sufficiently large. Moreover, because U is a Grosshans subgroup of G, both the algebras of U-invariants and $\hat{U}$-invariants of any positive tensor power of the linearisation $\mathcal{O}(c)_{(\chi^{[2]})} \to \mathbb{P}(V)$ are finitely generated $\mathbb{C}$-algebras, and the enveloping quotients

$$\mathbb{P}(V) \wr\wr U \cong (\mathbb{P}^2 \times \mathbb{P}(V)) /\!\!/_{\mathcal{P}'_N} G, \quad \mathbb{P}(V) \wr\wr \hat{U}^{[2\ell]} \cong (\mathbb{P}^2 \times \mathbb{P}(V)) /\!\!/_{\mathcal{P}'_N} (G \times (\hat{U}^{[2\ell]})_r)$$

are projective varieties.

By Theorem 1.26 the stable loci $\mathbb{P}(V)^{\mathrm{s},U}$ and $\mathbb{P}(V)^{\mathrm{s},\hat{U}}$ and finitely generated semi-stable loci $\mathbb{P}(V)^{ss,U}$ and $\mathbb{P}(V)^{ss,U}$ for the linearisation $\mathcal{O}(c)_{(\chi^{[2]})} \to \mathbb{P}(V)$ may be computed as the completely stable and completely semistable loci associated to the G or $G \times (\hat{U}^{[2\ell]})_r$-linearisation $\mathcal{P}'_N$, using the Hilbert-Mumford criteria. Note that under the isomorphism (5), the minimal $\mathbb{C}^*$-weight for the $\hat{U}^{[\ell]}$-action on V is

$$\omega_{\min} = \min\{(a_i - \ell\, l_i)/2 \mid i = 1, \ldots, q\}.$$

Let us temporarily call an index $i \in \{0, \ldots, q\}$ *exceptional* if $\omega_{\min} = (a_i - \ell\, l_i)/2$.

Lemma 2.9. *Stability and semistability are equivalent for the linear action $G \times (\hat{U}^{[2\ell]})_r \curvearrowright \mathcal{P}'_N \to \mathbb{P}^2 \times \mathbb{P}(V)$. Moreover a point $p = ([1 : w_1 : w_2], [v]) \in \mathbb{P}^2 \times \mathbb{P}(V)$ is stable if and only if $p \in (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}(V)$ and, when (5) is used to write $v = \sum_i z_i \otimes s_i \in \bigoplus_i \mathbb{C}^{(a_i)} \otimes \mathrm{Sym}^{l_i}\mathbb{C}^2$ with each $s_i \neq 0$, the following two conditions hold:*

- *there is an exceptional i such that $z_i \neq 0$ and s_i is not divisible by $(w_1, w_2) \in \mathbb{C}^2 \setminus \{0\}$; and*
- *either there is a non-exceptional i such that $z_i \neq 0$, or for each $(\tilde{w}_1, \tilde{w}_2) \in \mathbb{C}^2 \setminus \{0\}$ with $[\tilde{w}_1 : \tilde{w}_2] \neq [w_1 : w_2]$ as points in $\mathbb{P}^1$ there is an exceptional i such that $z_i \neq 0$ and $s_i \neq (\tilde{w}_1, \tilde{w}_2)^{l_i} \in \mathrm{Sym}^{l_i}\mathbb{C}^2$.*

Proof of Lemma 2.9. We shall deduce this by using the Hilbert-Mumford criteria as given in Proposition 1.4 using the maximal torus $T_1 \times T_2 \subseteq G \times (\hat{U}^{[2\ell]})_r$, where T_1 is the subgroup of diagonal matrices in G and $T_2 = \mathbb{C}^* = (\hat{U}^{[2\ell]})_r$. The group of characters of $T_1 \times T_2$ is identified with $\mathbb{Z} \times \mathbb{Z}$ in the natural way. Introduce the following notation: for $i = 1, \ldots, q$ let $e_{i,1} = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $e_{i,2} = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ be the standard basis of $\mathbb{C}^2$, so that

$$e_{i,1}^{l_i}, \ldots, e_{i,1}^{j} e_{i,2}^{l_i - j}, \ldots, e_{i,2}^{l_i} \in \mathrm{Sym}^{l_i}\mathbb{C}^2$$

is a basis of T_1-weight vectors in $\mathrm{Sym}^{l_i}\mathbb{C}^2$, and consider the basis

$$1 \otimes e_{i,1}^{l_i-j} e_{i,2}^{j}, \quad j = 0, \dots, l_i, \quad i = 1, \dots, q$$

of weight vectors for the $T_1 \times T_2$-action on V. Without loss of generality, we may apply the Hilbert-Mumford criteria by using the projective space into which $\mathbb{P}^2 \times \mathbb{P}(V)$ is embedded via $\mathcal{P}'_N$ and computing rational weights. The fixed points in $\mathbb{P}^2 \times \mathbb{P}(V)$ for the $T_1 \times T_2$-action, along with the corresponding rational weights with respect to the embedding defined by $\mathcal{P}'_N$, are given in Table 1.

Fixed point $(j = 0, \dots, l_i,\ i = 1, \dots, q)$	Rational weight in $\mathrm{Hom}\,(T_1 \times T_2, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q} \times \mathbb{Q}$
$([1:0:0], [1 \otimes e_{i,1}^{j} e_{i,2}^{l_i-j}])$	$(2j - l_i, a_i - 2\omega_{\min} - 2\epsilon)$
$([0:1:0], [1 \otimes e_{i,1}^{j} e_{i,2}^{l_i-j}])$	$(2j - l_i, a_i - 2\omega_{\min} - 2\epsilon) + (N, -\ell N)$
$([0:0:1], [1 \otimes e_{i,1}^{j} e_{i,2}^{l_i-j}])$	$(2j - l_i, a_i - 2\omega_{\min} - 2\epsilon) + (-N, -\ell N)$

TABLE 1. Rational weights of the fixed points of $T_1 \times T_2 \curvearrowright \mathbb{P}^2 \times \mathbb{P}(V)$ with respect to the linearisation $\mathcal{P}'_N$.

Consider the rational weight $\vartheta := (2j - l_i, a_i - 2\omega_{\min} - 2\epsilon)$ for the fixed point $([1 : 0 : 0], [1 \otimes e_{i,1}^{j} e_{i,2}^{l_i-j}])$. Note that either ϑ is contained in the interior of the cone

$$C := \{(c_1, c_2) \in \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} \mid \ell\, c_1 + c_2 \geq 0 \text{ and } -\ell\, c_1 + c_2 \geq 0\},$$

or ϑ lies outside C and i, j satisfy $\omega_{\min} = (a_i - \ell\, l_i)/2$ and $j \in \{0, l_i\}$: because $0 < \epsilon < 1/2$ we see that

$$\ell(2j - l_i) + (a_i - 2\omega_{\min} - 2\epsilon) \begin{cases} = -2\epsilon < 0 & \text{iff } j = 0 \text{ and } \omega_{\min} = (a_i - \ell\, l_i)/2 \\ > 0 & \text{otherwise} \end{cases}$$

while

$$-\ell(2j - l_i) + (a_i - 2\omega_{\min} - 2\epsilon) \begin{cases} = -2\epsilon < 0 & \text{iff } j = l_i \text{ and } \omega_{\min} = (a_i - \ell\, l_i)/2 \\ > 0 & \text{otherwise.} \end{cases}.$$

We also claim that $a_i - 2\omega_{\min} - 2\epsilon > 0$ for all $i = 1, \dots, q$. Indeed, suppose $a_i - 2\omega_{\min} - 2\epsilon \leq 0$ for some $i = 1, \dots, q$. Because $0 < 2\epsilon < 1$ and $a_i - 2\omega_{\min} \in \mathbb{Z}$, this is equivalent to $a_i - 2\omega_{\min} \leq 0$. But $2\omega_{\min} \leq a_i - \ell\, l_i$, so $\ell\, l_i \leq a_i - 2\omega_{\min} \leq 0$. Because $\ell > 0$ we must have $l_i = 0$, and by examining the above possible cases for the value of $\ell(2j - l_i) + (a_i - 2\omega_{\min} - 2\epsilon)$ we see that $\omega_{\min} = a_i/2$ and i is exceptional. This implies there is a line $\mathbb{C}^{(\omega_{\min})} = \mathbb{C}^{(\omega_{\min})} \otimes \mathrm{Sym}^0\mathbb{C}^2 \subseteq V_{\min}$ fixed by U, which contradicts the assumption that $V_{\min}$ does not contain a point fixed by the U-action. This verifies the claim.

We thus see that for sufficiently large $N > 0$ the weights for the rational $T_1 \times T_2$-linearisation $\mathcal{P}'_N \to \mathbb{P}^2 \times \mathbb{P}(V)$ are arranged in the fashion of Figure 1. (Notice that the only weights that lie outside the chambers are the extremal weights for rows corresponding to exceptional indices. This makes calculating semistability and stability for the torus $T_1 \times T_2$ easy.) In particular, the weight polytope $\Delta_p \subseteq \mathrm{Hom}\,(T_1 \times T_2, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ for a point $p = ([w_0 : w_1 : w_2], [v]) \in \mathbb{P}^2 \times \mathbb{P}(V)$

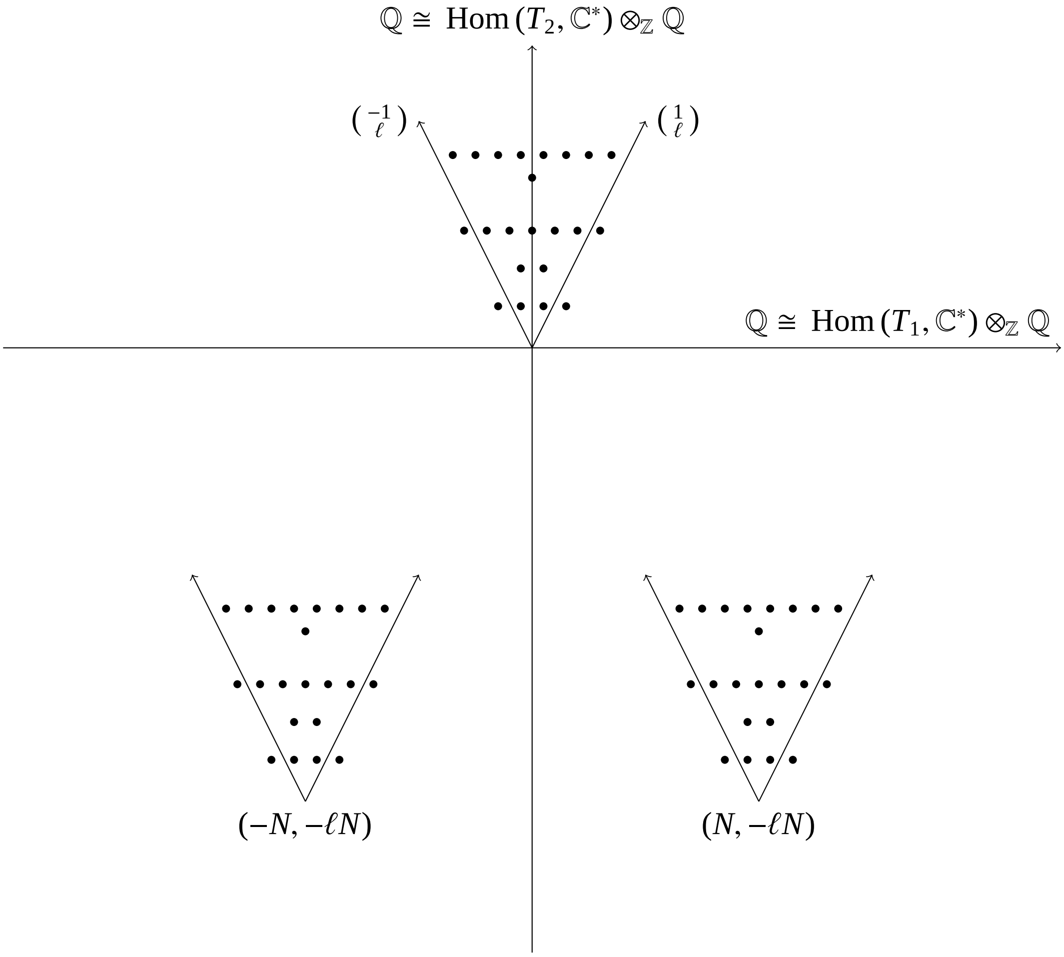

FIGURE 1. Example of distribution of rational weights for $T_1 \times T_2 \curvearrowright \mathcal{P}'_N \to \mathbb{P}^2 \times \mathbb{P}(V)$.

contains the origin precisely when the interior Δ_p° does and so semistability and stability for the rational linearisation $T_1 \times T_2 \curvearrowright \mathcal{P}'_N \to \mathbb{P}^2 \times \mathbb{P}(V)$ coincide. Using the isomorphism (5), write

$$v = \textstyle\sum_{i=1}^q z_i \otimes s_i, \quad z_i \in \mathbb{C}^{(a_i)}, \quad 0 \neq s_i = \sum_{j=0}^{l_i} v_{i,j} e_{i,1}^j e_{2,i}^{l_i-j}, \; v_{i,j} \in \mathbb{C}.$$

Then one finds that p is $T_1 \times T_2$-*unstable* precisely when $p \notin (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}(V)$ (i.e. $w_0 = 0$ or $w_1 = w_2 = 0$) or else by satisfying one of the following criteria, split into three cases:

Case $w_0 w_1 w_2 \neq 0$:

$$0 \notin \Delta_p \iff \begin{array}{l} \text{Either } v_{i,j} \neq 0 \implies (i \text{ exceptional and } j = 0), \\ \text{or } v_{i,j} \neq 0 \implies (i \text{ exceptional and } j = l_i). \end{array}$$

Case $w_0 w_1 \neq 0$, $w_2 = 0$:

$$0 \notin \Delta_p \iff \begin{array}{l} \text{Either } i \text{ exceptional} \implies v_{i,0} = 0, \\ \text{or } v_{i,j} \neq 0 \implies (i \text{ exceptional and } j = 0). \end{array}$$

Case $w_0 w_2 \neq 0$, $w_1 = 0$:

$$0 \notin \Delta_p \iff \begin{array}{l} \text{Either } i \text{ exceptional } \implies v_{i,l_i} = 0, \\ \text{or } v_{i,j} \neq 0 \implies (i \text{ exceptional and } j = l_i). \end{array}$$

By the Hilbert-Mumford criteria, the point p is (semi)stable for the $G \times (\hat{U}^{[2\ell]})_r$-linearisation if and only if $(g,t) \cdot p$ is $T_1 \times T_2$-(semi)stable for each $(g,t) \in G \times (\hat{U}^{[2\ell]})_r$. Thus stability and semistability are equivalent for $G \times (\hat{U}^{[2\ell]})_r \curvearrowright \mathcal{P}'_N \to \mathbb{P}^2 \times \mathbb{P}(V)$, and because G acts transitively on pairs of distinct points in $\mathbb{P}^1$ it follows that $p \in \mathbb{P}^2 \times \mathbb{P}(V)$ is stable precisely when $p \in (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}(V)$ and the two conditions in the statement of Lemma 2.9 are fulfilled. □

We are now in a position to complete the proof of Proposition 2.6. Lemma 2.9 tells us that for the linearisation $\mathcal{P}'_N$ we have

$$(\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{s},(\mathcal{P}'_N)} = (\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{ss}(\mathcal{P}'_N)} \subseteq (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}(V) \cong G \times_U \mathbb{P}(V),$$

so we have

$$G \times_U (\mathbb{P}(V)^{\bar{\mathrm{s}}}) = G \times_U (\mathbb{P}(V)^{\overline{\mathrm{ss}}}) = (\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{ss}(\mathcal{P}'_N)},$$

where $\mathbb{P}(V)^{\bar{\mathrm{s}}}$ and $\mathbb{P}(V)^{\overline{\mathrm{ss}}}$ are the completely stable and completely semistable locus, respectively, for the reductive envelope $(\mathbb{P}^2 \times \mathbb{P}(V), \beta, \mathcal{P}'_N)$. The GIT quotient map $(\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{ss}(\mathcal{P}'_N)} \to (\mathbb{P}^2 \times \mathbb{P}(V)) /\!\!/_{\mathcal{P}'_N} (G \times (\hat{U}^{[2\ell]})_r)$ is a geometric quotient with the inclusion $\beta \circ \alpha : \mathbb{P}(V)^{\bar{\mathrm{s}}} \hookrightarrow (\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{ss}(\mathcal{P}'_N)}$ inducing an isomorphism

$$\mathbb{P}(V)^{\bar{\mathrm{s}}} / \hat{U}^{[2\ell]} \cong (G \times_U (\mathbb{P}(V)^{\bar{\mathrm{s}}})) / (G \times (\hat{U}^{[2\ell]})_r) = (\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{ss}(\mathcal{P}'_N)} / (G \times (\hat{U}^{[2\ell]})_r).$$

Because $(\mathbb{P}^2 \times \mathbb{P}(V), \beta, \mathcal{P}'_N)$ is strong, by Theorem 1.26, Theorem 1.24 and Proposition 1.19 we have

$$\mathbb{P}(V)^{\mathrm{s},\hat{U}^{[2\ell]}} = \mathbb{P}(V)^{\bar{\mathrm{s}}} = \mathbb{P}(V)^{\overline{\mathrm{ss}}} = \mathbb{P}(V)^{ss,\hat{U}^{[2\ell]}},$$

and

$$(\mathbb{P}^2 \times \mathbb{P}(V)) /\!\!/_{\mathcal{P}'_N} (G \times (\hat{U}^{[2\ell]})_r) \cong \mathbb{P}(V) \wr\!\wr \hat{U}^{[2\ell]}$$

while the enveloping quotient map

$$\mathbb{P}(V)^{ss,\hat{U}^{[2\ell]}} = \mathbb{P}(V)^{\mathrm{s},\hat{U}^{[2\ell]}} \to \mathbb{P}(V) \wr\!\wr \hat{U}^{[2\ell]}$$

is a geometric quotient for the $\hat{U}^{[2\ell]}$-action on $\mathbb{P}(V)^{ss,\hat{U}^{[2\ell]}}$ onto a projective variety.

Proposition 2.6 (2) and (3) now follow, and to complete the proof of the proposition it remains to show that

$$\mathbb{P}(V)^{\mathrm{s},\hat{U}^{[\ell]},} = \mathbb{P}(V)^0_{\hat{U},\mathcal{O}(1)} \setminus (U \cdot \mathbb{P}(V_{\min})).$$

Recall that $\mathbb{P}(V)^{\mathrm{s},\hat{U}^{[\ell]}} = \mathbb{P}(V)^{\bar{\mathrm{s}}}$ is equal to the intersection of $\mathbb{P}(V)$ with $(\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{s}(\mathcal{P}'_N)}$ under the inclusion

$$\mathbb{P}(V) \hookrightarrow \mathbb{P}^2 \times \mathbb{P}(V), \quad [v] \mapsto ([1:1:0],[v]).$$

According to Lemma 2.9 we therefore have $[v] \in \mathbb{P}(V)^{\mathrm{s}(\hat{U}^{[\ell]},\mathcal{O}(1)^{(\chi)})}$ if, and only if, when one uses (5) to write $v = \sum_i z_i \otimes s_i \in \bigoplus_i \mathbb{C}^{(a_i)} \otimes \mathrm{Sym}^{l_i}\mathbb{C}^2$ with each $s_i \neq 0$, the following two conditions are satisfied:

- there is an exceptional i such that $z_i \neq 0$ and s_i is not divisible by $(1,0) \in \mathbb{C}^2$; and

- either there is a non-exceptional i such that $z_i \neq 0$, or for each $(w_1, w_2) \in \mathbb{C}^2 \setminus \{0\}$ with $[w_1 : w_2] \neq [1 : 0]$ as points in $\mathbb{P}^1$ there is an exceptional i such that $z_i \neq 0$ and $s_i \neq (w_1, w_2)^{l_i} \in \mathrm{Sym}^{l_i}\mathbb{C}^2$.

We can interpret each of these conditions geometrically, as follows. Under the isomorphism of vector spaces $V \cong \bigoplus_{i=1}^{q} \mathbb{C}^{(a_i)} \otimes \mathrm{Sym}^{l_i}\mathbb{C}^2$ the weight vectors for the induced $\mathbb{C}^* \subseteq \hat{U}^{[\ell]}$-action on V take the form $1 \otimes e_{1,i}^{j} e_{2,i}^{l_i - j}$, where $1 \leq i \leq q$ and $0 \leq j \leq l_i$, with the weight of $1 \otimes e_{1,i}^{j} e_{2,i}^{l_i-j}$ equal to $(a_i - \ell\, l_i + 2j)/2 \in \mathbb{Z}$. Moreover, the weight space $V_{\min}$ of minimal weight $\omega_{\min}$ is spanned by all $1 \otimes e_{2,i}^{l_i}$ with i an exceptional index, and the U-sweep $U \cdot V_{\min}$ of $V_{\min}$ is contained in the $\hat{U}^{[\ell]}$-subspace

$$\bigoplus_{i \text{ exceptional}} \mathbb{C}^{(a_i)} \otimes \mathrm{Sym}^{l_i}\mathbb{C}^2 \subseteq V.$$

Now, if $v = \sum_i z_i \otimes s_i$ with each $s_i \neq 0$, then the existence of an exceptional i with $z_i \neq 0$ and s_i not divisible by $(1, 0)$ is equivalent to $\lim_{t\to 0} t \cdot [v] \in \mathbb{P}(V_{\min})$ (where we take $t \in \mathbb{C}^* \subseteq \hat{U}^{[\ell]}$ in the limit). So the first of the above conditions is equivalent to requiring that $[v] \in \mathbb{P}(V)^0_{\hat{U},\mathcal{O}(1)}$. Now consider the second condition. The existence of a non-exceptional i such that $z_i \neq 0$ is equivalent to $v \notin \bigoplus_{i \text{ exceptional}} \mathbb{C}^{(a_i)} \otimes \mathrm{Sym}^{l_i}\mathbb{C}^2$, which itself implies $[v] \notin U \cdot \mathbb{P}(V_{\min})$. On the other hand, because of the transitivity of the U-action on $\mathbb{C} = \mathbb{P}^1 \setminus \{[1 : 0]\}$ and the fact that $V_{\min}$ is spanned by $1 \otimes e_{i,2}^{l_i}$ with i exceptional, we see that $[v] \in U \cdot \mathbb{P}(V_{\min})$ if and only if $v \in \bigoplus_{i \text{ exceptional}} \mathbb{C}^{(a_i)} \otimes \mathrm{Sym}^{l_i}\mathbb{C}^2$ and there is some $(w_1, w_2) \in \mathbb{C}^2 \setminus \{0\}$ with $[w_1 : w_2] \neq [1 : 0] \in \mathbb{P}^1$ such that $s_i = (w_1, w_2)^{l_i} \in \mathrm{Sym}^{l_i}\mathbb{C}^2$ for all exceptional i. Thus, the second condition is equivalent to demanding $[v] \notin U \cdot \mathbb{P}(V_{\min})$. It follows that

$$\mathbb{P}(V)^{s,\hat{U}^{[\ell]}} = \mathbb{P}(V)^0_{\hat{U},\mathcal{O}(1)} \setminus (U \cdot \mathbb{P}(V_{\min})),$$

as required.

This completes the proof of Proposition 2.6, and of Theorem 2.3 in the special case when $(X, L) = (\mathbb{P}(V), \mathcal{O}(1))$.

2.2. **Proof of Theorem 2.3 for general** (X, L)**.** Suppose now that $L \to X$ is a very ample line bundle over an irreducible projective variety equipped with a $\hat{U}$-linearisation, where $\hat{U}$ is a positive extension of $U = \mathbb{C}^+$, let $V = H^0(X, L)^*$ and let $\gamma : X \hookrightarrow \mathbb{P}(V)$ be the canonical closed immersion. Let $\omega_{\min}$ be the minimal weight for the induced $\mathbb{C}^*$-action on V and suppose the associated weight space $V_{\min}$ does not contain any fixed points for the U-action on V. Finally, let χ/c be a well adapted rational character.

By Proposition 2.6 the twisted linearisation $\hat{U} \curvearrowright \mathcal{O}_{\mathbb{P}(V)}(1)_{(\chi)} \to \mathbb{P}(V)$ has an enveloping quotient

$$q : \mathbb{P}(V)^{ss,\hat{U}} = \mathbb{P}(V)^{s,\hat{U}} \to \mathbb{P}(V) /\!\!/ \hat{U}$$

which is a geometric quotient for the $\hat{U}$-action on $\mathbb{P}(V)^{ss,\hat{U}}$, and the quotient $\mathbb{P}(V) /\!\!/_{\mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)}} \hat{U}$ is a projective variety. Furthermore,

$$\mathbb{P}(V)^{s,\hat{U}} = \mathbb{P}(V)^0_{\hat{U},\mathcal{O}(1)} \setminus (U \cdot \mathbb{P}(V_{\min})),$$

from which it follows that

$$\gamma^{-1}(\mathbb{P}(V)^{s,\hat{U}}) = X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}).$$

Thus $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ is an open subset of $X^{\mathrm{s}(L^{(\chi)})}$ whose image under the enveloping quotient

$$q : X^{ss,\hat{U}} \to X /\!\!/ \hat{U}$$

is a geometric quotient for the $\hat{U}$-action on $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ that embeds naturally as a closed subvariety of $\mathbb{P}(V)^{s,\hat{U}}/\hat{U} = \mathbb{P}(V) /\!\!/_{\mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)}} \hat{U}$. Hence $q(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}))$ is itself a projective variety. In particular it is complete, and since $X /\!\!/_{L^{(\chi)}} \hat{U}$ is separated over $\operatorname{Spec} \mathbb{C}$ it follows that the inclusion $q(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})) \hookrightarrow X /\!\!/_{L^{(\chi)}} \hat{U}$ is a closed map [37, Tag 01W0]. On the other hand, because X is irreducible $q(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}))$ is a dense open subset of $X /\!\!/ \hat{U}$, hence

$$q(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})) = X /\!\!/ \hat{U}.$$

In particular $X /\!\!/ \hat{U}$ is a projective variety, and Theorem 2.3 now follows from Theorem 1.26, Theorem 1.24 and Proposition 1.19, together with the case when $(X, L) = (\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$ already proved.

3. Actions of $\mathbb{C}^*$-extensions of unipotent groups

Now let U be any graded unipotent group; that is, U is a unipotent group with a one-parameter group of automorphisms $\lambda : \mathbb{C}^* \to \operatorname{Aut}(U)$ such that the weights of the induced $\mathbb{C}^*$ action on the Lie algebra $\mathfrak{u}$ of U are all strictly positive. Then we can form the semidirect product

$$\hat{U} = \mathbb{C}^* \ltimes U$$

given by $\mathbb{C}^* \times U$ with group multiplication

$$(z_1, u_1).(z_2, u_2) = (z_1 z_2, (\lambda(z_2^{-1})(u_1))u_2).$$

Definition 3.1. Let $\chi : \hat{U} \to \mathbb{C}^*$ be a character of $\hat{U}$. Note that its kernel must contain U; we will identify such characters χ with integers so that the integer 1 corresponds to the character which fits into the exact sequence $\{1\} \to U \to \hat{U} \to \mathbb{C}^* \to \{1\}$. Suppose that $\omega_{\min} < \omega_{\min+1} < \cdots < \omega_{\max}$ are the weights with which the one-parameter subgroup $\mathbb{C}^* \leq \hat{U}$ acts on the fibres of the line bundle $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*)}(1)$ over points of the connected components of the fixed point set $\mathbb{P}((H^0(X, L)^*)^{\mathbb{C}^*}$ for the action of $\mathbb{C}^*$ on $\mathbb{P}((H^0(X, L)^*)$; when L is very ample X embeds in $\mathbb{P}((H^0(X, L)^*)$ and the line bundle L extends to $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*)}(1)$. We will assume that there exist at least two distinct such weights since otherwise the action of U on X is trivial. Let c be a positive integer such that

$$\frac{\chi}{c} = \omega_{\min} + \epsilon$$

where $\epsilon > 0$ is a sufficiently small rational number; we will call rational characters χ/c with this property *well adapted* to the linear action of $\hat{U}$, and we will call the linearisation well adapted if the trivial character 0 is well adapted. The linearisation of the action of $\hat{U}$ on X with respect to the ample line bundle $L^{\otimes c}$ can be twisted by the character χ so that the weights ω_j are replaced with $\omega_j c - \chi$; let $L_\chi^{\otimes c}$ denote this twisted linearisation.

Let $X^{s,\mathbb{C}^*}_{\min +}$ denote the stable subset of X for the linear action of $\mathbb{C}^*$ with respect to the linearisation $L_\chi^{\otimes c}$ for any well adapted rational character χ/c; more precisely by VGIT (variation of GIT) [10, 38] if L is very ample we can take any rational character χ/c such that $\omega_{\min} < \chi/c < \omega_{\min +1}$ here. Let

$$X^{s,\hat{U}}_{\min +} = X \setminus \hat{U}(X \setminus X^{s,\mathbb{C}^*}_{\min +}) = \bigcap_{u \in U} u X^{s,\mathbb{C}^*}_{\min +}$$

be the complement of the $\hat{U}$-sweep (or equivalently the U-sweep) of the complement of $X^{s,\mathbb{C}^*}_{\min +}$, and let $X^{s,\hat{U}}_{L_\chi^{\otimes c}}$ and $X^{ss,\hat{U}}_{L_\chi^{\otimes c}}$ denote the stable and semistable subsets for the action of $\hat{U}$ on X with respect to the linearisation $L_\chi^{\otimes c}$. Let $X \wr\wr_{L_\chi^{\otimes c}} \hat{U}$ be the corresponding enveloping quotient.

Recall that the main theorem of this paper concerns a linear action of $\hat{U}$ on a projective variety X which is well adapted in the sense above and satisfies an additional condition to which we will refer as the condition that 'semistability coincides with stability'.

Definition 3.2. Any element ξ of the Lie algebra of U defines a derivation $\delta_\xi : H^0(X, L) \to H^0(X, L)$. We say that *semistability coincides with stability for the linear action of $\hat{U}$* if whenever U' is a subgroup of U normalised by $\mathbb{C}^*$ and ξ belongs to the Lie algebra of U but not the Lie algebra of U' and ξ is a weight vector for the action of $\mathbb{C}^*$, then the weight space with weight $-\omega_{\min}$ for the action of $\mathbb{C}^*$ on $H^0(X, L)$ is contained in the image $\delta_\xi(H^0(X, L)^{U'})$ of $H^0(X, L)^{U'}$ under the derivation δ_ξ.

We will say that the linear action of $\hat{U}$ on X is *very well adapted* if it is well adapted and semistability coincides with stability in this sense.

Lemma 3.3. *Suppose that a linear action of $\hat{U}$ on a projective variety X with respect to an ample line bundle L satisfies the condition that semistability equals stability. If $U_\dagger$ is a subgroup of U which is normal in $\hat{U}$ and $\hat{U}_\dagger = \mathbb{C}^* \ltimes U_\dagger$ is the subgroup of $\hat{U}$ generated by $U_\dagger$ and the one-parameter subgroup $\mathbb{C}^*$, then*

1. *the linear action of $\hat{U}$ on X with respect to any positive tensor power $L^{\otimes m}$ of L satisfies the condition that semistability equals stability;*
2. *the restriction to $\hat{U}_\dagger$ of the linear action of $\hat{U}$ on X with respect to any positive tensor power $L^{\otimes m}$ of L satisfies the condition that semistability equals stability;*
3. *if $c_\dagger$ is a sufficiently divisible positive integer then the induced linear action of $\hat{U}/U_\dagger$ on the closure $\overline{X/\!\wr U_\dagger}$ in $\mathbb{P}((H^0(X, L^{\otimes c_\dagger})^{U_\dagger})^*)$ of an inner enveloping quotient $X/\!\wr U_\dagger$ for the action of $U_\dagger$ on X satisfies the condition that semistability equals stability with respect to the ample line bundle determined by $L^{\otimes c}$.*

Proof: (1) Suppose that U' is a subgroup of U normalised by $\mathbb{C}^*$ and that a $\mathbb{C}^*$-weight vector ξ with weight a belongs to the Lie algebra of U but not the Lie algebra of U', with corresponding derivation $\delta_\xi = \delta : H^0(X, L) \to H^0(X, L)$. By abuse of notation let δ also denote the induced derivation on $H^0(X, L^{\otimes m})$. As X is $\mathbb{C}^*$-invariant, the minimum weight $\omega_{\min}^{L^{\otimes m}}$ with which the one-parameter subgroup $\mathbb{C}^* \leq \hat{U}$ acts on the fibres of the line bundle $\mathcal{O}_{\mathbb{P}((H^0(X,L^{\otimes m})^*)}(1)$ over points of the connected components of the fixed point set $\mathbb{P}((H^0(X, L^{\otimes m})^*)^{\mathbb{C}^*}$ for the action of $\mathbb{C}^*$ on $\mathbb{P}((H^0(X, L^{\otimes m})^*)$ is $m\omega_{\min}$. Suppose that $s \in H^0(X, L^{\otimes c})^{U'}$ is a weight vector with weight $\omega_{\min}^{L^{\otimes m}}$

for the action of $\mathbb{C}^*$. We want to show that there is some section $s' \in H^0(X, L^{\otimes m})^{U'}$ such that $\delta(s') = s$. Since $\omega_{\min}^{L^{\otimes m}} = m\omega_{\min}$ we can write s as a linear combination of monomials $s_1 \cdots s_m$ where $s_j \in H^0(X, L)$ is a weight vector with weight $\omega_{\min}$ for the $\mathbb{C}^*$ action, which implies that $\delta(s_j) = 0$ for $j = 1, \ldots, m$. As the linear action of $\hat{U}$ on X with respect to L satisfies the condition that semistability equals stability, there is $s_1' \in H^0(X, L)^{U'}$ such that $\delta(s_1') = s_1$. It follows that

$$\delta(s_1' s_2 \cdots s_m) = s_1 \cdots s_m$$

where $s_1' s_2 \cdots s_m \in H^0(X, L^{\otimes m})^{U'}$ as required.

(2) By (1) we can assume that $m = 1$ and then this follows straight from the definition of well adaptedness (Definition 3.2).

(3) A subgroup of $U/U_\dagger$ normalised by $\mathbb{C}^*$ has the form $U'/U_\dagger$ where U' is a subgroup of U containing $U_\dagger$ and normalised by $\mathbb{C}^*$. A weight vector in the Lie algebra of $U/U_\dagger$ which does not lie in the Lie algebra of $U'/U_\dagger$ can be represented by a weight vector ξ in the Lie algebra of U not lying in the Lie algebra of U', and the corresponding derivation on $H^0(X, L^{\otimes c_\dagger})^{U_\dagger}$ is the restriction of the derivation on $H^0(X, L^{\otimes c_{dagger}})$ determined by ξ, so (3) follows from (1). □

Our aim is to prove the following theorem, from which Theorem 0.1 and Corollary 0.4 will follow.

Theorem 3.4. *Let X be a complex projective variety equipped with a linear action (with respect to an ample line bundle L) of a unipotent group U with a one-parameter group of automorphisms such that the weights of the induced $\mathbb{C}^*$ action on the Lie algebra of U are all strictly positive. Suppose that the linear action of U on X extends to a linear action of the semi-direct product $\hat{U} = \mathbb{C}^* \ltimes U$. Suppose also that the linear action of $\hat{U}$ on X satisfies the condition that 'semistability coincides with stability' as above. If $\chi : \hat{U} \to \mathbb{C}^*$ is a character of $\hat{U}$ and c is a sufficiently divisible positive integer such that the rational character χ/c is well adapted for the linear action of $\hat{U}$ with respect to L, then after twisting this linear action by χ/c we have*

1. *the $\hat{U}$-invariant open subset $X^{s,\hat{U}}_{\min+}$ of X has a geometric quotient $\pi : X^{s,\hat{U}}_{\min+} \to X^{s,\hat{U}}_{\min+}/\hat{U}$ by the action of $\hat{U}$;*
2. *this geometric quotient $X^{s,\hat{U}}_{\min+}/\hat{U}$ is a projective variety and the tensor power $L^{\otimes c}$ of L descends to an ample line bundle $L_{(c,\hat{U})}$ on $X^{s,\hat{U}}_{\min+}/\hat{U}$;*
3. $X^{s,\hat{U}}_{L_\chi^{\otimes c}} = X^{ss,\hat{U}}_{L_\chi^{\otimes c}} = X^{s,\hat{U}}_{\min+}$;
4. *the geometric quotient $X^{s,\hat{U}}_{\min+}/\hat{U}$ is the enveloping quotient $X \wr\wr_{L_\chi^{\otimes c}} \hat{U}$;*
5. *the algebra of invariants $\bigoplus_{k\geq 0} H^0(X, L_{k\chi}^{\otimes ck})^{\hat{U}}$ is finitely generated and the enveloping quotient $X \wr\wr_{L_\chi^{\otimes c}} \hat{U} \cong \mathrm{Proj}(\oplus_{k\geq 0} H^0(X, L_{k\chi}^{\otimes ck})^{\hat{U}})$ is the associated projective variety.*

Remark 3.5. Note that, for any positive integer m, Theorem 3.4 holds for a linear action of $\hat{U}$ on X with respect to an ample line bundle L if it holds for the induced linearisation of the action of $\hat{U}$ with respect to the line bundle $L^{\otimes m}$. To see this, we use Lemma 3.3 and observe that almost all the ingredients of the theorem are unchanged when L is replaced with $L^{\otimes m}$. The only ingredients over which we still need to take care are the concept of well adaptedness and the definition of $X^{s,\hat{U}}_{\min+}$, which both depend on the weights of the action of $\mathbb{C}^*$ on $H^0(X, L)$. However since X is

$\mathbb{C}^*$-invariant the minimum weight $\omega_{\min}^{L^{\otimes m}}$ with which the one-parameter subgroup $\mathbb{C}^* \leq \hat{U}$ acts on the fibres of the line bundle $\mathcal{O}_{\mathbb{P}((H^0(X,L^{\otimes m})^*)}(1)$ over points of the connected components of the fixed point set $\mathbb{P}((H^0(X,L^{\otimes m})^*)^{\mathbb{C}^*}$ for the action of $\mathbb{C}^*$ on $\mathbb{P}((H^0(X,L^{\otimes m})^*)$ is $m\omega_{\min}$, and by variation of GIT for the reductive group $\mathbb{C}^*$ we know that if $\omega_{\min} < \chi/c < \omega_{\min+1}$ then the stable set $X^{s,\mathbb{C}^*}_{\min+}$ for the linear action of $\mathbb{C}^*$ with respect to the linearisation $L_\chi^{\otimes c}$ is the same as the stable set for the linear action of $\mathbb{C}^*$ with respect to the linearisation $(L_\chi^{\otimes c})^{\otimes m} = L_{m\chi}^{\otimes cm}$. So $X^{s,\mathbb{C}^*}_{\min+}$ and $X^{s,\hat{U}}_{\min+} = X \setminus \hat{U}(X \setminus X^{s,\mathbb{C}^*}_{\min+})$ are unchanged by replacing L with $L^{\otimes m}$. Finally

$$\chi/c = \omega_{\min} + \epsilon \quad \text{iff} \quad m\chi/c = m\omega_{\min} + m\epsilon = \omega_{\min}^{L^{\otimes m}} + m\epsilon$$

where $(L_{k\chi}^{\otimes ck})^{\otimes m} = L_{mk\chi}^{\otimes ckm} = (L^{\otimes m})_{km\chi}^{\otimes ck}$ for any $k \geq 0$. Thus χ/c is well adapted for the linear action of $\hat{U}$ with respect to L if and only if $m\chi/c$ is well adapted for the linear action of $\hat{U}$ with respect to $L^{\otimes m}$. Note however that it is not always true that

$$\omega_{\min} < \chi/c < \omega_{\min+1} \quad \text{iff} \quad \omega_{\min}^{L^{\otimes m}} < m\chi/c < \omega_{\min+1}^{L^{\otimes m}}$$

since in general $\omega_{\min+1}^{L^{\otimes m}} < m\omega_{\min+1}$ although $\omega_{\min}^{L^{\otimes m}} = m\omega_{\min}$.

Proof of Theorem 3.4: We will use induction on the dimension of U to prove a slightly stronger result including

(6) the tensor power $L^{\otimes c}$ of L induces a very ample line bundle on an inner enveloping quotient $X /\!\!\wr U$ for the action of U on X with a $\mathbb{C}^*$-equivariant embedding

$$X /\!\!\wr U \to \mathbb{P}((H^0(X, L^{\otimes c})^U)^*)$$

as a quasi-projective subvariety, containing the geometric quotient $X^{s,U}/U$ as an open subvariety, with closure $\overline{X /\!\!\wr U}$ in $\mathbb{P}((H^0(X, L^{\otimes c})^U)^*)$, and

(7) $X^{s,\hat{U}}_{\min+}$ is a U-invariant open subset of $X^{s,U}$ and has a geometric quotient $X^{s,\hat{U}}_{\min+}/U$ which is a $\mathbb{C}^*$-invariant open subset of $X^{s,U}/U$ and coincides with both the stable and semistable sets $(\overline{X /\!\!\wr U})^{s,\mathbb{C}^*} = (\overline{X /\!\!\wr U})^{ss,\mathbb{C}^*}$ for the $\mathbb{C}^*$ action with respect to the linearisation on $\mathcal{O}_{\mathbb{P}((H^0(X,L^{\otimes c})^U)^*)}(1)$ induced by $L_\chi^{\otimes c}$, so that the associated GIT quotient

$$\overline{X /\!\!\wr U} /\!/ \mathbb{C}^* \cong (X^{s,\hat{U}}_{\min+}/U)/\mathbb{C}^* \cong X^{s,\hat{U}}_{\min+}/\hat{U} = X /\!\!\wr_{L_\chi^{\otimes c}} \hat{U}.$$

When $\dim(U) = 1$ so that $U = \mathbb{C}^+$, this extended version of Theorem 3.4 including (6) and (7) follows immediately from Theorem 2.3.

Now suppose that $\dim(U) > 1$ and that the extended result is true for all strictly smaller values of $\dim(U)$. We can assume without loss of generality that U is nontrivial. The centre of U is then nontrivial and isomorphic to a product of copies of $\mathbb{C}^+$ on which $\mathbb{C}^*$ acts with positive weights. So U has a normal subgroup U_0 which is central in U and normal in $\hat{U}$ and is isomorphic to $\mathbb{C}^+$, such that the given one-parameter group $\mathbb{C}^* \leq \hat{U}$ of automorphisms of U preserves U_0 and acts on the Lie algebra of U_0 with positive weight. By induction on the dimension of U, we can now find a subgroup $U_\dagger$ of U which is normal in $\hat{U}$ and such that $U/U_\dagger$ is one-dimensional and so isomorphic to $\mathbb{C}^+$, while $\hat{U}/U_\dagger$ is a semidirect product of $U/U_\dagger$ by $\mathbb{C}^*$ where $\mathbb{C}^*$ acts on the Lie algebra of $U/U_\dagger$ with strictly positive weight. Let $\hat{U}_\dagger = \mathbb{C}^* \ltimes U_\dagger$ be the subgroup of $\hat{U}$ generated by $U_\dagger$ and the one-parameter subgroup $\mathbb{C}^*$.

By Lemma 3.3 the linear action of $\hat{U}_\dagger$ on X satisfies the condition that semistability is the same as stability. Thus by induction on the dimension of U we can assume that, for a sufficiently divisible positive integer $c_\dagger$,

(i) the $\hat{U}_\dagger$-invariant open subset $X^{s,\hat{U}_\dagger}_{\min+}$ of X has a geometric quotient $\pi : X^{s,\hat{U}_\dagger}_{\min+} \to X^{s,\hat{U}_\dagger}_{\min+}/\hat{U}_\dagger$ by the action of $\hat{U}_\dagger$;

(ii) this geometric quotient $X^{s,\hat{U}_\dagger}_{\min+}/\hat{U}_\dagger$ is a projective variety and the tensor power $L^{\otimes c_\dagger}$ of L descends to an ample line bundle $L_{(c_\dagger,\hat{U}_\dagger)}$ on $X^{s,\hat{U}_\dagger}_{\min+}/\hat{U}_\dagger$;

(iii) the tensor power $L^{\otimes c_\dagger}$ of L induces a very ample line bundle on an inner enveloping quotient $X /\!\!\wr U_\dagger$ for the action of $U_\dagger$ on X with a $\mathbb{C}^*$-equivariant embedding $X /\!\!\wr U_\dagger \to \mathbb{P}((H^0(X, L^{\otimes c_\dagger})^U_\dagger)^*)$ as a quasi-projective subvariety, containing the geometric quotient $X^{s,U_\dagger}/U_\dagger$ as an open subvariety, with closure $\overline{X/\!\wr U_\dagger}$ in $\mathbb{P}((H^0(X, L^{\otimes c_\dagger})^{U_\dagger})^*)$, and

(iv) $X^{s,\hat{U}_\dagger}_{\min+}$ is a $U_\dagger$-invariant open subset of $X^{s,U_\dagger}$ and has a geometric quotient $X^{s,\hat{U}_\dagger}_{\min+}/U_\dagger$ which is a $\mathbb{C}^*$-invariant open subset of $X^{s,U_\dagger}/U_\dagger$ and, if the rational character $\chi/c_\dagger$ is well adapted for the linear action of $\hat{U}_\dagger$ with respect to L, coincides with both the stable and semistable sets $(\overline{X/\!\wr U_\dagger})^{s,\mathbb{C}^*} = (\overline{X/\!\wr U_\dagger})^{ss,\mathbb{C}^*}$ for the $\mathbb{C}^*$ action with respect to the linearisation on $\mathcal{O}_{\mathbb{P}((H^0(X,L^{\otimes c_\dagger})^{U_\dagger})^*)}(1)$ induced by $L_\chi^{\otimes c_\dagger}$, so that the associated GIT quotient

$$\overline{X/\!\wr U_\dagger}/\!/\mathbb{C}^* \cong (X^{s,\hat{U}_\dagger}_{\min+}/U_\dagger)/\mathbb{C}^* \cong X^{s,\hat{U}_\dagger}_{\min+}/\hat{U}_\dagger = X /\!\!\wr_{L_\chi^{\otimes c_\dagger}} \hat{U}_\dagger.$$

Note that $X^{s,\hat{U}}_{\min+}$ is a $\hat{U}$-invariant open subset of $X^{s,\hat{U}_\dagger}_{\min+}$. We have an induced linear action of $\hat{U}/U_\dagger \cong \mathbb{C}^+ \rtimes \mathbb{C}^*$ on $\mathbb{P}((H^0(X, L^{\otimes c_\dagger})^{U_\dagger})^*)$ which restricts to a linear action on $\overline{X/\!\wr U_\dagger}$ and to the induced linear action of the open subset $X^{s,\hat{U}}_{\min+}/U_\dagger$ of $X^{s,\hat{U}_\dagger}_{\min+}/U_\dagger = (\overline{X/\!\wr U_\dagger})^{s,\mathbb{C}^*}$. We have $X^{s,\hat{U}}_{\min+} = \bigcap_{u\in U} X^{s,\mathbb{C}^*}_{\min+}$ so that

$$X^{s,\hat{U}}_{\min+}/U_\dagger = \bigcap_{u\in U/U_\dagger} u\,(X^{s,\hat{U}_\dagger}_{\min+}/U_\dagger) = \bigcap_{u\in U/U_\dagger} u\;(\overline{X/\!\wr U_\dagger})^{s,\mathbb{C}^*} = (\overline{X/\!\wr U_\dagger})^{s,\hat{U}/U_\dagger}_{\min+}.$$

By Lemma 3.3 we can apply Theorem 2.3 to the action of $\hat{U}/U_\dagger$ on the closure $\overline{X/\!\wr U_\dagger}$ in $\mathbb{P}((H^0(X, L^{\otimes c_\dagger})^{U_\dagger})^*)$ of the inner enveloping quotient $X/\!\wr U_\dagger$ for the action of $U_\dagger$ on X. It follows that $X^{s,\hat{U}}_{\min+}/U_\dagger = (\overline{X/\!\wr U_\dagger})^{s,\hat{U}/U_\dagger}_{\min+}$ has a geometric quotient

$$(X^{s,\hat{U}}_{\min+}/U_\dagger)/(\hat{U}/U_\dagger)$$

which is then a geometric quotient for the action of $\hat{U}$ on $X^{s,\hat{U}}_{\min+}$. Furthermore by Theorem 2.3 this geometric quotient $(X^{s,\hat{U}}_{\min+}/U_\dagger)/(\hat{U}/U_\dagger) = X^{s,\hat{U}}_{\min+}/\hat{U}$ is a projective variety and for a sufficiently divisible multiple c of $c_\dagger$ the tensor power $L^{\otimes c}$ of L descends to a very ample line bundle $L_{(c,\hat{U})}$ on $X^{s,\hat{U}}_{\min+}/\hat{U}$; in addition if $\chi/c = \omega_{\min} + \epsilon$ where $\epsilon > 0$ is sufficiently small then $X^{s,\hat{U}}_{\min+}/U_\dagger$ is the stable set for the $\hat{U}/U_\dagger$-action on $\overline{X/\!\wr U_\dagger}$ with respect to the linearisation induced by $L^{\otimes c}$ and twisted by the rational character χ/c, so

$$X^{s,\hat{U}}_{\min+} \subseteq X^{s,\hat{U}}_{L_\chi^{\otimes c}}$$

by Remark 1.18(ii).

Conversely if $\epsilon < \omega_{\min+1} - \omega_{\min}$ then $X^{s,\hat{U}}_{L_\chi^{\otimes c}}$ is a $\hat{U}$-invariant subset of $X^{s,\mathbb{C}^*}_{\min+} = X^{s,\mathbb{C}^*}_{L_\chi^{\otimes c}}$ by Remark 1.18(i), so

$$X^{s,\hat{U}}_{L_\chi^{\otimes c}} \subseteq \bigcap_{u \in U} u\, X^{s,\mathbb{C}^*}_{\min+} = X^{s,\hat{U}}_{\min+}$$

and hence $X^{s,\hat{U}}_{L_\chi^{\otimes c}} = X^{s,\hat{U}}_{\min+}$.

Since the geometric quotient $X^{s,\hat{U}}_{\min+}/\hat{U} = X^{s,\hat{U}}_{L_\chi^{\otimes c}}/\hat{U}$ is a projective variety with a very ample line bundle $L_{(c,\hat{U})}$ induced by the tensor power $L^{\otimes c}$ of L, it follows from Proposition 1.19 that $X^{s,\hat{U}}_{L_\chi^{\otimes c}} = X^{ss,\hat{U}}_{L_\chi^{\otimes c}}$, that this geometric quotient coincides with the enveloping quotient $X \wr\wr_{L_\chi^{\otimes c}} \hat{U}$, and that if c is replaced with a sufficiently divisible multiple then the algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L_{k\chi}^{\otimes ck})^{\hat{U}}$ is finitely generated and the enveloping quotient $X \wr\wr_{L_\chi^{\otimes c}} \hat{U} \cong \mathrm{Proj}(\oplus_{k \geq 0} H^0(X, L_{k\chi}^{\otimes ck})^{\hat{U}})$ is the associated projective variety.

It also follows by induction after applying Theorem 2.3 to the action of $\hat{U}/U_\dagger$ on the closure $\overline{X /\!\wr\, U_\dagger}$ in $\mathbb{P}((H^0(X, L^{\otimes c_\dagger})^{U_\dagger})^*)$ of the inner enveloping quotient $X /\!\wr\, U_\dagger$ for the action of $U_\dagger$ on X, that after replacing c with a sufficiently divisible multiple if necessary, we can assume that there is an inner enveloping quotient $X /\!\wr\, U$ for the linear action of U on X with respect to the linearisation $L_\chi^{\otimes c}$ obtained by considering the induced action of the subgroup $U/U_\dagger$ of $\hat{U}/U_\dagger$ on $\overline{X /\!\wr\, U_\dagger}$, and that the tensor power $L^{\otimes c}$ of L induces a very ample line bundle on $X /\!\wr\, U$ so that there is a $\mathbb{C}^*$-equivariant embedding

$$X /\!\wr\, U \to \mathbb{P}((H^0(X, L^{\otimes c})^U)^*)$$

of $X /\!\wr\, U$ as a quasi-projective subvariety, containing the geometric quotient $X^{s,U}/U$ as an open subvariety, with closure $\overline{X /\!\wr\, U}$ in $\mathbb{P}((H^0(X, L^{\otimes c})^U)^*)$, such that $X^{s,\hat{U}}_{\min+}$ is a U-invariant open subset of $X^{s,U}$ and has a geometric quotient $X^{s,\hat{U}}_{\min+}/U$ which is a $\mathbb{C}^*$-invariant open subset of $X^{s,U}/U$ and coincides with both the stable and semistable sets $(\overline{X /\!\wr\, U})^{s,\mathbb{C}^*} = (\overline{X /\!\wr\, U})^{ss,\mathbb{C}^*}$ for the $\mathbb{C}^*$ action with respect to the linearisation on $\mathcal{O}_{\mathbb{P}((H^0(X, L^{\otimes c})^U)^*)}(1)$ induced by $L_\chi^{\otimes c}$. It then follows that the associated GIT quotient

$$\overline{X /\!\wr\, U} /\!/ \mathbb{C}^* \cong (X^{s,\hat{U}}_{\min+}/U)/\mathbb{C}^* \cong X^{s,\hat{U}}_{\min+}/\hat{U} = X \wr\wr_{L_\chi^{\otimes c}} \hat{U}$$

and this completes the inductive proof. □

We have now proved Theorem 0.1 and Corollary 0.3, which follow immediately from Theorem 3.4. Corollary 0.4 follows directly as well, since if a complex linear algebraic group H with unipotent radical U acts on a complex algebra A in such a way that the algebra of U-invariants A^U is finitely generated, then there is an induced action on A^U of the reductive group $R = H/U$, and the algebra of H-invariants

$$A^H = (A^U)^R$$

is finitely generated since R is reductive. In the situation of Corollary 0.4 when A is the algebra $\oplus_{k \geq 0} H^0(X, L_{k\chi}^{\otimes ck}$ then the associated projective variety is the enveloping quotient $X \wr\wr H$, and this enveloping quotient is the GIT quotient of the enveloping quotient $X \wr\wr \hat{U}$ by the reductive subgroup of R which is its intersection with the kernel of the character χ, with respect to the induced linearisation. The result follows from combining Theorem 3.4 with classical GIT for the action of this reductive subgroup of R.

4. Automorphism groups of toric varieties

In this section we will observe that if Y is a complete simplicial toric variety then its automorphism group $Aut(Y)$ satisfies the conditions of Corollary 0.4, so that every well adapted linear action of $Aut(Y)$ on a projective variety X with respect to an ample line bundle for which semistability coincides with stability has finitely generated invariants and its enveloping quotient is a geometric quotient of X^{ss}.

For this we use the description of $Aut(Y)$ given in [6]. Let Y be a complete simplicial toric variety over $\mathbb{C}$ of dimension n, and let S be its homogeneous coordinate ring in the sense of [6]. Thus

$$S = \mathbb{C}[x_\rho : \rho \in \Delta(1)]$$

is a polynomial ring in $d = |\Delta(1)|$ variables x_ρ, one for each one-dimensional cone ρ in the fan Δ determining the toric variety Y. The homogeneous coordinate ring S is graded by setting the degree of a monomial $\prod_\rho x_\rho^{a_\rho}$ to be the class of the corresponding Weil divisor $\sum_\rho a_\rho D_\rho$ in the Chow group $A_{n-1}(Y)$, giving us the decomposition

$$S = \bigoplus_{\alpha \in A_{n-1}(Y)} S_\alpha$$

where S_α is spanned by the monomials of degree α. Then we have

$$S_\alpha = S'_\alpha \oplus S''_\alpha$$

where S'_α is spanned by the x_ρ of degree α and S''_α is spanned by the remaining monomials in S_α, each being a product of at least two variables.

Then by [6] Theorem 4.2 and Proposition 4.3, $Aut(Y)$ is an affine algebraic group fitting into an exact sequence

$$1 \to \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(Y), \mathbb{C}^*) \to \widetilde{Aut}(Y) \to Aut(Y) \to 1$$

with $\mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(Y), \mathbb{C}^*)$ isomorphic to a product of a finite group and a torus $(\mathbb{C}^*)^{d-n}$, and the identity component $\widetilde{Aut}^0(Y)$ of $\widetilde{Aut}(Y)$ satisfies

$$\widetilde{Aut}^0(Y) \cong U \rtimes \tilde{R}$$

for

$$\tilde{R} \cong \prod_\alpha GL(S'_\alpha)$$

and the unipotent radical U of $\widetilde{Aut}^0(Y)$ is given by

$$U = 1 + \mathcal{N}$$

where $\mathcal{N}$ is the ideal

$$\mathcal{N} = \bigoplus_{\alpha} \mathrm{Hom}_{\mathbb{C}}(S^{'}_{\alpha}, S^{''}_{\alpha})$$

in End(S). The reductive group $\tilde{R} \cong \prod_{\alpha} GL(S^{'}_{\alpha})$ acts in the obvious way on S by identifying S with the symmetric algebra on $\bigoplus_{\alpha} S^{'}_{\alpha}$, so that $r \in \tilde{R}$ acts on $\mathrm{Hom}_{\mathbb{C}}(S^{'}_{\alpha}, S^{''}_{\alpha}$ for each $\alpha \in A_{n-1}(Y)$, and thus on $\mathcal{N}$, via pre-composition with the action of r on $S^{'}_{\alpha}$ and post-composition with the induced action of r^{-1} on

$$S^{''}_{\alpha} \subseteq \bigoplus_{j\geq 2} \mathrm{Sym}^{\,j}(\bigoplus_{\alpha} S^{'}_{\alpha}).$$

It follows that if we embed $\mathbb{C}^*$ in $\tilde{R} = \prod_{\alpha} \mathrm{GL}(S^{'}_{\alpha})$ via

$$t \mapsto (t^{-1}\mathrm{id}_{S^{'}_{\alpha}})_{\alpha}$$

where $\mathrm{id}_{S^{'}_{\alpha}}$ is the identity in $\mathrm{GL}(S^{'}_{\alpha})$, then the weights of the action of $\mathbb{C}^*$ on the Lie algebra $\mathcal{N}$ of U are all of the form

$$t \mapsto t^{j-1}$$

for some $j \geq 2$, so that $j - 1 > 0$. Thus we obtain

Lemma 4.1. *If Y is a complete simplicial toric variety then $Aut(Y)$ is of the form*

$$Aut(Y) \cong U \rtimes R$$

where U is unipotent and R is reductive, and R contains a one-parameter subgroup $\mathbb{C}^* \leq R$ such that the action of $\mathbb{C}^*$ on the Lie algebra of U induced by its conjugation action on U has all weights strictly positive.

As an immediate consequence of this lemma and Corollary 0.4 we have

Corollary 4.2. *Any well adapted linear action of $H = Aut(Y)$ on a projective variety X with respect to an ample line bundle L, for which semistability coincides with stability for the action of its unipotent radical U extended by the central one-parameter subgroup of $Aut(Y)/U$ described above, has finitely generated invariants when L is replaced by a tensor power $L^{\otimes c}$ for a sufficiently divisible positive integer c. Furthermore its enveloping quotient $X \wr H$ is the associated projective variety and is a categorical quotient of X^{ss} by the action of H, while the canonical morphism $\phi : X^{ss} \to X \wr H$ is surjective with $\phi(x) = \phi(y)$ if and only if the closures of the H-orbits of x and y meet in X^{ss}.*

5. Jet differentials and generalised Demailly–Semple jet bundles

Our remaining aim is to apply our results to a family of examples involving non-reductive reparametrisation groups which arise in singularity theory and the study of jets of curves. We borrow notation from [7].

Let X be a complex n-dimensional manifold. Green and Griffiths in [14] introduced a bundle $J_k \to X$, the bundle of k-jets of germs of parametrised curves in X; that is, the fibre over $x \in X$ is the set of equivalence classes of holomorphic maps $f : (W, 0) \to (X, x)$ where W is an open neighbourhood of 0 in $\mathbb{C}$, with the equivalence relation $f \sim g$ if and only if the jth derivatives $f^{(j)}(0) = g^{(j)}(0)$ are equal for $0 \leq j \leq k$. If we choose local holomorphic coordinates $(z_1, \ldots, z_n)$

on an open neighbourhood $\Omega \subset X$ around x, the elements of the fibre $J_{k,x}$ are represented by the Taylor expansions

$$f(t) = f(0) + tf'(0) + \frac{t^2}{2!}f''(0) + \ldots + \frac{t^k}{k!}f^{(k)}(0) + O(t^{k+1})$$

up to order k at $t = 0$ of $\mathbb{C}^n$-valued holomorphic maps

$$f = (f_1, f_2, \ldots, f_n) : (\mathbb{C}, 0) \to (\mathbb{C}^n, x).$$

In these coordinates we have

$$J_{k,x} \cong \left\{(f'(0), \ldots, f^{(k)}(0)/k!)\right\} \cong (\mathbb{C}^n)^k,$$

which we identify with $\mathbb{C}^{nk}$. Note, however, that J_k is not a vector bundle over X, since the transition functions are polynomial, but not in general linear.

Let $\mathbb{G}_k$ be the group of k-jets of biholomorphisms

$$(\mathbb{C}, 0) \to (\mathbb{C}, 0);$$

that is, the k-jets at the origin of local reparametrisations

$$t \mapsto \varphi(t) = \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k, \quad \alpha_1 \in \mathbb{C}^*, \alpha_2, \ldots, \alpha_k \in \mathbb{C},$$

in which the composition law is taken modulo terms t^j for $j > k$. This group acts fibrewise on J_k by substitution. A short computation shows that the action on the fibre is linear:

$$f \circ \varphi(t) = f'(0) \cdot (\alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k) + \frac{f''(0)}{2!} \cdot (\alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k)^2 + \ldots$$
$$\ldots + \frac{f^{(k)}(0)}{k!} \cdot (\alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k)^k \text{ (modulo } t^{k+1})$$

so the linear action of φ on the k-jet $(f'(0), \ldots, f^{(k)}(0)/k!)$ is given by the following matrix multiplication:

$$(7) \qquad (f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!) \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \cdots & \alpha_1\alpha_{k-1} + \ldots + \alpha_{k-1}\alpha_1 \\ 0 & 0 & \alpha_1^3 & \cdots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \alpha_1^k \end{pmatrix}$$

with (i, j)th entry

$$\sum_{s_1 + \ldots + s_i = j} \alpha_{s_1} \ldots \alpha_{s_i}$$

for $i, j \leq k$.

There is an exact sequence of groups:

$$0 \to \mathbb{U}_k \to \mathbb{G}_k \to \mathbb{C}^* \to 0,$$

where $\mathbb{G}_k \to \mathbb{C}^*$ is the morphism $\varphi \to \varphi'(0) = \alpha_1$ in the notation used above, and

$$\mathbb{G}_k = \mathbb{U}_k \ltimes \mathbb{C}^*$$

is a semi-direct product of $\mathbb{U}_k$ by $\mathbb{C}^*$. With the above identification, $\mathbb{C}^*$ is the subgroup of diagonal matrices satisfying $\alpha_2 = \ldots = \alpha_k = 0$ and $\mathbb{U}_k$ is the unipotent radical of $\mathbb{G}_k$, i.e. the subgroup of matrices with $\alpha_1 = 1$. The action of $\lambda \in \mathbb{C}^*$ on k-jets is described by

$$\lambda \cdot (f'(0), f''(0), \ldots, f^{(k)}(0)) = (\lambda f'(0), \lambda^2 f''(0), \ldots, \lambda^k f^{(k)}(0)).$$

Let $\mathcal{E}^n_{k,m}$ denote the vector space of polynomials $Q(u_1, u_2, \ldots, u_k)$, of weighted degree m, with respect to this $\mathbb{C}^*$ action, where $u_i = f^{(i)}(0)$; that is, such that

$$Q(\lambda u_1, \lambda^2 u_2, \ldots, \lambda^k u_k) = \lambda^m Q(u_1, u_2, \ldots, u_k).$$

Elements of $\mathcal{E}^n_{k,m}$ have the form

$$Q(u_1, u_2, \ldots, u_k) = \sum_{|\alpha_1|+2|\alpha_2|+\ldots+k|\alpha_k|=m} u_1^{\alpha_1} u_2^{\alpha_2} \ldots u_k^{\alpha_k},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are multi-indices of length n.

$\mathcal{E}^n_{k,m}$ can be identified with the fibre of the vector bundle $E^{GG}_{k,m} \to X$ introduced by Green and Griffiths in [14], whose fibres consist of polynomials on the fibres of J_k of weighted degree m with respect to the fibrewise $\mathbb{C}^*$ action on J_k.

The action of $\mathbb{G}_k$ naturally induces an action on the vector space

$$\mathcal{E}^n_k = \bigoplus_{m\geq 0} \mathcal{E}^n_{k,m} = \mathcal{O}(J_{k,x})$$

of polynomial functions on $J_{k,x}$. Following Demailly ([7]), we define $\tilde{\mathcal{E}}^n_{k,m} \subset \mathcal{E}^n_{k,m}$ to be the vector space of $\mathbb{U}_k$-invariant polynomials of weighted degree m, i.e. those which satisfy

$$Q((f \circ \varphi)', (f \circ \varphi)'', \ldots, (f \circ \varphi)^{(k)}) = \varphi'(0)^m \cdot Q(f', f'', \ldots, f^{(k)}).$$

Thus $\tilde{\mathcal{E}}^n_k = \bigoplus_{m\geq 0} \tilde{\mathcal{E}}^n_{k,m} = \mathcal{O}(J_{k,x})^{\mathbb{U}_k}$ consists of the polynomials functions on $J_{k,x}$ which are invariant under the induced action of $\mathbb{U}_k$ on $\mathcal{O}(J_{k,x})$. The corresponding bundle of invariants is the Demailly-Semple bundle of algebras $E^n_k = \oplus_m E^n_{k,m} \subset \oplus_m E^{GG}_{k,m}$ with fibres $\tilde{\mathcal{E}}^n_k = \bigoplus_{m\geq 0} \tilde{\mathcal{E}}^n_{k,m} = \mathcal{O}(J_{k,x})^{\mathbb{U}_k}$.

This bundle of graded algebras $E^n_k = \oplus_m E^n_{k,m}$ has been an important object of study for a long time. The invariant jet differentials play a crucial role in the strategy developed by Green, Griffiths, Bloch, Ahlfors, Demailly, Siu and others to prove Kobayashi's 1970 hyperbolicity conjecture [1, 5, 7, 8, 9, 14, 24, 26, 34, 35, 36].

We can now apply Theorem 0.1, Corollary 0.2 and Remark 0.3 to linear action of $\mathbb{G}_k$ on the projective variety associated to $J_{k,x}$. In this case we can also apply the results of [4] since $\mathbb{G}_k$ is a subgroup of $\mathrm{GL}(k;\mathbb{C})$ which is 'generated along the first row' in the sense of [4], and the action of $\mathbb{G}_k$ extends to $\mathrm{GL}(k;\mathbb{C})$.

We can also consider a generalised version of the Demailly-Semple jet differentials to which the results of [4] do not apply. Instead of germs of holomorphic maps $\mathbb{C} \to X$, we now consider higher dimensional holomorphic objects in X, and therefore we fix a parameter $1 \leq p \leq n$, and study germs of holomorphic maps $\mathbb{C}^p \to X$.

Again we fix the degree k of these maps, and introduce the bundle $J_{k,p} \to X$ of k-jets of germs of holomorphic maps $\mathbb{C}^p \to X$. With respect to local holomorphic coordinates near $x \in X$ the fibre over x is identified with the set of equivalence classes of holomorphic maps $f : (\mathbb{C}^p, 0) \to$

$(\mathbb{C}^n, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ are equal for $0 \le j \le k$. Equivalently the elements of the fibre $J_{k,p,x}$ are the Taylor expansions

$$f(\mathbf{u}) = x + \mathbf{u} f'(0) + \frac{\mathbf{u}^2}{2!} f''(0) + \ldots + \frac{\mathbf{u}^k}{k!} f^{(k)}(0) + O(|\mathbf{u}|^{k+1})$$

around $\mathbf{u} = 0$ up to order k of $\mathbb{C}^n$-valued maps

$$f = (f_1, f_2, \ldots, f_n) : (\mathbb{C}^p, 0) \to (\mathbb{C}^n, x).$$

Here

$$f^{(i)}(0) \in \operatorname{Hom}(\operatorname{Sym}^i \mathbb{C}^p, \mathbb{C}^n)$$

so that in these coordinates the fibre is

$$J_{k,p,x} = \left\{ (f'(0), \ldots, f^{(k)}(0)/k!) \right\} = \mathbb{C}^{n(\binom{k+p}{k} - 1)}$$

which is a finite dimensional vector space.

Let $\mathbb{G}_{k,p}$ be the group of k-jets of germs of biholomorphisms of $(\mathbb{C}^p, 0)$, that is, the group of biholomorphic maps

$$\mathbf{u} \to \varphi(\mathbf{u}) = \Phi_1 \mathbf{u} + \Phi_2 \mathbf{u}^2 + \ldots + \Phi_k \mathbf{u}^k = \sum_{1 \le i_1 + \cdots + i_p \le k} a_{i_1 \ldots i_p} u_1^{i_1} \ldots u_p^{i_p} \tag{8}$$

for which $\Phi_i \in \operatorname{Hom}(\operatorname{Sym}^i \mathbb{C}^p, \mathbb{C}^p)$ and $\Phi_1 \in \operatorname{Hom}(\mathbb{C}^p, \mathbb{C}^p)$ is non-degenerate. Then $\mathbb{G}_{k,p}$ admits a natural fibrewise right action on $J_{k,p}$, which consist of reparametrizing the k-jets of holomorphic p-discs. A similar computation to at (7) shows that

$$f \circ \varphi(\mathbf{u}) = (f'(0)\Phi_1)\mathbf{u} + (f'(0)\Phi_2 + \frac{f''(0)}{2!}\Phi_1^2)\mathbf{u}^2 + \ldots + \ldots + \sum_{i_1 + \ldots + i_l = k} (\frac{f^{(l)}(0)}{l!} \Phi_{i_1} \ldots \Phi_{i_l})\mathbf{u}^l.$$

This is a linear action on the fibres $J_{k,p,x}$ with matrix given by

$$\begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \ldots & \Phi_k \\ 0 & \Phi_1^2 & \Phi_1\Phi_2 & \ldots & \\ 0 & 0 & \Phi_1^3 & \ldots & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \Phi_1^k \end{pmatrix}, \tag{9}$$

where Φ_i is a $p \times \dim(\operatorname{Sym}^i \mathbb{C}^p)$-matrix, the ith degree component of the map Φ and the $p \times p$matrix Φ_1 is invertible. Here $\Phi_{i_1} \ldots \Phi_{i_l}$ is the matrix of the map $\operatorname{Sym}^{i_1 + \ldots + i_l}(\mathbb{C}^p) \to \operatorname{Sym}^l \mathbb{C}^p$, which is induced by

$$\Phi_{i_1} \otimes \cdots \otimes \Phi_{i_l} : (\mathbb{C}^p)^{\otimes i_1} \otimes \cdots \otimes (\mathbb{C}^p)^{\otimes i_l} \to (\mathbb{C}^p)^{\otimes l}$$

The linear group $\mathbb{G}_{k,p}$ is generated along its first p rows, in the sense that the parameters in the first p rows are independent, and all the remaining entries are polynomials in these parameters. The only condition which the parameters must satisfy is that the determinant of the first diagonal $p \times p$ block is nonzero. Note that $\mathbb{G}_{k,p}$ is an extension of its unipotent radical $\mathbb{U}_{k,p}$ (given by $\Phi_1 = 1$ by $GL(p; \mathbb{C})$ (given by $\Phi_i = 0$ for $i > 1$), so we have an exact sequence

$$0 \to \mathbb{U}_{k,p} \to \mathbb{G}_{k,p} \to GL(p; \mathbb{C}) \to 0.$$

The central $\mathbb{C}^*$ of $GL(p;\mathbb{C})$ corresponds to the diagonal matrices with entries $t, t^2, \ldots, t^k$ for $t \in \mathbb{C}^*$ where t^i occurs $\dim(\operatorname{Sym}^i(\mathbb{C}^p))$ times, and these act by conjugation on the Lie algebra of $\mathbb{U}_{k,p}$ with weights $i-1$ for $2 \le i \le k$. Thus by Corollary 0.4 we have

Corollary 5.1. *Any linear action of $\mathbb{G}_{k,p}$ on a projective variety X with respect to an ample line bundle L for which semistability coincides with stability for the action of $\mathbb{U}_{k,p}$ extended by the central one-parameter subgroup of $\mathrm{GL}(p;\mathbb{C})$ has finitely generated invariants when L is replaced by a tensor power $L^{\otimes c}$ for a sufficiently divisible positive integer c and the linearisation is twisted by a well adapted rational character. Furthermore its enveloping quotient $X /\!\!/ \mathbb{G}_{k,p}$ is the associated projective variety and is a categorical quotient of X^{ss} by the action of $\mathbb{G}_{k,p}$, while the canonical morphism $\phi : X^{ss} \to X /\!\!/ \mathbb{G}_{k,p}$ is surjective with $\phi(x) = \phi(y)$ if and only if the closures of the $\mathbb{G}_{k,p}$-orbits of x and y meet in X^{ss}.*

Definition 5.2. *The generalized Demailly-Semple jet bundle $E_{k,p,m} \to X$ of invariant jet differentials of order k and weighted degree $(m, \ldots, m)$ has fibre at $x \in X$ consisting of complex-valued polynomials $Q(f'(0), f''(0)/2, \ldots, f^{(k)}(0)/k!)$ on the fibre $J_{k,p,x}$ of $J_{k,p}$, which transform under any reparametrization $\phi \in \mathbb{G}_{k,p}$ of $(\mathbb{C}^p, 0)$ as*

$$Q(f \circ \phi) = (J_\phi(0))^m Q(f) \circ \phi,$$

where $J_\phi(0)$ denotes the Jacobian at 0 of ϕ; that is, $J_\phi(0) = \det \Phi_1$ when ϕ is given as at (9). Thus the generalized Demailly-Semple bundle $E_{k,p} = \oplus E_{k,p,m}$ of invariant jet differentials of order k has fibre at $x \in X$ given by the generalized Demailly-Semple algebra $\mathcal{O}(J_{k,p,x})^{\mathbb{U}_{k,p} \rtimes SL(p;\mathbb{C})}$.

We can apply Corollary 5.1 to the linear action of $\mathbb{G}_{k,p}$ on the projective space $X = \mathbb{P}(J_{k,p,x})$ with respect to the line bundle $L = \mathcal{O}_{\mathbb{P}(J_{k,p,x})}(1)$ satisfying

$$\mathcal{O}(J_{k,p,x}) = \oplus_{j \ge 0} H^0(X, L^{\otimes j}).$$

As at Remark 0.3, by considering a diagonal action on $X \times \mathbb{P}^1$, we can deduce that the algebra $\oplus_{m=0}^{\infty} H^0(X \times \mathbb{P}^1, L_{m\chi}^{\otimes cm} \otimes \mathcal{O}_{\mathbb{P}^1}(M))^{\mathbb{G}_{k,p}}$ of $\mathbb{G}_{k,p}$-invariants on $X \times \mathbb{P}^1$ is finitely generated when $M >> 1$ and c is a sufficiently divisible positive integer and the linear action has been twisted by a suitable rational character χ/c. This finitely generated graded algebra can be identified with the subalgebra of the generalized Demailly-Semple algebra $\mathcal{O}(J_{k,p,x})^{\mathbb{U}_{k,p} \rtimes SL(p;\mathbb{C})}$ generated by the $\mathbb{U}_{k,p} \rtimes SL(p;\mathbb{C})$-invariants in $\oplus_{m=0}^{\infty} H^0(X, L^{\otimes cm})^{\mathbb{U}_{k,p} \rtimes SL(p;\mathbb{C})}$ which are weight vectors with non-negative weights for the action of the central one-parameter subgroup of GL_p after twisting by a suitable character χ. This twisting is such that the matrix (9) is replaced with its multiple by $(\det(\Phi_1))^{-(1/p)-\epsilon}$ for $0 < \epsilon << 1$, so the only weight vectors $\sigma \in H^0(X, L) = \bigoplus_{i=1}^k \operatorname{Sym}^i(\mathbb{C}^p)$ with non-negative weights are the sections σ in $\operatorname{Sym}^1(\mathbb{C}^p) = \mathbb{C}^p$, which have weight $p\epsilon$. It therefore follows that the localisation $\mathcal{O}(J_{k,p,x})_\sigma^{\mathbb{U}_{k,p} \rtimes SL(p;\mathbb{C})}$ of the generalized Demailly-Semple algebra $\mathcal{O}(J_{k,p,x})^{\mathbb{U}_{k,p} \rtimes SL(p;\mathbb{C})}$ at any such σ is finitely generated (cf. [8, 26]).

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