

ABSTRACT QUOTIENTS OF PROFINITE GROUPS, AFTER NIKOLOV AND SEGAL

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ABSTRACT. In this expanded account of a talk given at the Oberwolfach Arbeitsgemeinschaft “Totally Disconnected Groups”, October 2014, we discuss results of Nikolay Nikolov and Dan Segal on abstract quotients of compact Hausdorff topological groups, paying special attention to the class of finitely generated profinite groups. Our primary source is [15]. Sidestepping all difficult and technical proofs, we present a selection of accessible arguments to illuminate key ideas in the subject.

1. INTRODUCTION

§1.1. Many concepts and techniques in the theory of finite groups depend intrinsically on the assumption that the groups considered are a priori finite. The theoretical framework based on such methods has led to marvellous achievements, including – as a particular highlight – the classification of all finite simple groups. Notwithstanding, the same methods are only of limited use in the study of infinite groups: it remains mysterious how one could possibly pin down the structure of a general infinite group in a systematic way. Significantly more can be said if such a group comes equipped with additional information, such as a structure-preserving action on a notable geometric object. A coherent approach to studying restricted classes of infinite groups is found by imposing suitable ‘finiteness conditions’, i.e., conditions that generalise the notion of being finite but are significantly more flexible, such as the group being finitely generated or compact with respect to a natural topology.

One rather fruitful theme, fusing methods from finite and infinite group theory, consists in studying the interplay between an infinite group Γ and the collection of all its finite quotients. In passing, we note that the latter, subject to surjections, naturally form a lattice that is anti-isomorphic to the lattice of finite-index normal subgroups of Γ , subject to inclusions. In this context it is reasonable to focus on Γ being *residually finite*, i.e., the intersection $\bigcap_{N \trianglelefteq_f \Gamma} N$ of all finite-index normal subgroups of Γ being trivial. Every residually finite group Γ embeds as a dense subgroup into its profinite completion $\widehat{\Gamma} = \varprojlim_{N \trianglelefteq_f \Gamma} \Gamma/N$. The latter can be constructed as a closed subgroup of the direct product $\prod_{N \trianglelefteq_f \Gamma} \Gamma/N$ of finite discrete groups and thus inherits the structure of a *profinite group*, i.e., a topological group that is compact, Hausdorff and totally disconnected. The finite quotients of Γ are in one-to-one correspondence with the continuous finite quotients of $\widehat{\Gamma}$.

Profinite groups also arise naturally in a range of other contexts, e.g., as Galois groups of infinite field extensions or as open compact subgroups of Lie groups over non-archimedean local fields, and it is beneficial to develop the theory of profinite groups in some generality. At an advanced stage, one is naturally led to examine more closely the relationship between the underlying abstract group

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structure of a profinite group G and its topology. In the following we employ the adjective ‘abstract’ to emphasise that a subgroup, respectively a quotient, of G is not required to be closed, respectively continuous. For instance, as G is compact, an abstract subgroup H is open in G if and only if it is closed and of finite index in G . A profinite group may or may not have (normal) abstract subgroups of finite index that fail to be closed. Put in another way, G may or may not have non-continuous finite quotients. What can be said about abstract quotients of a profinite group G in general? When do they exist and how ‘unexpected’ can their features possibly be?

To the newcomer, even rather basic groups offer some initial surprises. For instance, consider for a prime p the procyclic group $\mathbb{Z}_p = \varprojlim_{k \in \mathbb{N}} \mathbb{Z}/p^k\mathbb{Z}$ of p -adic integers under addition. Of course, its proper continuous quotients are just the finite cyclic groups $\mathbb{Z}_p/p^k\mathbb{Z}_p$. As we will see below, the abelian group \mathbb{Z}_p fails to map abstractly onto \mathbb{Z} , but does have abstract quotients isomorphic to \mathbb{Q} .

§1.2. In [15], Nikolov and Segal streamline and generalise their earlier results that led, in particular, to the solution of the following problem raised by Jean-Pierre Serre in the 1970s: are finite-index abstract subgroups of an arbitrary finitely generated profinite group G always open in G ? Recall that a profinite group is said to be (topologically) *finitely generated* if it contains a dense finitely generated abstract subgroup. The problem, which can be found in later editions of Serre’s book on Galois cohomology [20, I.§4.2], was solved by Nikolov and Segal in 2003.

Theorem 1.1 (Nikolov, Segal [12]). *Let G be a finitely generated profinite group. Then every finite-index abstract subgroup H of G is open in G .*

Serre had proved this assertion in the special case, where G is a finitely generated pro- p group for some prime p , by a neat and essentially self-contained argument; compare Section 2. The proof of the general theorem is considerably more involved and makes substantial use of the classification of finite simple groups; the same is true for several of the main results stated below.

The key theorem in [15] concerns normal subgroups in finite groups and establishes results about products of commutators of the following type. There exists a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that, if $H \trianglelefteq \Gamma = \langle y_1, \dots, y_r \rangle$ for a finite group Γ , then every element of the subgroup $[H, \Gamma] = \langle [h, g] \mid h \in H, g \in \Gamma \rangle$ is a product of $f(r)$ factors of the form $[h_1, y_1][h_2, y_1^{-1}] \cdots [h_{2r-1}, y_r][h_{2r}, y_r^{-1}]$ with $h_1, \dots, h_{2r} \in H$. Under certain additional conditions on H , a similar conclusion holds based on the significantly weaker hypothesis that $\Gamma = H\langle y_1, \dots, y_r \rangle = \Gamma'\langle y_1, \dots, y_r \rangle$, where $\Gamma' = [\Gamma, \Gamma]$ denotes the commutator subgroup. A more precise version of the result is stated as Theorem 3.1 in Section 3. By standard compactness arguments, one obtains corresponding assertions for normal subgroups of finitely generated profinite groups; this is also explained in Section 3.

Every compact Hausdorff topological group G is an extension of a compact connected group G° , its identity component, by a profinite group G/G° . By the Levi–Mal’cev Theorem, e.g., see [9, Theorem 9.24], the connected component G° is essentially a product of compact Lie groups and thus relatively tractable. We state two results whose proofs require the new machinery developed in [15] that goes beyond the methods used in [12, 13, 14].

Theorem 1.2 (Nikolov, Segal [15]). *Let G be a compact Hausdorff topological group. Then every finitely generated abstract quotient of G is finite.*

Theorem 1.3 (Nikolov, Segal [15]). *Let G be a compact Hausdorff topological group such that G/G° is topologically finitely generated. Then G has a countably infinite abstract quotient if and only if G has an infinite virtually-abelian continuous quotient.*

In a recent paper, Nikolov and Segal generalised their results in [15] to obtain the following structural theorem.

Theorem 1.4 (Nikolov, Segal [16]). *Let G be a compact Hausdorff group such that G/G° is topologically finitely generated. Then for every closed normal subgroup $H \trianglelefteq_c G$ the abstract subgroup $[H, G] = \langle [h, g] \mid h \in H, g \in G \rangle$ is closed in G .*

There are many interesting open questions regarding the algebraic properties of profinite groups; the introductory survey [11] provides more information as well as a range of ideas and suggestions for further investigation. In Section 4 we look at one possible direction for applying and generalising the results of Nikolov and Segal, namely in comparing the abstract and the continuous cohomology of a finitely generated profinite group.

2. SERRE'S PROBLEM ON FINITE ABSTRACT QUOTIENTS OF PROFINITE GROUPS

§2.1. A profinite group G is said to be *strongly complete* if every finite-index abstract subgroup is open in G . Equivalently, G is strongly complete if every finite quotient of G is a continuous quotient. Clearly, such a group is uniquely determined by its underlying abstract group structure as it is its own profinite completion.

It is easy to see that there are non-finitely generated profinite groups that fail to be strongly complete. For instance, $G = \varprojlim_{n \in \mathbb{N}} C_p^n \cong C_p^{\aleph_0}$, the direct product of a countably infinite number of copies of a cyclic group C_p of prime order p , has $2^{2^{\aleph_0}}$ subgroups of index p , but only countably many of these are open. This can be seen by regarding the abstract group G as a vector space of dimension 2^{\aleph_0} over a field with p elements. Implicitly, this simple example uses the axiom of choice.

Without taking a stand on the generalised continuum hypothesis, it is slightly more tricky to produce an example of two profinite groups G_1 and G_2 that are non-isomorphic as topological groups, but nevertheless abstractly isomorphic. As an illustration, we discuss a concrete instance. In [7], Jonathan A. Kiehlmann classifies more generally countably-based abelian profinite groups, up to continuous isomorphism and up to abstract isomorphism.

Proposition 2.1. *Let p be a prime. The pro- p group $G = \prod_{i=1}^{\infty} C_{p^i}$ is abstractly isomorphic to $G \times \mathbb{Z}_p$, but there is no continuous isomorphism between the groups G and $G \times \mathbb{Z}_p$.*

Proof. The torsion elements form a dense subset in the group G , but they do not in $G \times \mathbb{Z}_p$. Hence the two groups cannot be isomorphic as topological groups.

It remains to show that G is abstractly isomorphic to $G \times \mathbb{Z}_p$. Let $\mathfrak{U} \subseteq \mathcal{P}(\mathbb{N}) = \{T \mid T \subseteq \mathbb{N}\}$ be a non-principal ultrafilter. This means: \mathfrak{U} is closed under finite intersections and under taking supersets; moreover, whenever a disjoint union $T_1 \sqcup T_2$ of two sets belongs to \mathfrak{U} , precisely one of T_1 and T_2 belongs to \mathfrak{U} ; finally, \mathfrak{U} does not contain any one-element sets. Then \mathfrak{U} contains all co-finite subsets of \mathbb{N} , and, in fact, one justifies the existence of \mathfrak{U} by enlarging the filter consisting of all co-finite subsets to an ultrafilter; this process relies on the axiom of choice.

Informally speaking, we use the ultrafilter \mathfrak{U} as a tool to form a consistent limit. Indeed, with the aid of \mathfrak{U} , we define

$$\begin{aligned} \psi: G \cong \prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z} &\longrightarrow \mathbb{Z}_p \cong \varprojlim_j \mathbb{Z}/p^j\mathbb{Z}, \\ x = (x_1 + p\mathbb{Z}, x_2 + p^2\mathbb{Z}, \dots) &\mapsto \varprojlim_j (x\psi_j), \end{aligned}$$

where

$$x\psi_j = a + p^j\mathbb{Z} \in \mathbb{Z}/p^j\mathbb{Z} \quad \text{if } U_j(a) = \{k \geq j \mid x_k \equiv_{p^j} a\} \in \mathfrak{U}.$$

Observe that $\mathfrak{U} \ni \{k \mid k \geq j\} = U_j(0) \sqcup \dots \sqcup U_j(p^j - 1)$ to see that the definition of ψ_j is valid. It is a routine matter to check that ψ is a (non-continuous) homomorphism from G onto \mathbb{Z}_p . Furthermore, G decomposes as an abstract group into a direct product $G = \ker \psi \times H$, where $H = \overline{\langle (1, 1, \dots) \rangle} \cong \mathbb{Z}_p$ is the closed subgroup generated by $(1, 1, \dots)$. Consequently, $\ker \psi \cong G/H$ as an abstract group. Finally, we observe that

$$\vartheta: G \cong \prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z} \longrightarrow G \cong \prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z}, \quad (x_i + p^i\mathbb{Z})_i \mapsto (x_i - x_{i+1} + p^i\mathbb{Z})_i$$

is a continuous, surjective homomorphism with $\ker \vartheta = H$. This shows that

$$G \cong G/H \times H \cong G \times \mathbb{Z}_p$$

as abstract groups. \square

Theorem 1.1 states that finitely generated profinite groups are strongly complete: there are no unexpected finite quotients of such groups. What about infinite abstract quotients of finitely generated profinite groups? In view of Theorem 1.1, the following proposition applies to all finitely generated profinite groups.

Proposition 2.2. *Let G be a strongly complete profinite group and $N \trianglelefteq G$ a normal abstract subgroup. If G/N is residually finite, then $N \trianglelefteq_c G$ and G/N is a profinite group with respect to the quotient topology.*

Proof. The group N is the intersection of finite-index abstract subgroups of G . Since G is strongly complete, each of these is open and hence closed in G . \square

Corollary 2.3. *A strongly complete profinite group does not admit any countably infinite residually finite abstract quotients.*

We remark that a strongly complete profinite group can have countably infinite abstract quotients. Indeed, we conclude this section with a simple argument, showing that the procyclic group \mathbb{Z}_p has abstract quotients isomorphic to \mathbb{Q} . Again, this makes implicitly use of the axiom of choice.

Proposition 2.4. *The procyclic group \mathbb{Z}_p maps non-continuously onto \mathbb{Q} .*

Proof. Clearly, \mathbb{Z}_p is a spanning set for the \mathbb{Q} -vector space \mathbb{Q}_p . Thus there exists an ordered \mathbb{Q} -basis $x_0 = 1, x_1, \dots, x_\omega, \dots$ for \mathbb{Q}_p consisting of p -adic integers; here ω denotes the first infinite ordinal number. Set

$$y_0 = x_0 = 1, \quad y_i = x_i - p^{-i} \text{ for } 1 \leq i < \omega \quad \text{and} \quad y_\lambda = x_\lambda \text{ for } \lambda \geq \omega.$$

The \mathbb{Q} -vector space \mathbb{Q}_p decomposes into a direct sum $\mathbb{Q} \oplus W = \mathbb{Q}_p$, where W has \mathbb{Q} -basis $y_1, \dots, y_\omega, \dots$, inducing a natural surjection $\eta: \mathbb{Q}_p \rightarrow \mathbb{Q}$ with kernel W . By construction, $\mathbb{Z}_p + W = \mathbb{Q}_p$ so that $\mathbb{Z}_p/(\mathbb{Z}_p \cap W) \cong (\mathbb{Z}_p + W)/W \cong \mathbb{Q}$. \square

To some extent this basic example is rather typical; cf. Theorems 1.2 and 1.3.

§2.2. We now turn our attention to finitely generated profinite groups. Around 1970 Serre discovered that finitely generated pro- p groups are strongly complete, and he asked whether this is true for arbitrary finitely generated profinite groups. Here and in the following, p denotes a prime.

Theorem 2.5 (Serre). *Finitely generated pro- p groups are strongly complete.*

It is instructive to recall a proof of this special case of Theorem 1.1, which we base on two auxiliary lemmata.

Lemma 2.6. *Let G be a finitely generated pro- p group and $H \leq_f G$ a finite-index subgroup. Then $|G : H|$ is a power of p .*

Proof. Indeed, replacing H by its core in G , viz., the subgroup $\bigcap_{g \in G} H^g \trianglelefteq_f G$, we may assume that H is normal in G . Thus G/H is a finite group of order m , say, and the set $X = G^{\{m\}} = \{g^m \mid g \in G\}$ of all m th powers in G is contained in H . Being the image of the compact space G under the continuous map $x \mapsto x^m$, the set X is closed in the Hausdorff space G .

Since G is a pro- p group, it has a base of neighbourhoods of 1 consisting of open normal subgroups $N \trianglelefteq_o G$ of p -power index. Hence G/N is a finite p -group for every $N \trianglelefteq_o G$. Let $g \in G$. Writing $m = p^r \tilde{m}$ with $p \nmid \tilde{m}$, we conclude that $g^{p^r} \in XN$ for every $N \trianglelefteq_o G$. Since X is closed in G , this yields

$$g^{p^r} \in \bigcap_{N \trianglelefteq_o G} XN = X \subseteq H.$$

Consequently, every element of G/H has p -power order, and $|G : H| = m = p^r$. \square

For any subset X of a topological group G we denote by \overline{X} the topological closure of X in G .

Lemma 2.7. *Let G be a finitely generated pro- p group. Then the abstract commutator subgroup $[G, G]$ is closed in G , i.e., $[G, G] = \overline{[G, G]}$.*

Proof. We use the following fact about finitely generated nilpotent groups that can easily be proved by induction on the nilpotency class:

(\dagger) if $\Gamma = \langle \gamma_1, \dots, \gamma_d \rangle$ is a nilpotent group then

$$[\Gamma, \Gamma] = \{[x_1, \gamma_1] \cdots [x_d, \gamma_d] \mid x_1, \dots, x_d \in \Gamma\}.$$

Suppose that G is topologically generated by a_1, \dots, a_d , i.e., $G = \overline{\langle a_1, \dots, a_d \rangle}$. Being the image of the compact space $G \times \cdots \times G$ under the continuous map $(x_1, \dots, x_d) \mapsto [x_1, a_1] \cdots [x_d, a_d]$, the set

$$X = \{[x_1, a_1] \cdots [x_d, a_d] \mid x_1, \dots, x_d \in G\} \subseteq [G, G]$$

is closed in the Hausdorff space G . Using (\dagger), this yields

$$\overline{[G, G]} = \bigcap_{N \trianglelefteq_o G} [G, G]N = \bigcap_{N \trianglelefteq_o G} XN = X \subseteq [G, G],$$

and thus $[G, G] = \overline{[G, G]}$. \square

The Frattini subgroup $\Phi(G)$ of a profinite group G is the intersection of all its maximal open subgroups. If G is a finitely generated pro- p group, then $\Phi(G) = \overline{G^p [G, G]}$ is open in G . Here $G^p = \langle g^p \mid g \in G \rangle$ denotes the abstract subgroup generated by the set $G^{\{p\}} = \{g^p \mid g \in G\}$ of all p th powers.

Corollary 2.8. *Let G be a finitely generated pro- p group. Then $\Phi(G)$ is equal to*

$$G^p[G, G] = G^{\{p\}}[G, G] = \{x^p y \mid x \in G \text{ and } y \in [G, G]\}.$$

Proof. Indeed, $G^p[G, G] = G^{\{p\}}[G, G]$, because $G/[G, G]$ is abelian. By Lemma 2.7, $[G, G]$ is closed in G . Being the image of the compact space $G \times [G, G]$ under the continuous map $(x, y) \mapsto x^p y$, the set $G^{\{p\}}[G, G]$ is closed in the Hausdorff space G . Thus $\Phi(G) = \overline{G^p[G, G]} = G^{\{p\}}[G, G]$. \square

Proof of Theorem 2.5. Let $H \leq_f G$. Replacing H by its core in G , we may assume that $H \leq_f G$. Using Lemma 2.6, we argue by induction on $|G : H| = p^r$. If $r = 0$ then $H = G$ is open in G . Now suppose that $r \geq 1$.

From Corollary 2.8 we deduce that $M = H\Phi(G)$ is a proper open subgroup of G . Since M is a finitely generated pro- p group and $H \leq_f M$ with $|M : H| < |G : H|$, induction yields $H \leq_o M \leq_o G$. Thus H is open in G . \square

For completeness, we record another consequence of the proof of Lemma 2.7 that can be regarded as a special case of Corollary 3.4 below.

Corollary 2.9. *Let G be a finitely generated pro- p group and $N \trianglelefteq G$ a normal abstract subgroup. If $N[G, G] = G$ then $N = G$.*

Proof. Suppose that $N[G, G] = G$. Then $N\Phi(G) = G$ and N contains a set of topological generators a_1, \dots, a_d of G . Arguing as in the proof of Lemma 2.7 we obtain $[G, G] \subseteq N$ and thus $N = N[G, G] = G$. \square

§2.3. There were several generalisations of Serre’s result to other classes of finitely generated profinite groups, most notably by Brian Hartley [5] dealing with polypronilpotent groups, by Consuelo Martínez and Efim Zelmanov [8] and independently by Jan Saxl and John S. Wilson [17], each team dealing with direct products of finite simple groups, and finally by Segal [18] dealing with prosoluble groups.

The result of Martínez and Zelmanov, respectively Saxl and Wilson, on powers in non-abelian finite simple groups relies on the classification of finite simple groups and leads to a slightly more general theorem. A profinite group G is called *semisimple* if it is the direct product of non-abelian finite simple groups.

Theorem 2.10 (Martínez and Zelmanov [8]; Saxl and Wilson [17]). *Let $G = \prod_{i \in \mathbb{N}} S_i$ be a semisimple profinite group, where each S_i is non-abelian finite simple. Then G is strongly complete if and only if there are only finitely many groups S_i of each isomorphism type.*

In particular, whenever a semisimple profinite group G as in the theorem is finitely generated, there are only finitely many factors of each isomorphism type and the group is strongly complete. We do not recall the full proof of Theorem 2.10, but we explain how the implication ‘ \Leftarrow ’ can be derived from the following key ingredient proved in [8, 17]:

(\ddagger) for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that for every non-abelian finite simple group S whose exponent does not divide n ,

$$S = \{x_1^n \cdots x_k^n \mid x_1, \dots, x_k \in S\}.$$

Supposed that $G = \prod_{i \in \mathbb{N}} S_i$ as in Theorem 2.10, with only finitely many groups S_i of each isomorphism type, and let $H \leq_f G$. Replacing H by its core we may assume that $H \trianglelefteq_f G$. Put $n = |G : H|$ and choose $k \in \mathbb{N}$ as in (\ddagger). Then H

contains $X = \{x_1^n \cdots x_k^n \mid x_1, \dots, x_k \in G\} \supseteq \prod_{i \geq j} S_i$, where $j \in \mathbb{N}$ is such that the exponent of S_i does not divide n for $i \geq j$, and hence H is open in G . The existence of the index j is guaranteed by the classification of finite simple groups.

Observe that implicitly we have established the following corollary.

Corollary 2.11. *Let G be a semisimple profinite group and $q \in \mathbb{N}$. Then $G^q = \langle g^q \mid g \in G \rangle$, the abstract subgroup generated by all q th powers, is closed in G .*

§2.4. The groups $[G, G]$ and G^q featuring in the discussion above are examples of verbal subgroups of G . We briefly summarise the approach of Nikolov and Segal that led to the original proof of Theorem 1.1 in [12]; the theorem is derived from a ‘uniformity result’ concerning verbal subgroups of finite groups.

Let $d \in \mathbb{N}$, and let $w = w(X_1, \dots, X_r)$ be a group word, i.e., an element of the free group on r generators X_1, \dots, X_r . A w -value in a group G is an element of the form $w(x_1, x_2, \dots, x_r)$ or $w(x_1, x_2, \dots, x_r)^{-1}$ with $x_1, x_2, \dots, x_r \in G$. The verbal subgroup $w(G)$ is the subgroup generated (algebraically, whether or not G is a topological group) by all w -values in G . The word w is d -locally finite if every d -generator group H satisfying $w(H) = 1$ is finite. Finally, a simple commutator of length $n \geq 2$ is a word of the form $[X_1, \dots, X_n]$, where $[X_1, X_2] = X_1^{-1} X_2^{-1} X_1 X_2$ and $[X_1, \dots, X_n] = [[X_1, \dots, X_{n-1}], X_n]$ for $n > 2$. It is well-known that the verbal subgroup corresponding to the simple commutator of length n is the n th term of the lower central series.

Suppose that w is d -locally finite or that w is a simple commutator. In [12], Nikolov and Segal prove that there exists $f = f(w, d) \in \mathbb{N}$ such that in every d -generator finite group G every element of the verbal subgroup $w(G)$ is equal to a product of f w -values in G . This ‘quantitative’ statement about (families of) finite groups translates into the following ‘qualitative’ statement about profinite groups. If w is d -locally finite, then in every d -generator profinite group G , the verbal subgroup $w(G)$ is open in G . From this one easily deduces Theorem 1.1. Similarly, considering simple commutators Nikolov and Segal prove that each term of the lower central series of a finitely generated profinite group G is closed.

By a variation of the same method and by appealing to Zelmanov’s celebrated solution to the restricted Burnside problem, one establishes the following result.

Theorem 2.12 (Nikolov, Segal [14]). *Let G be a finitely generated profinite group. Then the subgroup $G^q = \langle g^q \mid g \in G \rangle$, the abstract subgroup generated by q th powers, is open in G for every $q \in \mathbb{N}$.*

The approach in [12] is based on quite technical results concerning products of ‘twisted commutators’ in finite quasisimple groups that are established in [13]. The proofs in [14] make use of the full machinery in [12]. Ultimately these results all rely on the classification of finite simple groups.

Remark 2.13. An independent justification of Theorem 2.12 would immediately yield a new proof of Theorem 1.1. Furthermore, Andrei Jaikin-Zapirain has shown that the use of the positive solution of the restricted Burnside problem in proving Theorem 2.12 is to some extent inevitable; see [6, Section 5.1].

We refer to the survey article [23] for a more thorough discussion of the background to Serre’s problem and further information on finite-index subgroups and verbal subgroups in profinite groups. A comprehensive account of verbal width in groups is given in [19].

3. NIKOLOV AND SEGAL'S RESULTS ON FINITE AND PROFINITE GROUPS

§3.1. In this section we discuss the new approach in [15]. The key theorem concerns normal subgroups in finite groups. A finite group H is almost-simple if $S \trianglelefteq H \leq \text{Aut}(S)$ for some non-abelian finite simple group S . For a finite group Γ , let $d(\Gamma)$ denote the minimal number of generators of Γ , write $\Gamma' = [\Gamma, \Gamma]$ for the commutator subgroup of Γ and set

$$\begin{aligned} \Gamma_0 &= \bigcap \{T \trianglelefteq \Gamma \mid \Gamma/T \text{ almost-simple}\} \\ &= \bigcap \{C_\Gamma(M) \mid M \text{ a non-abelian simple chief factor}\}, \end{aligned}$$

where the intersection over an empty set is naturally interpreted as Γ . To see that the two descriptions of Γ_0 agree, recall that the chief factors of Γ arise as the minimal normal subgroups of arbitrary quotients of Γ . Further, we remark that Γ/Γ_0 is semisimple-by-(soluble of derived length at most 3), because the outer automorphism group of any simple group is soluble of derived length at most 3. This strong form of the Schreier conjecture is a consequence of the classification of finite simple groups.

For $X \subseteq \Gamma$ and $f \in \mathbb{N}$ we write $X^{*f} = \{x_1 \cdots x_f \mid x_1, \dots, x_f \in X\}$.

Theorem 3.1 (Nikolov, Segal [15]). *Let Γ be a finite group and $\{y_1, \dots, y_r\} \subseteq \Gamma$ a symmetric subset, i.e., a subset that is closed under taking inverses. Let $H \trianglelefteq \Gamma$.*

(1) *If $H \subseteq \Gamma_0$ and $H\langle y_1, \dots, y_r \rangle = \Gamma'\langle y_1, \dots, y_r \rangle = \Gamma$ then*

$$\langle [h, g] \mid h \in H, g \in \Gamma \rangle = \{[h_1, y_1] \cdots [h_r, y_r] \mid h_1, \dots, h_r \in H\}^{*f},$$

where $f = f(r, d(\Gamma)) = O(r^6 d(\Gamma)^6)$.

(2) *If $\Gamma = \langle y_1, \dots, y_r \rangle$ then the conclusion in (1) holds without assuming $H \subseteq \Gamma_0$ and with better bounds on f .*

While the proof of Theorem 3.1 is rather involved, the basic underlying idea is simple to sketch. Suppose that $\Gamma = \langle g_1, \dots, g_r \rangle$ is a finite group and M a non-central chief factor. Then the set $[M, g_i] = \{[m, g_i] \mid m \in M\}$ must be ‘relatively large’ for at least one generator g_i . Hence $\prod_{i=1}^r [M, g_i]$ is ‘relatively large’. In order to transform this observation into a rigorous proof of Theorem 3.1 one employs a combinatorial result, discovered by Timothy Gowers. Informally speaking, to show that a finite group is equal to a product of some of its subsets, it suffices to know that the cardinalities of these subsets are ‘sufficiently large’. A precise statement of this principle is the following.

Theorem 3.2 ([1, Corollary 2.6]). *Let Γ be a finite group, and let $\ell(\Gamma)$ denote the minimal dimension of a non-trivial \mathbb{R} -linear representation of Γ .*

If $X_1, \dots, X_t \subseteq \Gamma$, for $t \geq 3$, satisfy

$$\prod_{i=1}^t |X_i| \geq |\Gamma|^t \ell(\Gamma)^{2-t},$$

then $X_1 \cdots X_t = \{x_1 \cdots x_t \mid x_i \in X_i \text{ for } 1 \leq i \leq t\} = \Gamma$.

A short, but informative summary of the proof of Theorem 3.1 can be found in [11, §10].

§3.2. By standard compactness arguments, Theorem 3.1 yields a corresponding result for normal subgroups of finitely generated profinite groups. For a profinite

group G , let $d(G)$ denote the minimal number of topological generators of G , write $G' = [G, G]$ for the abstract commutator subgroup of G and set

$$G_0 = \bigcap \{T \trianglelefteq_o G \mid G/T \text{ almost-simple}\},$$

where the intersection over an empty set is naturally interpreted as G . As in the finite case, G/G_0 is semisimple-by-(soluble of derived length at most 3). For $X \subseteq G$ and $f \in \mathbb{N}$ we write $X^{*f} = \{x_1 \cdots x_f \mid x_1, \dots, x_f \in X\}$ as before, and the topological closure of X in G is denoted by \overline{X} .

Theorem 3.3 (Nikolov, Segal [15]). *Let G be a profinite group and $\{y_1, \dots, y_r\} \subseteq G$ a symmetric subset. Let $H \trianglelefteq_c G$ be a closed normal subgroup.*

(1) *If $H \subseteq G_0$ and $H\langle y_1, \dots, y_r \rangle = \overline{G'\langle y_1, \dots, y_r \rangle} = G$ then*

$$\langle [h, g] \mid h \in H, g \in G \rangle = \{[h_1, y_1] \cdots [h_r, y_r] \mid h_1, \dots, h_r \in H\}^{*f},$$

where $f = f(r, d(G)) = O(r^6 d(G)^6)$.

(2) *If y_1, \dots, y_r topologically generate G then the conclusion in (1) holds without assuming $H \subseteq G_0$ and better bounds on f .*

Proof. We indicate how to prove (1). The inclusion ‘ \supseteq ’ is clear. The inclusion ‘ \subseteq ’ holds modulo every open normal subgroup $N \trianglelefteq_o G$, by Theorem 3.1. Consider the set on the right-hand side, call it Y . Being the image of the compact space $H \times \cdots \times H$, with rf factors, under a continuous map, the set Y is closed in the Hausdorff space G . Hence

$$\langle [h, g] \mid h \in H, g \in G \rangle \subseteq \bigcap_{N \trianglelefteq_o G} YN = Y. \quad \square$$

In particular, the theorem shows that, if G is a finitely generated profinite group and $H \trianglelefteq_c G$ a closed normal subgroup, then the abstract subgroup

$$[H, G] = \langle [h, g] \mid h \in H, g \in G \rangle$$

is closed. Thus G' and more generally all terms $\gamma_i(G)$ of the abstract lower central series of G are closed; these consequences were already established in [12]. Theorem 1.4, stated in the introduction, generalises these results to more general compact Hausdorff groups.

Furthermore, one obtains from Theorem 3.3 the following tool for studying abstract normal subgroups of a finitely generated profinite group G , reducing certain problems more or less to the abelian profinite group G/G' or the profinite group G/G_0 which is semisimple-by-(soluble of derived length at most 3).

Corollary 3.4 (Nikolov, Segal [15]). *Let G be a finitely generated profinite group and $N \trianglelefteq G$ a normal abstract subgroup. If $NG' = NG_0 = G$ then $N = G$.*

Proof. Suppose that $NG' = NG_0 = G$, and let $d = d(G)$ be the minimal number of topological generators of G . Then there exist $y_1, \dots, y_{2d} \in N$ such that

$$G_0 \overline{\langle y_1, \dots, y_{2d} \rangle} = \overline{G'\langle y_1, \dots, y_{2d} \rangle} = G.$$

Applying Theorem 3.3, with $H = G_0$, we obtain

$$[G_0, G] \subseteq \langle [G_0, y_i] \cup [G_0, y_i^{-1}] \mid 1 \leq i \leq 2d \rangle \subseteq N,$$

hence $G = NG' = N[NG_0, G] = N$. □

Using Corollaries 3.4 and 2.11, it is not difficult to derive Theorem 1.1.

Proof of Theorem 1.1. Let $H \leq_f G$ be a finite-index subgroup of the finitely generated profinite group G . Then its core $N = \bigcap_{g \in G} H^g \trianglelefteq_f G$ is contained in H , and it suffices to prove that N is open in G . The topological closure \overline{N} is open in G ; in particular, \overline{N} is a finitely generated profinite group and without loss of generality we may assume that $\overline{N} = G$.

Assume for a contradiction that $N \not\cong G$. Using Corollary 3.4, we deduce that

$$(1) \quad NG' \not\cong G \quad \text{or} \quad NG_0 \not\cong G.$$

Setting $q = |G : N|$, we know that the abstract subgroup $G^q = \langle g^q \mid g \in G \rangle$ generated by all q th powers is contained in N . By Theorem 3.3, the abstract commutator subgroup $G' = [G, G]$ is closed, thus the subgroup $G'G^q$ is closed in G . Hence $G/G'G^q$, being a finitely generated abelian profinite group of finite exponent, is finite and discrete. As $NG'/G'G^q$ is dense in $G/G'G^q$, we deduce that $NG' = NG'G^q = G$.

It suffices to prove that $NG_0 = G$ in order to obtain a contradiction to (1). Factoring out by G_0 , we may assume without loss of generality that $G_0 = 1$. Then G has a semisimple subgroup $T \trianglelefteq_c G$ such that G/T is soluble. From $G = NG'$ we see that G/NT is perfect and soluble, and we deduce that

$$(2) \quad NT = G.$$

Corollary 2.11 shows that $T^q \leq_c T$. Factoring out by T^q (which is contained in N) we may assume without loss of generality that $T^q = 1$. The definition of G_0 shows that T is a product of non-abelian finite simple groups, each normal in G , of exponent dividing q . Using the classification of finite simple groups, one sees that the finite simple factors of T have uniformly bounded order. Thus $G/C_G(T)$ is finite. As $T \cap C_G(T) = 1$, we conclude that T is finite, thus $T \cap N \leq_c G$. This shows that

$$T = [T, G] = [T, \overline{N}] \leq \overline{[T, N]} \leq T \cap N,$$

and (2) gives $N = NT = G$. □

§3.3. The methods developed in [15] lead to new consequences for abstract quotients of finitely generated profinite groups and, more generally, compact Hausdorff topological groups. In the introduction we stated three such results: Theorems 1.2, 1.3 and 1.4.

We finish by indicating how Corollary 3.4 can be used to see that, for profinite groups, the assertion in Theorem 1.2 reduces to the following special case.

Theorem 3.5. *A finitely generated semisimple profinite group does not have countably infinite abstract images.*

Nikolov and Segal prove Theorem 3.5 via a complete description of the maximal normal abstract subgroups of a strongly complete semisimple profinite group; see [15, Theorem 5.12]. Their argument relies, among other things, on work of Martin Liebeck and Aner Shalev [10] on the diameters of non-abelian finite simple groups.

Proof of ‘Theorem 3.5 implies Theorem 1.2 for profinite groups’. For a contradiction, assume that $\Gamma = G/N$ is an infinite finitely generated abstract image of a profinite group G . Replacing G by the closed subgroup generated by any finite set mapping onto a generating set of Γ , we may assume that G is topologically finitely generated, hence strongly complete by Theorem 1.1. Let $\Gamma_1 = \bigcap_{\Delta \trianglelefteq_f \Gamma} \Delta$. Then Γ/Γ_1 is a countable residually finite image of G , and Corollary 2.3 shows

that $|\Gamma : \Gamma_1| < \infty$. Replacing Γ by Γ_1 and G by the preimage of Γ_1 in G , we may assume that Γ has no non-trivial finite images.

We apply Corollary 3.4 to G and the normal abstract subgroup $N \trianglelefteq G$ with $G/N = \Gamma$. Since Γ is finitely generated and has no non-trivial finite images, we conclude that $\Gamma' = \Gamma$ and hence $NG' = G$. Thus $NG_0 \not\cong G$. Consequently, G/G_0 maps onto the infinite finitely generated perfect quotient $\Gamma_0 = G/NG_0$ of Γ . We may assume that $G_0 = 1$ so that G is semisimple-by-soluble. Since soluble groups do not map onto perfect groups, we may assume that G is semisimple. \square

4. ABSTRACT VERSUS CONTINUOUS COHOMOLOGY

In this section we interpret Theorem 1.1 in terms of cohomology groups and indicate some work in preparation; for more details we refer to a forthcoming joint paper [2] with Yiftach Barnea and Jaikin-Zapirain.

§4.1. Let Γ be a dense abstract subgroup of a profinite group G , and let M be a continuous finite G -module. For $i \in \mathbb{N}_0$, the i -dimensional continuous cohomology group $H_{\text{cont}}^i(G, M)$ can be defined as the quotient $Z_{\text{cont}}^i(G, M)/B_{\text{cont}}^i(G, M)$ of continuous i -cocycles modulo continuous i -coboundaries with values in M . Alternatively, the continuous cohomology can be described via the direct limit

$$H_{\text{cont}}^i(G, M) = \varinjlim_{U \trianglelefteq_o G} H^i(G/U, M^U),$$

where U runs over all open normal subgroups, M^U denotes the submodule of invariants under U , and $H^i(G/U, M^U)$ denotes the ordinary cohomology of the finite group G/U . Restriction provides natural maps

$$(3) \quad H_{\text{cont}}^i(G, M) \rightarrow H^i(\Gamma, M), \quad i \in \mathbb{N}_0.$$

Here $H^i(\Gamma, M)$ is the ordinary cohomology, or equivalently the continuous cohomology of Γ equipped with the discrete topology.

In fact, we concentrate on the case $\Gamma = G$, regarded as an abstract group, and we write $H_{\text{disc}}^i(G, M) = H^i(\Gamma, M)$ for the cohomology of the abstract group G . Nikolov and Segal's solution to Serre's problem can be reformulated as follows.

Theorem 4.1 (Nikolov, Segal). *Let G be a finitely generated profinite group and M a continuous finite G -module. Then $H_{\text{cont}}^1(G, M) \rightarrow H_{\text{disc}}^1(G, M)$ is a bijection, and $H_{\text{cont}}^2(G, M) \rightarrow H_{\text{disc}}^2(G, M)$ is injective.*

It is natural to ask for analogous results in higher dimensions, but the situation seems to become rather complicated already in dimensions 2 and 3. Currently, very little is known, even at a conjectural level. Some basic results, regarding pro- p groups, can be found in [4]; see also the references therein.

§4.2. In [20, I.2.6], Serre introduced a series of equivalent conditions to investigate the maps (3), in a slightly more general setting. For our purpose the following formulation is convenient: for $n \in \mathbb{N}_0$ and p a prime, we define the property

$E_n(p)$: for every continuous finite G -module M of p -power cardinality, the natural map $H_{\text{cont}}^i(G, M) \rightarrow H_{\text{disc}}^i(G, M)$ is bijective for $0 \leq i \leq n$.

We say that G satisfies E_n if $E_n(p)$ holds for all primes p . Since cohomology 'commutes' with taking direct sums in the coefficients, G satisfies E_n if and only if, for every continuous finite G -module M , the maps (3) are bijections for $0 \leq i \leq n$. We say that G is *cohomologically p -good* if G satisfies $E_n(p)$ for all $n \in \mathbb{N}_0$.

The group G is called *cohomologically good* if it is cohomologically p -good for all primes p ; cf. [20, I.2.6].

The following proposition from [2] provides a useful tool for showing that a group has property $E_n(p)$.

Proposition 4.2. *A profinite group H satisfies $E_n(p)$ if there is an abstract short exact sequence*

$$1 \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow 1,$$

where N and G are finitely generated profinite groups satisfying $E_n(p)$.

We discuss a natural application. Recall that the *rank* of a profinite group G is

$$\text{rk}(G) = \sup\{d(H) \mid H \leq_c G\},$$

where $d(H)$ is the minimal number of topological generators of H . Using the classification of finite simple groups, it can be shown that if G is a profinite group all of whose Sylow subgroups have rank at most r then $\text{rk}(G) \leq r+1$. Furthermore, a pro- p group G has finite rank if and only if it is p -adic analytic; in this case the dimension of G as a p -adic manifold is bounded by the rank: $\dim(G) \leq \text{rk}(G)$.

The following result from [2] generalises [4, Theorem 2.10].

Theorem 4.3. *Soluble profinite groups of finite rank are cohomologically good.*

It is well-known that the second cohomology groups, and hence properties $E_2(p)$ and E_2 , are intimately linked to the theory of group extensions. Theorem 4.3 can be used to prove that every abstract extension of soluble profinite groups of finite rank carries again, in a unique way, the structure of a soluble profinite group of finite rank. We refer to [22, Chapter 8] for a general discussion of profinite groups of finite rank. An example of a soluble profinite group of finite rank that is not virtually pronilpotent can be obtained as follows: consider $G = \overline{\langle x \rangle} \rtimes A \cong \widehat{\mathbb{Z}} \rtimes \widehat{\mathbb{Z}}$, where conjugation by x on a Sylow pro- p subgroup $A_p \cong \mathbb{Z}_p$ of $A = \prod_p A_p$ is achieved via multiplication by a primitive $(p-1)$ th root of unity.

§4.3 Fix a prime p and consider the questions raised above for a finitely generated pro- p group G . By means of a reduction argument, one is led to focus on cohomology groups with coefficients in the trivial module \mathbb{F}_p , i.e., the groups

$$H_{\text{cont}}^i(G) = H_{\text{cont}}^i(G, \mathbb{F}_p) \quad \text{and} \quad H_{\text{disc}}^i(G) = H_{\text{disc}}^i(G, \mathbb{F}_p) \quad \text{for } i \in \mathbb{N}_0.$$

Indeed, G satisfies $E_n(p)$ if and only if, for $1 \leq i \leq n$, every cohomology $\alpha \in H_{\text{disc}}^i(U)$, where $U \leq_o G$, vanishes when restricted to a suitable smaller open subgroup. It is natural to start with the cohomology groups in dimension 2, and by virtue of Theorem 4.1, we may regard $H_{\text{cont}}^2(G)$ as a subgroup of $H_{\text{disc}}^2(G)$. It is well-known that these groups are intimately linked with continuous, respectively abstract, central extensions of a cyclic group C_p by G .

First suppose that the finitely generated pro- p group G is not finitely presented as a pro- p group. For instance, the profinite wreath product $C_p \wr \widehat{\mathbb{Z}}_p = \varprojlim_i C_p \wr C_{p^i}$ has this property. Then it can be shown that there exists a central extension $1 \rightarrow C_p \rightarrow H \rightarrow G \rightarrow 1$ such that H is not residually finite, hence cannot be a profinite group; compare [4, Theorem A]. It follows that $H_{\text{cont}}^2(G)$ is strictly smaller than $H_{\text{disc}}^2(G)$, and G does not satisfy $E_2(p)$. This raises a natural question.

Problem. Characterise finitely presented pro- p groups that satisfy $E_2(p)$.

Groups of particular interest, for which the situation is not yet fully understood, include on the one hand non-abelian free pro- p groups and on the other hand p -adic analytic pro- p groups. Indeed, a rather intriguing theorem of Aldridge K. Bousfield [3, Theorem 11.1] in algebraic topology shows that, for a non-abelian finitely generated free pro- p group F , the direct sum $H_{\text{disc}}^2(F) \oplus H_{\text{disc}}^3(F)$ is infinite. This stands in sharp contrast to the well-known fact that $H_{\text{cont}}^2(F) = H_{\text{cont}}^3(F) = 0$.

In [2] we establish the following result.

Theorem 4.4. *Let F be a finitely presented pro- p group satisfying $E_2(p)$. Then every finitely presented continuous quotient of F also satisfies $E_2(p)$.*

In particular, this shows that, if every finitely generated free pro- p group F has $H_{\text{disc}}^2(F) = H_{\text{cont}}^2(F) = 0$, then all finitely presented pro- p groups satisfy $E_2(p)$.

We close by stating a theorem from [2] regarding the cohomology of compact p -adic analytic groups that improves results of Balasubramanian Sury [21] on central extensions of Chevalley groups over non-archimedean local fields of characteristic 0.

Theorem 4.5. *Let Φ be an irreducible root system, and let K be a non-archimedean local field of characteristic 0 and residue characteristic p .*

Then every open compact subgroup H of the Chevalley group $\text{Ch}_{\Phi}(K)$ satisfies $E_2(p)$. Consequently, every central extension of C_p by H is residually finite and carries, in a unique way, the structure of a profinite group.

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