NO LIE p-ALGEBRAS OF COHOMOLOGICAL DIMENSION ONE

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Abstract. We prove that a Lie p-algebra of cohomological dimension one is one-dimensional, and discuss related questions.

0. Introduction

A *cohomological dimension* of a Lie algebra L over a field K, denoted by $\mathrm{cd}(L)$, is defined as the right projective dimension of the trivial L-module K, i.e., the minimal possible length of a finite projective resolution

$$(1) \qquad \dots \to P_2 \to P_1 \to P_0 \to K$$

consisting of right projective modules P_i over the universal enveloping algebra $U(L)$, or infinity if no such finite resolution exists. Since for every projective resolution (1) and every L-module M, the cohomology of the induced complex

$$0 \leftarrow M = \mathrm{Hom}_{U(L)}(K, M) \leftarrow \mathrm{Hom}_{U(L)}(P_1, M) \leftarrow \mathrm{Hom}_{U(L)}(P_2, M) \leftarrow \dots$$

of L-modules coincides with the Chevalley–Eilenberg cohomology $\mathrm{H}^\bullet(L, M)$, L has cohomological dimension n if and only if there is an L-module M such that $\mathrm{H}^{n-1}(L, M) \neq 0$, and the two equivalent conditions hold:

(i) $\mathrm{H}^i(L, M) = 0$ for any L-module M and any $i \geq n$;
(ii) $\mathrm{H}^n(L, M) = 0$ for any L-module M.

A similar notion may be defined for other classes of algebraic systems with good cohomology theory, e.g., for groups and associative algebras.

The Shapiro lemma about cohomology of a coinduced module implies that if S is a subalgebra of a Lie algebra L, then $\mathrm{cd}(S) \leq \mathrm{cd}(L)$. As cohomological dimension of the one-dimensional Lie algebra is equal to one, the cohomological dimension of any Lie algebra is ≥ 1. In particular, the class of Lie algebras of cohomological dimension one is closed with respect to subalgebras.

Due to the standard interpretation of the second cohomology, the condition for a Lie algebra L to be of cohomological dimension 1 is equivalent to the condition that each exact sequence

$$0 \to ? \to ? \to L \to 0$$

of L-modules splits. The latter condition holds for a free Lie algebra, due to its universal property, and hence a free Lie algebra (of any rank) has cohomological dimension one. The same is true for free groups and free associative algebras.

The celebrated Stallings–Swan theorem says that for groups the converse is true: a group of cohomological dimension one is free (cf., e.g., [Co]). A question by Bourbaki ([B, Chapitre II, §2, footnote to Exercice 9]) asks whether the same is true for Lie algebras, i.e., whether a Lie algebra of cohomological dimension one is free.

For a while, it was widely believed that the answer to the latter question is affirmative (the author has witnessed several attempts of the proof), until Mikhalev, Umirbaev and Zolotykh constructed an example of a non-free Lie algebra of cohomological dimension one over a field of characteristic > 2 (cf. [MUZ]; note that the cases of characteristic zero and characteristic 2 remain widely open). This example is not a p-algebra, and at the same paper they made the following conjecture: a Lie p-algebra of cohomological dimension one is a free Lie p-algebra ([MUZ, Conjecture 2]). As stated, the conjecture is somewhat misleading, for a free Lie p-algebra

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is not of cohomological dimension one: its cohomological dimension is equal to infinity. Indeed, for any element x of such an algebra, the elements $x, x^{[p]}, x^{[p]^2}, \dots$ span an infinite-dimensional abelian subalgebra, whose cohomological dimension is equal to infinity (cf. Lemma 1 below).

This conjecture may be repaired in two ways. First, one may merely ask about description of Lie p-algebras of cohomological dimension one. A (trivial) answer to this question is given in §1: such algebras are one-dimensional. Another possibility is to replace cohomological dimension with *restricted* cohomological dimension. The conjecture in this form is discussed in §2. §1 also contains an auxiliary result related to the old Jacobson conjecture about periodic Lie p-algebras.

1. Lie p-algebras of cohomological dimension one, and almost-periodic algebras

The following is elementary but useful.

Lemma 1.

(i) *Cohomological dimension of an abelian Lie algebra is equal to its dimension.*
(ii) *Cohomological dimension of the two-dimensional nonabelian Lie algebra is equal to* 2.

Proof. It is clear that cohomological dimension of a Lie algebra does not exceed its dimension.

(i) For an abelian Lie algebra, we have $\mathrm{H}^n(L,K) = (\bigwedge^n L)^\star$ for any n ($\star$ denotes the dual space).

(ii) Let L be a two-dimensional nonabelian Lie algebra with a basis $\{x,y\}$, $[x,y]=x$. For an one-dimensional module Kv with an L-action $x \bullet v = 0, y \bullet v = -v$, we have $\dim \mathrm{H}^2(L, Kv) = 1$. □

Corollary. *A Lie algebra of cohomological dimension one does not contain a two-dimensional subalgebra.*

Lemma 2. *Let x,y be two elements of a Lie algebra without two-dimensional subalgebras, such that $(\mathsf{ad}\, x)^n y = 0$ for some n. Then x,y are linearly dependent.*

Proof. Applying repeatedly the condition of absence of two dimensional subalgebras, we can lower the degree n. Indeed, $(\mathsf{ad}\, x)^n y = [(\mathsf{ad}\, x)^{n-1}(y), x] = 0$ implies $[(\mathsf{ad}\, x)^{n-2}(y), x] = (\mathsf{ad}\, x)^{n-1}(y) = \lambda x$ for some $\lambda \in K$, what, in turn, implies $\lambda = 0$. Repeating this process, we get eventually $[y,x]=0$, and hence x,y are linearly dependent. □

Lemma 3. *A Lie p-algebra of dimension > 1 over an algebraically closed field contains a two-dimensional subalgebra.*

This lemma may be considered as a generalization of an elementary fact that a finite-dimensional Lie algebra of dimension > 1 over an algebraically closed field contains a two-dimensional subalgebra. We do not assume finite-dimensionality, but the presence of p-structure is a condition strong enough to infer the same conclusion.

Proof. Let L be a Lie p-algebra without two-dimensional subalgebras. For any $x \in L$, we have $[x, x^{[p]}] = 0$, and hence

$$x^{[p]} = \lambda(x) x \tag{2}$$

for some $\lambda(x) \in K$.

If $\lambda(x)$ is a constant, then L is abelian by [J, Chapter V, Exercise 15], and hence is one-dimensional. The general case, however, requires a bit of extra work. We pause the proof of Lemma 3 for a moment to reflect on the condition (2). This condition reminds of various conditions on the p-map studied by Jacobson and others. The major open problem in this area is the conjecture of Jacobson that a periodic Lie p-algebra is abelian (cf. [J, Chapter V, Exercise 16]). Recall that a Lie algebra L is called *periodic* if for any $x \in L$ there is integer $n(x) > 0$ such that $x^{[p]^{n(x)}} = x$. The strongest result toward this conjecture belongs to Premet: a periodic *finite-dimensional* Lie algebra is abelian ([P, Corollary 1]).

Generalizing the condition of periodicity, let us call a Lie p-algebra L *almost periodic*, if for any $x \in L$, there is an integer $n(x) > 0$ and an element $\lambda(x) \in K$ such that

$$x^{[p]^{n(x)}} = \lambda(x)x. \tag{3}$$

The elements for which $\lambda(x) = 0$, i.e., $x^{[p]^{n(x)}} = 0$, will be called *p-nilpotent*.

Lemma 4. *Let L be an almost periodic Lie p-algebra of dimension > 1 over an algebraically closed field, with all $n(x)$'s bounded. Then L contains a nonzero p-nilpotent element.*

Note some other related results connecting properties of Lie (p-)algebras and its elements: Chwe proved in [Ch2] that a Lie p-algebra over an algebraically closed field with a nondegenerate p-map is abelian. Farnsteiner investigated in [F1] Lie p-algebras in which a certain power $[p]^n$ of the p-map is p^n-semilinear. The condition (3) is somewhat reminiscent of semilinearity (in some sense stronger, in some sense weaker). Finally, it is well-known that any finite-dimensional Lie algebra over an algebraically closed field contains a nilpotent element. (For Lie p-algebras, this follows from the Jordan–Chevalley decomposition, cf., e.g., [P, Proof of Theorem 3], and for a short elementary proof valid for arbitrary Lie algebras, cf. [BI]). Lemma 4 establishes a similar result for not necessary finite-dimensional Lie algebras, but subject to a strong condition of bounded p-periodicity.

Proof of Lemma 4. Since $n(x)$ are bounded, we may assume that

$$x^{[p]^n} = \lambda(x)x \tag{4}$$

for some fixed n (for example, by letting n to be the product of all distinct $n(x)$'s, and redenoting $\lambda(x)$'s appropriately).

Pick any two linearly independent elements $x, y \in L$, and set $\varphi_{xy}(t) = \lambda(x+ty)$, for $t \in K$. Using the well-known Jacobson binomial formula for the p-map (strictly speaking, its generalization for the nth power of the p-map – cf., e.g., [F1, §1]), we have

$$\varphi_{xy}(t)(x+ty) = (x+ty)^{[p]^n} = x^{[p]^n} + t^{p^n} y^{[p]^n} + \sum_{i=1}^{p^n-1} t^i s_i(x,y) = \lambda(x)x + t^{p^n}\lambda(y)y + \sum_{i=1}^{p^n-1} t^i s_i(x,y), \tag{5}$$

where $s_i(x,y)$ are certain Lie monomials in x, y. Completing x, y to a basis of L, writing $s_i(x,y)$'s as linear combinations of basis elements, and collecting all coefficients of x in (5), we get that $\varphi_{xy}(t)$ is a polynomial in t with the free term $\lambda(x)$.

Suppose that there is a pair x, y such that $\varphi_{xy}(t)$ is not constant. Since the ground field K is algebraically closed, $\varphi_{xy}(t)$ has a root ξ. This means that the nonzero element $x + \xi y$ is nilpotent.

Suppose now that for any pair $x, y \in L$, $\varphi_{xy}(t)$ is constant, i.e., $\varphi_{xy}(t) = \lambda(x)$. This means that $\lambda(x + ty) = \lambda(x)$ for any linearly independent $x, y \in L$, and any $t \in K$, and, consequently, $\lambda(x) = \lambda$ is constant. If $\lambda \neq 0$, then substituting in (4) αx instead of x, we get that $\alpha^{p^n - 1} = \alpha$ for any $\alpha \in K$, i.e., K is a finite field, a contradiction. Hence $\lambda = 0$, and every element of L is nilpotent. □

Continuation of the proof of Lemma 3. According to Lemma 4 (with $n(x) = 1$ for all x), L is either one-dimensional, or contains a nonzero nilpotent element. In the latter case by Lemma 2, L is one-dimensional too, a contradiction. □

Theorem. *A Lie p-algebra of cohomological dimension one is one-dimensional.*

Proof. As the property of being a p-algebra, and dimension (and, in particular, vanishing) of cohomology do not change under field extensions, we may assume that the ground field is algebraically closed. Then the claim follows from Corollary to Lemma 1, and Lemma 3. □

2. Lie p-algebras of restricted cohomological dimension one

When speaking about cohomological dimension, we consider the category of *all* Lie algebra modules, including infinite-dimensional ones. If we restrict ourselves with, say, finite-dimensional Lie algebras and the category of finite-dimensional modules, the whole subject, both in results and methods employed, becomes quite different. In fact, we cannot longer speak about cohomological dimension, as vanishing of all cohomology in a given degree does not imply vanishing in higher degrees. A sample of results in this domain: in characteristic zero, an "almost" converse of the classical Whitehead Lemmas holds ([Z1], [Z2]), and in positive characteristic, for any degree less than the dimension of the algebra, a module with non-vanishing cohomology exists ([D] and [FS]).

Still, instead of the category of all modules we can consider a smaller subcategory of modules with a good-behaving cohomology theory: for example, restricted modules with restricted cohomology. The definition of a *restricted cohomological dimension* of a Lie p-algebra L over a field of positive characteristic (notation: $\mathrm{cd}_*(L)$) repeats the definition of the ordinary cohomological dimension, with projective resolutions (1) are considered in the category of restricted modules over the restricted universal enveloping algebra $u(L)$.

As in the unrestricted case, Shapiro's lemma for restricted cohomology implies that the restricted cohomological dimension does not increase when passing to subalgebras. In particular, the class of Lie p-algebras of restricted cohomological dimension one is closed with respect to subalgebras. A free Lie p-algebra has restricted cohomological dimension one. In this context, we reformulate the Conjecture 2 from [MUZ]:

Conjecture. *A Lie p-algebra of restricted cohomological dimension one is a free Lie p-algebra.*

Let us establish some facts about Lie p-algebras of restricted cohomological dimension one, which provide some evidence in support of this conjecture.

The following fact was established in [Ch1, Theorem 5.1] using a not entirely trivial result from homological algebra due to Kaplansky. We give an alternative, more elementary proof – a mere reformulation of known (and easy) results about (co)homology of commutative associative algebras.

Lemma 5. *A finite-dimensional Lie p-algebra has infinite restricted cohomological dimension.*

Proof. As for a Lie p-algebra L, and an $u(L)$-bimodule M, we have

$$\mathrm{H}^n_*(L, M^{\mathsf{ad}}) \simeq \mathrm{HH}^n(u(L), M), \tag{6}$$

where H_* and HH stand for the restricted cohomology of a Lie p-algebra, and Hochschild cohomology of an associative algebra, respectively, and M^{ad} is a restricted L-module structure on M defined via $x \bullet m = xm - mx$ for $x \in L$, $m \in M$, it is enough, for each finite-dimensional L, to exhibit an $u(L)$-bimodule in which Hochschild cohomology does not vanish for an arbitrarily high degree.

Also, it is enough to prove the claim for an one-generated (abelian) Lie p-algebra, i.e., for a Lie algebra L with a basis of the form $\{x, x^{[p]}, x^{[p]^2}, \dots, x^{[p]^{n-1}}\}$, where an element x satisfies the relation of the form

$$\lambda_0 x + \lambda_1 x^{[p]} + \dots + \lambda_n x^{[p]^n} = 0 \tag{7}$$

for $\lambda_i \in K$, $\lambda_n \neq 0$. In this case $u(L) \simeq K[x]/(f)$, where the polynomial f is obtained from the left-hand side of (7) by replacing $[p]$-powers in a Lie p-algebra by the ordinary p-powers in a polynomial algebra: $f(t) = \lambda_0 + \lambda_1 t^p + \dots + \lambda_n t^{p^n}$.

The Hochschild cohomology of associative algebras of the form $K[x]/(f)$ was well studied in the literature. For example, in [H, Proposition 2.2] a periodic free resolution of such algebras is constructed, and it is proved that the Hochschild cohomology $\mathrm{HH}^n(K[x]/(f), K[x]/(f))$, for any f, does not vanish for an arbitrarily high n. As $K[x]/(f)$ is commutative, $K[x]/(f)^{\mathsf{ad}}$, as an L-module, is the direct sum of p^n copies of the trivial L-module K, and due to isomorphism (6), $\mathrm{H}^n_*(L, K)$ is nonzero for an arbitrarily high n. ☐

Proposition 1. *A p-subalgebra of a Lie p-algebra of finite restricted cohomological dimension is infinite-dimensional.*

Proof. Follows from Lemma 5. □

In particular, in a Lie p-algebra L of finite restricted cohomological dimension every nonzero element x is not p-algebraic, i.e., does not satisfy any relation of the form (7) (or, in other words, the p-envelope of x inside L is infinite-dimensional). This is a Lie-p-algebraic analog of the well-known fact that groups of finite cohomological dimension are torsion-free (cf., e.g., [Co, p. 6, Corollary 2]).

Similarly with the unrestricted case, we have:

Corollary. *The restricted cohomological dimension of any Lie p-algebra is ≥ 1.*

Proof. Take an arbitrary nonzero element x in a Lie p-algebra L. If x is p-algebraic, then by Proposition 1, L has infinite restricted cohomological dimension. If x is not p-algebraic, then it generates a free Lie p-subalgebra of rank 1. The restricted cohomological dimension of the latter algebra is equal to 1, and hence the restricted cohomological dimension of L is ≥ 1. □

Proposition 2. *An abelian p-subalgebra of a Lie p-algebra of restricted cohomological dimension one is isomorphic to the free Lie p-algebra of rank one.*

Proof. It is enough to prove that any two commuting elements, x and y, in a Lie p-algebra L of restricted cohomological dimension one, can be represented as p-polynomials of a third element. Suppose the contrary. By Proposition 1, each of x, y generate the free Lie p-algebra of rank one, and hence the restricted universal enveloping algebra of the p-subalgebra S of L generated by x, y, is isomorphic to the polynomial algebra in two variables $K[x,y]$. The latter algebra has non-vanishing 2nd Hochschild cohomology (for example, $\mathrm{HH}^2(K[x,y], K[x,y]) \simeq \bigwedge^2(\mathrm{Der}(K[x,y])) \otimes_{K[x,y]} K$ by the Hochschild–Kostant–Rosenberg theorem), and reasoning as in the end of the proof of Lemma 5, we get that $\mathrm{H}^2_*(S,K)$ does not vanish, whence $\mathrm{cd}_*(L) \geq 2$, a contradiction. □

The next lemma shows that the (ordinary) cohomology of Lie p-algebras of restricted cohomological dimension one behaves in a rather peculiar way.

Lemma 6. *Let L be a Lie p-algebra L of restricted cohomological dimension one, and M a restricted L-module M. Then*

$$\mathrm{H}^n(L,M) \simeq \Big(\Big(\bigwedge\nolimits^n L\Big)^\star \otimes M^L\Big) \oplus \Big(\Big(\bigwedge\nolimits^{n-1} L\Big)^\star \otimes \mathrm{H}^1_*(L,M)\Big) \tag{8}$$

for any $n \geq 1$.

Proof. This follows from a particular form of the Grothendieck spectral sequence relating restricted and ordinary cohomology. Namely, for a Lie p-algebra and a restricted L-module M, there is a spectral sequence with the E_2 term

$$\mathrm{E}_2^{st} = \mathrm{C}^t(L, \mathrm{H}^s_*(L,M)) \simeq \Big(\bigwedge\nolimits^n L\Big)^\star \otimes \mathrm{H}^s_*(L,M)$$

abutting to $\mathrm{H}^{s+t}(L,M)$ (cf. [FP, Proposition 5.3]; note that the standing assumption in [FP] of finite-dimensionality of algebras and modules is not relevant here; cf. also [F2, Theorem 4.1] and [M, Corollary 1.3]). Here $\mathrm{C}^n(V,W) \simeq (\bigwedge^n V)^\star \otimes W$ denotes, as usual, the space of skew-symmetric n-linear maps from one vector space to another.

If $\mathrm{H}^s_*(L,M) = 0$ for $s \geq 2$, the only nonvanishing E_2 terms are E_2^{0t} and E_2^{1t}. Hence the spectral sequence stabilizes at E_2, $\mathrm{H}^n(L,M) \simeq \mathrm{E}_2^{0n} \oplus \mathrm{E}_2^{1,n-1}$ for any $n \geq 1$, and (8) follows. □

Lemma 6 provides a yet another proof of the fact that a Lie p-algebra L of restricted cohomological dimension one is infinite-dimensional (a particular case of Lemma 5), without appealing to any computation of Hochschild cohomology. Indeed, suppose the contrary, and take in (8)

$n = \dim L + 1$. Then the left-hand side and the first direct summand at the right-hand side of the isomorphism vanish, and the second direct summand is isomorphic to $\mathrm{H}^1_*(L, M)$. Therefore, $\mathrm{H}^1_*(L, M) = 0$ for any restricted L-module M, i.e., L is of restricted cohomological dimension zero, a contradiction.

Moreover, a stronger statement holds:

Proposition 3. *A Lie p-algebra of restricted cohomological dimension one has infinite (ordinary) cohomological dimension.*

Proof. Let L be a Lie algebra of restricted cohomological dimension one. Taking in (8) $M = K$, we get

$$\mathrm{H}^n(L, K) \simeq \Big(\bigwedge\nolimits^n L\Big)^\star \oplus \Big(\Big(\bigwedge\nolimits^{n-1} L\Big)^\star \otimes \mathrm{H}^1_*(L, K)\Big).$$

Either by Lemma 5, or by the reasoning above, L is infinite-dimensional, and thus $\bigwedge^n L$, and hence $\mathrm{H}^n(L, K)$, does not vanish for any $n \geq 1$. $\square$

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References

[BI] G. Benkart and I.M. Isaacs, *On the existence of ad-nilpotent elements*, Proc. Amer. Math. Soc. **63** (1977), 39–40.

[B] N. Bourbaki, *Groupes et Algèbres de Lie*, Chapitres 2 et 3, Hermann, Paris, 1972; reprinted by Springer, 2006.

[Ch1] B.-S. Chwe, *Relative homological algebra and homological dimension of Lie algebras*, Trans. Amer. Math. Soc. **117** (1965), 477–493.

[Ch2] ———, *On the commutativity of restricted Lie algebras*, Proc. Amer. Math. Soc. **16** (1965), 547.

[Co] D.E. Cohen, *Groups of cohomological dimension one*, Lect. Notes Math. **245** (1972).

[D] A.S. Dzhumadil'daev, *Cohomology of truncated coinduced representations of Lie algebras of positive characteristic*, Mat. Sbornik **180** (1989), 456–468 (in Russian); Math. USSR Sbornik **66** (1990), 461–473 (English translation).

[F1] R. Farnsteiner, *Restricted Lie algebras with semilinear p-mappings*, Proc. Amer. Math. Soc. **91** (1984), 41–45.

[F2] ———, *Cohomology groups of reduced enveloping algebras*, Math. Z. **206** (1991), 103–117.

[FS] ——— and H. Strade, *Shapiro's lemma and its consequences in the cohomology theory of modular Lie algebras*, Math. Z. **206** (1991), 153–168.

[FP] E.M. Friedlander and B.J. Parshall, *Modular representation theory of Lie algebras*, Amer. J. Math. **110** (1988), 1055–1093.

[H] T. Holm, *Hochschild cohomology rings of algebras $k[X]/(f)$*, Beiträge Algebra Geom. **41** (2000), 291–301.

[J] N. Jacobson, *Lie Algebras*, Interscience Publ., 1962; reprinted by Dover, 1979.

[MUZ] A.A. Mikhalev, U.U. Umirbaev, and A.A. Zolotykh, *A Lie algebra with cohomological dimension one over a field of prime characteristic is not necessarily free*, First International Tainan-Moscow Algebra Workshop (ed. Y. Fong et al.), de Gruyter, 1996, 257–264.

[M] M. Muzere, *Relative Lie algebra cohomology revisited*, Proc. Amer. Math. Soc. **108** (1990), 665–671.

[P] A.A. Premet, *On Cartan subalgebras of Lie p-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), 788–800 (in Russian); Math. USSR Izvestija **29** (1987), 145–157 (English translation).

[Z1] P. Zusmanovich, *A converse to the Second Whitehead Lemma*, J. Lie Theory **18** (2008), 295–299; Erratum: **24** (2014), 1207–1208; arXiv:0704.3864.

[Z2] ———, *A converse to the Whitehead Theorem*, J. Lie Theory **18** (2008), 811–815; arXiv:0808.0212.

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