Removable isolated asymptotic singularities of solutions of the minimal graph equation in a 2−dimensional Cartan-Hadamard manifold

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Abstract

Let M be a 2−dimensional Cartan-Hadamard manifold with sectional curvature satisfying $-b^2 \le K \le -a^2 < 0$, $b \ge a > 0$. Denote by $\partial_\infty M$ the asymptotic boundary of M and by $\bar{M} := M \cup \partial_\infty M$ the geometric compactification of M with the cone topology. Given a finite number of points $p_1, ..., p_n \in \partial_\infty M$, it is proved that if $u \in C^\infty(M) \cap C^0\left(\bar{M} \backslash \{p_1, ..., p_n\}\right)$ is a solution of the minimal surface equation in M and if $u|_{\partial_\infty M \backslash \{p_1,...,p_n\}}$ extends continuously to p_i, $i = 1, ..., n$, then $u \in C^0\left(\bar{M}\right)$. We also prove the same result in arbitrary dimensions when the ambient space is the hyperbolic space.

1 Introduction

Let M be Cartan-Hadamard manifold (complete, connected, simply connected Riemannian manifold with non-positive sectional curvature). It is well-known that M can be compactified with the so called cone topology by adding a sphere at infinity, also called the asymptotic boundary of M; we refer to [3] for details. In the sequel, we will denote by $\partial_\infty M$ the sphere at infinity and by $\bar{M} = M \cup \partial_\infty M$ the compactification of M.

We recall that the asymptotic Dirichlet problem of a PDE $Q(u) = 0$ in M for a given asymptotic boundary data $\psi \in C^0(\partial_\infty M)$ consists in finding a solution $u \in C^0\left(\bar{M}\right)$ of $Q(u) = 0$ in M such that $u|_{\partial_\infty M} = \psi$, determining the uniqueness of u as well.

The asymptotic Dirichlet problem for the Laplacian PDE has been studied since the last 30 years and there is a vast literature in this case. More recentely, it has been studied in a larger class of PDE's which include the minimal graph PDE

$$\mathcal{M}(u) = \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0, \tag{1}$$

case that we are interested in the present work. We note that div and ∇ are the divergence and the gradient in M and it is worth to mention that the graph

$$G(r) = \{(x, u(x)) \mid \quad x \in M\}$$

of u is a minimal surface in $M \times \mathbb{R}$ if and only if u satisfies (1).

Antonio Galvez and Harold Rosenberg [5] proved that if $K_M \leq -k^2$, $k > 0$, then the asymptotic Dirichlet problem for $\mathcal{M}$ is solvable for any continuous asymptotic boundary data. The second author of the paper with Miriam Telichevesky [8] proved that it is also solvable if the metric of M is rotationally symmetric and if, fixing an origin $o \in M$, one has

$$\max_{C_r} K_M \leq -\frac{1+\varepsilon}{r^2 \log r} \tag{2}$$

for some $\epsilon > 0$, where $r(x) = d(x, o)$, $d =$ Riemannian distance, and C_r is the geodesic circle with radius r with origin o (it is known that the upper bound (2) is sharp for harmonic functions and not assuming the rotationally symmetry of the metric ([6]). For more general operators, including the minimal one, the sharpeness or not of the upper bound (2) is not known. For existence results on higher dimensions and to more general operators see [2], [9]).

A natural problem related to the asymptotic Dirichlet problem concerns the existence or not of solutions with isolated singularities at $\partial_\infty M$. We investigate here this problem for the minimal graph operator (1), proving that such a solutions cannot exist if M has negatively pinched curvature. Precisely, we prove:

Theorem 1.1. *Suppose that M is a $2-$dimensional Cartan-Hadamard manifold with secctional curvature satisfying $-b^2 \leq K \leq -a^2 < 0$, $b \geq a > 0$. Given a finite number of points $p_1, ..., p_n \in \partial_\infty M$, if $m \in C^\infty(M) \cap C^0\left(\bar{M} \backslash \{p_1, ..., p_n\}\right)$ is a solution of the minimal surface equation in M and if $m|_{\partial_\infty M \backslash \{p_1,...,p_n\}}$ extends continuously to p_i, $i = 1, ..., n$, then $m \in C^0\left(\bar{M}\right)$.*

We observe that a similar problem can obviously be posed to solutions of the minimal graph equation $\mathcal{M}(u) = 0$ on a bounded C^0 domain Ω of $\mathbb{R}^2$. In this case, from a classical and well-known result of R. Finn [4], it follows that if u is as in the above theorem with M replaced by Ω, ∂_∞ by ∂, and if there there is a solution $v \in C^\infty(\Omega) \cap C^0\left(\bar{\Omega}\right)$ such that

$$u|_{\partial\Omega\backslash\{p_1,\dots,p_n\}} = v|_{\partial\Omega\backslash\{p_1,\dots,p_n\}}$$

then $u = v$ and hence u extends continuously through the singularities. If the Dirichlet problem $\mathcal{M}(u) = 0$ on Ω is not solvable for the continuous boundary data $\phi := u|_{\partial\Omega}$ then the result is false, a known fact on the classical minimal surface theory (see [7], Chapter V, Section 3). We remark that even if the Dirichlet problem is not solvable there might exist smooth compact minimal surfaces which boundary is the graph of ϕ if ϕ and the domain are regular enough (see [1]).

Although under the hypothesis of Theorem 1.1 there exists a solution $v \in C^\infty(M) \cap C^0\left(\bar{M}\right)$ such that $u|_{\partial_\infty M\backslash\{p_1,\dots,p_n\}} = v|_{\partial_\infty M\backslash\{p_1,\dots,p_n\}}$, we felt necessary to use a different approach from Finn's since the boundeddness of the domain is fundamental to the arguments used in [4]. Our proof relies heavily on asymptotic properties of $2-$dimensional Cartan-Hadamard manifolds and does not use the existence of v. It is fundamentally based on the fact that a point p of the asymptotic boundary of M is an isolated point of the asymptotic boundary of a domain U such that $M \setminus U$ is convex. This property allows the construction of suitable barriers at infinity. Although the existence of U in the $n = 2$ dimensional case is trivial (for example, a domain which boundary are two geodesics asymptotic to p), we don't know if such an U exists in M if $n \geq 3$. Nevertheless, it is possible in the special case of the hyperbolic space to give an ad hoc proof of Theorem 1.1 using the symmetries of the space. Precisely, our result in $\mathbb{H}^n$ reads:

Theorem 1.2. *Let $\mathbb{H}^n$ be the hyperbolic space of constant section curvature -1. Given a finite number of points $p_1, ..., p_n \in \partial_\infty\mathbb{H}^n$, if $m \in C^\infty(\mathbb{H}^n) \cap C^0\left(\bar{\mathbb{H}^n}\backslash\{p_1, ..., p_n\}\right)$ is a solution of the minimal surface equation in M and if $m|_{\partial_\infty\mathbb{H}^n\backslash\{p_1,...,p_n\}}$ extends continuously to p_i, $i = 1, ..., n$, then $m \in C^0\left(\bar{\mathbb{H}^n}\right)$.*

2 Proof of the theorems

2.1 Proof of Theorem 1.1

We first claim that m is bounded: For each p_i, consider a geodesic Γ_i such that the asymptotic boundary of one of the connected components of $M \setminus \Gamma_i$, say X_i, does not contain p_j for $j \neq i$. Assume also that $p_i \in \operatorname{int} X_i$. Since $\Gamma_i(\pm\infty) \notin \{p_1, \dots p_n\}$, m is continuous at $\Gamma_i(\pm\infty)$ and therefore it is bounded on Γ_i. Let $S_i = \sup_{\Gamma_i} m$ for $i \in \{1, \dots n\}$, $S_0 = \sup m|_{\partial_\infty M \setminus \{p_1, \dots, p_n\}}$ and

$$S = \max\{S_0, S_1, \dots, S_n\}.$$

From the maximum principle, $m \leq S$ in $M \setminus \{X_1 \cup \dots \cup X_n\}$. To prove that $m \leq S$ in X_i, take a sequence of geodesics β_k such that the ending points $\beta_k(+\infty)$ and $\beta_k(-\infty)$ converge to p_i. Let Y_k be the connected component of $M \setminus \beta_k$ whose the asymptotic boundary does not contain p_i. Observe that $M \setminus X_i \subset Y_k$ for large k and $\cup Y_k = M$. Consider the Scherk surface that is the graph of a function w_k which is $+\infty$ on β_k and S at $\partial_\infty Y_k \setminus \{\beta_k(\pm\infty)\}$. Hence $w_k \geq S$ and therefore $w_k \geq m$ on $\Gamma_i = \partial X_i$, $w_k = S \geq m$ on $\partial_\infty (X_i \cap Y_k)$ and $w_k = +\infty > m$ on $\beta_k = \partial Y_k$. Then $w_k \geq m$ in $Y_k \cap X_i$ for large k. For any given $x \in M$, $x \in Y_k$ for large k. Hence, using that $w_k(x) \to S$, we have $m(x) \leq S$. In a similar way, we can conclude that m is bounded from below, proving the claim.

Assume that $m \leq S$. Denote by ϕ the continous extension of $m|_{\partial_\infty M \setminus \{p_1, \dots, p_n\}}$ to $\partial_\infty M$. Let $p \in \{p_1, ..., p_n\}$. Adding a constant to ϕ we may assume wlg that $\phi(p) = 0$. Let $0 < \delta \leq S$ be given.

By the continuity of ϕ, there is an open connected neighborhood U of p such that $\phi(q) \leq \delta$ for all $q \in U$. Moreover, we may assume that U does not contain another point x_i except p.

Let γ be a geodesic such that $\gamma(\infty) = p$. Set $\gamma = \gamma(\mathbb{R})$. Choose a point $q_0 \in \gamma$ and a geodesic α_0 orthogonal to γ at q_0 such that $\alpha_0(\pm\infty) \in U$. Let $q_k \in \gamma$ be a sequence converging to p, α_k the geodesic of M orthogonal to γ at q_k and A_k the domain of M bounded by α_0 and α_k.

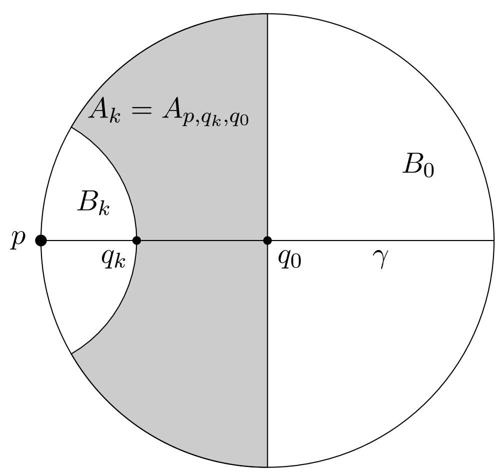

Fig. 1

By a simple adaptation of the proof of Theorem 4 of [9] one obtains the following lemma:

Lemma 2.1. *The Dirichlet problem*

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) & = & 0 & \text{in} & A_k \\ u & = & S & \text{on} & \partial A_k \\ u & = & \delta & \text{in} & \operatorname{int}\partial_\infty A_k \end{cases}$$

has a solution $u_k \in C^\infty(A_k) \cap C^0(\bar{A}_k \setminus \{\alpha_k(\pm\infty), \alpha_0(\pm\infty)\})$.

Since the sequence $\{u_k\}$ is uniformly bounded, interior gradient estimates (see [10]) assures that $\{u_k\}$ contains a subsequence converging uniformlly in the C^2 norm, on compact subsets of $A := \bigcup_{k=1} A_k$, to a solution $u \in C^\infty(A) \cap C^0(\bar{A} \setminus \{p\})$. Note that u satisfies:

$$\begin{cases} 0 \leq & u & \leq & S & \text{in} & A \\ & u & = & S & \text{on} & \partial A \\ & u & = & \delta & \text{in} & \operatorname{int}\partial_\infty A \backslash \{p\}. \end{cases}$$

Moreover, by the comparison principle, $m \leq u_k$ in A_k for all k so that $m \leq u$ in A.

We will prove that $K := \limsup_{x\to p} u(x) \leq \delta$. By contradiction assume that that $K > \delta$.

Let γ_i, $i \in \{1, 2\}$, be the geodesics with ending points at p and $p_1 := \alpha_0(\infty)$ and p and $p_2 := \alpha_0(-\infty)$, respectively. Denote by U_i the connected component of $M \setminus \gamma_i$ that does not contain α_0.

Let $Sh_i)$ satisfy

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) & \leq & 0 & \text{in} & U_i \\ u & = & +\infty & \text{on} & \gamma_i \\ u & = & \delta & \text{in} & \operatorname{int}\partial_\infty U_i. \end{cases}$$

Observe that $u < Sh_i$. Let c_i be the level set of Sh_i

$$c_i = \left\{ x \in M \ : \ Sh_i(x) = \frac{K}{2} + \frac{\delta}{2} \right\}$$

and

$$V_i = \left\{ x \in U_i \ : \ Sh_i(x) < \frac{K}{2} + \frac{\delta}{2} \right\}$$

Hence $u < K/2 + \delta/2$ on V_i. Let $V = A \backslash (V_1 \cup V_2)$.

Now, let W be a neighborhood of p (a ball centered at p) such that the asymptotic boundary of $W \cap V$ is $\{p\}$. Observe that for $R > 0$ and any point z on the boundary of $W \cap V$ there exist a ball of radius R, $B_R \subset M \backslash (W \cap V)$ such that $B_R \cap W \bar{\cap} V = \{z\}$. We consider $R = 1$.

There is $\rho > 0$ be such that

$$\operatorname{dist}(x, V_i) < \rho \qquad \text{for any } x \in W \cap V.$$

That is, for any $x \in W \cap V$, there is a ball B_ρ centered at some point of $\partial(V_1 \cup V_2) \cap W$ s.t. $x \in B_\rho$.

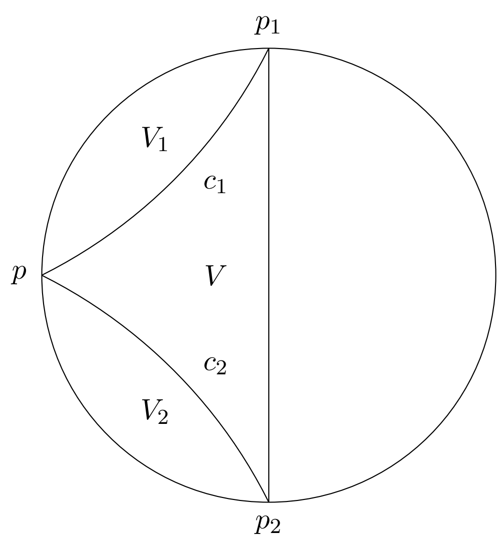

Fig. 2

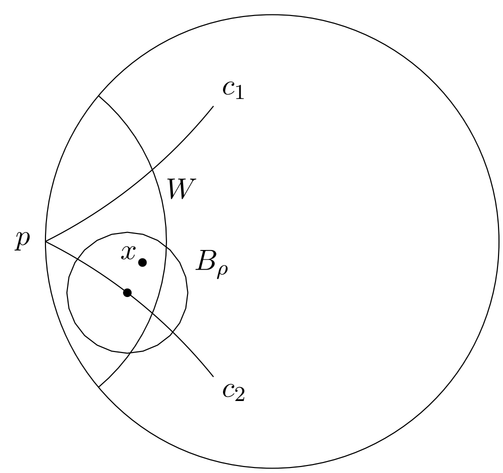

Fig.3

Lemma 2.2. *There exist h_0 and h_1 depending only on a and b, satisfying*

$$\delta < h_1 < h_0 < K/2 + \frac{\delta}{2}$$

such that, for any $y \in M$, the Dirichlet problem in the annulus $B_{2\rho+1}(y)\backslash\overline{B_1(y)}$

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) &= 0 \quad \text{in} \quad B_{2\rho+1}(y)\backslash\overline{B_1(y)} \\ u &= \delta \quad \text{on} \quad \partial B_1(y) \\ u &= h_0 \quad \text{on} \quad \partial B_{2\rho+1}(y) \end{cases}$$

has a supersolution $w_y(x)$ and $w_y(x) \le h_1$ if $\operatorname{dist}(x,y) < \rho+1$.

Proof. Let $f : [1,\infty) \to \mathbb{R}$ be the function defined by

$$f(r) = \delta + \int_1^r \frac{\sinh b\,\alpha}{\sqrt{(\sinh bs)^2 - (\sinh b\,\alpha)^2}}\, ds,$$

where $0 < \alpha \le 1$. Hence $f(1) = \delta$ and, choosing α sufficiently small, $f(2\rho+1) < K/2+\delta/2$. Let $h_0 = f(2\rho+1)$. Observe that the graphic of f is a minimal surface in the hyperbolic plane with constant negative sectional curvature $-b^2$, that is, f satisfies

$$\frac{f''(r)}{(1+(f'(r))^2)^{3/2}} + \frac{bf'(r)\coth br}{(1+(f'(r))^2)^{1/2}} = 0.$$

Moreover, from the Comparison Laplacian Theorem

$$\Delta d(x) \le \Delta r(\tilde{x}) = b \coth br,$$

where $d(x) = dist(x,y)$ and r is the distance in $\mathbb{H}^2(-b^2)$ from $\tilde{x}$ to a fixed point such that $d(x) = r(\tilde{x})$. Then, using these two relations and that $f' > 0$, we conclude that $w_y(x) := f(d(x))$ is a supersolution of the minimal equation.

Since $f(1) = \delta$ and $f(2\rho+1) = h_0$, $w_y(x)$ satisfies the required boundary conditions. Finally defining $h_1 := f(\rho+1)$, $w_y(x) \le h_1 < h_0$ in $B_{\rho+1}(y)$. $\square$

Let $\varepsilon < h_0 - h_1$ and $W_0 \subset W$ be a neighborhood of p (a ball centered at p) s.t.

$$u < K + \varepsilon \quad \text{in} \quad W_0.$$

Let $\tilde{W} \subset W_0$ be a neighborhood of p (a ball centered at p) s.t.

$$\operatorname{dist}(\partial W_0, \tilde{W}) > 3\rho + 2.$$

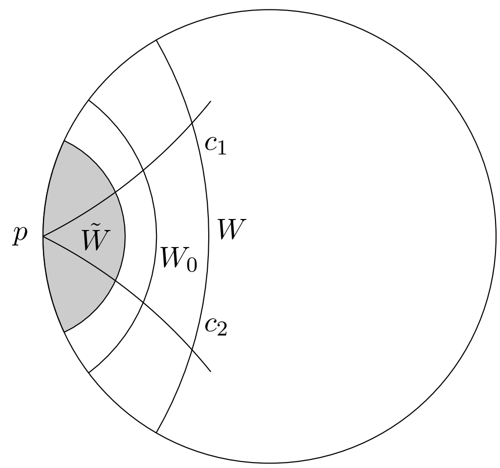

Fig. 4

We claim that

$$u < K + \delta + \varepsilon - h_0 + h_1 < K + \delta$$

in $\tilde{W}$.

Indeed: Let $x \in \tilde{W}$ and assume first that $x \in V$. As observed above, there is some $z \in \partial(V_1 \cup V_2)$, say $z \in \partial V_1$, s.t.

$$x \in B_\rho(z)$$

and there is $y \in V_1$ s.t.

$$B_1(y) \cap W \bar{\cap} v = \{z\}.$$

Therefore

$$\mathrm{dist}(x, y) < \rho + 1.$$

Using triangular inequality and that $\mathrm{dist}(\partial W_0, \tilde{W}) > 3\rho + 2$, we have

$$B_{2\rho+1}(y) \subset B_{3\rho+2}(x) \subset W_0.$$

Let w_y be the solution associated to the annulus $B_{2\rho+1}(y) \backslash B_1(y)$ given by Lemma 2.2. Define

$$w = w_y + K + \delta + \varepsilon - h_0$$

Then, using that $B_1(y) \subset V_1$,

$$w = 0 + K + \delta + \varepsilon - h_0 > K + \delta + \varepsilon - K/2 > K/2 + \delta > u \quad \text{on} \quad \partial B_1(y)$$

and, from $B_{2\rho+1}(y) \subset W_0$,

$$w = h_0 + K + \delta + \varepsilon - h_0 = K + \delta + \varepsilon > u \quad \text{on} \quad \partial B_{2\rho+1}(y).$$

From the comparison principle,

$$u < w \quad \text{in} \quad B_{2\rho+1}(y) \backslash B_1(y)$$

and, therefore

$$u < w_y + K + \delta + \varepsilon - h_0 < h_1 + K + \delta + \varepsilon - h_0 \quad \text{in} \quad B_{\rho+1}(y) \backslash B_1(y).$$

Since $\operatorname{dist}(x, y) < \rho + 1$, then $x \in B_{\rho+1}(y)$. Hence $x \in B_{2\rho+1}(y) \backslash B_1(y)$, if $x \notin V_1$. In this case, $u(x) < h_1 + K + \delta + \varepsilon - h_0$. Finally, if $x \in V_1$, then $u(x) < K/2 + \delta < K + \delta + \varepsilon - h_0 + h_1$ proving the claim.

To conclude with the proof of the theorem, note that $\nu := -\varepsilon + h_0 - h_1 > 0$, since $\varepsilon < h_0 - h_1$. Then

$$K + \nu + \varepsilon - h_0 + h_1 = K + \delta - \nu$$

and, from the above claim,

$$u < K - \nu < K \quad \text{in} \quad \tilde{W}.$$

Hence

$$\limsup_{x \to p} u(x) \leq K - \nu < K$$

leading a contradiction.

2.2 Proof of Theorem 1.2.

We first introduce some terminology. By a totally geodesic hyperball of $\mathbb{H}^n$ we mean a domain in $\mathbb{H}^n$ whose boundary is a totally geodesic hypersurface of $\mathbb{H}^n$. Given p in $\partial_\infty \mathbb{H}^n$, some geodesic γ that has an endpoint at p, two disjoint totally geodesic hyperballs B_0 and B orthogonal to γ at the points q_0 and q, respectively, and such that p belongs to the asymptotic boundary of B, we define the totally geodesic hyperannulus A_{p,q,q_0} by

$$A_{p,q,q_0} = \mathbb{H}^n \backslash (B \cup B_0).$$

We may eventually refer to totally geodesic hyperball and hyperannulus simply as hyperball and hyperannulus.

Lemma 2.3. *For real numbers L and M, there exists a function $w = w_{p,q,q_0}$ defined in A_{p,q,q_0} that is the solution of the minimal graph PDE (1) satisfying $w = M$ on $\partial A_{p,q,q_0}$ and $w = L$ in the interior of $\partial_\infty A_{p,q,q_0}$. Furthermore, there exists also a solution $\tilde{w} = \tilde{w}_{p,q,q_0}$ such that $w = M$ on ∂B_0 and $w = L$ on $\partial B \cup \operatorname{int}\partial_\infty A_{p,q,q_0}$.*

Proof. Consider uniformly bounded sequences of functions

$$f_k^\pm \in C^0\left(\partial A_{p,q,q_0} \cup \partial_\infty A_{p,q,q_0}\right)$$

that converges, uniformly on the the C^0 norm on compact subsets of $\partial A_{p,q,q_0} \cup \operatorname{Int}\left(\partial_\infty A_{p,q,q_0}\right)$, monotonically from above and from below, to a function $f \in C^0\left(\partial A_{p,q,q_0} \cup \operatorname{Int}\left(\partial_\infty A_{p,q,q_0}\right)\right)$ such that $f|_{\partial A_{p,q,q_0}} = M$ and $f|_{\operatorname{Int}\left(\partial_\infty A_{p,q,q_0}\right)} = L$. Clearly A_{p,q,q_0} is mean convex and, since M has negative pinched curvature, A_{p,q,q_0} is also strictly convex at infinity (see [11], [9]). It follows from Theorem 5 of [11] the existence of solutions $u_k^\pm \in C^0\left(\overline{A_{p,q,q_0}}\right)$ of (1) such that $u_k^\pm|_{\partial A_{p,q,q_0} \cup \partial_\infty A_{p,q,q_0}} = f_k^\pm$. Since $u_k^\pm$ has uniformly bounded C^0 norm, interior gradient estimates (see [10]) and linear elliptic PDE theory implies that $u_k^\pm$ has equibounded $C^{2,\alpha}$ norm on compact subsets of A_{p,q,q_0}. Arzela-Ascoli theorem and the diagonal method implies that $u_k^\pm$ contains subsequences converging uniformly on the C^2 norm on compact subsets of A_{p,q,q_0} to solutions $u^\pm$ of (1) in A_{p,q,q_0}. The comparison principle implies that $u^\pm$ extend continuously to $\partial A_{p,q,q_0} \cup \operatorname{Int}\left(\partial_\infty A_{p,q,q_0}\right)$ and that $u^\pm|_{\partial A_{p,q,q_0}} = M$ and $u^\pm|_{\operatorname{Int}\left(\partial_\infty A_{p,q,q_0}\right)} = L$. We may then choose $w = u^+$. The existence of $\widetilde{w}$ is proved in a similar way. □

Lemma 2.4. *Let $p \in \partial_\infty \mathbb{H}^n$, γ be a geodesic that has an endpoint at p, $q_0 \in \gamma$, and (q_k) be a sequence of points in γ such that $q_k \to p$ in the cone topology. If w_k is the solution of*

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & \text{in } A_k \\ u = M & \text{on } \partial A_k \\ u = L & \text{in } \operatorname{int}\partial_\infty A_k \end{cases}$$

where $A_k = A_{p,q_k,q_0}$, then w_k converges to the solution of the minimal surface equation v, defined in $A = \cup_{k=1}^\infty A_k$, that satisfies $v = M$ in ∂A and $v = L$ on int $\partial_\infty A$.

Proof. We can suppose that $L = 0$ and $M > 0$, otherwise we can consider $w_k - L$ if $M > L$ or $-w_k + L$ if $M < L$. Furthermore, since $q_k \to p$, we can extract a subsequence $q_j = q_{k_j}$ such that

$$B_{j+1} \subset B_j,$$

where B_j is the totally geodesic hyperball that is orthogonal to γ at q_j and whose its asymptotic boundary contains p. Then $A_{j+1} \supset A_j$.

Observe that the maximum principle implies that $0 \leq w_j \leq M$ in A_j. Hence, using that $\partial A_j \subset \partial A_{j+1} \cup A_{j+1}$, we have that $w_{j+1} \leq M = w_j$ on ∂A_j and, by comparison principle, $w_{j+1} \leq w_j$ in A_j. Since w_j is uniformaly bounded, interior gradient estimates imply that, up to a subsequence, w_j converges uniformly on the C^2 norm on compact subsets of A, to a C^∞ solution v defined in A such that $0 \leq v \leq w_j$ in A_j for any j. Thus

$$v = 0 \quad \text{on} \quad \bigcup_{j=1}^{\infty} (\text{int } \partial_\infty A_j) = \text{int } \partial_\infty A \backslash \{p\}.$$

We have also that

$$w_j \geq \tilde{w}_1 \quad \text{in} \quad A_1,$$

where $\tilde{w}_1 = \tilde{w}_{p,q_1,q_0}$ is the solution in A_1, given by Lemma 2.3, that satisfies $\tilde{w}_1 = M$ on ∂B_0 and $\tilde{w}_1 = 0$ on $\partial B \cup \text{int}\partial_\infty A$. Thus $v \geq \tilde{w}_1$ and, therefore, $v = M$ on ∂B_0. Finally we have to prove that

$$\lim_{x \to p} v(x) = 0.$$

Let

$$K = \limsup_{x \to p} v(x).$$

From $0 \leq v \leq w_j \leq M$ in A_j it follows that $0 \leq K \leq M$. Suppose that $K > 0$. Let V_m be a decreasing sequence of neighborhood of p such that

$$\bigcap \overline{V}_m = \{p\} \quad \text{and} \quad \sup_{x \in V_m} v(x) < K + 1/m.$$

We can suppose that each V_m is a hyperball centered at p. For each m, let $\tilde{V}_m \subset V_m$ be a hyperball centered at p such that

$$dist(\partial \tilde{V}_m, \partial V_m) \geq m \quad \text{and} \quad \sup_{x \in \tilde{V}_m} v(x) > K - 1/m.$$

Then there exists a sequence (x_m) that satisfies $x_m \in \tilde{V}_m$ and

$$K - 1/m < v(x_m) < K + 1/m.$$

It is well known that there exists an isometry $T_m : \mathbb{H}^n \to \mathbb{H}^n$ (a transvection along γ) that preserves p, $T_m(\gamma) = \gamma$, $T_m(\tilde{V}_m) \supset A$ and $y_m := T_m(x_m) \in \partial B_0$. Observe that

$$w_m = v \circ T_m^{-1}$$

is a solution to the minimal surface problem defined in $T_m(A)$ and satisfies

$$\sup_{T_m(V_m)} w_m < K + 1/m \quad \text{and} \quad T_m(y_m) > K - 1/m. \tag{3}$$

Moreover $\bigcup T_m(V_m) = \mathbb{H}^n$, since $\tilde{V}_m \subset V_m \subset A \subset T(\tilde{V}_m)$ and, therefore,

$$\begin{aligned} dist(\partial T_m(V_m), A) &\geq dist(\partial T_m(V_m), T_m(\tilde{V}_m)) \\ &= dist(\partial V_m, \tilde{V}_m) \geq m \to \infty. \end{aligned}$$

We also notice that (w_m) is a decreasing sequence as a result of the comparison principle for minimal surfaces, $w_{m+1} \leq M = w_m$ on $\partial T_m(A)$ and $w_{m+1} = w_m = 0$ in $\mathrm{int}\partial_\infty T_m(A)$. Hence, using that $0 \leq w_m \leq M$ is a sequence of solutions, we have that $w_m \to w$ uniformly in compacts of $\mathbb{H}^n$. From (3), it follows that $0 \leq w \leq K$ and w is a solution to the minimal surface equation. The monotonicity of (w_m) implies that

$$w_1(y_m) \geq w_m(y_m) = v(T_m^{-1}(T_m(x_m)) = v(x_m) > K - 1/m.$$

Since $y_m \in \partial B_0$, w_1 is continuous and zero on $\partial_\infty B_0 \cap \partial_\infty A$, we conclude that (y_m) is a bounded sequence in ∂B_0. Hence there exists some convergent subsequence, we name (y_m), such that $y_m \to y \in \partial B_0$ and, therefore, $w(y) = \lim w_m(y_m) \geq K$. Thus w is not constant, since $w \leq w_1 = 0$ on $\partial_\infty A \backslash \{p\}$. But this contradicts the maximum principle, $0 \leq w \leq K$ and $w(y) = K$. Therefore $K = 0$, completing the proof. □

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