# A BIJECTIVE PROOF OF VERSHIK'S RELATIONS FOR THE KOSTKA NUMBERS

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Abstract. We turn the tableau insertion algorithm for semistandard tableaux to a sequence called a buming route. We also define a reverse bumping route by the reverse insertion algorithm. These sequences are used to give a bijective proof of Vershik's relations for the Kostka numbers.

## 1. Introduction

Throughout this paper, $n$ will denote a positive integer. We write $\lambda \vDash n$ if $\lambda$ is a composition of $n$, that is, a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ of nonnegative integers such that $|\lambda| = \sum_{i=1}^{h} \lambda_i = n$. In particular, if a sequence $\lambda$ is non-increasing and $\lambda_i > 0$ for all $1 \le i \le h$, then we write $\lambda \vdash n$ and say that $\lambda$ is a partition of $n$. Given a partition $\mu = (\mu_1, \dots, \mu_k) \vdash n$ and a composition $\lambda = (\lambda_1, \dots, \lambda_h) \vDash n$, we denote by $D_\mu$ the Young diagram of $\mu$, and by $\mathrm{STab}(\mu, \lambda)$ the set of all semistandard tableaux of shape $\mu$ and weight $\lambda$. More precisely,

$$\begin{aligned} D_\mu &= \{(i,j) \in \mathbb{Z}^2 \mid 1 \le i \le k,\ 1 \le j \le \mu_i\}, \\ \mathrm{STab}(\mu,\lambda) &= \{T \mid T : D_\mu \to \{i \mid \lambda_i > 0\},\ T(i,j) \le T(i,j+1), \\ &\qquad T(i,j) < T(i+1,j),\ (|T^{-1}(\{i\})|)_{i \ge 1} = \lambda\}. \end{aligned}$$

The Kostka number $K(\mu, \lambda)$ is defined to be the cardinality of $\mathrm{STab}(\mu, \lambda)$. We denote by $\tilde{\lambda}$ the partition obtained by rearranging components of $\lambda$, and by $\lambda^{(i)}$ the composition of $n-1$ defined by $\lambda^{(i)}_i = \lambda_i - 1$, and $\lambda^{(i)}_j = \lambda_j$ otherwise. For $\lambda = (\lambda_1, \dots, \lambda_h) \vdash n$ and $\gamma \vdash n-1$, we write $\gamma \preceq \lambda$ if $\gamma_i \le \lambda_i$ for all $i$ with $1 \le i \le h$, and define

$$C(\lambda, \gamma) = |\{i \mid 1 \le i \le h,\ \widetilde{\lambda^{(i)}} = \gamma\}|.$$

Vershik's relations for the Kostka numbers is as follows:

**Theorem 1** ([1, p.143, Theorem 3.6.13] and [4, Theorem 4]). *For any $\lambda \vdash n$ and $\rho \vdash n-1$, we have*

$$\sum_{\substack{\mu \vdash n \\ \mu \succeq \rho}} K(\mu, \lambda) = \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} C(\lambda, \gamma) K(\rho, \gamma).$$

Theorem 1 can be proved using representation theory. As previously stated, since $K(\mu, \lambda) = |\mathrm{STab}(\mu, \lambda)|$, it is natural to expect a bijective proof of Theorem 1. In

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fact, Vershik [4, Theorem 4] claims to give a bijection from

$$\mathcal{L} = \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \mathrm{STab}(\mu, \lambda)$$

to

$$\mathcal{R} = \bigcup_{1 \le x \le h} \mathrm{STab}(\rho, \lambda^{(x)}),$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \vdash n$ and $\rho \vdash n-1$. In order to explain his proof, we call a tableau in $\mathrm{STab}(\mu, \lambda)$ a $\mu$-tableau, and a tableau in $\mathrm{STab}(\rho, \lambda^{(x)})$ a $\rho$-tableau. Since $\mu$-tableaux have one more box than $\rho$-tableaux, Vershik [4, Theorem 4] claims that removable of one box from $\mu$-tableaux gives a bijection from $\mathcal{L}$ to $\mathcal{R}$. Vershik [4, Section 4] gives examples, each of which comes with a bijection. However, if $\lambda = (3,3,2) \vdash 8$ and $\rho = (4,3) \vdash 7$ then there is no bijection from $\mathcal{L}$ to $\mathcal{R}$ arising from removable of one box. More precisely, we consider two tableaux in $\mathcal{L}$ as follows:

$$A = \begin{array}{ccccc} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & & \end{array}, \quad E = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & \\ 3 & & & \end{array}.$$

The only $\rho$-tableau obtainable from $A$ by removing one box is

$$Q = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & \end{array}.$$

Similarly, the only $\rho$-tableau obtainable from $E$ by removing one box is $Q$.

In this paper, we describe a bijection between $\mathcal{R}$ and $\mathcal{L}$ using tableau insertion and reverse insertion algorithms (see [2] and [3]). We note that, in our bijection, $\lambda$ is allowed to be a composition which is not necessarily a partition.

This paper is organized as follows. In Section 2, we define a bumping route using the tableau insertion algorithm. Similarly, we define a reverse bumping route using the reverse insertion algorithm in Section 3. Finally, in Section 4, we prove Theorem 1 by showing that the tableau insertion algorithm gives a bijection.

## 2. Insertion

Throughout this paper, $\lambda \vDash n$. For a positive integer $i$, we define $\lambda^i \vDash n+1$ as follows:

$$\lambda^i_j = \begin{cases} \lambda_j + 1 & \text{if } j = i, \\ \lambda_j & \text{otherwise.} \end{cases}$$

In this section, we let $\mu \vdash n$ and $T \in \mathrm{STab}(\mu, \lambda)$. We also let $x$ be a positive integer.

**Definition 2.** The *bumping route* of $(T, x)$ is defined as the sequence $\overrightarrow{R}(T, x) = (j_1, j_2, \ldots)$ with integer entries defined as follows: first, $\overrightarrow{R}(T, x) = 0$ if $\{q \mid T(1,q) > x\} = \emptyset$. Otherwise, $j_1 = \min\{q \mid T(1,q) > x\}$ and for $p \ge 2$,

$$j_p = \begin{cases} 0 & \text{if } j_{p-1} = 0 \text{ or } T(p, \mu_p) \le T(p-1, j_{p-1}), \\ \min\{q \mid T(p,q) > T(p-1, j_{p-1})\} & \text{otherwise.} \end{cases}$$

If $j_p = 0$ for $p > l$ then we write $\overrightarrow{R}(T, x) = (j_1, j_2, \ldots, j_l)$. We denote by $l(\overrightarrow{R}(T, x))$ the length of $\overrightarrow{R}(T, x)$, that is, $l(\overrightarrow{R}(T, x)) = \max\{l \mid j_l \neq 0\}$. Note that if $\overrightarrow{R}(T, x) = 0$, then we define $l(\overrightarrow{R}(T, x)) = 0$.

By previous definition, clearly, we have

$$T(1, j_1 - 1) \leq x < T(1, j_1), \tag{1}$$

$$T(p, j_p - 1) \leq T(p-1, j_{p-1}) < T(p, j_p) \ (2 \leq p \leq l), \tag{2}$$

$$T(l+1, \mu_{l+1}) \leq T(l, j_l), \tag{3}$$

whenever $T(\cdot, \cdot)$ is defined. Moreover, we get $j_p \geq j_{p+1}$ for all positive integers $p$. Indeed, we may assume $p < l$. Then we have $T(p+1, j_p) > T(p, j_p)$. Thus

$$\begin{aligned} j_p &\geq \min\{q \mid T(p+1, q) > T(p, j_p)\} \\ &= j_{p+1}. \end{aligned}$$

For the remainder of this section, we let $\overrightarrow{R}(T, x) = (j_1, j_2, \ldots, j_l)$, where $l = l(\overrightarrow{R}(T, x))$.

**Lemma 3.** *We have $\mu_{l+1} < j_l$. In particular, $\mu^{l+1} \vdash n + 1$.*

*Proof.* If $l = 0$, then it is obvious. Suppose $l \geq 1$. Then by (3) we have $(l+1, j_l) \notin D_\mu$. Since $(l, j_l) \in D_\mu$, we have $\mu_{l+1} < j_l$. In particular, $\mu_{l+1} < \mu_l$. □

**Definition 4.** We define a *insertion* or *bumping* tableau $T_x$ of shape $\mu^{l+1}$ and weight $\lambda^x$ as follows: if $l = 0$, then define $T_x$ by

$$T_x(p, q) = \begin{cases} T(p, q) & \text{if } (p, q) \in D_\mu, \\ x & \text{if } (p, q) = (1, \mu_1 + 1). \end{cases} \tag{4}$$

Otherwise, define $T_x$ by

$$T_x(p, q) = \begin{cases} x & \text{if } (p, q) = (1, j_1), \\ T(p-1, j_{p-1}) & \text{if } q = j_p,\ 2 \leq p \leq l, \\ T(l, j_l) & \text{if } (p, q) = (l+1, \mu_{l+1} + 1), \\ T(p, q) & \text{otherwise.} \end{cases} \tag{5}$$

**Lemma 5.** *We have $T_x(p, j_p) < T(p, j_p)$ for all $p$ with $1 \leq p \leq l$.*

*Proof.* We have

$$\begin{aligned} T_x(p, j_p) &= \begin{cases} x & \text{if } p = 1, \\ T(p-1, j_{p-1}) & \text{if } 2 \leq p \leq l \end{cases} && \text{(by (5))} \\ &< T(p, j_p) && \text{(by (1) and (2)).} \end{aligned}$$

□

**Lemma 6.** *We have $T_x \in \mathrm{STab}(\mu^{l+1}, \lambda^x)$.*

*Proof.* By Lemma 3, we have $\mu^{l+1} \vdash n+1$. To show that $T_x$ is a semistandard tableau, it suffices to prove

$$T_x(p, j_p - 1) \leq T_x(p, j_p) \leq T_x(p, j_p + 1),$$

$$T_x(l+1,\mu_{l+1}) \le T_x(l+1,\mu_{l+1}+1),$$
$$T_x(p-1,j_p) < T_x(p,j_p) < T_x(p+1,j_p),$$
$$T_x(l,\mu_{l+1}+1) < T_x(l+1,\mu_{l+1}+1), \tag{6}$$

whenever $T_x(\cdot,\cdot)$ is defined.

For all $p$ with $1 \le p \le l$, we have

$$\begin{aligned} T_x(p,j_p-1) &= T(p,j_p-1) \\ &\le \begin{cases} x & \text{if } p=1, \\ T(p-1,j_{p-1}) & \text{if } 2 \le p \le l \end{cases} && \text{(by (1) and (2))} \\ &= T_x(p,j_p) \\ &< T(p,j_p) && \text{(by Lemma 5)} \\ &\le T(p,j_p+1) \\ &= T_x(p,j_p+1). \end{aligned}$$

Also,

$$\begin{aligned} T_x(l+1,\mu_{l+1}) &= T(l+1,\mu_{l+1}) \\ &\le \begin{cases} x & \text{if } \overrightarrow{R}(T,x)=0, \\ T(l,j_l) & \text{otherwise} \end{cases} && \text{(by (3))} \\ &= T_x(l+1,\mu_{l+1}+1) && \text{(by (4) and (5)).} \end{aligned}$$

For all $p$ with $2 \le p \le l$, we have

$$\begin{aligned} T_x(p-1,j_p) &= \begin{cases} x & \text{if } p=2,\ j_1=j_2, \\ T(1,j_2) & \text{if } p=2,\ j_1>j_2, \\ T(p-2,j_{p-2}) & \text{if } 3 \le p \le l,\ j_{p-1}=j_p, \\ T(p-1,j_p) & \text{if } 3 \le p \le l,\ j_{p-1}>j_p \end{cases} \\ &< \begin{cases} T(1,j_1) & \text{if } p=2, \\ T(p-1,j_{p-1}) & \text{if } 3 \le p \le l \end{cases} && \text{(by (1) and (2))} \\ &= T_x(p,j_p) && \text{(by (5)).} \end{aligned}$$

For all $p$ with $1 \le p \le l$, we have

$$\begin{aligned} T_x(p,j_p) &< T(p,j_p) && \text{(by Lemma 5)} \\ &\le \begin{cases} T(p,j_p) & \text{if } j_p=j_{p+1}, \\ T(p+1,j_p) & \text{if } j_p>j_{p+1} \end{cases} \\ &= T_x(p+1,j_p) && \text{(by (5)).} \end{aligned}$$

Finally, we have $\mu_{l+1} < j_l$ by Lemma 3, so

$$\begin{aligned} T_x(l,\mu_{l+1}+1) &= \begin{cases} T(l-1,j_{l-1}) & \text{if } \mu_{l+1}+1=j_l, \\ T(l,\mu_{l+1}+1) & \text{if } \mu_{l+1}+1<j_l \end{cases} \\ &< T(l,j_l) && \text{(by (2))} \end{aligned}$$

$$= T_x(l+1, \mu_{l+1}+1) \qquad \text{(by (5))}.$$

Thus we proved (6). □

## 3. Reverse insertion

In this section, let $\rho \vdash n-1$ and let $l$ be a nonnegative integer such that $\rho^{l+1} \vdash n$. We also let $S \in \mathrm{STab}(\rho^{l+1}, \lambda)$.

**Definition 7.** The *reverse bumping route* of $(S, l)$ is defined as the sequence $\overleftarrow{R}(S,l) = (j'_1, j'_2, \ldots, j'_l)$ with integer entries defined as follows: first, $\overleftarrow{R}(S,l) = 0$ if $l = 0$. Otherwise, $j'_l = \max\{q \mid S(l,q) < S(l+1, \rho_{l+1}+1)\}$ and

$$j'_p = \max\{q \mid S(p,q) < S(p+1, j'_{p+1})\}$$

for all $p$ with $1 \leq p < l$.

By previous definition, clearly, we have

$$S(p, j'_p) < S(p+1, j'_{p+1}) \leq S(p, j'_p+1) \ (1 \leq p < l), \tag{7}$$

$$S(l, j'_l) < S(l+1, \rho_{l+1}+1) \leq S(l, j'_l+1). \tag{8}$$

Moreover, we get $j'_p \geq j'_{p+1}$ for all $p$ with $1 \leq p < l$. Indeed, since $S(p, j'_{p+1}) < S(p+1, j'_{p+1})$, we have

$$\begin{aligned} j'_p &= \max\{q \mid S(p,q) < S(p+1, j'_{p+1})\} \\ &\geq j'_{p+1}. \end{aligned}$$

For the remainder of this section, we let $\overleftarrow{R}(S,l) = (j'_1, j'_2, \ldots, j'_l)$.

**Definition 8.** We define

$$x(S,l) = \begin{cases} S(1, \rho_1+1) & \text{if } l = 0, \\ S(1, j'_1) & \text{otherwise.} \end{cases}$$

We define a *reverse insertion* or *reverse bumping* tableau $S^l$ of shape $\rho$ and weight $\lambda^{(x(S,l))}$ as follows: if $l = 0$, then define $S^l = S|_{D_\rho}$. Otherwise, define $S^l$ by

$$S^l(p,q) = \begin{cases} S(p+1, j'_{p+1}) & \text{if } q = j'_p,\ 1 \leq p < l, \\ S(l+1, \rho_{l+1}+1) & \text{if } (p,q) = (l, j'_l), \\ S(p,q) & \text{otherwise.} \end{cases} \tag{9}$$

**Lemma 9.** *We have $S(p, j'_p) < S^l(p, j'_p)$ for all $p$ with $1 \leq p \leq l$.*

*Proof.* We have

$$\begin{aligned} S(p, j'_p) &< \begin{cases} S(p+1, j'_{p+1}) & \text{if } 1 \leq p < l, \\ S(l+1, \rho_{l+1}+1) & \text{if } p = l \end{cases} && \text{(by (7) and (8))} \\ &= S^l(p, j'_p) && \text{(by (9))}. \end{aligned}$$

□

**Lemma 10.** *Let $x = x(S,l)$. Then*

(i) $S^l \in \mathrm{STab}(\rho, \lambda^{(x)})$,

(ii) $\overrightarrow{R}(S^l, x) = \overleftarrow{R}(S, l)$,
(iii) $(S^l)_x = S$.

*Proof.* (i) If $l = 0$ then $S^l = S|_{D_\rho}$ and $x = S(1, \rho_1 + 1)$, so $S^l \in \mathrm{STab}(\rho, \lambda^{(x)})$. Suppose $l \geq 1$. By Definition 8, $S^l$ is a tableau of shape $\rho$ and weight $\lambda^{(x)}$. It suffices to prove that

$$\begin{aligned} &S^l(p, j'_p - 1) \leq S^l(p, j'_p) \leq S^l(p, j'_p + 1), \\ &S^l(p-1, j'_p) < S^l(p, j'_p) < S^l(p+1, j'_p), \end{aligned} \tag{10}$$

whenever $S^l(\cdot, \cdot)$ is defined.

For all $p$ with $1 \leq p \leq l$, we have

$$\begin{aligned} S^l(p, j'_p - 1) &= S(p, j'_p - 1) \\ &\leq S(p, j'_p) \\ &< S^l(p, j'_p) && \text{(by Lemma 9)} \\ &= \begin{cases} S(p+1, j'_{p+1}) & \text{if } 1 \leq p < l, \\ S(l+1, \rho_{l+1} + 1) & \text{if } p = l \end{cases} && \text{(by (9))} \\ &\leq \begin{cases} S(p, j'_p + 1) & \text{if } 1 \leq p < l, \\ S(l, j'_l + 1) & \text{if } p = l \end{cases} && \text{(by (7) and (8))} \\ &= S^l(p, j'_p + 1) && \text{(by (9)).} \end{aligned}$$

For $2 \leq p \leq l$, we have

$$\begin{aligned} S^l(p-1, j'_p) &= \begin{cases} S(p-1, j'_p) & \text{if } j'_{p-1} > j'_p, \\ S(p, j'_p) & \text{if } j'_{p-1} = j'_p \end{cases} && \text{(by (9))} \\ &\leq S(p, j'_p) \\ &< S^l(p, j'_p) && \text{(by Lemma 9).} \end{aligned}$$

If $l = 1$, then $\rho_2 + 1 \leq j'_1$ since $S(1, \rho_2 + 1) < S(2, \rho_2 + 1)$, so $(2, \rho_2 + 1) \notin D_\rho$. Suppose $l \geq 2$. For $1 \leq p < l$, we have

$$\begin{aligned} S^l(p, j'_p) &= S(p+1, j'_{p+1}) \\ &< \begin{cases} S(p+1, j'_p) & \text{if } j'_p > j'_{p+1},\ 1 \leq p \leq l-2, \\ S(p+2, j'_{p+2}) & \text{if } j'_p = j'_{p+1},\ 1 \leq p \leq l-2, \\ S(l, j'_{l-1}) & \text{if } j'_p > j'_{p+1},\ p = l-1, \\ S(l+1, \rho_{l+1} + 1) & \text{if } j'_p = j'_{p+1},\ p = l-1 \end{cases} && \text{(by (7) and (8))} \\ &= S^l(p+1, j'_p). \end{aligned}$$

Thus we proved (10).

(ii) Let $\overrightarrow{R}(S^l, x) = (j_1, \ldots, j_{l'})$ where $l' = l(\overrightarrow{R}(S^l, x))$. If $l = 0$ then $x = x(S, l) = S(1, \rho_1 + 1)$ and $S^l = S|_{D_\rho}$. Since $S^l \in \mathrm{STab}(\rho, \lambda^{(x)})$, we have $S^l(1, q) = S(1, q) \leq x$ for all $q$ with $1 \leq q \leq \rho_1$, so $\{q \mid S^l(1, q) > x\} = \emptyset$. Thus $\overrightarrow{R}(S^l, x) = 0$.

Suppose $l \geq 1$. Note that, for all $p$ with $1 \leq p \leq l$, we have

$$S^l(p, j'_p - 1) = S(p, j'_p - 1) \leq S(p, j'_p). \tag{11}$$

We prove $j_p = j'_p$ by induction on $p$. If $p = 1$ then

$$\begin{aligned} j_1 &= \min\{q \mid S^l(1,q) > x\} \\ &= \min\{q \mid S^l(1,q) > S(1,j'_1)\} \\ &= j'_1 && \text{(by Lemma 9 and (11)).} \end{aligned}$$

Assume $j_{p-1} = j'_{p-1}$ for some $2 \le p \le l$. Then

$$\begin{aligned} j_p &= \min\{q \mid S^l(p,q) > S^l(p-1,j_{p-1})\} \\ &= \min\{q \mid S^l(p,q) > S^l(p-1,j'_{p-1})\} \\ &= \min\{q \mid S^l(p,q) > S(p,j'_p)\} && \text{(by (9))} \\ &= j'_p && \text{(by Lemma 9 and (11)).} \end{aligned}$$

Since

$$\begin{aligned} S^l(l+1,\rho_{l+1}) &= S(l+1,\rho_{l+1}) && \text{(by (9))} \\ &\le S(l+1,\rho_{l+1}+1) \\ &= S^l(l,j'_l) && \text{(by (9))} \\ &= S^l(l,j_l), \end{aligned}$$

we have $j_p = 0$ for $p > l$, so $l' = l$.

(iii) By (ii), we have $\overrightarrow{R}(S^l,x) = \overleftarrow{R}(S,l)$. Then $(S^l)_x \in \mathrm{STab}(\rho^{l+1},\lambda)$ by (i) and Lemma 6. Suppose first $l = 0$. For $(p,q) \in D_{\rho^1}$, we have

$$\begin{aligned} (S^l)_x(p,q) &= \begin{cases} S^l(p,q) & \text{if } (p,q) \in D_\rho, \\ x & \text{if } (p,q) = (1,\rho_1+1) \end{cases} \\ &= \begin{cases} S(p,q) & \text{if } (p,q) \in D_\rho, \\ S(1,\rho_1+1) & \text{if } (p,q) = (1,\rho_1+1) \end{cases} \\ &= S(p,q). \end{aligned}$$

Suppose $l \ge 1$. For $(p,q) \in D_{\rho^{l+1}}$, we have

$$\begin{aligned} (S^l)_x(p,q) &= \begin{cases} x & \text{if } (p,q) = (1,j_1), \\ S^l(p-1,j_{p-1}) & \text{if } q = j_p,\ 2 \le p \le l, \\ S^l(l,j_l) & \text{if } (p,q) = (l+1,\rho_{l+1}+1), \\ S^l(p,q) & \text{otherwise} \end{cases} \\ &= \begin{cases} x & \text{if } (p,q) = (1,j'_1), \\ S^l(p-1,j'_{p-1}) & \text{if } q = j'_p,\ 2 \le p \le l, \\ S^l(l,j'_l) & \text{if } (p,q) = (l+1,\rho_{l+1}+1), \\ S^l(p,q) & \text{otherwise} \end{cases} \\ &= S(p,q). \end{aligned}$$

$\square$

## 4. Vershik's relations for the Kostka numbers

**Lemma 11.** *Let $\mu \vdash n$, $\lambda \vDash n$ and $T \in \mathrm{STab}(\mu, \lambda)$. Let $x$ be a positive integer and $l = l(\overrightarrow{R}(T, x))$. Then*

(i) $\overleftarrow{R}(T_x, l) = \overrightarrow{R}(T, x)$.

(ii) $(T_x)^l = T$.

*Proof.* If $l = 0$, then (i) clearly holds, and $(T_x)^l = T_x|_{D_\mu} = T$. Suppose $l \geq 1$ and let $\overleftarrow{R}(T_x, l) = (j'_1, \ldots, j'_l)$ and $\overrightarrow{R}(T, x) = (j_1, \ldots, j_l)$.

(i) Note that, for all $p$ with $1 \leq p \leq l$, we have

$$T_x(p, j_p + 1) = T(p, j_p + 1) \geq T(p, j_p). \tag{12}$$

We prove $j'_p = j_p$ by induction on $l - p$. If $p = l$ then

$$\begin{aligned} j'_l &= \max\{q \mid T_x(l, q) < T_x(l+1, \mu_{l+1} + 1)\} \\ &= \max\{q \mid T_x(l, q) < T(l, j_l)\} && \text{(by (5))} \\ &= j_l && \text{(by Lemma 5 and (12)).} \end{aligned}$$

Assume $j'_{p+1} = j_{p+1}$ for some $1 \leq p < l$. Then

$$\begin{aligned} j'_p &= \max\{q \mid T_x(p, q) < T_x(p+1, j'_{p+1})\} \\ &= \max\{q \mid T_x(p, q) < T_x(p+1, j_{p+1})\} \\ &= \max\{q \mid T_x(p, q) < T(p, j_p)\} && \text{(by (5))} \\ &= j_p && \text{(by Lemma 5 and (12)).} \end{aligned}$$

(ii) By (i), we have $\overleftarrow{R}(T_x, l) = \overrightarrow{R}(T, x)$. Then $x(T_x, l) = T_x(1, j'_1) = T_x(1, j_1) = x$, so $(T_x)^l \in \mathrm{STab}(\mu, \lambda)$ by Lemma 10 (i). For $(p, q) \in D_\mu$, we have

$$\begin{aligned} (T_x)^l(p, q) &= \begin{cases} T_x(p+1, j'_{p+1}) & \text{if } q = j'_p,\ 1 \leq p < l, \\ T_x(l+1, \mu_{l+1} + 1) & \text{if } (p, q) = (l, j'_l), \\ T_x(p, q) & \text{otherwise} \end{cases} && \text{(by (9))} \\ &= \begin{cases} T_x(p+1, j_{p+1}) & \text{if } q = j_p,\ 1 \leq p < l, \\ T_x(l+1, \mu_{l+1} + 1) & \text{if } (p, q) = (l, j_l), \\ T_x(p, q) & \text{otherwise} \end{cases} \\ &= T(p, q). \end{aligned}$$

□

Before proving the main result, for $\lambda \vDash n$, we let

$$\mathrm{Supp}(\lambda) = \{i \mid \lambda_i > 0\}.$$

**Theorem 12.** *Let $\lambda \vDash n$ and $\rho \vdash n - 1$. Then the map*

$$\begin{array}{ccc} \bigcup_{x \in \mathrm{Supp}(\lambda)} \mathrm{STab}(\rho, \lambda^{(x)}) & \to & \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \mathrm{STab}(\mu, \lambda) \\ T & \mapsto & T_x \end{array}$$

*is a bijection.*

*Proof.* The map is well-defined by Lemma 6. Suppose $S \in \mathrm{STab}(\mu, \lambda)$ for some $\mu \vdash n$ with $\mu \succeq \rho$. Then $\mu = \rho^{l+1}$ for some $l$. Set $x = x(S, l)$. By Lemma 10, $S^l \in \mathrm{STab}(\rho, \lambda^{(x)})$ and $(S^l)_x = S$, so the map is a surjection.

Let $T \in \mathrm{STab}(\rho, \lambda^{(x)})$ and $S \in \mathrm{STab}(\rho, \lambda^{(x')})$. Suppose $T_x = S_{x'} \in \mathrm{STab}(\mu, \lambda)$ for some $\mu \vdash n$ with $\mu \succeq \rho$. Then $\mu = \rho^{l+1}$ for some $l$, so $l = l(\overrightarrow{R}(T, x)) = l(\overrightarrow{R}(S, x'))$. By Lemma 11, we have $T = (T_x)^l = (S_{x'})^l = S$, so the map is an injection. □

**Remark 13.** Let $\mu \vdash n$ and let $X$ be a set of positive integers. Define

$$\mathcal{W}(X, n) = \{\lambda \vDash n \mid \lambda_i \geq 0,\ \mathrm{Supp}(\lambda) \subseteq X\},$$
$$\mathrm{STab}_X(\mu) = \bigcup_{\lambda \in \mathcal{W}(X,n)} \mathrm{STab}(\mu, \lambda).$$

For $\rho \vdash n - 1$, the map

$$\begin{array}{ccc} \mathrm{STab}_X(\rho) \times X & \to & \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \mathrm{STab}_X(\mu) \\ (T, x) & \mapsto & T_x \end{array} \tag{13}$$

is a bijection (see [3, p.399, 10.60]). This follows from Theorem 12. Indeed, collecting the bijections of Theorem 12 for all $\lambda \in \mathcal{W}(X, n)$, we obtain a bijection

$$\begin{array}{ccc} \bigcup_{\lambda \in \mathcal{W}(X,n)} \bigcup_{x \in \mathrm{Supp}(\lambda)} \mathrm{STab}(\rho, \lambda^{(x)}) \times \{x\} & \to & \bigcup_{\lambda \in \mathcal{W}(X,n)} \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \mathrm{STab}(\mu, \lambda) \\ (T, x) & \mapsto & T_x \end{array}. \tag{14}$$

Then the codomain of the bijection (14) equals that of (13), while

$$\begin{aligned} \bigcup_{\lambda \in \mathcal{W}(X,n)} \bigcup_{x \in \mathrm{Supp}(\lambda)} \mathrm{STab}(\rho, \lambda^{(x)}) \times \{x\} &= \bigcup_{\nu \in \mathcal{W}(X,n-1)} \mathrm{STab}(\rho, \nu) \times X \\ &= \mathrm{STab}_X(\rho) \times X. \end{aligned}$$

**Corollary 14** (Vershik's relations for the Kostka numbers)**.** *For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h) \vdash n$ and $\rho \vdash n - 1$, we have*

$$\sum_{\substack{\mu \vdash n \\ \mu \succeq \rho}} K(\mu, \lambda) = \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} C(\lambda, \gamma) K(\rho, \gamma).$$

*Proof.*

$$\begin{aligned} \sum_{\substack{\mu \vdash n \\ \mu \succeq \rho}} K(\mu, \lambda) &= \sum_{\substack{\mu \vdash n \\ \mu \succeq \rho}} |\,\mathrm{STab}(\mu, \lambda)| \\ &= \sum_{1 \leq x \leq h} |\,\mathrm{STab}(\rho, \lambda^{(x)})| && \text{(by Theorem 12)} \\ &= \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} \sum_{\substack{1 \leq x \leq h \\ \widetilde{\lambda^{(x)}} = \gamma}} |\,\mathrm{STab}(\rho, \lambda^{(x)})| \\ &= \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} \sum_{\substack{1 \leq x \leq h \\ \widetilde{\lambda^{(x)}} = \gamma}} |\,\mathrm{STab}(\rho, \gamma)| && \text{(by [1, Lemma 3.7.1])} \end{aligned}$$

$$= \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} |\{x \mid 1 \leq x \leq h,\ \widetilde{\lambda^{(x)}} = \gamma\}||\,\mathrm{STab}(\rho,\gamma)|$$

$$= \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} C(\lambda,\gamma)K(\rho,\gamma).$$

□

Now, we compare Vershik's bijection with ours using [4, Example 1].

**Example 15** ([4, Example 1])**.** Let $\lambda = (3,2,1) \vdash 6$ and $\rho = (4,1) \vdash 5$. Then

$$\mu\text{-tableaux}: A = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 3 & & & & \end{array},\ B = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 2 & & & & \end{array},\ C = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 3 & & \end{array},$$

$$D = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & & \end{array},\ E = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & & & \\ 3 & & & \end{array};$$

$$\rho\text{-tableaux}: L = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 3 & & & \end{array},\ M = \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & & & \end{array},\ N = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 3 & & & \end{array},$$

$$P = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & & & \end{array},\ Q = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & & & \end{array}.$$

We remove one box from the first row in $A$ and $B$, one box from the second row in $C$ and $D$, and one box $(3,1)$ in $E$ in order to obtain $\rho$-tableaux. Then we have a bijection as follows:

$$A \leftrightarrow L; \quad B \leftrightarrow M; \quad C \leftrightarrow N; \quad D \leftrightarrow P; \quad E \leftrightarrow Q.$$

The bijection given by Theorem 12 is:

$$\begin{aligned} L &\leftrightarrow L_1 = E; \quad M \leftrightarrow M_1 = D; \\ N &\leftrightarrow N_2 = A; \quad P \leftrightarrow P_2 = C; \\ Q &\leftrightarrow Q_3 = B. \end{aligned}$$

Finally, we give an example, for which there is no bijection arising from removable of one box.

**Example 16.** Let $\lambda = (3,3,2) \vdash 8$ and $\rho = (4,3) \vdash 7$. Then

$$\mu\text{-tableaux}: A = \begin{array}{ccccc} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & & \end{array},\ B = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & & \end{array},\ C = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & & \end{array},$$

$$D = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \end{array},\ E = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & \\ 3 & & & \end{array},\ F = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & \\ 3 & & & \end{array};$$

$$\rho\text{-tableaux}: L = \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & \end{array},\ M = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & \end{array},\ N = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & \end{array},$$

$$P = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 3 & 3 & \end{array},\ Q = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & \end{array},\ R = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & \end{array}.$$

As mentioned in Section 1, $\mu$-tableaux $A$ and $E$ result in $\rho$-tableau $Q$, so there is no bijection between $\mu$-tableaux and $\rho$-tableaux arising from removable of one box. The bijection given by Theorem 12 is:

$$L \leftrightarrow L_1 = E; \quad M \leftrightarrow M_1 = F;$$
$$N \leftrightarrow N_2 = D; \quad P \leftrightarrow P_2 = C;$$
$$Q \leftrightarrow Q_3 = A; \quad R \leftrightarrow R_3 = B.$$

## References


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