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Analytical solutions and genuine multipartite entanglement of the three-qubit Dicke model

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We present analytical solutions to three qubits and a single-mode cavity coupling system beyond the rotating-wave approximation (RWA). The zeroth order approximation gives correct solutions when the qubits are far detuned from the cavity. The first order approximation, called generalized rotating-wave approximation (GRWA), produces an effective solvable Hamiltonian with the same form as the ordinary RWA one and exhibits substantial improvements of energy levels over the RWA even on resonance. Based on these analytical eigen-solutions, we study both the bipartite entanglement and genuine multipartite entanglement (GME). The dynamics of the concurrence and the GME using the GRWA are in consistent with the numerical ones. Interestingly, the well known sudden death of entanglement occurs in the bipartite entanglement dynamics but not in the GME dynamics.

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I. INTRODUCTION

The interaction between qubits and a cavity is ubiquitous in several branches of physics ranging from quantum optics [1] to condensed matter and widespread application to quantum information [2, 3]. In early work on cavity quantum electrodynamics (QED), the qubit-cavity coupling strength was much smaller than the cavity transition frequency, the rotating-wave approximation (RWA) can be applied to produce a solvable treatment [4]. With recent advances in the circuit QED using superconducting qubit circuits, it is possible to engineer systems for which the qubits are so far detuned from the cavity, or are coupled to the cavity in a ultra-strong coupling regime where the coupling strength is comparable to the cavity transition frequency, that the RWA fails to describe the system correctly [5–10].

Due to the breakdown of the RWA, the counter-rotating-wave (CRW) interactions in the qubit-cavity systems are expected to be taken into account, producing non-conserved excitation number. Under the RWA, the ground state of the qubit-cavity system consists of a product of the qubits' ground state and the cavity's vacuum state. An inclusion of the CRW interactions leads to a squeezed vacuum state containing virtual photons [11, 12]. It is challenging to give an analytical exact treatment for the qubit-cavity system. There have been numerous theoretical studies on one- and two-qubit and cavity systems finding new phenomena in the ultra-strong coupling regime, including the adiabatic approximation [13, 14], a Bargmann space technique [15, 16], an extended coherent state method [17–20]. Motivated by experimental developments and the importance of understanding collective quantum behavior, we will investigate an analytical solution to a three-qubit Dicke model [21], which describes the interaction of three qubits with a single-mode cavity.

In recent years it has turned out that theoretical characterization of entanglement has attracted much attentions. Most of the existing studies of entanglement focus on bipartite entanglement [22, 23], which can be quantified through the concurrence characterizing qubit-qubit entanglement [24–26]. However, bipartite entanglement can only give a partial characterization, since multipartite entanglement is known to be different from entanglement between all bipartitions [27]. There is on going interest in the genuine multipartite entanglement (GME) of the Dicke states for collective qubits systems [27, 28, 30, 31]. It was found that the symmetric three-qubit state is relatively robust to decoherence [32]. On the other hand, the bipartite entanglement decoherence has been studied in connection with a phenomenon termed entanglement sudden death, indicating that the bipartite entanglement can decay to zero abruptly during a finite period of time [37]. Whether this properties occurs for the dynamics of GME remain unexplored. So it is highly desirable to study both the bipartite entanglement and the GME for the multipartite entanglement in the more than two qubits system, where the three qubits and cavity coupling system can be served as the most simple paradigm.

The paper is outlined as follows. In Sec. II, we map the three-qubit Dicke model with CRW interactions into a solvable Hamiltonian by the zeroth and first order approximation, giving the analytical expression of the energy levels. In Sec.III, we discuss dynamics of the GME for the multipartite entanglement, and the concurrence of qubit-qubit entanglement by our method. Finally, a brief summary is given in Sec. IV.

II. AN ANALYTICAL TREATMENT TO THE THREE-QUBIT CAVITY SYSTEM

The Hamiltonian of the three-qubit Dicke model, which describes three identical qubits couple to a common harmonic cavity, is written as ($\hbar = 1$)

$$H = -\Delta J_z + \omega a^\dagger a + \frac{g}{2}(a^\dagger + a)(J_+ + J_-), \qquad (1)$$

where a and $a^\dagger$ are, respectively, the annihilation and creation operators of the harmonic cavity with frequency ω, $J_i\,(i = z, \pm)$ are the angular momentum operators, describing the three qubits of level-splitting Δ in terms of a pseudospin of length $J = 3/2$, g denotes the collective qubit-cavity coupling strength.

In the RWA, the CRW terms $a^\dagger J_+$ and aJ_- are neglected, the Hamiltonian becomes

$$H_{\texttt{RWA}} = -\Delta J_z + \omega a^\dagger a + \frac{g}{2}(a^\dagger J_- + aJ_+)$$

which is restricted to relatively weak coupling strength $g \ll \omega$, and to qubit-cavity near resonance, $\Delta \approx \omega$. Now, the interaction couples only $|-\frac{3}{2}\rangle|n+2\rangle$, $|-\frac{1}{2}\rangle|n+1\rangle$, $|\frac{1}{2}\rangle|n\rangle$, $|\frac{3}{2}\rangle|n-1\rangle$ for each n and no other states. These states form a subspace where the Hamiltonian can be diagonalized analytically. It is easy to write the following tri-diagonal matrix form

$$H_{\texttt{RWA}} = \begin{pmatrix} \omega(n+2)+\frac{3\Delta}{2} & T_{n+1,n+2} & 0 & 0 \\ T_{n+1,n+2} & \omega(n+1)+\frac{\Delta}{2} & T_{n,n+1} & 0 \\ 0 & T_{n,n+1} & \omega n-\frac{\Delta}{2} & T_{n-1,n} \\ 0 & 0 & T_{n-1,n} & \omega(n-1)-\frac{3\Delta}{2} \end{pmatrix}. \qquad (2)$$

where

$$\begin{aligned} T_{n+1,n+2} &= g\sqrt{3(n+2)}/4, T_{n,n+1} = g\sqrt{n+1}/4, \\ T_{n-1,n} &= g\sqrt{3n}/4. \end{aligned}$$

If CRW terms $a^\dagger J_+$ and aJ_- are included, the Hilbert space can not be decomposed into the finite dimensional spaces, because the total excitation number $N = a^\dagger a + J_z + 3/2$ is non-conserved and the subspace for different index n defined above is highly correlated. So analytical solutions in this case should be highly non-trivial.

The full Hamiltonian (1) can be rewritten in the J_x-representation as

$$H = \Delta J_x + \omega a^\dagger a + g(a^\dagger + a)J_z \qquad (3)$$

By a rotation around y axis with the angle $\pi/2$. Introducing an unitary transformation $U = \exp\left[\frac{g}{\omega}J_z\left(a^\dagger - a\right)\right]$, one can obtain $H = H_0 + H_1$ where

$$\begin{aligned} H_0 &= \omega a^\dagger a - \frac{g^2}{\omega}J_z^2 \qquad (4) \\ H_1 &= \Delta\left\{J_x \cosh\left[\frac{g}{\omega}\left(a^\dagger - a\right)\right] + iJ_y \sinh\left[\frac{g}{\omega}\left(a^\dagger - a\right)\right]\right\} \qquad (5) \end{aligned}$$

Since $\cosh(y)$ and $\sinh(y)$ are the even and odd functions respectively, which can be expanded as $\cosh\left[\frac{g}{\omega}\left(a^\dagger - a\right)\right] = G_0\left(a^\dagger a\right) + G_1\left(a^\dagger a\right)\left(a^\dagger\right)^2 + a^2 G_1\left(a^\dagger a\right) + ...$ and $\sinh\left[\frac{g}{\omega}\left(a^\dagger - a\right)\right] = F_1\left(a^\dagger a\right)a^\dagger - aF_1\left(a^\dagger a\right) + F_2\left(a^\dagger a\right)\left(a^\dagger\right)^3 - a^3 F_2\left(a^\dagger a\right) + ...$, containing powers of the number operator $a^\dagger a$. Here $G_i(a^\dagger a)(i = 0, 1, ...)$ and $F_j(a^\dagger a)(j = 1, 2, ...)$ are the coefficients that depend on the cavity number operator $\widehat{n} = a^\dagger a$. Different order of approximations can then be performed by neglecting some terms in the expansions.

Zeroth order approximation: In the zeroth order approximation, we only keep the first term $G_0\left(a^\dagger a\right)$ in Eq. (5), and have

$$H^{0th} = \omega a^\dagger a - \frac{g^2}{\omega}J_z^2 + \Delta J_x G_0\left(a^\dagger a\right). \qquad (6)$$

In the basis of the photonic number state, we can easily find that the term $G_0\left(a^\dagger a\right)$ only has nonvanishing diagonal elements. Keeping the terms containing the number operator $\widehat{n}$ in $\cosh\left[\frac{g}{\omega}\left(a^\dagger - a\right)\right]$, we can evaluate

$$G_0(n) = \langle n|\cosh\left[\frac{g}{\omega}\left(a^\dagger - a\right)\right]|n\rangle = e^{-\frac{g^2}{2\omega^2}}L_n(\frac{g^2}{\omega^2}),$$

where Laguerre polynomials $L_n^{m-n}(x) = \sum_{i=0}^{\min\{m,n\}}(-1)^{n-i}\frac{m!x^{n-i}}{(m-i)!(n-i)!i!}$. Note that only the oscillator number operator $\widehat{n} = a^\dagger a$ appears, so the Hilbert space can be decomposed into same n manifolds spanned by the spin and cavity basis of $|-\frac{3}{2}\rangle|n\rangle$, $|-\frac{1}{2}\rangle|n\rangle$, $|\frac{1}{2}\rangle|n\rangle$ and $|\frac{3}{2}\rangle|n\rangle$. In the subspace contain only the n-th manifold, the Hamiltonian takes the form

$$H^{0th}=\left(\begin{array}{cccc} \omega n-\frac{9g^2}{4\omega} & \frac{\sqrt{3}}{2}\Delta G_0(n) & 0 & 0\\ \frac{\sqrt{3}}{2}\Delta G_0(n) & \omega n-\frac{g^2}{4\omega} & \Delta G_0(n) & 0\\ 0 & \Delta G_0(n) & \omega n-\frac{g^2}{4\omega} & \frac{\sqrt{3}}{2}\Delta G_0(n)\\ 0 & 0 & \frac{\sqrt{3}}{2}\Delta G_0(n) & \omega n-\frac{9g^2}{4\omega}\end{array}\right). \tag{7}$$

The corresponding eigenvalues $\varepsilon_{k,n}$ ($k=1,2,3,4$) are easily given by

$$\begin{aligned}\varepsilon_{1,n} &= \omega n+5A-\frac{1}{2}B_n-2\chi_{1,n},\\ \varepsilon_{2,n} &= \omega n+5A+\frac{1}{2}B_n-2\chi_{2,n},\\ \varepsilon_{3,n} &= \omega n+5A-\frac{1}{2}B_n+2\chi_{1,n},\\ \varepsilon_{4,n} &= \omega n+5A+\frac{1}{2}B_n+2\chi_{2,n},\end{aligned} \tag{8}$$

and eigenvectors $|\varphi_{k,n}\rangle$ are

$$\begin{aligned}|\varphi_{1,n}\rangle &\propto \left(\begin{array}{c}-1\\ K_{1,n}\\ -K_{1,n}\\ 1\end{array}\right),|\varphi_{2,n}\rangle\propto\left(\begin{array}{c}1\\ -K_{2,n}\\ -K_{2,n}\\ 1\end{array}\right),\\ |\varphi_{3,n}\rangle &\propto \left(\begin{array}{c}-1\\ K_{3,n}\\ -K_{3,n}\\ 1\end{array}\right),|\varphi_{4,n}\rangle\propto\left(\begin{array}{c}1\\ -K_{4,n}\\ -K_{4,n}\\ 1\end{array}\right),\end{aligned} \tag{9}$$

where

$$\begin{aligned}K_{i,n} &= \frac{1}{\sqrt{3}B}(-\frac{2g^2}{\omega}-(-1)^i\Delta G_0(n)+4\chi_{i,n}),(i=1,2)\\ K_{i,n} &= \frac{1}{\sqrt{3}B}(-\frac{2g^2}{\omega}(-1)^i\Delta G_0(n)-4\chi_{i-2,n}),(i=3,4)\end{aligned} \tag{10}$$

with

$$\chi_{i,n}=\sqrt{\frac{g^4}{4\omega^2}+(-1)^i\frac{g^2}{4\omega}\Delta G_0(n)+\frac{[\Delta G_0(n)]^2}{4}},(i=1,2)$$

We can discuss the analytical expression of the ground state, $n=0$. For a weak coupling strength, $A=-\frac{g^2}{4\omega}\sim 0$ and $K_{1,0}=\sqrt{3}$, the ground state is given explicitly as $|\varphi_{1,0}\rangle=|0\rangle(-|-\frac{3}{2}\rangle+\sqrt{3}|-\frac{1}{2}\rangle-\sqrt{3}|\frac{1}{2}\rangle+|\frac{3}{2}\rangle)/2\sqrt{2}$, corresponding to the ground state in the J_z-represention as

$$|\psi_{1,0}\rangle=|-\frac{3}{2}\rangle|0\rangle. \tag{11}$$

The Dicke state $|-\frac{3}{2}\rangle$ corresponds that three qubits are all in the spin-down state in the weak coupling regime. Similarly, in the strong coupling regimes, $K_{1,0}\sim 0$, the ground state in the J_x-represention is approximated as a three-qubit GHZ state $\frac{1}{\sqrt{2}}(|-\frac{3}{2}\rangle-|\frac{3}{2}\rangle)|0\rangle$.

The zeroth order approximation can yield good approximate results if the qubits are far detuned from the cavity, $\Delta\ll\omega$. In the zero detuning limiting, $\Delta=0$, within the same manifold n, $|\pm\frac{3}{2}\rangle|n\rangle$ and $|\pm\frac{1}{2}\rangle|n\rangle$ are nearly degenerate in the ultra-strong coupling regime. For a finite and small detuning $\Delta\ll\omega$, it is reasonable to consider transitions between the four states that belong to the same manifold, resulting the eigenstates in Eq.(9). For a weak coupling strength $g/\omega\ll 1$, the analytical eigen-energies (8) are simplified as $n\omega\pm 3B_n/2$ and $n\omega\pm B_n/2$.

Energy levels by the zeroth order approximation are plotted in Fig. 1 with blue dotted lines. In small detuning regime $\Delta/\omega=0.1$, the zeroth order results agree well with the numerical ones even for strong coupling strength in Fig. 1 (a). But the RWA fails to give correct energies. It exhibits improvements of the zeroth order approximation over the RWA. It ascribes to the cavity states are displaced Fock states $|n\rangle_j=\exp[\frac{jg}{\omega}(a^\dagger-a)]|n\rangle(j=\pm\frac{3}{2},\pm\frac{1}{2})$ in the zeroth order approximation. However, there is a noticeable deviation of the zeroth order approximated results for the resonance case $\Delta/\omega=1$, indicating that the higher order terms in Eq.(5) should be taken into account. Physically, states with different oscillator excitations manifolds should be coupled.

First-order approximation: Keeping the linear terms in a and $a^\dagger$ and neglecting all higher order terms in Eq. (5) gives

$$H_1=\Delta\{J_xG_0\left(a^\dagger a\right)+iJ_y[F_1\left(a^\dagger a\right)a^\dagger-aF_1\left(a^\dagger a\right)]\}. \tag{12}$$

The term $F_1\left(a^\dagger a\right)a^\dagger$ describes the photon hopping from state $|n\rangle$ to $|n+1\rangle$. Setting

$$\begin{aligned}R_{n+1,n} &= \langle n+1|\sinh\left[\frac{g}{\omega}\left(a^\dagger-a\right)\right]|n\rangle/\sqrt{n+1}\\ &= \frac{1}{n+1}\frac{g}{\omega}e^{-\frac{g^2}{2\omega^2}}L_n^1(\frac{g^2}{\omega^2}),\end{aligned} \tag{13}$$

$R_{n+1,n}a^\dagger$and $F_1\left(a^\dagger a\right)a^\dagger$ play the same role in physics processes. While, the term $aF_1\left(a^\dagger a\right)$ only has value in $\langle n|n+1\rangle$. It follows that the term $F_1\left(a^\dagger a\right)a^\dagger$ creates and $aF_1\left(a^\dagger a\right)$ eliminates a single photon of the cavity in $\sinh\left[\frac{g}{\omega}\left(a^\dagger-a\right)\right]$, similar to the process described in

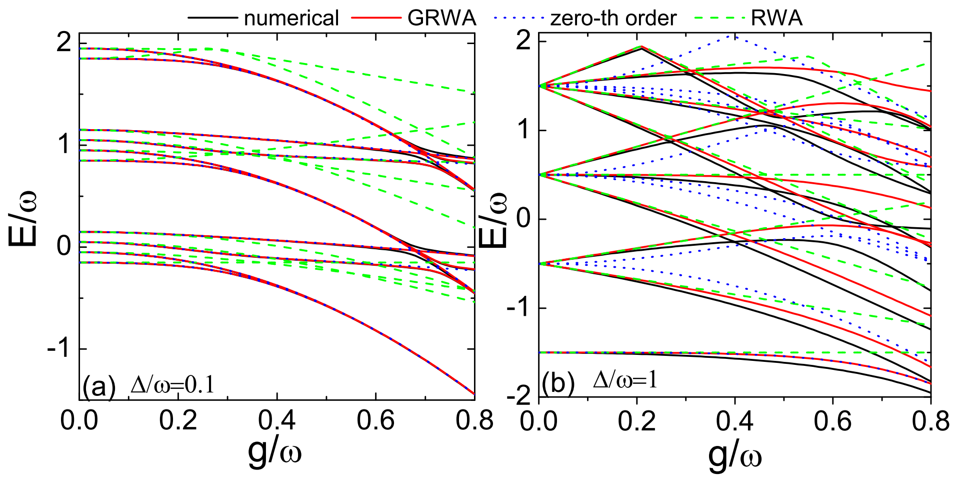

FIG. 1: (Color online) Energy levels obtained by the GRWA method (red solid lines) for different $\Delta/\omega = 0.1$ (a), and $\Delta/\omega = 1$ (b). The energies by the numerically exact diagonalization (black solid lines), results of RWA (green dashed lines) and results obtained by the zeroth-order approximation (blue dotted lines) are plotted for comparison.

the RWA model, which facilitates the further analytic treatment.

The renormalized Hamiltonian is $H = H_0^{'} + H_1^{'}$:

$$\begin{aligned} H_0^{'} &= \omega a^\dagger a - \frac{g^2}{\omega} J_z^2 + \Delta\beta J_x, \qquad (14)\\ H_1^{'} &= \Delta J_x [G_0 \left(a^\dagger a\right) - \beta] + i J_y \Delta [F_1 \left(a^\dagger a\right) a^\dagger - a F_1 \left(a^\dagger a\right)] \end{aligned}$$

with $\beta = G_0(0) = e^{-\frac{g^2}{2\omega^2}}$. Since the qubit and cavity in noninteracting part $H_0^{'}$ are decoupled, we apply a unitary transformation S to diagonalize the qubit part in $H_0^{'}$

$$S = \begin{pmatrix} -\frac{1}{C_1} & \frac{1}{C_2} & -\frac{1}{C_3} & \frac{1}{C_4} \\ \frac{K_1}{C_1} & -\frac{K_2}{C_2} & \frac{K_3}{C_3} & -\frac{K_4}{C_4} \\ -\frac{K_1}{C_1} & -\frac{K_2}{C_2} & -\frac{K_3}{C_3} & -\frac{K_4}{C_4} \\ \frac{1}{C_1} & \frac{1}{C_2} & \frac{1}{C_3} & \frac{1}{C_4} \end{pmatrix}, \qquad (15)$$

where K_i has been defined in Eq.(10) for $n = 0$, and the normalized parameter is $C_i = \sqrt{2 + 2K_i^2}$.

The effective Hamiltonian of the three-qubit Dicke model by the transformation S can be approximated as

$$\begin{aligned} H_{\texttt{GRWA}} &= \omega a^\dagger a + \mu_1 |-\frac{3}{2}\rangle\langle -\frac{3}{2}| + \mu_2 |-\frac{1}{2}\rangle\langle -\frac{1}{2}| + \mu_3 |\frac{1}{2}\rangle\langle \frac{1}{2}| + \mu_4 |\frac{3}{2}\rangle\langle \frac{3}{2}| \\ &+ \Delta F_1 \left(a^\dagger a\right) [\frac{-\sqrt{3}K_2 + K_1(\sqrt{3} + 2K_2)}{C_1 C_2} (a| -\frac{1}{2}\rangle\langle -\frac{3}{2}| + h.c) \\ &+ \frac{-\sqrt{3}K_3 + K_2(\sqrt{3} - 2K_3)}{C_2 C_3} (a|\frac{1}{2}\rangle\langle -\frac{1}{2}| + h.c) \\ &+ \frac{-\sqrt{3}K_4 + K_3(\sqrt{3} + 2K_4)}{C_3 C_4} (a|\frac{3}{2}\rangle\langle \frac{1}{2}| + h.c)], \qquad (16) \end{aligned}$$

where $\mu_i(a^\dagger a) = \varepsilon_{i,0} - \Delta \frac{2K_i[\sqrt{3} - (-1)^i K_i]}{C_i^2} [G_0 \left(a^\dagger a\right) - \beta]$. There are only the energy-conserving terms $(a|-\frac{1}{2}\rangle\langle -\frac{3}{2}| + h.c)$, $(a|\frac{1}{2}\rangle\langle -\frac{1}{2}| + h.c)$ and $(a|\frac{3}{2}\rangle\langle \frac{1}{2}| + h.c)$ with renormalized coefficients. The dominated effect of the original CRW terms are considered here. Because it is three-qubit

Dicke model Hamiltonian in the RWA with renormalized coefficients, and thus called as generalized rotating-wave approximation (GRWA).

Note that the individual bosonic creation (annihilation) operator $a^\dagger$ (a) also appears in the GRWA, so the transitions between states belonging to 4 different manifolds should be involved. In the basis of $|-\frac{3}{2}\rangle|n+2\rangle$, $|-\frac{1}{2}\rangle|n+1\rangle$, $|\frac{1}{2}\rangle|n\rangle$ and $|\frac{3}{2}\rangle|n-1\rangle$ $(n>0)$, the Hamiltonian can be written in the matrix form as

$$H_{\texttt{GRWA}}=\begin{pmatrix} \omega(n+2)+\mu_1(n+2) & R'_{n+1,n+2} & 0 & 0 \\ R'_{n+1,n+2} & \omega(n+1)+\mu_2(n+1) & R'_{n,n+1} & 0 \\ 0 & R'_{n,n+1} & \omega n+\mu_3(n) & R'_{n-1,n} \\ 0 & 0 & R'_{n-1,n} & \omega(n-1)+\mu_4(n-1) \end{pmatrix}, \tag{17}$$

with $R'_{n+1,n+2}=\frac{-\sqrt{3}K_2+K_1(\sqrt{3}+2K_2)}{C_1C_2}R_{n+1,n+2}\sqrt{n+2}$, $R'_{n,n+1}=\frac{-\sqrt{3}K_3+K_2(\sqrt{3}-2K_3)}{C_2C_3}R_{n,n+1}\sqrt{n+1}$ and $R'_{n-1,n}=\frac{-\sqrt{3}K_4+K_3(\sqrt{3}+2K_4)}{C_3C_4}R_{n-1,n}\sqrt{n}$.

To this end, the GRWA can be also analytically performed without more efforts than that in the original RWA study in the three-qubit Dicke model. The displaced Fock states in the cavity $|n\rangle_m$, $|n\pm1\rangle_m$ and $|n+2\rangle_m$ depend upon the Dicke state $|j,m\rangle$, which are definitely different from the RWA ones. Different manifold $n+2$, $n\pm1$ and n are coupled in the GRWA, in contrast with the only one manifold in the zeroth order approximation.

The ground-state energy for the ground state $|-\frac{3}{2}\rangle|0\rangle$ is

$$E_0=5A-\frac{B_0}{2}-2\chi_{1,0}. \tag{18}$$

The first and second excited energies $\{E_0^k\}$ $(k=1,2)$ can be given by expanding the GRWA Hamiltonian in the basis $|-\frac{3}{2}\rangle|1\rangle$, $|-\frac{1}{2}\rangle|0\rangle$

$$H_{\texttt{GRWA}}=\begin{pmatrix} \omega+\mu_1(1) & R'_{0,1} \\ R'_{0,1} & \mu_2(0) \end{pmatrix}. \tag{19}$$

Similarly, $H_{\texttt{GRWA}}$ is given in terms of $|-\frac{3}{2}\rangle|2\rangle$, $|-\frac{1}{2}\rangle|1\rangle$, $|\frac{1}{2}\rangle|0\rangle$ as

$$H_{\texttt{GRWA}}=\begin{pmatrix} 2\omega+\mu_1(2) & R'_{1,2} & 0 \\ R'_{1,2} & \omega+\mu_2(1) & R'_{0,1} \\ 0 & R'_{0,1} & \mu_3(0) \end{pmatrix}, \tag{20}$$

which provides three analytical excited energies $\{E_0^k\}$ $(k=3,4,5)$.

Energies obtained by the GRWA are presented in red solid lines in Fig. 1. Especially, for the resonance case $\Delta=\omega$ in Fig. 1(b), the GRWA results are much better than the zeroth order results (blue dotted lines) due to the coupling between states belonging to different oscillator manifolds, $|n\rangle$, $|n\pm1\rangle$ and $|n+2\rangle$. The level crossing is present in both the GRWA results and the exact ones. The GRWA includes the dominant contribution of the GRW terms by the first order approximation, exhibiting substantial improvement of energy levels over the RWA one. The RWA fails in particular to describe the eigenstates, which should be more sensitive in the quantum entanglement presented in the next section.

III. QUANTUM ENTANGLEMENT

We study the GME of three qubits, and the concurrence for the bipartite entanglement in this model. Since a fully separable three-particle state contains no entanglement. If the state is not fully separable, then it contains some entanglement, but it might be still separable with respect to two-party configurations. For genuine multiparticle entangled states, all particles are entangled and therefore GME is also interesting if not most important among various entanglement.

The GME measuring the multipartite entanglement has been detected efficiently using positive partial transpose (PPT) mixtures [33]. If a bipartite state is separable, its partial transpose is positive semidefinite. The separable state is PPT. Similar to the definition of a separable state, a PPT mixture of a three-party state is defined as a convex combination of PPT states. The set of PPT mixtures contains the set of biseparable states, combining states which are biseparable with respect to a specific bipartition. Consequently, if a state is not a PPT mixture, it is genuinely multipartite entangled. PPT mixtures can be fully characterized by the method of semidefinite programming (SDP) [34], which makes it an easy-to-implement criterion to detect GME. A state ρ is a PPT mixture if and only if the following optimization problem,

$$\texttt{minTr}(W\rho), \tag{21}$$

where W is an operator as a decomposable entanglement witness for any semidefinite. For solving the minimum Eq.(21) by the SDP, we use the semidefinite programs, which are freely available [35, 36]. For a negative minimum, the state ρ is not a PPT mixture and is genuinely multipartite entangled.

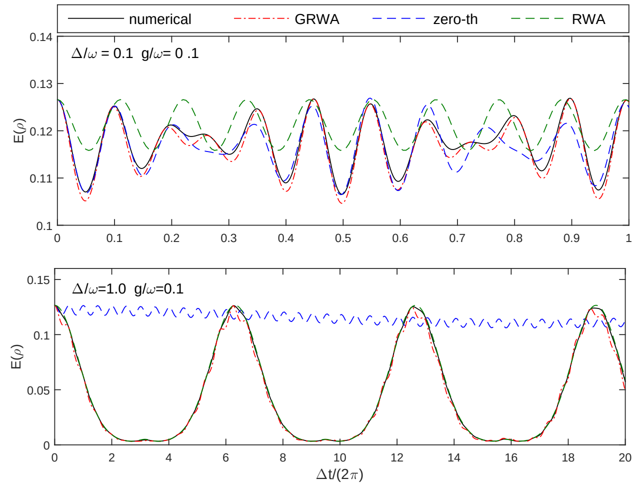

FIG. 2: (Color online) Dynamics of the GME for three-qubit entanglement with the initial W state for the ultra-strong coupling strength $g/\omega = 0.1$ with the different detuning $\Delta/\omega = 0.1$ (a) and $\Delta/\omega = 1$ (b) by the GRWA method (red dashed dotted lines), numerically exact diagonalization (black solid lines), RWA (green dashed lines), and by the zeroth-order approximation (blue dashed lines).

There is ongoing interest in the dynamics of the bipartite entanglement and multipartite entanglement. We choose the initial three-qubit state as W state with respected to the original Hamiltonian (1) in the J_z-representation

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle), \tag{22}$$

which can be expressed as the Dicke state $|D_3\rangle = |-\frac{1}{2}\rangle$. In the J_x-representation by the unitary transformation $e^{-i\pi J_y/2}$, the initial three-qubit Dicke state can be written as

$$|D_3\rangle = \frac{1}{\sqrt{8}}[-\sqrt{3}|-\frac{3}{2}\rangle - |-\frac{1}{2}\rangle + |\frac{1}{2}\rangle + \sqrt{3}|\frac{3}{2}\rangle]. \tag{23}$$

And the initial cavity state is the vacuum state $|0\rangle$. Based on the eigenstates $\left\{|\varphi_n\rangle^k\right\}$ and eigenvalues $\left\{E_n^k\right\}$ by the GRWA and the zeroth order approximation, the wavefunction evolves from the initial W state as $|\phi(t)\rangle = \sum_n e^{-iE_n^k t}|\varphi_n^k\rangle\langle\varphi_n|D_3\rangle$. To calculate the GME dynamics by solving the minimum Eq.(21), the three-qubit reduced state $\rho(t)$ can be given by tracing out the cavity degrees of freedom $\rho(t) = Tr_{\texttt{cavity}}(|\phi(t)\rangle\langle\phi(t)|)$.

Fig. 2 shows evolution of the GME $E(\rho)$ for the three-qubit entanglement for a ultra-strong coupling strength $g/\omega = 0.1$ for different detuning case. For comparison, results from numerical exact diagonalization and RWA are also shown. We observe a quasi-periodic behavior of the GME dynamics. $E(\rho)$ decays from the initial entangled W state and fall off to non-zero minimum value. The GME dynamics obtained by the GRWA are consistent with the numerical results, while the RWA results are qualitatively incorrect for the off-resonance case $\Delta/\omega = 0.1$ in Fig. 2 (a). The validity of the GRWA ascribes to the inclusion of the CRW interaction $iJ_yF_1\left(a^\dagger a\right)\left(a^\dagger - a\right)$, which leads to transitions between states in the different photon manifold spanned by $|n\rangle$, $|n\pm1\rangle$ and $|n+2\rangle$. For the zeroth order approximation, where only states within the same manifold are included, works well for the off-resonance case $\Delta = 0.1$ in Fig. 2 (a) but not for the on-resonance case as shown in Fig. 2 (b). The onset of the decay of the entanglement is due to the information loss of qubit dynamics to the cavity. On the other hand, it is the interaction with the cavity that lead to the entanglement resurrection. The lost information will come back to the qubit subsystem in finite times. Fig. 2 (b) shows the GME $E(\rho)$ for the three-qubit entanglement recovers from a non-zero minimum entanglement after a period time.

For the bipartite entanglement, the concurrence characterizes the entanglement between two qubits. Due to the symmetric Dicke states in the three-qubit collective model, the concurrence is evaluated in terms of the expectation values of the collective spin operators as $C = \max\{0, C_y, C_z\}$, where the quantity C_n is defined for a given direction $n(=y,z)$ as $C_n = \frac{1}{2N(N-1)}\{N^2 - 4\langle S_n^2\rangle - \sqrt{[N(N-2)+4\langle S_n^2\rangle]^2 - [4(N-1)\langle S_n\rangle]^2}\}$ [25]. From the dynamical wavefunction $|\phi(t)\rangle$, it is easily to evaluate the coefficients for the qubit to remain in the $|j,m\rangle$ state as

$$P_m^{0th} = \sum_{n=0}^{\infty}\sum_{k=1}^{4} f_n(t)e^{-iE_n^k t}|n\rangle_m, \tag{24}$$

with the zeroth order approximation and as

$$\begin{aligned} P_m^{\texttt{GRWA}} &= f_0(t)e^{-iE_0 t}|0\rangle + \sum_{k=3}^{5} f_0^k(t)e^{-iE_0^k t}|2\rangle_m \\ &+ \sum_{k=1}^{5} f_0^k(t)e^{-iE_0^k t}(|0\rangle_m + |1\rangle_m) \\ &+ \sum_{n>1}^{\infty}\sum_{k=1}^{4} f_n^k(t)(e^{-iE_{n-2}^k t} + e^{-iE_{n-1}^k t} \\ &+ e^{-iE_n^k t} + e^{-iE_{n+1}^k t})|n\rangle_m, \end{aligned} \tag{25}$$

with the GRWA. $f_n^k(t)$ is a dynamical parameter associated with the initial state and the k-th eigenstates for each n. It is easily to obtain the concurrence C in terms of the average value of collective spin operators, such as $4\langle S_y^2\rangle = 4\sqrt{3}({}_{-\frac{3}{2}}\langle n-2|n\rangle_{\frac{1}{2}}P_{-\frac{3}{2}}P_{\frac{1}{2}} + {}_{-\frac{1}{2}}\langle n-1|n+1\rangle_{\frac{3}{2}}P_{-\frac{1}{2}}P_{\frac{3}{2}}) - 4(P_{-\frac{1}{2}}^2 + P_{\frac{1}{2}}^2) + 3$. Since four basis states $|n\rangle$, $|n\pm1\rangle$ and $|n+2\rangle$ involved in the GRWA, we could expect energy transitions among E_{n-2}^k, $E_{n\pm1}^k$ and E_n^k, which produce essential improvement over the zeroth order ones.

We plot the dynamics of the concurrence in the ultra-strong coupling strength $g/\omega = 0.1$ for two detunings

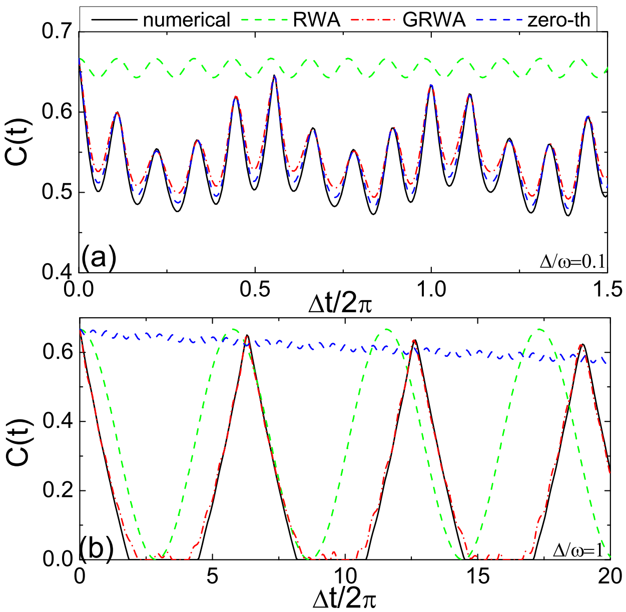

FIG. 3: (Color online) Dynamics of the concurrence for the qubit-qubit entanglement with the initial W state for the ultrastrong coupling strength $g/\omega = 0.1$. The parameters are the same as in Fig. 2.

$\Delta/\omega = 0.1$ and 1 in Fig. 3. The initial W state gives the maximum pairwise concurrence of any Dicke state $C = 2/3$. Fig. 3 (a) shows that dynamics of the concurrence by the zeroth order approximation are similar to the numerical ones in the off-resonance case $\Delta/\omega = 0.1$, in which the RWA results are invalid. The sudden death of the bipartite entanglement is observed in the resonance case in Fig. 3 (b). The dynamics of the concurrence obtained by the GRWA are similar to the numerical results, exhibiting an disappearance of the entanglement for a period of time. However, there is no sudden death of the entanglement in the RWA study, indicating that RWA can not display qualitatively correct dynamics of the concurrence for the ultra-strong coupling $g/\omega = 0.1$.

Interestingly, the entanglement by the numerical method can fall abruptly to zero, and will return zero for a period of time before entanglement revivals. The vanishment of entanglement implies that the state stay in the disentangled separable state. It is in sharp contrast with dynamics of the GME for the three-qubit entanglement. It is shown in Fig. 2(b) that GME never vanishes. More interestingly, during the vanishment of concurrence, the GME is also generally small, but still finite. This is one advantage to use GME as a quantum information resource.

IV. CONCLUSION

In this work, we analytically study the three-qubit Dicke model in the ultra-strong coupling regime where the RWA is invalid. For the large detuning $\Delta \ll \omega$, the zeroth order approximation is suited for all coupling strengths. The first-order approximation, also called GRWA, can describe this model almost for the whole parameter space

By the newly proposed GRWA scheme, we have also calculated the dynamics of concurrence for the bipartite entanglement and the GME for the multipartite entanglement. The quasi-periodic behavior of the dynamics of the concurrence displays the sudden death of the entanglement, and revivals due to the interaction of the cavity. It is distinguished from the GME dynamics, which exhibits the non-zero minimum entanglement. It manifest that there still contains some entanglement in multipartite system even through there is no bipartite entanglement. It is found that the GME is a strong entanglement criteria to detect multiparticle entanglement. The dynamical behaviors for two kinds of entanglement may be explored in the multi-qubits realized in the recent circuit QED systems in the ultra-strong coupling.

In the end of the preparation of the present work, we noted a recent paper by Mao et al [38] for the same model. We should say that the approach used there is the adiabatic approximation of the present work, i.e. the zeroth order approximation.

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