Weak lower semicontinuity of integral functionals and applications

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"Nothing takes place in the world whose meaning is not that of some maximum or minimum."
Leonhard Paul Euler (1707–1783)

Abstract. In the occassion of the fifty-year anniversary of N.G. Meyers's paper [84] which significantly extended weak lower semicontinuity statements for integral functionals depending on maps and their gradients available at that time, we recapitulate up-to-date results on this topic. Special attention is paid to signed integrands and to applications to continuum mechanics of solids. In particular, we review existing results for polyconvex simple/nonsimple materials and related statements about weak sequential continuity of minors. These are non-coercive and belong precisely to the class of integrands studied by Meyers in his seminal work. Besides, we emphasize some recent progress in lower semicontinuity of functionals along sequences satisfying differential and algebraic constraints which have applications in continuum mechanics of solids to ensure injectivity and orientation-preservation of elastic deformations. Finally, we outline generalization of these results to more general first-order partial differential operators.

1. Introduction. The observation that continuous functions attain extreme values on compact sets goes back to Bernard Bolzano who proved it in his work "Function Theory" in 1830. This result is called the *Extreme Value Theorem*. Later on, it was independently shown by Karl Weierstrass around 1860. The main ingredient of the proof, namely the fact that one can extract a convergent subsequence from a closed bounded interval on reals, is nowadays known as the Bolzano-Weierstrass theorem. While Riesz and Hilbert already used the weak topology on Hilbert spaces from the beginning of the 20th century, Stefan Banach defined it on other normed spaces around 1929 [96, 120] and opened the possibility to extend Bolzano's Extreme Value Theorem to more general situations and, in particular, to the calculus of variations. This mathematical discipline has in its background minimization problems of the type

$$y \mapsto \int_a^b v(x,y,y')\,\mathrm{d}x \to \inf \ \text{ with } y(a)=y_a \text{ and } y(b)=y_b \ .$$

It includes, for example, the *brachistochrone problem*, i.e., the problem of curves with a minimum time of descent. Foundations of the calculus of variations were laid down in the 18th century by L.P. Euler and J.L. Lagrange who also realized its important connections to physics and to mechanics.

Lower semicontinuity plays a fundamental role in the *direct method of the calculus of variations*, an algorithm, proposed by David Hilbert around 1900, to show (in a non-constructive way) the existence of a solution to the minimization problem

$$\text{find minimum of } I \text{ on } \mathcal{A}\ .$$

It consists of three steps. First, we find a minimizing sequence along which I converges to its infimum on $\mathcal{A}$. The second step is to show that a subsequence of the minimizing sequence converges to an element of $\mathcal{A}$ in some topology τ. Finally, it remains to show that this limiting element is a minimizer. This is easily done if I is (sequentially) lower semicontinuous with respect to the topology τ. In the most typical situation the topology τ is either the weak or the weak* one; thus, we shall also limit our view to this case.

Definition 1.1. *Let $\mathcal{A}$ be a subset of a Banach space. We say that the functional $I : \mathcal{A} \to \mathbb{R}$ is (sequentially) weakly/weakly* lower-semicontinuous on $\mathcal{A}$ if for any sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{A}$ converging weakly/weakly* to $u \in \mathcal{A}$, we have that*

$$I(u) \le \liminf_{k\to\infty} I(u_k).$$

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While the first two steps of the direct method can be satisfied by assuming coercivity of I and by choosing a sufficiently weak topology on $\mathcal{A}$ the last step essentially relies on fine properties of I as convexity, for instance.

Let us point out that early studies of minimizers of integral functionals of the form

$$y \mapsto \int_a^b v(x, y, y')\,\mathrm{d}x$$

naturally relied on smoothness properties of v when calculating variations of this integral; see for example the book by Bolza [22]. On the other hand, the direct method is not based on calculating derivatives and thus it is natural to expect that it will cope also with non-smooth and possibly also partially non-continuous functionals. This expectation is indeed true and relaxing smoothness/continuity assumptions of f will be a re-occurring theme throughout this review.

As indicated above, many phenomena in nature are successfully modeled by solving a minimization problem for a suitably chosen (energy) functional. A prominent example is found in continuum mechanics of solid media, where minimization of the stored energy

$$\mathcal{E}(y) := \int_\Omega W(\nabla y(x))\,\mathrm{d}x \ , \tag{1.1}$$

determines stable states of the system. Here, W is the stored energy density and the map $y : \Omega \to \mathbb{R}^3$, with Ω a bounded domain, is the deformation of the modeled medium.

Naturally, the question arises under what conditions on W minimizers of (1.1) exist on a suitable function space $\mathcal{A}$. In view of the direct method described above, this particularly includes the study of *weak lower semicontinuity* of functionals $\mathcal{E}$.

Even if the study of weak lower semicontinuity is motivated by understanding minimization problems, it has become an independent subject in mathematical literature that is studied for its own right. In 1920, Tonelli [116] showed that if $n = m = 1$ and

$$I(u) := \int_\Omega v(x, u(x), \nabla u(x))\,\mathrm{d}x \ , \tag{1.2}$$

for $\Omega = (a, b)$ and $u \in W^{1,\infty}(a, b)$ then I is weakly lower semicontinuous if and only if v is convex in its last variable, i.e., in the derivative $\nabla u = u'$. Later, several authors generalized this result to $n > 1$, see for example Serrin [104], where differentiability properties of v were removed from assumptions, and Marcellini and Sbordone [82]. On the other hand, if we allow the function u to be vector-valued, i.e., besides $n > 1$ also $m > 1$, then the convexity hypothesis turns out to be sufficient but unnecessary. A suitable condition, termed quasiconvexity, was introduced by Morrey [87].

Definition 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We say that a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex if for any $A \in \mathbb{R}^{m\times n}$ and any $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$*

$$f(A)|\Omega| \le \int_\Omega f(A + \nabla\varphi(x))\,\mathrm{d}x \ . \tag{1.3}$$

He showed that, under strong regularity assumptions on v, I is weakly lower semicontinuous in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ if and only if v is quasiconvex in the last variable (i.e. in the gradient). All the above characterizations turn the problem of sequential weak lower semicontinuity of I into conditions on the integrand. Obviously, this is a much more *explicit characterization* which does not need to deal with weakly convergent sequences.

These results were generalized fifty years ago, in 1965, by Norman G. Meyers in his seminal paper [84]. There he investigated $W^{k,p}$-weak (weak* if $p = +\infty$) lower semicontinuity of integral functionals of the form

$$I(u) := \int_\Omega v(x, u(x), \nabla u(x), \ldots, \nabla^k u(x))\,\mathrm{d}x \ , \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $u : \Omega \to \mathbb{R}^m$ is a mapping possessing (weak) derivatives up to the order $k \in \mathbb{N}$. The function v was supposed to be continuous in all its arguments. Since now higher gradients than the first ones are considered, the definition of quasiconvexity also needs to be generalized accordingly (see Section 2 for the notation).

Definition 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We say that a function $f : X \to \mathbb{R}$ is k-quasiconvex[3] if for any $A \in X$ and any $\varphi \in W_0^{k,\infty}(\Omega; \mathbb{R}^m)$*

$$f(A)|\Omega| \leq \int_\Omega f(A + \nabla^k \varphi(x))\,\mathrm{d}x\ . \tag{1.5}$$

Thus, more precisely, k-quasiconvexity of v (i.e. quasiconvexity with respect to the k-th gradient) means that $A^k \mapsto v(x, A^{[k-1]}, A^k)$ is quasiconvex for all fixed $(x, A^{[k-1]}) \in \Omega \times Y(m, n, k-1)$; here, we already used the notation which will be introduced in Section 2.

Remark 1.1. *In fact, it was shown in [32] that if $k = 2$ and if f satisfies a (slightly) stronger version of 2-quasiconvexity then 2-quasiconvexity coincides with 1-quasiconvexity. See [26] for an analogous result with general k.*

However, more generally than in Morrey's work, the function v is not necessarily bounded from bellow in [84]. From this, additional difficulties arise and, in fact, quasiconvexity is no longer a sufficient condition for weak lower semicontinuity (cf. Section 3).

The motivation for studying functionals of the type (1.4) is twofold: from the point of view of applications in continuum mechanics it is reasonable to let v depend also on higher-order gradients since their appearance in the energy usually models interfacial energies or multipolar elastic materials [53]. Another reason might be to consider deformation-gradient dependent surface loads [11]. On the other hand, not assuming a lower bound on v is important to consider for mathematical completeness. Moreover, integrands of the type $v(A) := \det A$, which are unbounded from below, are of crucial importance in continuum mechanics.

Meyers' main result are necessary and sufficient conditions on v so that I is weakly lower semicontinuous on $W^{k,p}(\Omega; \mathbb{R}^m)$. We review these results in Section 3. He first discusses the problem $p = +\infty$, where quasiconvexity in the highest-order gradient (cf. Theorem 3.1) turns out to be a necessary and sufficient condition for weak*-lower semicontinuity. For the case $1 \leq p < +\infty$, the situation is, however, much more subtle and an additional condition (cf. Theorem 3.3 and Section 3.1) is needed.

Since the appearance of Meyers' work, significant progress has been achieved with respect to the characterization of weak lower semicontinuity of functionals of the type (1.1) or (1.4). In particular, for $k = 1$ in (1.4) the additional condition for sequential weak lower semicontinuity was characterized more explicitly and results relaxing Meyer's continuity assumptions were obtained for functionals bounded from below; cf. Section 3.

Moreover, it has been identified for which functions v the functional I in (1.4) is even weakly continuous (see Section 4)–these functions are the so-called null Lagrangians–and this knowledge led to the notion of polyconvexity (see Section 5) that is sufficient for weak lower semicontinuity and of particular importance in mathematical elasticity. In fact, quasiconvexity, which is, for a large class of integrands, the necessary and sufficient condition for weak lower semicontinuity is not well-suited for elasticity. We explain this issue in Section 6 and review some recent progress in this field. Null Lagrangians and weak continuity has also been identified for functional defined on the boundary (see Section 7). Finally, we review weak lower semicontinuity results for functionals depending on maps that satisfy general differential constraints in Section 8.

2. Notation. In this section we summarize the notation that shall be used throughout the paper. It largely coincides with the one used in [11]. In what follows, $\Omega \subset \mathbb{R}^n$ is a bounded domain whose boundary is Lipschitz or smoother. This domain is mapped to a set in $\mathbb{R}^m$ by means of a mapping $u : \Omega \to \mathbb{R}^m$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If $J := (j_1, \ldots, j_n) \in \mathbb{N}_0^n$ and $K := (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ are two multiindices we define $J \pm K := (j_1 \pm k_1, \ldots, j_n \pm k_n)$, further $|J| = \sum_{i=1}^n j_i$, $J! := \Pi_{i=1}^n j_i!$, and

[3] In the original paper [84] quasiconvexity with respect to the k-th gradient is also referred to as quasiconvexity.

we say that $J \le K$ if $j_i \le k_i$ for all i. Then we also define $\binom{J}{K} := J!/K!/(J-K)!$, $\partial u^j_K := \frac{\partial^{k_1}\dots\partial^{k_n}}{\partial x_1^{k_1}\dots\partial x_n^{k_n}} u^j$, $x_K = x^K := x_1^{k_1}\dots x_n^{k_n}$, and $(-D)^K := \frac{(-\partial)^{k_1}\dots(-\partial)^{k_n}}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}$.

We will work with the space of matrices $X = X(n,m,k)$ with the dimension $m\binom{n+k-1}{k}$, too. This is the space of matrices $M = (M^i_K)$ for $1 \le i \le m$ and $|K| = k$. Similarly, $Y = Y(n,m,k)$ is a space of matrices $M = (M^i_K)$ for $1 \le i \le m$ and $|K| \le k$. Its dimension is $m\binom{n+k}{k}$. We denote the elements of $X(n,m,k)$ by A^k while the $A^{[k]} = (A, A^2, \dots, A^k)$ is an element of $Y(n,m,k)$. We use an analogous notation also for gradients; thus, if $x \in \Omega$, then $\nabla^k u(x) \in X(n,m,k)$ while $\nabla^{[l]} u(x) \in Y(n,m,k)$.

We denote by $B(x_0, r)$ the ball of origin x_0 with the radius r while $D_\rho(x_0, r)$ is the half-ball with ρ being the normal of the planar component of the boundary; i.e.

$$D_\rho(x_0, r) := \{x \in B(x_0, r) : (x - x_0)\cdot \varrho < 0\},$$

and we write $D_\rho := D_\rho(0,1)$.

We we shall use the standard notation for the Lebesgue spaces $L^p(\Omega;\mathbb{R}^m)$ and Sobolev spaces $W^{k,p}(\Omega;\mathbb{R}^m)$. Moreover, $\mathrm{BV}(\Omega;\mathbb{R}^m)$ is the space of functions of a bounded variation. If Ω is a bounded open domain we denote $\mathcal{M}(\Omega)$ the space of Radon measures on Ω. If $n = m$ and $F \in \mathbb{R}^{n\times n}$ is invertible, we denote $\mathrm{Cof}F := (\det F)F^{-\top}$ the cofactor matrix. Rotation matrices with determinants equal one are denoted $\mathrm{SO}(n)$.

3. A review of Meyers' results.

Meyers studies in [84] weak lower semicontinuity of (1.4) on a fairly general class of integrands. In particular, for weak lower semicontinuity on $W^{k,p}(\Omega,\mathbb{R}^m)$ with $1 \le p < +\infty$ he introduces the class $\mathcal{F}_p(\Omega)$ (cf. [84, Def. 4] and Definition 3.2 below). On $W^{k,\infty}(\Omega,\mathbb{R}^m)$, any continuous integrand is admitted. On $W^{k,\infty}(\Omega,\mathbb{R}^m)$, Meyers recovers an analogous result to the one found in the original work of Morrey [87]:

THEOREM 3.1. *Let Ω be a bounded domain and v a continuous function. Then I from* (1.4) *is weakly* lower semicontinuous in $W^{k,\infty}(\Omega;\mathbb{R}^m)$ if and only if it is k-quasiconvex.*

Nevertheless, when it comes to the case of $W^{k,p}(\Omega;\mathbb{R}^m)$ with p finite the situation is substantially more subtle; particularly, because the considered integrands are not bounded from below. This can be seen from the definition of the class $\mathcal{F}_p(\Omega)$:

DEFINITION 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. A continuous integrand $v : \Omega \times Y \to \mathbb{R}$ is said to be in the class $\mathcal{F}_p(\Omega)$ for $1 \le p < +\infty$ if ($C > 0$ is a constant depending only on v)*

(i) $v(x, A^{[k]}) \le C\big(1 + |A^{[k]}|\big)^p$,

(ii) $|v(x, A^{[k]} + B^{[k]}) - v(x, A^{[k]})| \le C\big(1 + |A^{[k]}| + |B^{[k]}|\big)^{p-\gamma}|B^{[k]}|^\gamma$, *where* $0 < \gamma \le 1$,

(iii) $|v(x+y, A^{[k]}) - v(x, A^{[k]})| \le (1 + |A^{[k]}|)^p \eta(|y|)$ *with* $\eta : [0;+\infty) \to [0;+\infty)$ *continuous, increasing and vanishing at zero.*

Indeed, when setting $A^{[k]} = 0$ in (ii) we get that $|v(x, B^{[k]})| \le C(1 + |B^{[k]}|)^p$ and thus the class $\mathcal{F}_p(\Omega)$ contains also noncoercive integrands and, in particular, those which decay as $-|\cdot|^p$.

This decay is problematic with respect to weak lower semicontinuity, because then, along concentrating sequences of gradients[4], energy may be *gained* and hence the lower semicontinuity is destroyed. On the boundary of the domain this effect cannot be excluded by quasiconvexity as the following example shows.

EXAMPLE 3.1 (following [76], [6]). *Choose $\Omega = (0,1)$ and define a sequence on $\mathrm{BV}((0,1))$, i.e. the functions of bounded variations, defined through $u_n := \chi_{(0,\frac{1}{n})}$ so that $Du_n = -\delta_{\frac{1}{n}}$. Further let us choose and $v(x,\xi) := \xi$; i.e. v is a linear function and so quasiconvex. Nevertheless, $I(u_n) = -1$ for all n, but $u_n \overset{*}{\rightharpoonup} 0$ in $\mathrm{BV}((0,1))$ and $I(0) = 0 > -1$.*

The example illustrates the above mentioned effect that a sequence concentrating on the boundary (such as u_n) may actually lead to an energy gain in the limit. While the above example is in $\mathrm{BV}((0,1))$, because this

[4] We say that a sequence bounded in L^1 is concentrating if it converges weak* in measures but not weakly in L^1.

allows us to take a linear, and thus a particularly easy, integrand in (1.4) *appropriate nonlinear integrands lead to the same effect in* $W^{k,p}(\Omega;\mathbb{R}^m)$ *with* $p>1$*; cf. Example 3.2 below.*

Meyers hence introduced an additional condition to ensure sequential weak lower semicontinuity and proved the following.

THEOREM 3.3. *Let* Ω *be a bounded domain and* $v\in\mathcal{F}_p(\Omega)$*. Then* I *from* (1.4) *is weakly lower semicontinuous on* $W^{k,p}(\Omega;\mathbb{R}^m)$ *with* $1lep<\infty$ *if and only if the following two conditions hold simultaneously:*

(i) $v(x,A^{[k-1]},\cdot)$ *is* k*-quasiconvex for all values of* $(x,A^{[k-1]})$*,*

(ii) $\liminf_{j\to\infty} I(u_j,\Omega')\ge-\mu(|\Omega'|)$ *for every subdomain* $\Omega'\subset\Omega$ *and every sequence* $\{u_j\}_{j\in\mathbb{N}}\subset W^{k,p}(\Omega;\mathbb{R}^m)$ *such that* $u_j=u$ *on* $\Omega\setminus\Omega'$ *and* $u_j\rightharpoonup u$ *in* $W^{k,p}(\Omega;\mathbb{R}^m)$*. Here* μ *is an increasing continuous function with* $\mu(0)=0$ *which only depends on* u *and on* $\limsup_{j\to\infty}\|u_j\|_{W^{k,p}}$*.*

Above, $I(\cdot,\Omega')$ denotes the functional I when the integration domain Ω is replaced by Ω'.

We immediately see that condition (ii) is satisfied, for example, if $v\ge 0$. To see why this condition excludes the effect of concentrations on the boundary, take a sequence of "rings" Ω'_j around the boundary of Ω. The measure of such rings converges to zero and so, also $\mu(|\Omega'_j|)$ tends to zero. But if $\{|\nabla^k u_j|^p\}$ is a concentrating sequence which converges to a measure supported on $\partial\Omega$ then $I(u_j,\Omega'_j)$ may take a fixed negative value and thus it *violates* condition (ii) from Theorem 3.3.

Since condition (ii) in Theorem 3.3 is connected with concentrations on the boundary, Meyers conjectured that it can be dropped if $\partial\Omega$ is "smooth enough" or a "smooth enough" function is prescribed on the boundary *as the datum.* While the second part of the conjecture turned out to be true at least if $k=1$ in (1.4) and the integrand does not depend on u (cf. Theorem 3.5 below) the first part is *not* as the following example illustrates:

EXAMPLE 3.2 (See [15]). *Let* $n=m=p=2$*,* $0<a<1$*,* $\Omega:=(0,a)^2$ *and for* $x\in\Omega$ *define*

$$u_j(x_1,x_2)=\frac{1}{\sqrt{j}}(1-|x_2|)^j(\sin jx_1,\cos jx_1).$$

We see that $\{u_j\}_{j\in\mathbb{N}}$ *converges weakly in* $W^{1,2}(\Omega;\mathbb{R}^2)$ *as well as pointwise to zero. Moreover, we calculate for* $j\to\infty$

$$\int_0^a\int_0^a \det\nabla u_j(x)\,\mathrm{d}x\to\frac{-a}{2}<0\ .$$

Hence, we see that $I(u):=\int_\Omega\det\nabla u(x)\,\mathrm{d}x$ *is not weakly lower semicontinuous in* $W^{1,2}(\Omega;\mathbb{R}^2)$*. This example can be generalized to arbitrary dimensions* $m=n\ge 2$*. Indeed, take* $u\in W^{1,n}_0(B(0,1);\mathbb{R}^n)$ *and extend* u *by zero to the whole* $\mathbb{R}^n$*. We get that* $\int_{B(0,1)}\det\nabla u(x)\,\mathrm{d}x=0$ *because of the zero Dirichlet boundary conditions on* $\partial B(0,1)$*. Take* $\varrho\in\mathbb{R}^n$ *a unit vector* ρ *such that* $\int_{D_\varrho}\det\nabla u(x)\,\mathrm{d}x<0$*. Notice that this condition can be fulfilled, if we take* u *suitably.*

Denote $u_j(x):=j^{-1}u(jx)$ *for all* $j\in\mathbb{N}$*; then,* $u_j\rightharpoonup 0$ *in* $W^{1,n}(B(0,1);\mathbb{R}^n)$ *(even in measure) but* $\int_{D_\varrho}\det\nabla u_j(x)\,\mathrm{d}x\to\int_{D_\varrho}\det\nabla u(x)\,\mathrm{d}x<0$ *by our construction. The same conclusion can be drawn if we take* $\Omega\subset\mathbb{R}^n$ *with arbitrarily smooth boundary and such that* $0\in\partial\Omega$*. Let* ϱ *be the outer unit normal to* $\partial\Omega$ *at zero. Then we have for the same sequence as before*

$$\lim_{j\to\infty}\int_\Omega\det\nabla u_j(x)\,\mathrm{d}x=\lim_{j\to\infty}\int_{B(0,1)\cap\Omega}\det\nabla u_j(x)\,\mathrm{d}x$$
$$=\lim_{j\to\infty}\int_{B(0,1)\cap\Omega}j^n\det\nabla(u(jx)\,\mathrm{d}x=\int_{D_\varrho}\det\nabla u(y)\,\mathrm{d}y<0\ .$$

3.1. Understanding condition (ii) in Theorem 3.3. As already outlined above, condition (ii) in Theorem 3.3 is rather implicit and thus hard to verify. Nevertheless, as Examples 3.1 and 3.2 show, it should be linked to concentrations on the boundary of the domain. To our best knowledge, this link has been fully drawn only in the case $l=1$ and for integrands $v(x,u,\nabla u):=v(x,\nabla u)$.

First, we present a result showing that indeed concentrations are the key issue.

Theorem 3.4 (taken from [65]). *Let $v \in C(\bar{\Omega} \times \mathbb{R}^{m\times n})$, $|v| \le C(1 + |\cdot|^p)$, $C > 0$, $v(x,\cdot)$ quasiconvex for almost all $x \in \Omega$, and $1 < p < +\infty$. Then the functional $I(u) := \int_\Omega v(x, \nabla u(x))\,\mathrm{d}x$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ if and only if for any bounded sequence $\{u_j\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\nabla u_j \to 0$ in measure we have $\liminf_{j\to\infty} I(u_j) \ge I(0)$.*

Recall that two effects may cause a sequence $\{u_j\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ to converge weakly but not strongly to some limit function u: *oscillations* and *concentrations*. The above theorem then states that a functional with a quasiconvex integrand is lower semicontinuous along any weakly converging sequences if it is so along *purely concentrating* ones. Indeed, realize that a purely concentrating sequence converges to zero in measure.

The next theorem shows that weak lower semicontinuity of (1.2) for quasiconvex v can be proved, for example, if the negative part of v has sub-critical growth or Dirichlet boundary conditions are fixed.

Theorem 3.5 (taken from [65]). *Let the assumptions of Theorem 3.4 hold. Let further $\{u_j\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$, $u_j \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and at least one of the following conditions be satisfied:*

(i) for any subsequence of $\{u_j\}$ (not relabeled) such that $|\nabla u_j|^p \overset{*}{\rightharpoonup} \sigma$ in $\mathcal{M}(\bar{\Omega})$ it holds that $\sigma(\partial\Omega) = 0$,

(ii) $\lim_{|A|\to\infty} \frac{v^-(x,A)}{1+|A|^p} = 0$ for all $x \in \bar{\Omega}$ where $v^- := \max\{0, -v\}$,

(iii) $u_j = u$ on $\partial\Omega$ for any $j \in \mathbb{N}$ and Ω is Lipschitz.
Then $I(u) \le \liminf_{j\to\infty} I(u_j)$.

Notice that (ii) is satisfied for example, if $v \ge 0$ or if $v^- \le C(1 + |\cdot|^q)$ for some $1 \le q < p$ in which case $-C(1 + |A|^q) \le v(s) \le C(1 + |A|^p)$, $C > 0$. This result can be found e.g. in [31]. In 1990, Ball and Zhang [17] considered the following bound on a Carathéodory integrand $(x,(s,A)) \to v(x,s,A)$

$$|v(x,s,A)| \le a(x) + C(|s|^p + |A|^p)\ . \tag{3.1}$$

Under (3.1), we cannot expect weak lower semicontinuity of I along generic sequences. Indeed, they proved the following weaker result.

Theorem 3.6 (Ball and Zhang [17]). *Let $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$, $v(x,s,\cdot)$ be quasiconvex for all $s \in \mathbb{R}^m$ and almost all $x \in \Omega$, and let (3.1) hold. Then there exist a sequence of sets $\{\Omega_j\}_{j\in\mathbb{N}} \subset \Omega$ such that $\Omega_{j+1} \subseteq \Omega_j$ for all $j \ge 1$, and $\lim_{j\to\infty} \mathcal{L}^n(\Omega_j) = 0$ such that for all $j \ge 1$*

$$\int_{\Omega\setminus\Omega_j} v(x,u(x),\nabla u(x))\,\mathrm{d}x \le \liminf_{k\to\infty} \int_{\Omega\setminus\Omega_j} v(x,u_k(x),\nabla u_k(x))\,\mathrm{d}x\ . \tag{3.2}$$

We immediately see that if $v \ge 0$ then the statement holds for $\Omega_1 = \Omega_j = \emptyset$, i.e., that weak lower semicontinuity is recovered. The sets $\{\Omega_j\}$ which must be removed (bitten) from Ω are the sets where possible concentration effects of the bounded sequence $\{v(x,u_k,\nabla u_k)\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ take place. Thus, $\{\Omega_j\}$ depends on $\{u_k\}$ and Ω_j are not known apriori. The main tool of the proof is the Biting Lemma due to Chacon [25, 16].

Lemma 3.7 (Biting lemma). *Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set. Let $\{z_k\} \subset L^1(\Omega;\mathbb{R}^m)$ be bounded. Then there is a (non-relabeled) subsequence of $\{z_k\}$'s, $z \in L^1(\Omega;\mathbb{R}^m)$ and a nonincreasing sequence of sets $\{\Omega_j\}_{j\in\mathbb{N}} \subset \Omega$ with $|\Omega_j| \to 0$ for $j \to \infty$ such that $z_k \rightharpoonup z$ in $L^1(\Omega\setminus\Omega_j;\mathbb{R}^m)$ for $k \to \infty$ and any $j \in \mathbb{N}$.*

Let us return to the issue of understanding better condition (ii) in Theorem 3.3. It has been understood in [77] that a suitable *growth* from below of the whole functional in (1.4) (which does not necessarily imply a lower bound on the integrand v itself) equivalently replaces this condition. First, let us illustrate that some form of boundedness from below is indeed necessary for weak lower semicontinuity.

Example 3.3. *Take $u \in W^{1,p}_0(B(0,1);\mathbb{R}^m)$ $(1 < p < \infty)$ and extend it by zero to the whole of $\mathbb{R}^n$. Define for $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$ $u_j(x) = j^{\frac{n-p}{p}} u(jx)$, i.e., $u_j \rightharpoonup 0$ in $W^{1,p}(B(0,1);\mathbb{R}^m)$ and consider a smooth* *domain $\Omega \in \mathbb{R}^n$ such that $0 \in \partial\Omega$; denote by ρ the outer unit normal to $\partial\Omega$ at 0. Moreover, take a function $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ that is positively p-homogeneous, i.e., $v(\alpha\xi) = \alpha^p v(\xi)$ for all $\alpha \geq 0$. If*

$$I(u) = \int_\Omega v(\nabla u(x))\,\mathrm{d}x$$

is weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ then

$$\begin{aligned}(3.3)\qquad 0 = I(0) &\leq \liminf_{j\to\infty} \int_\Omega v(\nabla u_j(x))\,\mathrm{d}x = \liminf_{j\to\infty} \int_{B(0,1/j)\cap\Omega} v(\nabla u_j(x))\,\mathrm{d}x \\ &= \liminf_{j\to\infty} \int_{B(0,1/j)\cap\Omega} j^n v(\nabla u(jx))\,\mathrm{d}x = \int_{D_\rho} v(\nabla u(y))\,\mathrm{d}y\ .\end{aligned}$$

Thus, we see that

$$(3.4)\qquad 0 \leq \int_{D_\rho} v(\nabla u(y))\,\mathrm{d}y$$

for all $u \in W_0^{1,p}(B(0,1);\mathbb{R}^m)$ forms a necessary condition for weak lower semicontinuity of I whenever v is positively p-homogeneous.*

For functions that are not p-homogeneous, S. Krömer [77] generalized (3.4) as follows.

Definition 3.8 (following [77][5]). *Assume that $\Omega \subset \mathbb{R}^n$ has a smooth boundary and let ρ be a unit outer normal to $\partial\Omega$ at x_0. We say that a function $v : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is of* p-quasi-subcritical growth from below *(p-qsb) if for every $x_0 \in \partial\Omega$*

for every $\varepsilon > 0$, there exists $C_\varepsilon \geq 0$ such that

$$(3.5)\qquad \int_{D_\rho} v(x_0,\nabla u(x))\,dx \geq -\varepsilon \int_{D_\rho} |\nabla u(x)|^p\,dx - C_\varepsilon \quad \text{for all } u \in W_0^{1,p}(B(0,1);\mathbb{R}^m),$$

It has been proved in [77] that the p-quasi-subbcritical growth from below of the function $v := v(x,\nabla u)$ equivalently replaces (ii) in Theorem 3.3.

Notice that (3.5) is expressed only in terms of v and that it is local in x. It also shows that, at least in the case when v does depend only on the first gradient of u but not on u itself, only *concentrations at the boundary* may interfere with weak lower semicontinuity of functionals involving quasiconvex functions. This means that concentrations inside of the domain Ω are already "taken care of" by the quasiconvexity itself.

Remark 3.4. *Let us realize that* (3.5) *implies* (3.4) *if v is positively p-homogeneous and independent of x. To this end, we use, for $t \geq 0$, $u = t\tilde{u}$ in* (3.5) *to see that*

$$0 \leq \frac{1}{t^p}\left(\int_{D_\varrho} v(t\nabla\tilde{u}(x))\,dx + \varepsilon|t\nabla\tilde{u}(x)|^p\mathrm{d}x + C_\varepsilon\right).$$

Letting now $t \to \infty$ gives that $C_\varepsilon = 0$. Then, we may also send $\varepsilon \to 0$ to get (3.4)*.*

Since only concentration effects play a role for (ii) in Theorem 3.3, it is natural to use the *recession function* of the function v, if it admits one. Recall, that we say that the functions $v_\infty : \overline{\Omega} \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a recession function for $v : \overline{\Omega} \times \mathbb{R}^{m\times n} \to \mathbb{R}$ if

$$\lim_{|A|\to\infty} \frac{v(x,A) - v_\infty(x,A)}{|A|^p} = 0.$$

[5]In [77] this condition is actually not referred to as p-quasi-subcritical growth from below but is introduced in Theorem 1.6 (ii).

Note that v_∞ is necessarily positively p-homogeneous; i.e. $v_\infty(x,\lambda p) = \lambda^p v_\infty(x,p)$ for all $\lambda \geq 0$.

It follows from Remark 3.9 in [77] that if v admits a recession function, then quasi-subcritical growth from below is equivalent to (3.4) for v_∞.

Since weak lower semicontinuity is connected to quasiconvexity and to condition (ii) in Theorem 3.3 which is connected to effects at the boundary, it is reasonable to ask whether the two ingredients can be combined. Indeed, so-called *quasiconvexity at the boundary* was introduced in [13] to study necessary conditions satisfied by local minimizers of variational problems – we also refer to [107, 85, 105] where this condition is analyzed, too. In order to define quasiconvexity at the boundary, we put for $1 \leq p \leq +\infty$

$$W^{1,p}_{\Gamma_\rho}(D_\rho;\mathbb{R}^m) := \{u \in W^{1,p}(D_\rho;\mathbb{R}^m);\ u = 0 \text{ on } \partial D_\rho \setminus \Gamma_\rho\}\ , \tag{3.6}$$

where Γ_ρ is the planar part of ∂D_ρ.

Definition 3.9 (taken from [13]). *Let $\varrho \in \mathbb{R}^n$ be a unit vector. A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called quasiconvex at the boundary at the point $A \in \mathbb{R}^{m\times n}$ with respect to ρ if there is $q \in \mathbb{R}^m$ such that for all $\varphi \in W^{1,\infty}_{\Gamma_\rho}(D_\rho;\mathbb{R}^m)$ it holds*

$$\int_{\Gamma_\rho} q\cdot\varphi(x)\,\mathrm{d}S + f(A)|D_\varrho| \leq \int_{D_\rho} f(A+\nabla\varphi(x))\,\mathrm{d}x\ . \tag{3.7}$$

Let us remark that, analogously to quasiconvexity, we may generalize quasiconvexity at the boundary to $W^{1,p}$-quasiconvexity at the boundary (for $1 < p < \infty$) by using all $u \in W^{1,p}_{\Gamma_\varrho}(D_\rho;\mathbb{R}^m)$ as test functions in (3.7). For functions with p-growth these two notions coincide.

Remark 3.5. *Let us give an intuition on the above definition. Take a smooth convex function v : $\mathbb{R}^{m\times n} \to \mathbb{R}$, then we know that*

$$v(A+\nabla\varphi(x)) \geq \phi(A) + \frac{\mathrm{d}v}{\mathrm{d}A}(A) : \nabla\varphi(x);$$

integrating this expression over Ω then gives

$$\int_\Omega v(A+\nabla\varphi(x))\mathrm{d}x \geq \int_\Omega v(A) + \frac{\mathrm{d}v}{\mathrm{d}A}(A) : \nabla\varphi\mathrm{d}x = |\Omega|v(A) + \int_{\partial\Omega}\Big(\frac{\mathrm{d}v}{\mathrm{d}A}(A)\nu\Big)\cdot\varphi\mathrm{d}S,$$

where ν is the outer normal to $\partial\Omega$. Now when setting $q := \frac{\mathrm{d}v}{\mathrm{d}A}(A)$ we obtained the definition of the quasiconvexity at the boundary. Notice also that if φ is zero at the whole boundary we recover the definition of classical quasiconvexity, too.

Remark 3.6. *It is possible to work with more general domains that half-balls in Definition 3.9; namely with so-called* standard boundary domains. *We say that $\tilde{D}_\rho$ is a standard boundary domain with the normal ρ if there is $a \in \mathbb{R}^n$ such that $\tilde{D}_\rho \subset H_{a,\rho} := \{x \in \mathbb{R}^n;\ \rho\cdot x < a\}$ and the $(n-1)$- dimensional interior Γ_ρ of $\partial\tilde{D}_\rho \cap \partial H_{a,\rho}$ is nonempty. Roughly speaking, this means that the boundary of $\tilde{D}_\rho$ should contain a planar part.*

As with standard quasiconvexity, if (3.7) holds for one standard boundary domain it holds for other standard boundary domains, too.

Remark 3.7. *If $p > 1$, $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is positively p-homogeneous*[6]*, continuous, and $W^{1,p}$-quasiconvex at the boundary at $(0,\rho)$ then $q = 0$ in (3.7). Indeed, we have $v(0) = 0$ and suppose that $\int_{D_\rho} v(\nabla u(x))\,\mathrm{d}x < 0$ for some $u \in W^{1,\infty}_{\Gamma_\rho}(D_\rho;\mathbb{R}^m)$. By (3.7), we must have for all $\lambda > 0$*

$$0 \leq \lambda^p \int_{D_\rho} v(\nabla u(x))\,\mathrm{d}x - \lambda\int_{\Gamma_\rho} q\cdot u(x)\,\mathrm{d}S\ .$$

[6]Recall that this means that $v(\lambda A) = \lambda^p v(A)$ for all $A \in \mathbb{R}^{m\times n}$

However, this is not possible for $\lambda > 0$ large enough and therefore for all $\varphi \in W^{1,\infty}_{\Gamma_\varrho}(D_\varrho;\mathbb{R}^m)$ it holds that $\int_{D_\varrho} v(\nabla\varphi(x))\,\mathrm{d}x \geq 0$. Thus, we can take $q = 0$.

From the above remark and from (3.4), we have the following lemma:

Lemma 3.10. *If a function $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ that is $W^{1,p}$-quasiconvex at the boundary at zero and every $\varrho \in \mathbb{R}^n$ a unit normal vector to $\partial\Omega$ then it is also of p-subcritical growth from below. The two notions become equivalent if v is also positively p-homogeneous. Here Ω must have a smooth boundary.*

3.2. Integrands bounded from below. As already mentioned, condition (ii) in Theorem 3.3 is automatically satisfied if the integrand in (1.4) is bounded from below. Moreover, in this case, the continuity assumptions stated in Definition 3.2 can be considerably weakened. In fact, the Carathéodory property is sufficient in case $k = 1$ in (1.4) as the following famous result due to E. Acerbi and N. Fusco [1] shows.

Theorem 3.11 (Acerbi and Fusco [1]). *Let $k = 1$, $\Omega \subset \mathbb{R}^n$ be an open, bounded set, and let $v : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to [0;+\infty)$ be a Carathéodory integrand, i.e., $v(\cdot, s, A)$ is measurable for all $(s, A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n}$ and $v(x, \cdot, \cdot)$ is continuous for almost all $x \in \Omega$. Let further $v(x, s, \cdot)$ be quasiconvex for almost all $x \in \Omega$ and all $s \in \mathbb{R}^m$, and suppose that for some $C > 0$, $1 \leq p < +\infty$, and $a \in L^1(\Omega)$ we have that*

$$0 \leq v(x, s, A) \leq a(x) + C(|s|^p + |A|^p)\ . \tag{3.8}$$

Then $I : W^{1,p}(\Omega;\mathbb{R}^m) \to [0;+\infty)$ given in (1.2) is weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$.

Interestingly, the paper by Acerbi and Fusco [1] already implicitly contained a result known in literature as the *Decomposition lemma*; cf. Lemma 3.12 below. Later on, it was proved by Kristensen in [74] and by Fonseca, Müller, and Pedregal in [45].

Lemma 3.12 (Decomposition lemma). *Let $1 < p < +\infty$ and $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ be bounded. Then there is a subsequence $\{u_j\}_{j\in\mathbb{N}}$ and a sequence $\{z_j\}_{j\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that*

$$\lim_{j\to\infty} |\{x \in \Omega;\ z_j(x) \neq u_j(x) \text{ or } \nabla z_j(x) \neq \nabla u_j(x)\}| = 0 \tag{3.9}$$

and $\{|\nabla z_j|^p\}_{j\in\mathbb{N}}$ is relatively weakly compact in $L^1(\Omega)$.

This lemma allows us, in the case of non-negative functionals, to replace a general minimizing sequence bounded in $W^{1,p}$ by the another one which is p-equiintegrable, i.e. for which $\{|\nabla z_j|^p\}$ is relatively weakly compact in $L^1(\Omega)$. Thus, we decompose $u_k = z_k + w_k$, where $w_k := u_k - z_k$ and tends to zero in measure for $k \to \infty$; i.e. it is a purely concentrating sequence. Roughly speaking, this means that any weakly converging sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$ can be decomposed into a purely oscillating and a purely concentrating one. Notice that Lemma 3.12 inherited its name exactly from this decomposition.

However, it is unclear whether the lemma can be generalized if the sign on the determinant should be preserved. This is particularly important in elasticity–see also Section 6.

Open problem 3.13. *Can one choose $\{z_j\}$ in the decomposition lemma in such a way that $\det\nabla z_j$ and $\det\nabla u_j$ have the same sign almost everywhere in Ω for all $j \in \mathbb{N}$.*

Marcellini [81] proved, by a different technique of constructing a suitable non-decreasing sequence of approximations, a very similar result to Theorem 3.11 allowing also for a slightly more general growth condition

$$-c_1|A|^r - c_2|s|^t - c_3(x) \leq v(x, s, A) \leq g(x, s)\big(1 + |A|^p\big), \tag{3.10}$$

where $c_1, c_2 \geq 0$, $c_3 \in L^1(\Omega)$; g is Carathéodory but otherwise arbitrary and for the exponents we have that $p \geq 1$, $1 \leq r < p$ (but $r = 1$ if $p = 1$) and if $p < n$ $1 \leq t < np/(n-p)$, otherwise $t \geq 1$.

Note that the growth condition (3.10) actually allows for integrands unbounded from below but the exponent r determining this growth is strictly smaller than p. Such integrands are of *sub-critical growth* and for integrand of the class $\mathcal{F}_p(\Omega)$ weak lower semicontinuity under this growth follows also from Theorem 3.5(ii).

Acerbi and Fusco [1, p. 127] remarked that "*...using more complicated notations as in [11], [84], our results can be extended to the case of functionals of the type* (1.4)".This extension has been considered by Fusco [50] for the case $p = 1$ an later by Guidorzi and Poggilioni [54] who rewrote functional (1.4) as (using the notation from Section 2)

$$I(u) = \int_\Omega v(x, \nabla^{[k-1]}u(x), \nabla^k u(x))\mathrm{d}x \tag{3.11}$$

and proved the following.

Proposition 3.14 ([54]). *Let $v : \Omega \times Y(n, m, k-1) \times X(n, m, k) \to \mathbb{R}$ be a Carathéodory k-quasiconvex function satifying*

$$0 \le v(x, H, A) \le g(x, H)(1 + |A|)^p$$
$$|v(x, H, A) - v(x, H, B)| \le C(1 + |A|^{p-1} + |B|^{p-1})|A - B|$$

where g is a Carathéodory function and $C \ge 0$. Then the functional from (3.11) *is weakly lower semicontinuous in $W^{k,p}(\Omega; \mathbb{R}^n)$ for $1 \le p < \infty$ and $k \in \mathbb{N}$.*

Note that in this result the continuity of the integrand in the space variable x could be omitted, which is, roughly speaking, due to the fact that quasiconvexity is enough to handle the concentration effects. On the other hand, the continuity assumption from Definition 3.2(ii) still remains present (with $\gamma = 1$). A similar result can be drawn from the more general setting of $\mathcal{A}$-quasiconvexity (which we review in Section 8 below) considered in [23].

While the above results handle also weak lower semicontinuity on $W^{k,1}(\Omega; \mathbb{R}^m)$ with respect to the standard weak convergence of this space, it is more suitable to investigate lower semicontinuity with respect to the strong convergence in $W^{k-1,1}(\Omega; \mathbb{R}^m)$. This is due to the fact that $W^{k,1}(\Omega; \mathbb{R}^m)$ is not reflexive and therefore suitable coercivity of (1.4) does not allow us to select a minimizing sequence that would be weakly convergent in $W^{k,1}(\Omega; \mathbb{R}^m)$ but the strong convergence in $W^{k-1,1}(\Omega; \mathbb{R}^m)$ can be assured.

The case for $k = 1$ was treated by Fonseca and Müller [43] who considered continuous integrands under mild growth conditions. The result was later generalized by Fonseca, Leoni, Malý and Paroni [42] not only with respect to the continuity of the integrand that could be partially dropped, but also to arbitrary k. We give the result in Theorem 3.15.

Theorem 3.15 (taken from [42]). *Let v in* (1.4) *be a Borel integrand that is moreover continuous in the following sense: For all $\varepsilon > 0$ and $(x_0, H_0) \in \Omega \times Y(n, m, k-1)$ there exist $\delta > 0$ and a modulus of continuity $\rho(s)$ with the property that, for some $C > 0$, $\rho(s) \le C(1 + s), s > 0$ such that*

$$v(x_0, H_0, A) - v(x, H, A) \le \varepsilon(1 + v(x, H, A)) + \rho(|H_0 - H|),$$

for all $x \in \Omega$ satisfying $|x - x_0| \le \delta$ and for all $H \in Y(n, m, k-1)$ and all $A \in X(n, m, k)$. Suppose further that v is k-quasiconvex and satisfies[7]

$$\frac{1}{c}|A| - c \le v(x_0, H_0, A) \le c(1 + |A|),$$

for some $c > 0$ and all $A \in X(n, m, k)$.

Then, v in (1.4) *is lower semicontinuous with respect to the strong convergence in $W^{k-1,1}(\Omega; \mathbb{R}^m)$.*

For the functions $v : X(m, n, k) \to \mathbb{R}$, i.e. those depending only on the highest gradient, an analogous result has been obtained in [4].

[7] If $k = 1$ the growth condition can be relaxed to

$$0 \le v(x_0, s, A) \le c(1 + |A|) \qquad \forall A \in \mathbb{R}^{m \times n}$$

and it can be even omitted if v is convex in its last variable.

4. Null Lagrangians. In this section, we study under which conditions on v the functional (1.4) is not only weakly lower semicontinuous but actually *weakly continuous* in $W^{k,p}(\Omega;\mathbb{R}^m)$. This question is tightly connected (cf. Theorem 4.3 below) to the study of so-called null Lagrangians. We start the discussion by presenting definitions of null Lagrangians of the first and higher order.

DEFINITION 4.1. *We say that a continuous map $L:\mathbb{R}^{m\times n}\to\mathbb{R}$ is a null Lagrangian of the first order, if for every $u\in C^1(\bar\Omega;\mathbb{R}^m)$ and every $\varphi\in C^1_0(\Omega;\mathbb{R}^m)$ it holds that*

$$\int_\Omega L(\nabla(u(x)+\varphi(x)))\,\mathrm{d}x=\int_\Omega L(\nabla u(x))\,\mathrm{d}x\ . \tag{4.1}$$

Notice that the definition is independent of the particular Lipschitz domain Ω. In fact, if (4.1) holds for one domain Ω it also holds for all other (Lipschitz) domains.

REMARK 4.1. *The name "null Lagragians" comes from the fact that, if L is even smooth so that the Gateaux derivative of $J(u):=\int_\Omega L(\nabla u(x))\,\mathrm{d}x$ can be evaluated, it easily follows from* (4.1) *that J satisfies $J'(u)=0$ for all $u\in C^1(\bar\Omega;\mathbb{R}^m)$. In other words, the Euler-Lagrange equations of J are fulfilled identically in the sense of distributions.*

REMARK 4.2. *Let us notice that, if L is a null Lagrangian, the value of $J(u)=\int_\Omega L(\nabla u(x))\,\mathrm{d}x$ is only dependent on the boundary values of $u(x)$. This can be seen from* (4.1) *as the value remains unchanged even if we add arbitrary functions vanishing on the boundary.*

It is straightforward to generalize (4.1) also to higher order problems.

DEFINITION 4.2. *Let $k\ge 2$. We say that $L:X\to\mathbb{R}$ is a (higher-order) null Lagrangian if*

$$\int_\Omega L(\nabla^k(u(x)+\varphi(x))\,\mathrm{d}x=\int_\Omega L(\nabla^k(u(x))\,\mathrm{d}x \tag{4.2}$$

for all $u\in C^k(\bar\Omega;\mathbb{R}^m)$ and all $\varphi\in C^k_0(\Omega;\mathbb{R}^m)$.

Similarly as in the first-order gradient case, the definition is independent of the particular (Lipschitz) domain Ω. In the same way as in the first order case, it follows that Euler-Lagrange equations

$$\sum_{|K|\le l}(-D)^K\frac{\partial L}{\partial u^i_I}(\nabla^l u)=0 \tag{4.3}$$

are satisfied in the sense of distributions for arbitrary $u\in C^k(\bar\Omega;\mathbb{R}^m)$.

REMARK 4.3. *It is natural to generalize the notion of null Lagrangians to functionals of the type* (1.4)*, i.e. those depending also on lower order gradients, in the following way: We say that the function $L:\Omega\times Y(n,m,k)\to\mathbb{R}$ is a null Lagrangian for the functional* (1.4) *if for all $u\in C^k(\Omega;\mathbb{R}^m)$ and all $\varphi\in C^k_0(\Omega;\mathbb{R}^m)$ it holds that*

$$J(u+\varphi)=J(u)\qquad\textit{and}\qquad J(u)=\int_\Omega L(x,u(x),\nabla u(x),\dots,\nabla^k u(x))\,\mathrm{d}x.$$

We shall see in the end of the section that null Lagrangians for these types of functionals are actually determined by higher order null Lagrangians at least if $k=1$.

The following result characterizes null Lagrangians (of first and higher order) by means of a few equivalent statements. In particular, it shows that null Lagrangians are the only integrands along which $\int_\Omega v(\nabla^l(u(x))\,\mathrm{d}x$ is weakly continuous. It is taken from [11].

THEOREM 4.3 (Characterization of (higher-order) null Lagrangians). *Let $L:X(n,m,k)\to\mathbb{R}$ be continuous. Then the following statements are mutually equivalent:*

(i) L is a null Lagrangian,

(ii) $\int_\Omega L(A+\nabla^k\varphi(x))\,\mathrm{d}x = \int_\Omega L(A)\,\mathrm{d}x$ *for every* $\varphi \in C_0^\infty(\Omega;\mathbb{R}^m)$ *and every* $A \in X(n,m,k)$ *and every open subset* $\Omega \subset \mathbb{R}^n$,

(iii) L *is continuously differentiable and* (4.3) *holds in the sense of distributions,*

(iv) *The map* $u \mapsto L(\nabla^k u)$ *is sequentially weakly* continuous from* $W^{k,\infty}(\Omega;\mathbb{R}^m)$ *to* $L^\infty(\Omega)$*. This means that if* $u_j \overset{*}{\rightharpoonup} u$ *in* $W^{k,\infty}(\Omega;\mathbb{R}^m)$ *as* $j\to\infty$ *then* $L(\nabla^k u_j) \overset{*}{\rightharpoonup} L(\nabla^k u)$ *in* $L^\infty(\Omega)$,

(v) L *is a polynomial of degree* p *and the map* $u \mapsto L(\nabla^k u)$ *is sequentially weakly* continuous from* $W^{k,p}(\Omega;\mathbb{R}^m)$ *to* $\mathcal{D}'(\Omega)$*. This means that if* $u_j \rightharpoonup u$ *in* $W^{k,p}(\Omega;\mathbb{R}^m)$ *as* $j\to\infty$ *then* $L(\nabla^k u_j) \rightharpoonup L(\nabla^k u)$ *in* $\mathcal{D}'(\Omega)$,

While Theorem 4.3 provides us with very useful properties of null Lagrangians it is interesting to note that that they are known *explicitely* in the first as well as in the higher order. In fact, null Lagrangians are formed by minors or sub-determinants of the gradient entering the integrand in J.

4.1. Explicit characterization of null Lagrangians of the first order. Let us start with the first order case: If $A \in \mathbb{R}^{m\times n}$ we denote by $\mathbb{T}_i(A)$ the vector of all subdeterminants of A of order i for $1 \le i \le \min(m,n)$. Notice that the dimension of $\mathbb{T}_i(A)$ is $d(i) := \binom{m}{i}\binom{n}{i}$, hence the the number of all subdeterminants of A is $\sigma := \binom{m+n}{n} - 1$. Finally, we write $\mathbb{T} := (\mathbb{T}_1,\ldots,\mathbb{T}_{\min(m,n)})$. For example, if $m=1$ or $n=1$ then $\mathbb{T}(A)$ consists only of entries of A, if $m=n=2$ then $\mathbb{T}(A)=(A,\det A)$ and for $m=n=3$ we obtain $\mathbb{T}(A) = (A,\mathrm{Cof}A,\det A)$.

Clearly, linear maps are weakly continuous. Yet, it has been known at least since [87, 100, 7] that also minors have this property (see Theorem 4.4 below). This result, usually called *(sequential) weak continuity of minors* is unexpected because if $i>1$ then $A\mapsto \mathbb{T}_i(A)$ is a nonlinear polynomial of the i-th order. As it is well-known, weak convergence generically does not commute with nonlinear mappings.

Theorem 4.4 (Weak continuity of minors/null Lagrangians (see e.g. [31])). *Let* $\Omega\subset\mathbb{R}^n$ *be a bounded Lipschitz domain. Let* $1\le i\le \min(m,n)$*. Let* $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ *be such that* $u_k \rightharpoonup u$ *in* $W^{1,p}(\Omega;\mathbb{R}^m)$ *for* $p>i$*. Then* $\mathbb{T}_i(\nabla u_k)\rightharpoonup \mathbb{T}_i(\nabla u)$ *in* $L^{p/i}(\Omega;\mathbb{R}^{d(i)})$*.*

It follows from Theorem 4.7 below (see also [31]) that minors are the only mappings depending exclusively on ∇u which have this property. Thus in view of Theorem 4.3, any null Lagrangian can be written as an affine combination of elements of $\mathbb{T}$, i.e., for any $A\in\mathbb{R}^{m\times n}$

$$L(A) = c_0 + c\cdot \mathbb{T}(A)\ ,$$

where $c_0\in\mathbb{R}$ and $c\in\mathbb{R}^\sigma$ are arbitrary constants. Let us note however, that it has been realized independently in e.g. [37, 38] that minors are the only maps for which the Euler-Lagrange equation of $J(u)=\int_\Omega L(\nabla u)\mathrm{d}x$ is satisfied identically.

As we saw in Example 3.2, Theorem 4.4 fails if $p=i$. Nevertheless, the results can be much improved if we additionally assume that for every $k\in\mathbb{N}$ $\mathbb{T}_i(\nabla u_k)\ge 0$ almost everywhere in Ω. Indeed, Müller [90] proved the following result.

Proposition 4.5 (Higher integrability of determinant). *Assume that* $\omega\subset\Omega\subset\mathbb{R}^n$ *is compact,* $u\in W^{1,n}(\Omega;\mathbb{R}^n)$*, and that* $\det\nabla u\ge 0$ *almost everywhere in* Ω*. Then*

$$\|(\det\nabla u)\ln(2+\det\nabla u)\|_{L^1(\omega)} \le C(\omega,\|u\|_{W^{1,n}(\Omega;\mathbb{R}^n)}) \tag{4.4}$$

for some $C(\omega,\|u\|_{W^{1,n}(\Omega;\mathbb{R}^n)})>0$ *a constant depending only on* ω *and the Sobolev norm of* u *in* Ω*.*

This proposition results in the following Corollary:

Corollary 4.6 (Uniform integrability of determinant). *If* $\{u_k\}_{k\in\mathbb{N}}\subset W^{1,n}(\Omega;\mathbb{R}^n)$ *is bounded and* $\det\nabla u_k\ge 0$ *almost everywhere in* Ω *for all* $k\in\mathbb{N}$ *then* $\det\nabla u_k\rightharpoonup\det\nabla u$ *in* $L^1(\omega)$ *for every compact set* $\omega\subset\Omega$*.*

A related statement was achieved by Kinderlehrer and Pedregal in [67]. It says that under the assumptions of Corollary 4.6 and if $u_k=u$ on $\partial\Omega$ for all $k\in\mathbb{N}$ the statement of Corollary 4.6 holds for $\omega:=\Omega$.

4.2. Explicit characterization of null Lagrangians of higher order. Null-Lagrangians of higher order are of the same structure as those of the first order. Indeed, they also correspond to minors. In order to make the statement more precise, we assume that $K := (k_1, \ldots, k_r)$ is such that $1 \leq k_i \leq n$ and denote by $\alpha := (\nu_1, J_1; \nu_2, J_2; \ldots; J_r, \nu_r)$ with $|J_i| = k - 1$ and where $1 \leq \nu_i \leq m$. We define the k-th order Jacobian determinant $J_K^\alpha : X \to \mathbb{R}$ by the formula

$$J_K^\alpha(\nabla u) = \frac{\partial(\partial u_{J_1}^{\nu_1}, \ldots, \partial u_{J_r}^{\nu_r})}{\partial(x^{k_1}, \ldots, x^{k_r})} = \det\left(\frac{\partial u_{J_i}^{\nu_i}}{\partial x^{k_j}}\right) .$$

Then any null Lagrangian of higher order is just an affine combination of J_K^α, i.e.,

Theorem 4.7 (See [11]). *Let $L \in C(X(n, m, k))$. Then L is a null Lagrangian if and only if it is an affine combination of k-th order Jacobian determinant, i.e.,*

$$L = C_0 + \sum_{\alpha, K} C_K^\alpha J_K^\alpha$$

for some constants C_0 and C_K^α.

Remark 4.4. *The maximum degree of nonzero $J_K^\alpha(\nabla^k y)$ is denoted by R. It can be shown that $R = \min(m, n)$ if $k = 1$ and $R := n$ for $k > 1$.*

4.3. Generalization to functionals (1.4)**.** As pointed out in Remark 4.3, the notion of null Lagrangians can be generalized also to functionals of the type (1.4); i.e. those containing also lower order terms. A characterization of these null Lagrangians is due to Olver and Sivaloganathan [94] who considered the first order case; i.e., null Lagrangians for those functionals which can also depend on x and u.

Based on Olver's results [93], they showed in [94] that such null Lagrangians are given by the formula

$$\tilde{L}(x, u, \nabla u) = C_0(x, u) + \sum_i C_i(x, u)\mathbb{T}_i(\nabla u) ,$$

where C_0 is a real-valued C^1-function and C_i are C^1-functions of its arguments for $1 \leq i \leq \min(m, n)$ with values in $R^{d(i)}$, $i > 0$. This means that they are determined by the already known null Lagrangians of the first order. Let us remark, that it is noted in [94] that the result generalizes analogously to the higher order case.

5. Polyconvexity and applications to hyperelasticity. We saw that, at least for integrands bounded from below and satisfying (i) in Definition 3.2, quasiconvexity is an equivalent condition for weak lower semicontinuity. This presents an *explicit* characterization of the latter since it is not necessary to examine all weakly converging sequences. Nevertheless, in practice quasiconvexity is almost impossible to verify since, in a sense, its verification calls for solving a minimization problem itself. Therefore, it is desirable to find at least *sufficient* conditions for weak lower semicontinuity that can be easily verified. Such a notion, called *polyconvexity* introduced by J.M. Ball, can be designed by employing the null Lagrangians introduced in the last section.

We start with the definition of polyconvexity for first order functionals $I(u) = \int_\Omega v(\nabla u(x))\mathrm{d}x$.

Definition 5.1 (Due to [7]). *We say that $v : \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ is polyconvex if there exists a convex function $h : \mathbb{R}^\sigma \to \mathbb{R} \cup \{+\infty\}$ such that $v(A) = h(\mathbb{T}(A))$[8] for all $A \in \mathbb{R}^{m\times n}$.*

Remark 5.1. *Interestingly, already Morrey in [87, Thm. 5.3] proved that one-homogeneous convex functions depending on minors are quasiconvex.*

It is a straigthforward idea to generalize polyconvexity to higher-order variational problems, i.e., those that depend on higher-order gradients of a mapping. The attractiveness of such problems for applications is clear. Suitably chosen terms depending on higher-order gradients allow for compactness of a minimizing

[8]Recall that $\mathbb{T}(A)$ denotes the vector of all minors of A.

sequence in some stronger topology than the weak one on $W^{1,p}$ which enable us to pass to a limit in lower-order terms without restrictive assumptions on their convexity properties. Thus, for example, models of shape memory alloys can be treated by this approach; cf. e.g. [91, 92].

Thus, we extend the notion of polyconvexity to higher order problems (1.4) and it employs the notion of null Lagrangians of higher order and is due to to Ball, Currie, and Olver [11].

Definition 5.2 (Higher-order polyconvexity). *Let $1 \le r \le R$ where R is defined in Remark 4.4. Let $U \subset X(n,m,k)$ be open and $\mathrm{Co}(U)$ its convex hull. A function $G : U \to \mathbb{R}$ is r-polyconvex if there exists a convex function $h : \mathrm{Co}(J^{[r]}(U)) \to \mathbb{R}$ such that $v(A) = h(J^{[r]}(A))$ for all $A \in U$. G is polyconvex if it is R-polyconvex. Here $J^r(H) := (J^{r,1}(H), \dots, J^{r,N_r}(H))$ is a N_r-tuple with the property that any Jacobian determinant of degree r can be written as a linear combination of elements of J^r. Consequently, $J^{[r]} := (J^1, \dots, J^r)$.*

If h above is affine then we call v polyaffine. In this case, $v(A)$ is a linear combination of all minors of A and a real constant. Consequently, any polyconvex function is bounded from below by a polyaffine function. Similarly, as in the convex case, a polyconvex function is found by forming the supremum of all polyaffine functions lying below it see e.g. [31, Rem. 6.7]; i.e., we have the following lemma.

Lemma 5.3. *The function $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is polyconvex if and only if*

$$v(A) = \sup\{\varphi(A); \varphi \text{ polyaffine and } \varphi \le v\}.$$

Since polyconvexity implies qusiconvexity, we may deduce by the results in Section 3 that polyconvex functions in the class $\mathcal{F}_p(\Omega)$ (from Definition 3.2) are weakly lower semicontinuous. Yet, weak lower semicontinuity can be proved for wider class of polyconvex functions than those in $\mathcal{F}_p(\Omega)$; in particular, the functions do not have to be of p-growth. This is of great importance in elasticity as explained later in this section.

The proof of weak lower semicontinuity of polyconvex functions is actually based on *convexity* and weak continuity of null Lagrangians. Thus, since weak lower semicontinuity can be shown for arbitrarily growing convex functions, it generalizes to polyconvex ones, too. The following result for convex functions can be found in [11, Thm. 5.4] and is based on results by Eisen [36] who proved this theorem for $\Phi < +\infty$.

Theorem 5.4 (weak lower semicontinuity). *Let $\Phi : \Omega \times \mathbb{R}^s \times \mathbb{R}^\sigma \to \mathbb{R} \cup \{+\infty\}$ satisfy the following properties*

(i) $\Phi(\cdot, z, a) : \Omega \to \mathbb{R} \cup \{+\infty\}$ is measurable for all $(z,a) \in R^s \times \mathbb{R}^\sigma$,
(ii) $\Phi(x, \cdot, \cdot) : \mathbb{R}^s \times \mathbb{R}^\sigma \to \mathbb{R} \cup \{+\infty\}$ is continuous for almost every $x \in \Omega$,
(iii) $\Phi(x, z, \cdot) : \mathbb{R}^\sigma \to \mathbb{R} \cup \{+\infty\}$ is convex.

Assume further that for all $(z,a) \in R^s \times \mathbb{R}^\sigma$ $\Phi(\cdot, z, a) \ge \phi$ for some $\phi \in L^1(\Omega)$. Let $z_k \to z$ almost everywhere in Ω and let $a_k \rightharpoonup a$ in $L^1(\Omega; R^\sigma)$. Then

$$\int_\Omega \Phi(x, z(x), a(x))\,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega \Phi(x, z_k(x), a_k(x))\,\mathrm{d}x\ .$$

Using this theorem, we may easily deduce weak lower semicontinuity of polyconvex functions. For the sake of clarity, let us start with first order problems. Then, consider $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ as $k \to \infty$ where $r > \min(m,n)$. Then $u_k \to u$ in $L^p(\Omega;\mathbb{R}^m)$, so, for a (non-relabeled) subsequence, even $u_k \to u$ almost everywhere in Ω. Hence, we can apply Theorem 5.4 with $z_k := u_k$, $a_k := \mathbb{T}(\nabla u_k)$ and $v(x, y, \nabla y) := \Phi(x, y, \mathbb{T}(\nabla y))$ to obtain the following corollary:

Corollary 5.5. *Let $v : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ satisfy the following properties*

(i) $v(\cdot, z, A) : \Omega \to \mathbb{R} \cup \{+\infty\}$ is measurable for all $(z, A) \in R^m \times \mathbb{R}^{m\times n}$,
(ii) $v(x, \cdot, \cdot) : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ is continuous for almost every $x \in \Omega$,
(iii) $v(x, z, A) = \Phi(x, z, \mathbb{T}(A))$ where Φ satisfies (i)–(iii) from Theorem 5.4.

If $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ as $k\to\infty$ where $r>\min(m,n)$ then

$$\int_\Omega v(x,u(x),\nabla u(x))\,\mathrm{d}x \le \liminf_{k\to\infty}\int_\Omega f(x,u_k(x),\nabla u_k(x))\,\mathrm{d}x\ .$$

Similarly as in the case of first order problems, we can exploit (v) of Theorem 4.3 and Theorem 5.4 to show the existence of minimizers to energy functionals (1.4). Let us present the result just for functionals (1.4) with $k=2$; generalizations for higher k are straightforward and can be found in [11].

Corollary 5.6 (after [11]). *Assume that $\Omega\subset\mathbb{R}^n$ is a bounded smooth domain and that $1\le r\le R$. Let $v:\Omega\times Y(n,m,2)\to\mathbb{R}\cup\{+\infty\}$ in*

$$I=\int_\Omega v(x,u,\nabla u,\nabla^2 u)\mathrm{d}x$$

satisfy the following assumptions:

(i) $v(x,H,A)=h(x,H,J^{[r]}(A))$, where $h(x,\cdot,\cdot):\mathbb{R}^{m\times n}\times J^{[r]}(X(n,m,2))\to\mathbb{R}\cup\{+\infty\}$ is continuous for almost every $x\in\Omega$,

(ii) $h(\cdot,H,J^{[r]}(A)):\Omega\to\mathbb{R}\cup\{+\infty\}$ is measurable for all $(H,J^{[r]}(A))$,

(iii) $h(x,H,\cdot):J^{[r]}(X)\to\mathbb{R}\cup\{+\infty\}$ is convex for almost all $x\in\Omega$ and all $F\in\mathbb{R}^{m\times n}$,

(iv) $W(x,H,A)\ge C(-1+|A|^p)$ for some $C>0$, $p>n$, almost all $x\in\Omega$ and all $A\in\mathbb{R}^{m\times n}$,

(v) let (5.18) hold for $u\in W^{2,p}(\Omega;\mathbb{R}^m)$.

Let further for some $u_0,u_1\in W^{2,p}(\Omega;\mathbb{R}^m)$

$$\mathcal{A}:=\{u\in W^{2,p}(\Omega;\mathbb{R}^m):\ u=u_0 \text{ on } \Gamma_\mathrm{D}\ ,\ \nabla u=\nabla u_1 \text{ on } \Gamma_\mathrm{D}\}\neq\emptyset$$

and such that $\inf_\mathcal{A}\mathcal{E}<+\infty$. Then there is a minimum of $\mathcal{E}$ on $\mathcal{A}$.

It is important to realize that main strength of polyconvexity consists in the fact that convexity in subdeterminants can be advantageously combined with the Mazur lemma to show weak lower semicontinuity is a way similar to the proof for mere convex and lower semicontinuous integrands. This contrasts with proofs available for quasiconvex integrands where manipulations with boundary conditions are usually needed to prove the result. It is already clearly visible in Meyers paper [84]. These manipulations, however, typically destroy any pointwise constraints on the determinant of ∇y, which, however, are crucial in elasticity. We shall return to this issue in Section 6.

5.1. Rank-1 convexity. Since polyconvexity is an explicit sufficient condition for quasiconvexity, we may ask if similarly a simpler necessary condition can be found. This is indeed so, the sought notion of convexity is *rank-1 convexity*:

Definition 5.7 (Due to [88]). *We say that $v:\mathbb{R}^{m\times n}\to\mathbb{R}$ is rank-1 convex if*

$$v(\lambda A_1+(1-\lambda)A_2)\le\lambda v(A_1)+(1-\lambda)v(A_2). \tag{5.1}$$

for all $\lambda\in[0,1]$ and all A_1, A_2 such that $\mathrm{rank}(A_1-A_2)\le 1$.

The relations among the introduced notions of convexity is as follows:

convexity ⇒ polyconvexity ⇒ quasiconvexity ⇒ rank-1 convexity;

however, none of the converse implications holds if $v:\mathbb{R}^{m\times n}\to\mathbb{R}$ and $m>2$ and $n\ge 2$. To see that polyconvexity does not imply convexity (even for $m,m>1$) just consider the function $v(F):=\det(F)$ which is even polyaffine but not convex. Also quasiconvexity does not imply polyconvexity even for $m,n>1$ as was shown in e.g. [3, 115]. Šverák's important counter example [112] is a construction of a function that is rank-1 convex, but not quasiconvex and holds for $m\ge 3$ and $n\ge 2$. For $m=2$ and $n\ge 2$ the question of equivalence between quasiconvexity and rank-1 convexity is still unsolved. On the other hand, it was shown in [119] that rank-one convexity coincides with quasiconvexity for quadratic forms in any dimension.

Open problem 5.8. *Let $m = 2$ and $n \geq 2$. Does rank-1 convexity imply quasiconvexity for $v : \mathbb{R}^{m\times n} \to \mathbb{R}$.*

Notice that, if $m = 1$ or $n = 1$ all the generalized notions of convexity trivially coincide with standard convexity itself.

An equivalent to rank-1 convexity can also be defined for higher-order problems–the corresponding notion is called Λ-convexity. Following [11], we define a nonconvex cone $\Lambda \subset X(n, m, k)$ as $\Lambda := \{a \otimes^l b : a \in \mathbb{R}^m,\ b \in \mathbb{R}^n\}$ where $(a \otimes^l b)^i_K = a^i b_K$.

Definition 5.9. *A function $f : X \to \mathbb{R}$ is called Λ-convex if $t \mapsto f(A + tB) : \mathbb{R} \to \mathbb{R}$ is convex for any $A \in X(n, m, k)$ and any $B \in \Lambda$.*

Notice that for $l = 1$ Λ-convexity coincides with rank-1 convexity. If f is twice continuously differentiable then Λ-convexity is equivalent to the Legendre-Hadamard condition

$$\sum_{j,k=1}^{m} \sum_{|J|=|K|=l} \frac{\partial^2 f(A)}{\partial A^j_J \partial A^k_K} a^j a^k b_J b_K \geq 0$$

for all $A \in X(n, m, k)$, $a \in \mathbb{R}^m$, and $b \in \mathbb{R}^n$.

Proposition 5.10 (see [11]). *Continuous and k-quasiconvex functions $f : X(n, m, k) \to \mathbb{R}$ are Λ-convex .*

Hence, Λ-convexity forms a necessary condition for (k-)quasiconvexity. This proposition was first proved by Meyers [84, Thm. 7] for smooth functions and then generalized in [11] to the continuous case. The opposite assertion does not hold. Indeed, if $n = l = 2$ and $m = 3$ then we have the following example due to Ball, Currie, and Olver for $f : X \to \mathbb{R}$

$$v(\nabla^2 u) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \frac{\partial^2 u^i}{\partial x_1^2} \frac{\partial^2 u^j}{\partial x_1 \partial x_2} \frac{\partial^2 u^k}{\partial x_3^2} \ .$$

This function is even Λ-affine (i.e., both $\pm v$ are Λ-convex) but not a null Lagrangian and it is not quasiconvex. As Λ-convexity replaces rank-one convexity in the current setting we see, that this example is a reminiscent of Šverák's example mentioned above.

5.2. Applications to hyperelasticity in the first order setting. In elasticity, one is interested in modeling the response of a rubber-like material to the action of applied outer forces. This response is obtained by solving a minimization problem; to be more specific, we are to minimize the free energy of the material. We will see that polyconvexity is perfectly fitted to the setting in elasticity and that existence of minimizers can be assured for polyconvex energies. We give a short introduction to this matter in this section and refer the reader e.g. to the monographs [55, 56, 107] for more details on the physical modeling.

Take a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ which, for $n = 3$, plays a role of a reference configuration of an elastic material. For given applied loads, we search for a mapping $y : \Omega \to \mathbb{R}^m$, the *deformation* of the material, which describes the new "shape" $y(\Omega)$ of the body. The mapping y is found by solving the following system of equations

$$-\operatorname{div} S = f \qquad \text{in } \Omega, \tag{5.2}$$

$$S\nu = g \qquad \text{on } \Gamma_{\mathrm{N}}, \tag{5.3}$$

$$y = y_0 \qquad \text{on } \Gamma_{\mathrm{D}}. \tag{5.4}$$

Here, (5.2) is the reduced version of Newton's law of motion for the (quasi)static case, f is the applied volume force. Further, (5.3) represents the action of applied surface forces g (ν denotes the outer unit normal vector to Γ_{N}) and (5.4) models that the body may be clamped at some part of the boundary to a prescribed shape y_0. We shall require that $\Gamma_{\mathrm{D}} \subset \partial\Omega$ is disjoint from Γ_{N} and of positive $(n-1)$-dimensional Lebesgue measure.

The material properties of the specimen are encoded in the first Piola-Kirchhoff stress tensor $S : \Omega \to \mathbb{R}^{m\times n}$ in (5.2) and (5.3). The form of the Piola-Kirchhoff stress tensor cannot be deduced from first principles within continuum mechanics[9] but has to be prescribed phenomenologically. The prescription for S is called the *constitutive relation* of the given material. In the easiest case, we assume the form $S(x) = \hat{S}(x, \nabla y(x))$ for some given $\hat{S}$. Materials for which this assumption is adequate are sometimes referred to a *simple* materials as opposed to non-simple materials for which $\hat{S}$ may depend also higher gradients of y. Later, in subsection 5.3, we will consider also these sophisticated constitutive relations.

Hyperelasticity is a part of elasticity where an *additional assumption* is made; namely, that S has a potential $W : \Omega \times \mathbb{R}^{3\times3} \to [0; +\infty]$ such that

$$S_{ij}(x) = \frac{\partial W(x, F)}{\partial F_{ij}}|_{F=\nabla y(x)} \ .$$

This assumption emphasizes the idea that there are no energy losses in elasticity and all work, made by external forces and/or Dirichlet boundary conditions, stored in the material can be fully exploited.

In the following, let us restrict our attention to *deformations of bulks*, i.e. we do not treat plates and rods, and set thus $m = n$. In order to fulfill the basic physical requirements, W has to satisfy the following relations:

$$W(x, RF) = W(x, F) \text{ for a.a. } x \in \Omega, \text{ all } z \in \mathbb{R}^n \text{ and for all } R \in \mathrm{SO}(n) \tag{5.5}$$

$$W(x, F) = +\infty \text{ for a.a. } x \in \Omega, \text{ all } z \in \mathbb{R}^n \text{ if } \det F \le 0, \text{ and} \tag{5.6}$$

$$\lim_{\det F \to 0_+} W(x, F) = +\infty \quad \text{for a.a. } x \in \Omega. \tag{5.7}$$

Indeed, assumption (5.5) is a consequence of the *axiom of frame indifference* [28]; in other words the assumptions assures that material properties are independent of the position of the observer. Conditions (5.6) and (5.7) ensure, respectively, that the material does not locally penetrate itself and that compression of a finite volume of the specimen into zero volume is not possible. These conditions, however, do not assure that the body does not penetrate through itself which is also natural to assume from a physical point of view. Nevertheless, we shall see in the end of this section that with additional assumptions on the growth of the energy and e.g. the boundary conditions even complete non-interpenetration can be assured.

The assumptions (5.5)-(5.7) rule out that $W(x, \cdot)$ can be convex. Moreover, due to (5.6)-(5.7) even if $W(x, \cdot)$ was quasiconvex, we could not apply the theorems in Section 3 since W cannot be an element of the class $\mathcal{F}_p(\Omega)$. Nevertheless, *polyconvexity* is fully compatible with these assumptions.

The mechanical model is that stable states of the system are found by minimizing the overall free energy

$$\mathcal{E}(y) = \int_\Omega W(x, \nabla y(x))\, \mathrm{d}x, \tag{5.8}$$

subject to (5.4). Smooth minimizers fulfill the balance equations (5.2)-(5.3); however, even in the smooth case there might exist solutions to (5.2)-(5.3) which are not minimizers of (5.8). Nevertheless, such solutions are thought to be metastable and hence left after a small perturbation. Thus, minimizing (5.8) is the proper way to find indeed stable states.

Remark 5.2. *Let us note that, since the minimizers of* (5.8) *might be non-smooth, it is not guaranteed that they will satisfy the Euler-Lagrange equations either in strong or weak form. Indeed, in [14] even one-dimensional examples of smooth W were given such that the minimizer does not fulfill the Euler-Lagrange equation.*

One of the reasons why deducing the Euler-Lagrange equation might be difficult is that even the calculation of the variation of $\mathcal{E}$ itself can pose difficulties. Indeed, due to (5.6)*, the minimizer y might be such that $\mathcal{E}(y+t\varphi)$ is infinite for all small enough $t > 0$ and a large class of φ. Let us refer to [14] for explicit examples in which this situation occurs.*

Remark 5.3. *Let us notice that the condition* (5.6) *is really necessarily to be stated explicitly. Namely, from the physical point of view, the frame-indifference* (5.5) *requires that $W(x, z, F) := \tilde{W}(x, C)$ where*

[9] Though it may be deduced from first principles when, e.g., working on a lattice of atoms and sending the number of atoms to infinity. If some cases, one may perform this *discrete-to-continuum* transition rigorously by means of a so-called Γ-limit; cf. e.g. [2, 24] or [79] for details

$C := F^\top F$ is the so-called right Cauchy-Green strain tensor. Note that $F^\top Q^\top QF = F^\top F$ for any orthogonal matrix Q. Hence, pointwise minimizers the energy density $W(x,z,\cdot)$ contain the set $\{QF_0 : Q \in \mathrm{O}(3)\}$ for some given matrix F_0 with $\det F_0 > 0$ which is a pointwise minimizer itself. Besides the physically acceptable energy wells $\{RF_0 : R \in SO(3)\}$ other minimizers live on a "dark" wells $\{RF_0 : R \in \mathrm{O}(3) \setminus \mathrm{SO}(3)\}$ which is not mechanically admissible. Those wells are excluded by (5.6).

In order to prove existence of stable states, that is minimizers of (5.8), we assume suitable growth of the energy density:

$$W(x,F) \geq C(-1+|F|^p) \text{ for a.a. } x\in\Omega, \text{ all } z\in\mathbb{R}^n \text{ and for some } C>0, \tag{5.9}$$

The existence theorem follows then directly from Corollary 5.5.

THEOREM 5.11. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz bounded domain, $r > 3$, $y_0 \in W^{1,p}(\Omega;\mathbb{R}^3)$, and $\Gamma_\mathrm{D} \subset \partial\Omega$ have a finite two-dimensional Lebesgue measure. Let W satisfy (i)-(iii) from Corollary 5.5 with $m = n = 3$. Let further* (5.5)–(5.7) *and* (5.9) *hold. If*

$$\mathcal{A} := \{y \in W^{1,p}(\Omega;\mathbb{R}^3) : \, y = y_0 \text{ on } \Gamma_\mathrm{D}\}$$

is such that $\inf_\mathcal{A} I < +\infty$ then there is a minimizer of I on $\mathcal{A}$.

This result can be generalized for different growth conditions like the one we consider in (5.13) below. Even more general settings can be found in [28] where various additional requirements on minimizers, as e.g. conditions ensuring a friction-less contact (Signorini problem); are included, too.

Let us mention a few important examples of polyconvex stored energy densities. Contrary to nontrivial examples of quasiconvex functions, it is relatively easy to design a polyconvex function. To ease our notation we only define the densities for matrices of positive determinant. Otherwise, it is implicitly extended by infinity.

EXAMPLE 5.4 (*Compressible Mooney-Rivlin material.*). *This material has a stored energy of the form*

$$W(F) = a|F|^2 + b|\mathrm{Cof} F|^2 + \gamma(\det F)\,, \tag{5.10}$$

where $a, b > 0$ and $\gamma(\delta) = c_1\delta^2 - c_2 \log\delta$, $c_1, c_2 > 0$.

It can be shown that for $n = 3$

$$W(F) = \frac{\lambda}{2}(\mathrm{tr}E)^2 + \mu|E|^2 + \mathcal{O}(|E|^3)\ ,\ E = (C-\mathbb{I})/2$$

where λ and μ are the usual Lamé constants, and $\mathbb{I}$ denotes the identity matrix. Indeed, it is a matter of a tedious computation to show that, given λ, μ, the following equations must be fulfilled by a, b, c_1, c_2: $c_2 := (\lambda + 2\mu)/2$ $2a + 2b = \mu$, and $4b + 4c_1 = \lambda$.

EXAMPLE 5.5 (Compressible neo-Hookean material.). *This material has a stored energy of the form*

$$W(F) = a|F|^2 + \gamma(\det F) \tag{5.11}$$

with the same constants as for the compressible Mooney-Rivlin materials.

EXAMPLE 5.6 (Ogden material.). *This material has a stored energy of the form (recall that $C = F^\top F$)*

$$W(F) = \sum_{i=1}^{M} a_i \mathrm{tr}\, C^{\gamma_i/2} + \sum_{i=1}^{N} b_i \mathrm{tr}(\mathrm{Cof}\, C)^{\delta_i/2} + \gamma(\det F) \tag{5.12}$$

and $a_i, b_i > 0$, $\lim_{\delta\to 0_+}\gamma(\delta) = +\infty$ for $\gamma : \mathbb{R}_+ \to \mathbb{R}$ convex growing suitably at infinity.

If W satisfies conditions (5.6)-(5.7) then any $y \in C^1(\Omega,\mathbb{R}^3)$ for which $\mathcal{E}(y)$ from (5.8) is finite is also locally invertible. This follows from the standard inverse function theorem. Nevertheless, what is actually desired for a physical deformation is that it is *injective* [28]. Indeed, non-injectivity of the deformation would mean that two material points from the reference configuration would be mapped to just one in the deformed configuration which means that the specimen penetrated through itself. Thus, additional assumptions to (5.6)-(5.7) on W are needed to assure *global invertibility of* y. Preferably, these assumptions should be compatible with polyconvexity and weak lower semicontinuity.

Take a diffeomorphism $y : \Omega \to y(\Omega)$ with $\det\nabla y > 0$ on Ω. Then, we have by the change of variables formula for $p > 1$

$$\int_{y(\Omega)} |\nabla y^{-1}(w)|^p \mathrm{d}w = \int_\Omega |\nabla y^{-1}(y(x))|^p \det(\nabla y(x))\mathrm{d}x = \int_\Omega |(\nabla y(x))^{-1}|^p \det\nabla y(x)\mathrm{d}x = \int_\Omega \frac{|\mathrm{Cof}^\top \nabla y(x)|^p}{(\det \nabla y(x))^{p-1}}\mathrm{d}x$$

where we used that $\nabla y^{-1}(y(x)) = (\nabla y(x))^{-1}$ for all x in Ω and that for any invertible matrix the relation $A^{-1} = \frac{\mathrm{Cof}^\top A}{\det A}$ holds.

Therefore, for energies satisfying a stricter growth condition than (5.9) in the form of

$$W(x,F) \geq C(-1 + |F|^p + \frac{|\mathrm{Cof}^\top A|^p}{(\det A)^{p-1}} \text{ for a.a. } x \in \Omega \text{ and for some } C > 0, \tag{5.13}$$

one could rather expect that deformations on which $\mathcal{E}(y)$ is finite are invertible. This is indeed so, as the Theorem 5.12 (below) shows.

Nevertheless, before proceeding to the theorem, let us point out that the new growth condition (5.13) is *fully compatible with polyconvexity.* Indeed, since the function $g(x,y) = \frac{x^p}{y^{p-1}}$ is convex for $r > 1$ on the set $\{(x,y) \in \mathbb{R}^2; y > 0\}$, $\frac{|\mathrm{cof}^\top A|^p}{(\det A)^{p-1}}$ is polyconvex on the set of matrices having a positive determinant.

THEOREM 5.12 (Taken from [8]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $y_0 : \overline{\Omega} \to \mathbb{R}^n$ be continuous in $\overline{\Omega}$ and one-to-one in Ω such that $y_0(\Omega)$ is also bounded and Lipschitz. Let $y \in W^{1,p}(\Omega;\mathbb{R}^n)$ for some $p > n$, $y(x) = y_0(x)$ for all $x \in \partial\Omega$, and let* $\det\nabla y > 0$ *a.e. in Ω. Finally, assume that for some $q > n$*

$$\int_\Omega |(\nabla y(x))^{-1}|^q \det \nabla y(x)\,\mathrm{d}x < +\infty\ . \tag{5.14}$$

Then $y(\overline{\Omega}) = y_0(\overline{\Omega})$ and y is a homeomorphism of Ω onto $y_0(\Omega)$. Moreover, the inverse map $y^{-1} \in W^{1,q}(u_0(\Omega);\mathbb{R}^n)$ and $\nabla y^{-1}(w) = (\nabla y(x))^{-1}$ for $w = y(x)$ and a.a. $x \in \Omega$.

Let us note that the growth conditions prescribed in the theorem have been weakened later in [111]. Indeed, in this work it was shown that an inverse to deformation can be defined even for $p > n-1$ and $q \geq \frac{p}{p-1}$.

Theorem 5.12 assures injectivity of y under the growth (5.13) if a up-to-the-boundary injective Dirichlet condition is prescribed. This, however, has the disadvantage that we could not model situations in which hard loads (Dirichlet boundary conditions) are prescribed only on a part on the boundary. Moreover, also the possibility of contact on the boundary (which is physically well possible) is excluded.

One possible remedy is to minimize $\mathcal{E}$ along with the so-called *Ciarlet-Nečas* condition

$$\int_\Omega \det\nabla y(x)\mathrm{d}x \leq |y(\Omega)|, \tag{5.15}$$

that was introduced in [29] in order to assure *global injectivity* of deformations. It was shown in [29] that C^1-functions satisfying (5.15) and that $\det\nabla y > 0$ are actually injective. The result generalizes to $W^{1,p}$-functions as well, but injectivity is obtained only almost everywhere in the deformed configuration; i.e., almost every point in the deformed configuration has only one pre-image.

REMARK 5.7. *Maps that are injective almost everywhere in the deformed configuration still include rather unphysical situations. For example a dense, countable set of points could be mapped to one point. This can be prevented if the deformation is* injectetive everywhere.

Using condition (5.15)*, this can be achieved for finite deformations of the energy $\mathcal{E}$ with a density W satisfying* (5.13) *for $p = r = n = 2$. This setting is the most explored one due to its relations to* *quasiconformal maps (see Section 6). Such deformations are open (that is they maps open sets to open sets) and discrete (the set of pre-images for any point does not accumulate) and, moreover, satisfy the Lusin N-condition (i.e. they map sets of zero measure again to sets of zero measure); cf. e.g. [58]).*

Then, we have by the area formula

$$\int_\Omega \det \nabla y \mathrm{d}x = \int_{\mathbb{R}^n} N(y, \Omega, z)\mathrm{d}z = \int_{y(\Omega)} N(y, \Omega, z)\mathrm{d}z$$

where $N(y, \Omega, z)$ is defined as the number of pre-images of $z \in y(\Omega)$ in Ω. So the Ciarlet-Nečas condition is satisfied if and only if $N(y, \Omega, z) = 1$ almost everywhere on $y(\Omega)$. Also we can immediately see that the reverse inequality to (5.15) always holds.

Further, if there existed $z \in y(\Omega)$ that had at least to two pre-images x_1 and x_2 then we could find an $\varepsilon > 0$ such that $B(x_1, \varepsilon) \cap B(x_2, \varepsilon) = \emptyset$ and $B(x_j, \varepsilon) \subset \Omega$ for $j = 1, 2$. On the other hand, for the images we have that $y(B(x_1, \varepsilon) \cap y(B(x_2, \varepsilon)) \neq \emptyset$. In fact, $y(B(x_1, \varepsilon)) \cap y(B(x_2, \varepsilon))$ is of positive measure since both $y(B(x_j, \varepsilon))$ are open. Therefore, there exists a set of positive measure where $N(y, \Omega, z)$ is at least two; a contradiction to (5.15).

5.3. Applications to hyperelasticity in the higher order setting. Let us now turn our attention to models of hyperelastic materials depending on higher-order gradients. Such materials are called non-simple of grade N, where N refers to the highest derivatives appearing in the stored energy density. The concept of such materials has been developing for long time, since the work by R.A. Toupin [117], under various names as non-simple materials as e.g. in [49, 69, 99, 106] or multipolar materials (in particular fluids). Here, we will consider only second-grade non-simple materials, i.e., those for which second-order deformation gradients (first-order strain gradients) are involved. The main mathematical advantage of nonsimple materials is that higher-order deformation gradients bring additional regularity of deformations and, possibly, also compactness of the set of admissible deformations in a stronger topology. Moreover, there the stored energy can be even convex in the highest derivatives of the deformation which is helpful in proving existence of minimizers.

The downside of this approach is that there are not many physically justified models of non-simple materials and material constants are rarely available.

We can now define an energy functional

$$\mathcal{E}(y) := \int_\Omega W(x, \nabla y(x), \nabla^2 y(x))\, \mathrm{d}x - \int_\Omega f(x) \cdot y(x)\, \mathrm{d}x - \int_{\Gamma_\mathrm{N}} \left(g(x) \cdot y(x) + \hat{g}_1(x) \cdot \frac{\partial y(x)}{\partial \nu} \right) \mathrm{d}S \ , \tag{5.16}$$

where $\hat{g}_1 : \Gamma_\mathrm{N} \to \mathbb{R}^n$ is the surface density of (hypertraction) forces balancing the *hyperstress*

$$x \mapsto \frac{\partial}{\partial G_{ijk}} W(x, F, G)|_{F=\nabla y(x),\, G=\nabla^2 y(x)} \ . \tag{5.17}$$

The corresponding first Piola-Kirchhoff stress tensor is constructed as follows.

Denote for $i, j \in \{1, \dots, n\}$

$$H_{ij}(x, F, G) := \sum_{k=1}^n \frac{\partial}{\partial G_{ijk}} W(x, F, G) \ .$$

Then for $x \in \Omega$, $F := \nabla y(x)$, and $G := \nabla^2 y(x)$ we evaluate the first Piola-Kirchhof stress tensor as

$$S_{ij}(x) = \frac{\partial W(x, F, G)}{\partial F_{ij}} - H_{ij}(x, F, G) \ .$$

We will assume that

$$y \mapsto \int_\Omega f(x) \cdot y(x)\, \mathrm{d}x + \int_{\Gamma_\mathrm{N}} (g(x) \cdot y(x) + \hat{g}_1(x) \cdot \frac{\partial y(x)}{\partial \nu})\, \mathrm{d}S \tag{5.18}$$

is a linear functional evaluating the work of external forces on the specimen. The other terms containing f and g are volume and surface forces. Here we, however, assume for simplicity that f, $\hat{g}_1$, and g depend only on $x \in$ and $x \in \Gamma_{\rm N}$, respectively.

Notice that existence of minimizers of $\mathcal{E}(y)$ is guaranteed by Corollary 5.6.

Similarly, as in the case of simple materials, it allows for formal derivation of Euler-Lagrange equations for minimizers of I. Again, the approach is far from being rigorous because, in particular, we should compose deformations rather than to add them together. Contrary to the simple-material situation, here the smoothness of $\partial\Omega$ is important because the mean curvature κ of the boundary enters the equations. Details on surface differential operators can be found, for example in [98].

6. Weak lower semicontinuity in general hyperelasticity. We have seen in the last section that polyconvexity is relatively easy to be verified and it ensures weak lower semicontinuity of the corresponding energy functional. Nevertheless, there are materials that cannot be modeled by polyconvex energy densities.

A prototypical example are systems featuring phase transition with each phase characterized by some specific deformation of the underlying atomic lattice. This setup is for example found in *shape-memory alloys* (see e.g. the monographs [21, 35, 47, 48, 97]). Shape memory alloys are intermetalic materials which have a high-temperature highly symmetric phase called ausenite and a low temperature phase called martensite which can, however, exist in several variants. Such systems are (for a suitable temperature range) typically modeled by a multi-well stored energy of the form

$$\begin{cases} W(QU_i) = 0 & \forall i = 1\dots M, \quad \forall Q \in \mathrm{SO}(n), \\ W(F) > 0 & \forall F \neq QU_i \;\; \forall i = 1\dots M, \quad \forall Q \in \mathrm{SO}(n), \end{cases} \tag{6.1}$$

where $U_1 \dots U_M$ is a given set of matrices representing the phases found in the material and $\mathrm{SO}(n)$ is the set of rotations in $\mathbb{R}^{n\times n}$. These materials form complicated patterns (microstructures) composed from different variants of martensite cf. Figure 1.

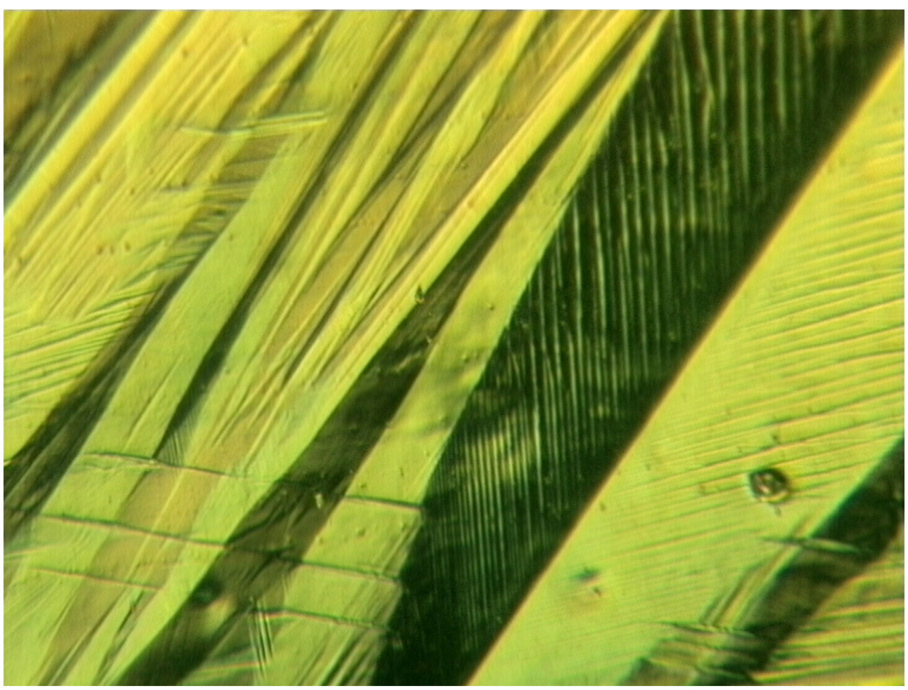

FIGURE 1. *Laminated microstructure in CuAlNi. Courtesy of P. Šittner (Inst. of Physics, CAS, Prague)*

Now, an energy density as given in (6.1) is neither polyconvex nor quasiconvex. Therefore, for constructing an appropriate model one is to find the *weakly lower semicontinuous envelope* of (1.1) with an energy density given by (6.1); in other words, one seeks the supremum of weakly lower semicontinuous functionals lying below the given energy. Let us remark that hese difficulties persist even if we used a geometrically linear description of energy wells; see [27], for instance.

Remark 6.1. *Notice that if* (6.1) *is additively enriched by a convex term of the form* $\varepsilon \int_\Omega |\nabla^2 y|^p \mathrm{d}x$, *which is usually interpreted as some kind of interfacial energy of the microstructure, Corollary 5.6 can be readily applied to show the existence of minimizers for* $\mathcal{E}$.

Let us also point out that a different approach has been proposed recently [108, 109]. There, a new notion of interface polyconvexity has been introduced which enables to prove existence of minimizers for simple materials with additional phase field variable.

In order to find the weakly lower semicontinuous envelope of (6.1), a precise charaterization of weak lower semicontinuity in terms of convexity conditions on W is needed. We have found these conditions in Section 3; however, only under the growth condition (i) in Definition 3.2. Yet, this is incompatible with the physical assumptions formulated in (5.6)-(5.7).

For such energies it is no longer known that quasiconvexity implies weak lower semicontinuity. Indeed, this is one of the standing problems in elasticity, which was formulated by Ball in the following way:

Open problem 6.1 (Problem 1 in [10]). *"Prove the existence of energy minimizers for elastostatics for quasiconvex stored-energy functions satisfying* (5.7)*."*

Remark 6.2. *It has been pointed out in [10] that one of the reasons why this problem is hard to solve is the fact that quasiconvexity possesses no local characterization [75].*

Let us stress that Problem 6.1 is an important attempt towards combining quasiconvexity and elasticity but additional steps are still required. Namely, if $u : \Omega \to \mathbb{R}^m$ entering (1.1) ought to represent a deformation of a physical body, it should be *injective* and *orientation-preserving*. Notice that this is not automatically satisfied for all maps on which the functional (1.1) is finite even if W fulfills (5.6)-(5.7). However, we may rely on Theorem 5.12 to assure this, provided suitable coercivity of the energy.

An alternative (and related approach) is to study directly weak lower semicontinuity along sequences found in a suitable class of mappings that are injective and orientation-preserving. As a first step, one may study classes of functions that fulfill some constraint on the Jacobian, e.g. that $\det\nabla u > 0$.

Even though Problem 6.1 remains widely open to date, it has been approached it from different perspectives recently. We review the results within this section.

In [70, 71], the authors study weak lower semicontinuity along sequences in $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ with $p < n$ satisfying that $\det\nabla u_k > 0$. They proved that (1.2) with $v = v(x, \nabla u)$ is weak lower semi-continuous along such sequences if and only if it is $W^{1,p}$*-orientation preserving quasiconvex*, i.e.

$$v(x, A) \leq \frac{1}{\Omega} \int_\Omega v(x, \nabla\varphi(x))\mathrm{d}x,$$

for all A with $\det(A) > 0$, all $\varphi \in W^{1,p}(\Omega;\mathbb{R}^m)$ satisfying that $\varphi(x) = Ax$ on $\partial\Omega$ and $\det(\nabla\varphi(x)) > 0$ for a.a. $x \in \Omega$.

However, in [71] the authors also show that, in fact, for $p < n$ *no* $W^{1,p}$-orientation preserving quasiconvex integrands exist that would satisfy the natural coercivity/growth condition

$$\frac{1}{C}\big(|A|^p + \kappa(\det A)\big) \leq v(x, A) \leq C\big(|A|^p + \kappa(\det A)\big)$$

for almost all $x \in \Omega$. Here, κ is a convex function satisfying that $\lim_{s\to 0} \kappa(s) = \infty$, $\kappa(s) = \infty$ for $s \leq 0$ and $\limsup_{s\to\infty} \frac{\kappa(s)}{s^{p/n}} < \infty$. Notice that this growth condition is compatible with (5.6)-(5.7).

To the best of our knowledge, the only works in which the authors actually considered equivalent characterization of weak lower semicontinuity for *injective maps* are [18] and [19] where the authors studied bi-Lipschitz and quasiconformal maps *in the plane*, respectively.

Here, by bi-Lipschitz maps the following set is meant

$$\begin{aligned} W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2) = \Big\{ & y : \Omega \mapsto y(\Omega) \text{ an orientation preserving homeomorphism;} \\ & y \in W^{1,\infty}(\Omega;\mathbb{R}^2) \text{ and } y^{-1} \in W^{1,\infty}(y(\Omega);\mathbb{R}^2) \Big\}, \end{aligned} \tag{6.2}$$

while quasiconformal maps are introduced as follows

$$\mathcal{QC}(\Omega;\mathbb{R}^2) = \Big\{y \in W^{1,2}(\Omega;\mathbb{R}^2):\ y \text{ is a homeomorphism and } \exists K \geq 1 \text{ such that}$$
$$|\nabla y|^2 \leq K \det \nabla y \text{ a.e. in } \Omega\Big\}. \tag{6.3}$$

It is natural to expect that weak lower semicontinuity of the functional

$$I(y) = \int_\Omega v(\nabla y)\mathrm{d}x,$$

along sequences in $W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$ or $\mathcal{QC}(\Omega;\mathbb{R}^2)$ is connected to a suitable notion of quasiconvexity of v. One even expects a *weaker* notion than the one from Definition 1.2 since the set of possible sequences along which semicontinuity is studied is restricted. Indeed, the perfectly fitted notion to this setting seems to be an alternation of Definition 1.2 where only function from $W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$ or $\mathcal{QC}(\Omega;\mathbb{R}^2)$ enter as test funtions. Exactly this result has been achieved in [18] and [19]; we review the result in Proposition 6.3.

First, let us introduce a notion of weak convergence on $W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$ and $\mathcal{QC}(\Omega;\mathbb{R}^2)$. We say that $y_k \overset{*}{\rightharpoonup} y$ in $W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$ if the sequence has uniformly bounded bi-Lipschitz constants[10] and $y_k \overset{*}{\rightharpoonup} y$ in $W^{1,\infty}(\Omega;\mathbb{R}^2)$. Note that the weak limit is bi-Lipschitz, too.

For a sequence $\{y_k\}_{k\in\mathbb{N}} \subset \mathcal{QC}(\Omega;\mathbb{R}^2)$, we say that it converges weakly to $y \in W^{1,2}(\Omega;\mathbb{R}^2)$ in $\mathcal{QC}(\Omega;\mathbb{R}^2)$ if $y_k \rightharpoonup y$ in $W^{1,2}(\Omega;\mathbb{R}^2)$, there exists a $K \geq 1$ such that the y_k are all K-quasiconformal and $y(x)$ is non-constant. Here it is important to assume that the limit function is non-constant for otherwise the limit function may not quasiconformal.[11]

Moreover, let us introduce the notions of *bi-quasiconvexity* and *quasiconformal quasiconvexity.*

Definition 6.2. *We say that a Borel measurable and bounded from below function $f : \mathbb{R}^{2\times 2} \to \Omega$ is bi-quasiconvex if*

$$|\Omega| f(A) \leq \int_\Omega f(\nabla\varphi(x))\,\mathrm{d}x \tag{6.5}$$

for all $\varphi \in W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$, $\varphi = Ax$ on $\partial\Omega$ and all A with $\det(A) > 0$.

We say that f is quasiconformally quasiconvex if (6.5) *holds for all A with* $\det(A) > 0$. *and all $\varphi \in \mathcal{QC}(\Omega;\mathbb{R}^2)$ such that $\varphi(x) = Ax$ on $\partial\Omega$.*

Then we have the following result:

Proposition 6.3 (from [18] and [19]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Let v be continuous on the set of matrices with a positive determinant. Then v is bi-quasiconvex if and only if*

$$y \mapsto I(y) = \int_\Omega v(\nabla y(x))\,\mathrm{d}x$$

is sequentially weakly lower semicontinuous on $W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$.*

Moreover, let v satisfy

$$0 \leq v(A) \leq c(1+|A|^2) \qquad \textit{with } c>0$$

on the set of matrices with a positive determinant. Then v is quasiconformally quasiconvex if and only if I is weakly lower semicontinuous on $\mathcal{QC}(\Omega;\mathbb{R}^2)$.

---

[10]Notice that a function $y \in W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$ satisfies for all $x_1, x_2 \in \Omega$

$$\frac{1}{L}|x_1 - x_2| \leq |y(x_1) - y(x_2)| \leq L|x_1 - x_2|\ . \tag{6.4}$$

for some $L \geq 1$. This L is then called the bi-Lipschitz constant of y.

[11] Because a sequence of uniformly K-quasiconformal maps converges locally uniformly either to K-quasiconfomal function or a constant and the locally uniform convergence is implied by the notion of weak convergence in $\mathcal{QC}(\Omega;\mathbb{R}^2)$ [5].

At the heart of the proof of Proposition 6.3 is the construction of a suitable cutoff method that is compatible with the bi-Lipschitz or quasiconformal setting. Notice that the standard cutoff method cannot be used since it relies on convex averaging. Thus, as neither $W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$ or $\mathcal{QC}(\Omega;\mathbb{R}^2)$ are convex, we may "fall out" from these sets when relying on the standard cutoff method.

The approach taken in [18] and [19] is based on the characterization of the trace operator on $W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$ as well als $\mathcal{QC}(\Omega;\mathbb{R}^2)$ due to [34, 118] and [20], respectively.

Even though Proposition 6.3 provides us with an weak lower semicontinuity result, this is not yet enough to prove existence of minimizers for functionals with densities from some suitable class. This is so, because bi-Lipschitz as well as quasiconformal maps include a L^∞-type constraint which can be enforced by letting the stored energy density be finite only on a suitable subset of $\mathbb{R}^{2\times 2}$; yet, this subset is usually left when employing cutoff methods—this happens even in the standard cases [31]. Thus letting v being infinite on some set of matrices is incompatible with the proof of Proposition 6.3.

The usual remedy for proving existence of minimizers or relaxation results is to work with L^p-type (with p finite) constraints only. In the setting from above this would mean to work with so-called *bi-Sobolev* classes (see e.g. [60]) for $1 < p < \infty$:

$$W_+^{1,p,-p}(\Omega;\mathbb{R}^2) = \Big\{ y : \Omega \mapsto y(\Omega) \text{ an orientation preserving homeomorphism;} \\ y \in W^{1,p}(\Omega;\mathbb{R}^2) \text{ and } y^{-1} \in W^{1,p}(y(\Omega);\mathbb{R}^2) \Big\}\ .$$

However, for these classes of functions, the approach from [18] and [19] cannot be adopted since a complete characterization of the trace operator on these classes is missing to date. In fact, we have the following

Open problem 6.4. *Characterize the class of functions $\mathcal{X}(\partial\Omega;\mathbb{R}^2)$ such that*

$$\mathrm{Tr} : W_+^{1,p,-p}(\Omega;\mathbb{R}^2) \overset{onto}{\longrightarrow} \mathcal{X}(\partial\Omega;\mathbb{R}^2)$$

at least for Ω being the unit square.

Let us note that the above problem may play a role also when smooth approximation (by diffemorphisms) of deformations in elasticity is concerned. Indeed, the standard techniques of smoothing Sobolev functions (by a mollification kernel) fail under the injectivity requirement since they essentially rely on convex averaging and injectivity is a non-convex constraint.

Recently, several results on smoothing even under these constraints appeared [62, 33, 86, 59, 61] using completely different techniques and limiting their scope to planar deformations (as everywhere in this section). In particular, in [62] the authors could prove that a homeomorphism in $W^{1,p}(\Omega;\mathbb{R}^2)$ can be strongly approximated by diffeomorphisms in the $W^{1,p}$-norm for $p > 1$. For $p = 1$ this result has recently been extended in [61].

Nevertheless, in elasticity, one might rather be interested in approximating a function in $W_+^{1,p,-p}(\Omega;\mathbb{R}^2)$ *together with its inverse*. To the authors knowledge, the only result in this direction is the one by [33] who showed that bi-Lipschitz maps can we strongly approximated together with their inverse in the $W^{1,p}$-norm for any finite p. Yet, for functions in $W_+^{1,p,-p}(\Omega;\mathbb{R}^2)$ with $p < \infty$ the problem remains largely open as mentioned also in [62].

To end this section let us remark (by formulating several open problems) that the relation of bi-quasiconvexity to the standard notions of convexity mentioned in this paper is still unexplored. We focus here only on bi-quasiconvexity but similar problems could be formulated also for quasiconformal quasiconvexity, too.

It is clear from the definitions that any function that is quasiconvex on the set of matrices with a positive determinant is also bi-quasiconvex. Moreover, bi-quasiconvexity implies, at least in the plane, rank-1 convexity on the set of matrices with a positive determinant.

Remark 6.3. *To see why bi-quasiconvexity implies rank-1 convexity on the set of matrices with a positive determinant, we proceed as follows. First, notice that the determinant changes affinely on rank-1 lines due to the formula*

$$\det(A + \lambda a \otimes n) = \det A\big(1 + \lambda n\cdot(A^{-1}a)\big), \tag{6.6}$$

where a and n are some arbitrary vectors. Therefore, rank-1 convexity on the set of matrices with a positive determinant is really meaningfull, since all matrices on a rank-1 line between two matrices with a positive determinant have this property, too.

Next we mimic the proof from [31, Lemma 3.11 and Theorem 5.3] showing that quasiconvexity implies rank-1 convexity. Without loss of generalization, we suppose that Ω is the unit square and that we want to show rank-1 convexity along the line $A + a \otimes e_1$ with e_1 the unit vector in the first coordinate. Then we consider the following sequence of mappings

$$y_n(x) = y_n(x_1, x_2) = \begin{cases} Ax & \text{for } x \in \left[\frac{k}{n}, \frac{k}{n} + \lambda\frac{1}{n}\right) \text{ for } k = 0 \dots n-1, \\ (A + a \otimes e_1)x & \text{for } x \in \left[\frac{k}{n} + \lambda\frac{1}{n}, \frac{k+1}{n}\right) \text{ for } k = 0 \dots n-1, \end{cases}$$

with some $\lambda \in [0,1]$. Notice that $\{y_n\}$ are Lipschitz, injective and that $(\nabla y)^{-1}$ is uniformly bounded and $\det(\nabla y)$ is bounded away from zero. Thus, $\{y_n\}$ is a sequence of uniformly bi-Lipschitz maps that converges weakly to $\lambda Ax + (1-\lambda)(A + a \otimes e_1)x$. We may therefore use the cut-off technique from [18] to modify the sequence in such a way that it attains exactly the value of the weak limit at the boundary. Then, the same procedure as in [31, Theorem 5.3] gives the rank-1 convexity.

In summary, we have the following series of implications

$$\text{quasiconvexity on } \mathbb{R}^{2\times 2}_+ \Rightarrow \text{bi-quasiconvexity} \Rightarrow \text{rank-1 convexity on } \mathbb{R}^{2\times 2}_+,$$

where we denoted by $\mathbb{R}^{2\times 2}_+$ the two-times-two matrices with positive determinant. But it is unclear whether some of the converse implications holds, too. We have the following:

Open problem 6.5. *Does rank-1 convexity on $\mathbb{R}^{2\times 2}_+$ imply bi-quasiconvexity?*

Open problem 6.6. *Does bi-quasiconvexity imply quasiconvexity on $\mathbb{R}^{2\times 2}_+$?*

7. Null Lagrangians at the boundary. Null Lagrangians at the boundary form a subset of null Lagrangians depending of the first order which have the following remarkable properties: If $\mathcal{N}$ is a null Lagrangian at the boundary then it is a polynomial of order p, say. If, additionally $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ converges weakly to $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ then $\{\mathcal{N}(\nabla u_k)\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ weak* converges to $\mathcal{N}(\nabla u)$ in $\mathcal{M}(\bar\Omega)$, i.e., in measures on the closure of the domain. This means that the L^1-bounded sequence $\{\mathcal{N}(\nabla u_k)\}$ converges to a Radon measure whose singular part vanishes. Null Lagrangians at the boundary can be also used to construct functions quasiconvex at the boundary; cf. Definition 3.9.

We first give a definition of null Lagrangians at the boundary.

Definition 7.1. *Let $\varrho \in \mathbb{R}^n$ be a unit vector and let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a given function.*

(i) f is called a null Lagrangian at the boundary at given $A \in \mathbb{R}^{m\times n}$ if both f and $-f$ are quasiconvex at the boundary at A in the sense of Definition 3.9; cf. [107]. This means that there is $q \in \mathbb{R}^m$ such that for all $\varphi \in W^{1,\infty}_{\Gamma_\rho}(D_\rho;\mathbb{R}^m)$ it holds

$$\int_{\Gamma_\rho} q \cdot \varphi(x)\, \mathrm{d}S + f(A)|D_\varrho| = \int_{D_\rho} f(A + \nabla\varphi(x))\, \mathrm{d}x \ . \tag{7.1}$$

(ii) If v is a null Lagrangian at the boundary at every $F \in \mathbb{R}^{m\times n}$, we call it a null Lagrangian at the boundary.

The following theorem explicitly characterizes all possible null Lagrangians at the boundary. It was first proved by P. Sprenger in his thesis [105, Satz 1.27] written in German. Later on, the proof was slightly simplified in [64]. Before stating the result we recall that $\mathrm{SO}(n) := \{R \in \mathbb{R}^{n\times n};\ R^\top R = RR^\top = \mathbb{I}\ ,\ \det R = 1\}$ denotes the set of orientation-preserving rotations and if we write $A = (B|\varrho)$ for some $B \in \mathbb{R}^{n\times(n-1)}$ and $\varrho \in \mathbb{R}^n$ then $A \in \mathbb{R}^{n\times n}$, its last column is ϱ and $A_{ij} = B_{ij}$ for $1 \le i \le n$ and $1 \le j \le n-1$.

Theorem 7.2. *Let $\varrho \in \mathbb{R}^n$ be a unit vector and let $\mathcal{N} : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a given continuous function. Then the following three statements are equivalent.*

(i) $\mathcal{N}$ *satisfies* (7.1) *for every* $F \in \mathbb{R}^{m\times n}$*;*
(ii) $\mathcal{N}$ *satisfies* (7.1) *for* $F = 0$,
(iii) There are constants $\tilde{\beta}_s \in \mathbb{R}^{\binom{m}{s}\times\binom{n-1}{s}}$, $1 \le s \le \min(m, n-1)$, *such that for all* $H \in \mathbb{R}^{m\times n}$,

$$\mathcal{N}(H) = \mathcal{N}(0) + \sum_{i=1}^{\min(m,n-1)} \tilde{\beta}_i \cdot \mathbb{T}_i(H\tilde{R}), \tag{7.2}$$

where $\tilde{R} \in \mathbb{R}^{n\times(n-1)}$ *is a matrix such that* $R = (\tilde{R}|\varrho)$ *belongs to* $\mathrm{SO}(n)$*;*
(iv) $\mathcal{N}(F + a \otimes \varrho) = \mathcal{N}(F)$ *for every* $F \in \mathbb{R}^{m\times n}$ *and every* $a \in \mathbb{R}^m$.

If $m = n = 3$ the only nonlinear null Lagrangian at the boundary with the normal ϱ is $\mathcal{N}(F) := \mathrm{Cof}\, F \cdot (a \otimes \varrho) = a \cdot \mathrm{Cof}\, F\varrho$ where $a \in \mathbb{R}^3$ is fixed; see [107]. In the following theorem we let ϱ freely move along the boundary which introduces a x-dependence to the problem. It first appeared in [78].

THEOREM 7.3. *Let* $\Omega \subset \mathbb{R}^3$ *be a smooth bounded domain. Let* $\{u_k\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *be such that* $u_k \rightharpoonup u$ *in* $W^{1,2}(\Omega;\mathbb{R}^3)$*. Let* $h(x, F) := \mathrm{Cof}\, F \cdot (a(x) \otimes \varrho(x))$*, where* $a, \varrho \in C(\bar{\Omega};\mathbb{R}^3)$*,* ϱ *coincides at* $\partial\Omega$ *with the outer unit normal to* $\partial\Omega$*. Then for all* $g \in C(\bar{\Omega})$

$$\lim_{k\to\infty} \int_\Omega g(x)h(x, \nabla u_k(x))\,\mathrm{d}x = \int_\Omega g(x)h(x, \nabla u(x))\,\mathrm{d}x \ . \tag{7.3}$$

If, moreover, for all $k \in \mathbb{N}$ $h(\cdot, \nabla u_k) \ge 0$ *almost everywhere in* Ω *then* $h(\cdot, \nabla u_k) \rightharpoonup h(\cdot, \nabla u)$ *in* $L^1(\Omega)$.

Notice that even though $\{h(\cdot, \nabla u_k)\}_{k\in\mathbb{N}}$ is bounded merely in $L^1(\Omega)$ its weak* limit in measures is $h(\cdot, \nabla u) \in L^1(\Omega)$, i.e., a measure which is absolutely continuous with respect to the Lebesgue measure on Ω. This holds independently of $\{\nabla u_k\}$. Therefore, the fact that h is a null Lagrangian at the boundary automatically improved regularity of the limit measure, namely its singular part vanishes. In order to understand why this happens, denote $\mathbb{P}(x) := \mathbb{I} - \varrho(x) \otimes \varrho(x)$ the orthogonal projector on the plane with the normal $\varrho(x)$, i.e., a tangent plane to $\partial\Omega$ at $x \in \partial\Omega$. Then

$$\mathrm{Cof}(F\mathbb{P}) = \mathrm{Cof}F\mathrm{Cof}\mathbb{P} = \mathrm{Cof}F\varrho \otimes \varrho \ .$$

Consequently,

$$\mathrm{Cof}(F\mathbb{P})\varrho = (\mathrm{Cof}F)\varrho \ ,$$

and if F is a placeholder for ∇u we see that $h(x, \cdot)$ only depends on the surface gradient of u. In other words, concentrations in the sequence of normal derivatives, $\{\nabla u_k \cdot (\varrho \otimes \varrho)\}_{k\in\mathbb{N}}$, are filtered out. The following two statements describing weak sequential continuity of null Lagrangians at the boundary can be found in [64]. They apply to cases in which the condition (ii) from Theorem 3.3 is always satisfied.

THEOREM 7.4 (see [64]). *Let* $m, n \in \mathbb{N}$ *with* $n \ge 2$*, let* $\Omega \subset \mathbb{R}^n$ *be open and bounded with a boundary of class* C^1*, and let* $x : \overline{\Omega} \times \mathbb{R}^{m\times n} \to \mathbb{R}$ *be a continuous function. In addition, suppose that for every* $x \in \Omega$*,* $v(x, \cdot)$ *is a null Lagrangian and for every* $x \in \partial\Omega$*,* $v(x, \cdot)$ *is a null Lagrangian at the boundary with respect to* $\varrho(x)$*, the outer normal to* $\partial\Omega$ *at* x*. Hence, by Theorem 7.2,* $v(x, \cdot)$ *is a polynomial, the degree of which we denote by* $d_v(x)$*. Finally, let* $p \in (1,\infty)$ *with* $p \ge d_v(x)$ *for every* $x \in \overline{\Omega}$ *and let* $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ *be a sequence such that* $u_k \rightharpoonup u$ *in* $W^{1,p}$*. If*

$$v(x, \nabla u_k(x)) \ge 0 \ \text{ for every } k \in \mathbb{N} \text{ and a.e. } x \in \Omega,$$

then $v(\cdot, \nabla u_n) \rightharpoonup v(\cdot, \nabla u)$ *weakly in* $L^1(\Omega)$.

THEOREM 7.5 (see [64]). *Let* $h : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *be such that* $h(\cdot, s)$ *is measurable for all* $s \in \mathbb{R}$ *and* $h(x, \cdot)$ *is convex for almost all* $x \in \Omega$*. Let* v *and* d_v *be as in Theorem 7.4. Then* $\int_\Omega h(x, v(x, \nabla u(x)))\,\mathrm{d}x$ *is weakly lower semicontinuous on the set* $\{u \in W^{1,p}(\Omega;\mathbb{R}^m); v(\cdot, \nabla u) \ge 0 \text{ in } \Omega\}$.

Let us finally point out that $A \mapsto h(\mathcal{N}(A))$ for a convex function h is quasiconvex at the boundary [13].

8. $\mathcal{A}$-quasiconvexity. In this section, we summarize results about weak lower semicontinuity of integral functionals along sequence which satisfy a first-order linear differential constraint. Clearly, gradients as curl-free fields are included in this setting and therefore this is really a generalization of (some) previously mentioned results. As emphasized by L. Tartar, besides curl-free fields there are also other PDE important constraints on possible minimizers. Such a setting naturally arises in electromagnetism, linearized elasticity or even higher-order gradients, to name a few. Tartar's program was materialized by Dacorogna in [30] and then studied by many other authors, too.

The problem studied in this section can be formulated as follows: Having a sequence $\{u_k\} \subset L^p(\Omega;\mathbb{R}^m)$, $1 < p < +\infty$ such that each member satisfies a linear differential constraint $\mathcal{A}u_k = 0$ ($\mathcal{A}$-free sequence), or $\mathcal{A}u_k \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^n)$ (asymptotically $\mathcal{A}$-free sequence), what conditions on v precisely ensure weak lower semicontinuity of integral functionals in the form

$$\mathcal{I}(u) := \int_\Omega v(x,u(x))\,\mathrm{d}x\ . \tag{8.1}$$

Here $\mathcal{A}$ is a first-order linear differential operator.

To the best of our knowledge, the first result of this type was proved in [44] for nonnegative integrands. In this case, the crucial necessary and sufficient condition ensuring weak lower semicontinuity of $\mathcal{I}$ in (8.1) is the so-called $\mathcal{A}$-quasiconvexity; cf. Def. 8.2 below. However, if we refrain from considering only nonnegative integrands, this condition is not necessarily sufficient as we already observed in the case $\mathcal{A}$:=curl.

If $v(x,\cdot)$ has a recession function $v_\infty(x,\cdot)$ for all $x \in \bar{\Omega}$ we also define

$$\mathcal{I}_\infty(u) := \int_\Omega f_\infty(x,u(x))\,\mathrm{d}x\ . \tag{8.2}$$

8.1. The operator $\mathcal{A}$ and $\mathcal{A}$-quasiconvexity. Following [44], we consider linear operators $A^{(i)}$: $\mathbb{R}^m \to \mathbb{R}^d$, $i = 1,\ldots,n$, and define $\mathcal{A} : L^p(\Omega;\mathbb{R}^m) \to W^{-1,p}(\Omega;\mathbb{R}^d)$ by

$$\mathcal{A}u := \sum_{i=1}^n A^{(i)}\frac{\partial u}{\partial x_i}\ , \text{where}\ \ u:\Omega\to\mathbb{R}^m\ ,$$

i.e., for all $w \in W_0^{1,p'}(\Omega;\mathbb{R}^d)$

$$\langle \mathcal{A}u, w\rangle = -\sum_{i=1}^n \int_\Omega A^{(i)}u(x)\cdot\frac{\partial w(x)}{\partial x_i}\,\mathrm{d}x\ .$$

For $w \in \mathbb{R}^n$ we define the linear map

$$\mathbb{A}(w) := \sum_{i=1}^n w_i A^{(i)}\ :\ \mathbb{R}^m \to \mathbb{R}^d\ .$$

In this review, we assume that there is $r \in \mathbb{N}\cup\{0\}$ such that

$$\operatorname{rank}\mathbb{A}(w) = r \text{ for all } w \in \mathbb{R}^n\ , |w| = 1\ , \tag{8.3}$$

i.e., $\mathcal{A}$ has the so-called *constant-rank property*. Below we use $\ker\mathcal{A}$ to denote the set of all locally integrable functions u such that $\mathcal{A}u = 0$ in the sense of distributions, i.e., $\int_\Omega u\cdot\mathcal{A}^*w\,\mathrm{d}x = 0$ for all $w \in C^\infty$ compactly supported in Ω. Here, $\mathcal{A}^* = -\sum_{i=1}^n \left(A^{(i)}\right)^T\frac{\partial u}{\partial x_i}$ is the formal adjoint of $\mathcal{A}$. Of course, $\ker\mathcal{A}$ depends on the considered domain Ω, which always should be clear from the context below. In particular, a periodic function u in the space

$$L^p_\#(\mathbb{R}^n;\mathbb{R}^m) := \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m):\ u \text{ is } Q\text{-periodic}\}$$

is in $\ker\mathcal{A}$ if and only if $\mathcal{A}u = 0$ again in the sense of distributions Here and in the following, Q denotes the unit cube $(-1/2,1/2)^n$ in $\mathbb{R}^n$, and we say that $u:\mathbb{R}^n\to\mathbb{R}^m$ is *Q-periodic* if for all $x \in \mathbb{R}^n$ and all $z \in \mathbb{Z}^n$

$$u(x+z) = u(x)\ .$$

We will use the following lemmas proved in [44, Lemma 2.14] and [44, Lemma 2.15], respectively.

Lemma 8.1 (projection onto $\mathcal{A}$-free fields in the periodic setting). *There is a linear bounded operator $\mathcal{T} : L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m) \to L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)$ that vanishes on constant functions, with the property that $\mathcal{T}(\mathcal{T}u) = \mathcal{T}u$ for all $u \in L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)$, and $\mathcal{T}u \in ker\mathcal{A}$. Moreover, for all $u \in L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)$ with $\int_Q u(x)\,\mathrm{d}x = 0$ it holds that*

$$\|u - \mathcal{T}u\|_{L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)} \le C\|\mathcal{A}u\|_{W^{-1,p}_{\#}(\mathbb{R}^n;\mathbb{R}^m)} ,$$

where $C > 0$ is a constant independent of u and $W^{-1,p}_{\#}(\mathbb{R}^n;\mathbb{R}^m)$ denotes the dual space of $W^{1,p'}_{\#}(\mathbb{R}^n;\mathbb{R}^m)$ ($\frac{1}{p'} + \frac{1}{p} = 1$), the Q-periodic functions in $W^{1,p'}_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ equipped with the norm of $W^{1,p'}(Q;\mathbb{R}^m)$.

Definition 8.2 (cf. [44, Def. 3.1, 3.2]). *We say that a continuous function $f : \mathbb{R}^m \to \mathbb{R}$, satisfying that $|f(s)| \le C(1+|s|^p)$ for some $C > 0$, is $\mathcal{A}$-quasiconvex if for all $s_0 \in \mathbb{R}^m$ and all $\varphi \in L^p_{\#}(Q;\mathbb{R}^m) \cap ker\mathcal{A}$ with $\int_Q \varphi(x)\,\mathrm{d}x = 0$ it holds*

$$f(s_0) \le \int_Q f(s_0 + \varphi(x))\,\mathrm{d}x .$$

Fonseca and Müller [44] proved the following result linking $\mathcal{A}$-quasiconvexity and weak lower semicontinuity. Notice that the integrand is more general than that one in (8.1).

Theorem 8.3. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $v : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [0;+\infty)$ be a Carathéodory integrand. Let*

$$0 \le v(x,z,u) \le a(x,z)(1 + |u|^p)$$

for almost every $x \in \Omega$ and all $(z,u) \in \mathbb{R}^d \times \mathbb{R}^m$, $1 < p < +\infty$, and some $0 \le a \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$. Assume that $z_k \to z$ in measure and that $u_k \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^d)$,$\|\mathcal{A}u_k\|_{W^{-1,p}(\Omega;\mathbb{R}^m)} \to 0$.

Then

$$\liminf_{k\to\infty} \int_\Omega v(x,z_k,u_k)\,\mathrm{d}x \ge \int_\Omega v(x,z,u)\,\mathrm{d}x$$

if and only if $v(x,z,\cdot)$ is $\mathcal{A}$-quasiconvex for almost all $x \in \Omega$ and all $z \in \mathbb{R}^d$.

However, if we refrain from considering only nonnegative integrands, the $\mathcal{A}$-quasiconvexity condition is not necessarily sufficient (as we already observed in the case $\mathcal{A}$:=curl) and the whole situation is much more involved. We will review the currently available results in the following. We start by defining the necessary notions.

Definition 8.4 (See [72]). *We say that a sequence $\{u_k\} \in L^p(\Omega;\mathbb{R}^m)$ is asymptotically $\mathcal{A}$-free if $\|\mathcal{A}u_k\|_{W^{-1,p}(\Omega;\mathbb{R}^m)} \to 0$ as $k \to \infty$.*

A functional $\mathcal{I}$ as in (8.1) is called weakly sequentially lower semicontinuous (wslsc) along asymptotically $\mathcal{A}$-free sequences in $L^p(\Omega;\mathbb{R}^m)$ if $\liminf_{k\to\infty} \mathcal{I}(u_k) \ge \mathcal{I}(u)$ for all such sequences $\{u_k\}$ that weakly converge to some limit u in L^p.

Analogously, we say that a functional $\mathcal{I}$ is weakly sequentially lower semicontinuous (wslsc) along $\mathcal{A}$-free sequences in $L^p(\Omega;\mathbb{R}^m)$ if

$$\liminf_{k\to\infty} I(u_k) \ge I(u) \textit{ for all } \{u_k\} \subset L^p(\Omega;\mathbb{R}^m) \cap ker\mathcal{A}.$$

We have the following result which was proved in [41, Theorem 2.4] in a slightly less general version. However, its original proof directly extends to this setting.

Theorem 8.5. *Let $v : \bar{\Omega} \times \mathbb{R}^m \to \mathbb{R}$ be continuous such that $v(x,\cdot)$ has a recession function for all $x \in \bar{\Omega}$ and let $v(x,\cdot)$ be $\mathcal{A}$-quasiconvex for almost every $x \in \Omega$, $1 < p < +\infty$. Then $\mathcal{I}$ is sequentially weakly lower semicontinuous in $L^p(\Omega;\mathbb{R}^m) \cap ker\mathcal{A}$ if and only if*

$$\liminf_{k\to\infty} \mathcal{I}_\infty(w_k) \ge \mathcal{I}_\infty(0) = 0 \tag{8.4}$$

for every bounded sequence $\{w_k\} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker\mathcal{A}$ *with* $w_k \to 0$ *in measure.*

The statement of Theorem 8.5 remains valid if we replace sequences in $\ker\mathcal{A}$ with asymptotically $\mathcal{A}$-free sequences. In the following definition, the space $L^p_0(B(x_0,\delta);\mathbb{R}^m)$ denotes the space of L^p functions compactly supported on $B(x_0,\delta)$ and $C^p_{\mathrm{hom}}(\mathbb{R}^m)$ the space of positively p-homogeneous functions on $\mathbb{R}^m$.

Definition 8.6 (See [72]). *We say that* $f_\infty \in C(\bar\Omega; C^p_{hom}(\mathbb{R}^m))$ *is* $\mathcal{A}$*-quasiconvex at the boundary* ($\mathcal{A}$*-qcb) at* $x_0 \in \partial\Omega$ *if for every* $\varepsilon > 0$ *there are* $\delta > 0$ *and* $\alpha > 0$ *such that*

$$\int_{B(x_0,\delta)\cap\Omega} f_\infty(x,u(x)) + \varepsilon|u(x)|^p\,\mathrm{d}x \geq 0 \tag{8.5}$$

for every $u \in L^p_0(B(x_0,\delta);\mathbb{R}^m)$ *with* $\|\mathcal{A}u\|_{W^{-1,p}(\mathbb{R}^n;\mathbb{R}^d)} < \alpha\|u\|_{L^p(B(x_0,\delta)\cap\Omega;\mathbb{R}^m)}$.

The next notion, defined in [72], is intimately related to weak lower semicontinuity along asymptotically $\mathcal{A}$-free sequences. Notice that the *only but crucial* difference between Definitions 8.6 and 8.7 is the norm used to measure $\mathcal{A}u$.

Definition 8.7. *We say that* $f_\infty \in C(\bar\Omega; C^p_{hom}(\mathbb{R}^m))$ *is strongly* $\mathcal{A}$*-quasiconvex at the boundary (strongly-*$\mathcal{A}$*-qcb) at* $x_0 \in \partial\Omega$ *if for every* $\varepsilon > 0$ *there are* $\delta > 0$ *and* $\alpha > 0$ *such that*

$$\int_{B(x_0,\delta)\cap\Omega} f_\infty(x,u(x)) + \varepsilon|u(x)|^p\,\mathrm{d}x \geq 0 \tag{8.6}$$

for every $u \in L^p_0(B(x_0,\delta);\mathbb{R}^m)$ *with* $\|\mathcal{A}u\|_{W^{-1,p}(\Omega;\mathbb{R}^d)} < \alpha\|u\|_{L^p(B(x_0,\delta)\cap\Omega;\mathbb{R}^m)}$.

We now focus on the link between (strong) $\mathcal{A}$-quasiconvexity at the boundary and weak lower semicontinuity along (asymptotically) $\mathcal{A}$-free sequences.

8.2. Asymptotically $\mathcal{A}$-free sequences. We have the following result proved in [72].

Proposition 8.8. *Let* $v_\infty \in C(\bar\Omega; C^p_{hom}(\mathbb{R}^m))$*. Then* $\mathcal{I}_\infty(u)$ *from* (8.2) *is weakly sequentially lower semicontinuous along asymptotically* $\mathcal{A}$*-free sequences in* $L^p(\Omega;\mathbb{R}^m)$ *if and only if*
(i) v_∞ *is strongly-*$\mathcal{A}$*-qcb at every* $x_0 \in \partial\Omega$ *and*
(ii) $v_\infty(x,\cdot)$ *is* $\mathcal{A}$*-quasiconvex at almost every* $x \in \Omega$*.*

8.3. Genuinely $\mathcal{A}$-free sequences. We now focus on weak lower semicontinuity along sequences $\{u_k\}$ that satisfy $\mathcal{A}u_k = 0$ for each $k \in \mathbb{N}$. The main difference is that for the link to $\mathcal{A}$-quasiconvexity at the boundary ($\mathcal{A}$-qcb) as introduced in Def. 8.6, more precisely, for its sufficiency, we rely on an extension property [72].

Definition 8.9 ($\mathcal{A}$-free extension domain). *We say that* Ω *is an* $\mathcal{A}$-free extension domain *if there exists a larger domain* Ω' *with* $\Omega \subset\subset \Omega'$ *and an associated* $\mathcal{A}$-free extension operator*, i.e., a bounded linear operator* $E : L^p(\Omega;\mathbb{R}^m) \cap \ker\mathcal{A} \to L^p(\Omega';\mathbb{R}^m) \cap \ker\mathcal{A}$ *such that* $Eu = u$ *on* Ω*.*

As mentioned before, the existence of an $\mathcal{A}$-free extension operator not only depends on the smoothness of $\partial\Omega$, but also on $\mathcal{A}$ itself. On the one hand, if $\partial\Omega$ is Lipschitz, extension operators are available for $\mathcal{A} = \operatorname{curl}$ and $\mathcal{A} = \operatorname{div}$ (essentially using a partition of unity and an extension by a suitable reflection), but on the other hand, if we choose $\mathcal{A}$ to be the differential operator of the Cauchy–Riemann system ($n = m = 2$, identifying $\mathbb{C}$ with $\mathbb{R}^2$), no such extension operator exists even for very smooth domains, since holomorphic functions with singularities at the boundary of Ω can never be extended to holomorphic functions on a larger set including the singular point.

With the help of the extension property and the projection $\mathcal{T}$ from Lemma 8.1, Proposition 8.8 can be adapted to the setting of genuinely $\mathcal{A}$-free sequences.

Proposition 8.10. *Suppose that* Ω *is an* $\mathcal{A}$*-free extension domain and let* $v_\infty \in C(\bar\Omega; C^p_{hom}(\mathbb{R}^m))$*. Then* $\mathcal{I}_\infty(u)$ *from* (8.2) *is weakly sequentially lower semicontinuous along* $\mathcal{A}$*-free sequences in* $L^p(\Omega;\mathbb{R}^m)$ *if and only if*
(i) v_∞ *is* $\mathcal{A}$*-qcb at every* $x_0 \in \partial\Omega$ *and*

(ii) $v_\infty(x,\cdot)$ *is* $\mathcal{A}$*-quasiconvex at almost every* $x\in\Omega$*.*

As it is shown in [44] higher-order gradients can be written as certain $\mathcal{A}$-free mappings. Thus, our results have a direct application to weak lower semicontinuity of integral functionals of the form $u\mapsto\int_\Omega f(x,\nabla^k u(x))\,\mathrm{d}x$ for $u\in W^{k,p}(\Omega)$ and $f\in C(\bar\Omega;\mathbb{R}^k)$ such that $|f(x,\cdot)|\le C(1+|\cdot|^p)$ for some $C>0$ and all $x\in\bar\Omega$.

Indeed, the following example taken from [72] shows that $I(u):=\int_\Omega \det\nabla^2 u(x)\,\mathrm{d}x$ is not weakly lower semicontinuous on $W^{2,2}(\Omega)$. Consequently, the determinant is not $\mathcal{A}$-qcb for suitably defined $\mathcal{A}$. As to the definition of $\mathcal{A}$, we recall from [44] that the functional I fits into our framework, if instead of on $\nabla^2 u$, we define I on fields $v=(v)_{ij}$, $1\le i\le j\le n$, in L^2, satisfying $\mathcal{A}v:=\mathrm{curl}\ v=0$, with the understanding that for each x, $v(x)$ (the upper triangular part of a matrix) is identified with a symmetric matrix in $\mathbb{R}^{n\times n}$ still denoted v, both for the application of the (row-wise) curl and the evaluation of I, where $\nabla^2 u$ is replaced by v. One can check that $\mathcal{A}v=0$ if and only if there exists a scalar-valued $u\in W^{2,2}$ with $v=\nabla^2 u$, at least as long as the domain is simply connected.

Example 8.1. *Consider* $\Omega:=(-1,1)^2$ *and for* $F\in\mathbb{R}^{2\times2}$ *the function* $v_\infty(F):=\det F$ *and the operator* $\mathcal{A}$ *such that* $\mathcal{A}w=0$ *if and only if for some* $u\in W^{2,2}(\Omega)$*,* w *is the upper (or lower) triangular part of* $\nabla^2 u$*, which takes values in the symmetric matrices; cf. [44, Example 3.10(d)]. Here* $\nabla^2 u$ *denotes the Hessian matrix of* u*. Then* v_∞ *is not* $\mathcal{A}$*-quasiconvex at the boundary. Indeed, take* $u\in W^{2,2}_0(\Omega)$ *extended by zero to the whole* $\mathbb{R}^2$*. Define* $u_k(x):=k^{-1}u(kx)$*. Then* $u_k\rightharpoonup 0$ *in* $W^{2,2}(\Omega)$*. We have that*

$$\lim_{k\to\infty}\int_{(0,1)\times(-1,1)}\det\nabla^2 u_k(x)\,\mathrm{d}x=\int_{(0,1)\times(-1,1)}\det\nabla^2 u(y)\,\mathrm{d}y\ . \tag{8.7}$$

Hence, it remains to find u *for which the integral on the right-hand side is negative which is certainly possible.*

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