Competing states for the fractional quantum Hall effect in the 1/3-filled second Landau level

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In this work, we investigate the exact nature of the fractional quantum Hall (FQH) state in the 1/3-filled second Landau level (SLL) at filling factor $\nu = 7/3$ via exact diagonalization in the torus as well as the spherical geometries. Specifically, we compute the overlap between the exact $7/3$ ground state and various competing states including (i) the Laughlin state, (ii) the fermionic Haffnian state, (iii) the antisymmetrized product state of two composite fermion (CF) seas at $1/6$ filling, and (iv) the particle-hole (PH) conjugate of the Z_4 parafermion state. All these trial states are constructed according to a guiding principle called the bilayer mapping approach, where a trial state is obtained as the antisymmetrized projection of a bilayer quantum Hall state with interlayer distance d as a variational parameter. The Z_4 parafermion state is obtained as the antisymmetrized projection of the Halperin (330) state. Similarly, it is proved in this work that the fermionic Haffnian state can be obtained as the antisymmetrized projection of the Halperin (551) state. It is shown that, while extremely accurate at large positive Haldane pseudopotential variation $\delta V_1^{(1)}$, the Laughlin state loses almost entirely its overlap with the exact 7/3 state around the Coulomb point, where $\delta V_1^{(1)} = 0$. At slightly negative $\delta V_1^{(1)}$, it is shown that the PH conjugate of the Z_4 parafermion state has a substantial overlap with the exact 7/3 state, which is the highest among those of the above four trial states. Around the Coulomb point, the energy spectrum exhibits an intriguing change from the Laughlin-type spectrum with a well-developed magnetoroton structure to the spectrum with a quasidegeneracy in the ground state energy that is characteristic to the PH conjugate of the Z_4 parafermion state.

I. INTRODUCTION

The fractional quantum Hall (FQH) states occurring in the second Landau level (SLL) have been attracting intense interest due to their possibility as exotic topological states with non-Abelian quasiparticle statistics. This possibility is in stark contrast to the fact that the major FQH states in the lowest Landau level (LLL) at filling factor $\nu = n/(2pn \pm 1)$ (where p and n are positive integers) and their particle-hole (PH) conjugates can be understood as the weakly-interacting integer quantum Hall (IQH) states of composite fermion (CF) at effective filling factor $\nu^* = n$, where quasiparticles satisfy Abelian statistics [1, 2]. The weakly-interacting CF theory serves as an excellent guiding principle for the FQH states in the LLL. Other minor unconventional FQH states in the LLL, for example, occurring at $\nu = 4/11$ and $5/13$ can be understood within the extended framework of the CF theory, where CFs form their own FQH states with mixed "vorticity flavor" with some carrying two vortices and the other four [3]. The FQH state at $\nu = 3/8$ [4, 5] is highly peculiar, but actually related with those occurring at even-denominator filling factors in the SLL [6], which requires a new guiding principle as explained below.

The situation is rather complicated in the SLL, where the FQH states are relatively rare and fragile in comparison with the LLL [7]. On the surface, the weakly-interacting CF theory seems to work very well. All odd-denominator FQH fractions in the SLL are well captured by the usual CF sequence $\nu = 2 + n/(2pn \pm 1)$ and its PH conjugates $\nu = 3 \pm n/(2pn \pm 1)$ except for a few, but robust even-denominator FQH states occurring in the half-filled SLL at $\nu = 5/2$ $(= 2 + 1/2)$ and $7/2$ $(= 3+1/2)$ [8–10], which can be understood as the paired states of CFs [11]. Note that the 7/2 state is the PH conjugate of the 5/2 state in the limit of zero Landau level mixing, in which case the same physics governs both states.

The 5/2 (7/2) state has attracted intense interest for the possibility that it may host non-Abelian statistics for low-energy quasiparticles. This possibility is largely based on the observation that the exact 5/2 state obtained from various finite-size numerical studies is well described by the Moore-Read (MR) Pfaffian state [12–14] or its PH conjugate state known as the anti-Pfaffian state [15, 16] in a certain range of parameters. Previous finite-size numerical studies utilized various numerical techniques such as exact diagonalization (ED) in the spherical [17–25] and the torus geometries [18, 20, 25–29], and the density matrix renormalization group (DMRG) method in the spherical [30] and the cylindrical [31] geometries.

The true nature of the 5/2 state, however, still remains elusive in part due to the fact that the MR Pfaffian/anti-Pfaffian state breaks the PH symmetry, while the Coulomb interaction preserves it [21, 26, 27]. The PH symmetry can be broken either spontaneously in a modified Coulomb interaction [18, 21] or externally via Landau level mixing [23–25, 28, 31–35]. While the PH symmetry issue may be resolved somehow, it is still problematic that the MR Pfaffian/anti-Pfaffian state has a rather low overlap with the exact 5/2 state at the pure Coulomb interaction, which is actually further reduced with the increase of Landau level mixing strength [25]. Recently, based on exact diagonalization in the torus geometry, it has been pro-

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posed by two of the authors [29] that the exact 5/2 state at the pure Coulomb interaction may be better described by the antisymmetrized product state of two CF seas at quarter filling. The antisymmetrized product state has an additional advantage over the MR Pfaffian/anti-Pfaffian state in that it is susceptible to an anisotropic instability, which is consistent with recent experimental observations [36, 37]. This state was constructed according to a guiding principle called the "bilayer mapping" approach, which is explained in detail below.

In addition to the $5/2$ state, the FQH states at $\nu = 12/5$ $(= 2 + 2/5)$ and $13/5$ $(= 3 - 2/5)$ [10, 38, 39] have attracted much attention in the context of non-Abelian statistics. There are several proposed trial states; (i) the (PH conjugate of) Z_3 parafermion state [40] (which is discussed in detail below), (ii) the Bonderson-Slingerland state [41, 42], (iii) the hierarchy state [43, 44], (iv) the weakly-interacting CF state [2], and (v) the multipartite CF states [45, 46]. Recent numerical studies using the DMRG method in the spherical [47] and cylindrical [48] geometries as well as ED in the torus geometry [48] seem to suggest that the $13/5$ $(12/5)$ state is in the same universality class as the (PH conjugate of) Z_3 parafermion state, which hosts non-Abelian statistics.

In contrast to the 5/2 (7/2) and 12/5 (13/5) states, the FQH state in the 1/3-filled SLL at $\nu = 7/3$ (8/3) has attracted relatively little attention. One of the main reasons for such a negligence was that the 7/3 state was generally believed to be in the same universality class as the Laughlin state occurring in the LLL [49]. When scrutinized, however, hard numerical evidence is not so conclusive. According to an early study using exact diagonalization in the torus geometry with hexagonal unit cell [50], the 7/3 state at $N = 6$ turns out to be compressible for the pure Coulomb interaction and undergoes a first-order transition to the Laughlin state as the hard-core component of the Haldane pseudopotential is increased. Moreover, overlap studies between the exact and the Laughlin state in the spherical geometry [51, 52] have shown that the square of overlap for the pure Coulomb interaction is very low, typically being below 40% in finite-size systems with $N \leq 15$.

On the other hand, the entanglement spectrum [53] obtained via ED in the spherical geometry [52, 54] and the DMRG method in the infinite cylindrical geometry [31] provides evidence supporting that the $7/3$ state has the Laughlin-type edge excitation spectrum. It was argued [52, 54] that the apparent discrepancy between the low ground-state overlap and the Laughlin-type entanglement spectrum could be caused by a significantly larger size of quasiparticle/hole in the 7/3 state, which can be well captured by the variational Monte Carlo simulation [52] as well as the DMRG method [54] , while not by ED in relatively small finite-size systems.

Despite this argument, however, the substantially low overlap between the exact 7/3 and the Laughlin state is alarming and demands a search for a better state. In this context, an important question is what guiding principle should be used to generate the FQH trial state in the SLL. Considering that the 5/2 state is generated by a mechanism involving the pairing of composite fermions, it is plausible that the 7/3 state might be also generated by a similar "pairing" mechanism. As a generalization of the pairing mechanism responsible for the MR Pfaffian state, Read and Rezayi [40] proposed a guiding principle for the FQH trial states in higher Landau levels, according to which the FQH ground states are generated at $\nu = k/(k+2)$ as the zero-energy state of a $k+1$-body δ-function interaction, where k is a positive integer. The $k = 2$ case corresponds to the MR Pfaffian state. In general, the ground state obtained at a given k is called the Z_k parafermion state, which includes the previously-mentioned Z_3 parafermion state at $\nu = 12/5$ (13/5). Physically speaking, the Z_k parafermion state involves k-particle clusters, generalizing the pairs in the MR Pfaffian state.

Under this guiding principle, a trial state at $\nu = 8/3$ $(= 2 + 2/3)$ is generated to be the Z_4 parafermion state. In the presence of the PH symmetry, a similar trial state can be obtained at $\nu = 7/3$ by applying the PH conjugation operator onto the Z_4 parafermion state. In fact, a recent numerical work based on ED in the spherical geometry [55] has shown that the Z_4 parafermion state has a significant overlap with the exact ground state at $\nu = 8/3$ in the limit of zero Landau level mixing. This suggests also a similar overlap between the 7/3 state and the PH conjugate of the Z_4 parafermion state.

We would like to test a different guiding principle called the "bilayer mapping" approach [29]. According to this approach, a trial state is constructed as the antisymmetrized projection of a bilayer quantum Hall state with interlayer distance d as a variational parameter. As mentioned above, this approach had been applied to the 5/2 problem. The MR Pfaffian state is obtained as the antisymmetrized projection of the Halperin (331) state [56], which occurs at $d/l_B \simeq 1$. Another trial state is the antisymmetrized product state of two CF seas at quarter filling, which occurs at $d/l_B \to \infty$. The usual CF sea state at half filling is obtained at $d/l_B \to 0$. It has been found that the antisymmetrized product state of two CF seas at quarter filling has a substantially higher overlap with the exact 5/2 state than the MR Pfaffian state [29]. This leads to an intriguing question if the bilayer mapping approach can be also applied to the 7/3 problem.

To see what trial states can be generated at $\nu = 7/3$ via the bilayer mapping, let us examine what bilayer quantum Hall ground states can occur as a function of d/l_B. Figure 1 shows schematic phase diagrams of the bilayer quantum Hall ground state as a function of d/l_B at $\nu = 1/3$ [57] and $\nu = 2/3$ [58]. Scarola and Jain [57] studied the phase diagram of the bilayer quantum Hall ground state at $\nu = 1/3$ by computing the energies of various trial states as a function of d/l_B. As a result, it was shown that (i) the Laughlin state, Ψ_{333}, had the lowest energy at $0 \leq d/l_B \lesssim 2$, (ii) the Jastrow-correlated product state of two CF seas at quarter filling, $\Psi_{^4\mathrm{CFS}\otimes{}^4\mathrm{CFS}}^{\mathrm{Jastrow-corr}}$, at $2 \lesssim d/l_B \lesssim 3$, (iii) the Halperin (551) state, Ψ_{551}, at $3 \lesssim d/l_B \lesssim 3.5$, and finally (iv) the product state of two CF seas at 1/6 filling, $\Psi_{^6\mathrm{CFS}\otimes{}^6\mathrm{CFS}}$, at $d/l_B \gg 1$. McDonald and Haldane [58] performed ED to determine the phase diagram of the bilayer quantum Hall ground state at $\nu = 2/3$. As a result, it was shown that (i) the pseudospin singlet state, Ψ_{singlet}, occured at $d/l_B \ll 1$, and (ii) the Halperin (330) state, Ψ_{330}, at $d/l_B \gg 1$. Out of these six bilayer quantum Hall ground states, we focus on three states

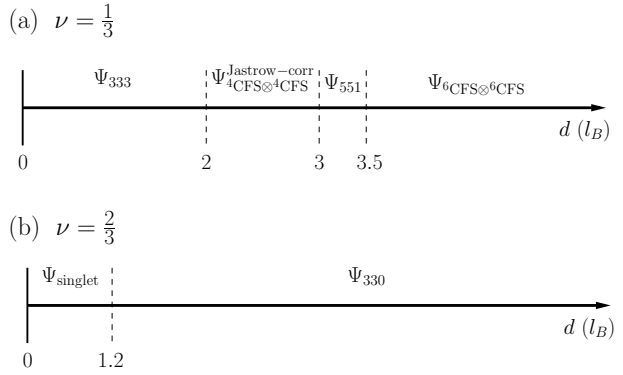

FIG. 1. Schematic phase diagrams of the bilayer quantum Hall ground state as a function of interlayer distance d/l_B at (a) $\nu = 1/3$ and (b) $\nu = 2/3$.

at $\nu = 1/3$, which are Ψ_{333}, Ψ_{551}, and $\Psi_{^6\text{CFS}\otimes{}^6\text{CFS}}$, and one state at $\nu = 2/3$, which is Ψ_{330}. Note that we do not pay attention to Ψ_{singlet} since it is completely annihilated upon applying the antisymmetrization operator. Meanwhile, we do not discuss $\Psi^{\text{Jastrow}-\text{corr}}_{^4\text{CFS}\otimes{}^4\text{CFS}}$ since it turns out that the antisymmetrized projection of this state has a negligible overlap with the exact 7/3 state. We construct the final four trial states for the FQH state at $\nu = 7/3$ by applying the antisymmetrization operator onto Ψ_{333}, Ψ_{551}, $\Psi_{^6\text{CFS}\otimes{}^6\text{CFS}}$, and by applying the antisymmetrization operator and then the PH conjugation operator onto Ψ_{330}. In summary, the four trial states obtained by the bilayer mapping approach are as follows: (i) Ψ_{333}, (ii) $\mathcal{A}\Psi_{551}$, (iii) $\mathcal{A}\Psi_{^6\text{CFS}\otimes{}^6\text{CFS}}$, and (iv) $\mathcal{C}_{\text{PH}}\mathcal{A}\Psi_{330}$, where $\mathcal{A}$ and $\mathcal{C}_{\text{PH}}$ are the antisymmetrization and the PH conjugation operators, respectively. Here, note that $\mathcal{A}\Psi_{333} = \Psi_{333}$ since the Laughlin state is already antisymmetrized.

Actually, there is an intriguing connection between the two guiding principles of the Z_k parafermion and the bilayer mapping approaches. It has been shown previously [59, 60] that the Z_4 parafermion state is entirely equivalent to the antisymmetrized projection of the Halperin (330) state:

$$\Psi_{Z_4} = \mathcal{A}\Psi_{330}. \tag{1}$$

Originally, this identity was derived in an attempt to generate non-Abelian states in bilayer quantum Hall systems as an alternative to the Z_k parafermion approach, which uses the unrealistic $k+1$-body δ-function interaction. In this point of view, $\mathcal{A}\Psi_{330}$ can be regarded as the single-layer limit of the bilayer quantum ground state Ψ_{330}, which may be obtained in the limit of large interlayer tunneling, while the interlayer Coulomb interaction is set equal to zero. Unfortunately, it was shown by a numerical calculation that taking the limit of large interlayer tunneling could be actually different from applying the antisymmetrized projection [61].

Similarly, it turns out that there is also an intriguing connection between $\mathcal{A}\Psi_{551}$ and a previously-known trial state called the fermionic Haffnian state [62],

$$\Psi_{\text{Hf}} = \Psi_{333}\text{Det}\left(\frac{1}{z_i - z_j}\right), \tag{2}$$

which can be regarded as the d-wave paired state of composite fermions. It is proved in this work that the fermionic Haffnian state is entirely equivalent to the antisymmetrized projection of the Halperin (551) state:

$$\Psi_{\text{Hf}} = \mathcal{A}\Psi_{551}. \tag{3}$$

It is interesting to note that there is a similar connection between the MR Pfaffian state and the antisymmetrized projection of the Halperin (331) state [56]:

$$\Psi_{\text{Pf}} = \mathcal{A}\Psi_{331}. \tag{4}$$

Physically, the MR Pfaffian state can be regarded as the p-wave paired state of composite fermions.

To summarize, the following four trial states are generated via the bilayer mapping; (i) the Laughlin state, (ii) the fermionic Haffnian state, (iii) the antisymmetrized product state of two CF seas at 1/6 filling, and (iv) the PH conjugate of the Z_4 parafermion state. To investigate which trial state is most relevant at $\nu = 7/3$, in this work, we compute the overlap between the exact $7/3$ ground state and the above four trial states by using ED up to $N = 12$ in the torus and the spherical geometries. As a result, it is shown that the Z_4 parafermion state has a substantial overlap with the exact 7/3 state around the Coulomb point, which is the highest among those of the four trial states.

The rest of this paper is organized as follows. In Sec. II, we provide analytic expressions of the FQH Hamiltonians both in the torus and the spherical geometries, which can be formulated in terms of the Haldane pseudopotentials. In Sec. III, we provide concrete mathematical forms of the trial states and explain how to obtain them by applying the antisymmetrization and the PH conjugation operators onto various bilayer quantum Hall ground states. In particular, it is proved that the antisymmetrized projection of the Halperin (551) state is entirely equivalent to the fermionic Haffnian state. In Sec. IV, we provide the results for the overlap between the exact 7/3 state and the above four trial states. In Sec. V, we examine the excitation spectrum, which exhibits an intriguing transition from the Laughlin-type spectrum with a well-developed magnetoroton structure to the spectrum with a quasidegeneracy in the ground state energy that is characteristic to the PH conjugate of the Z_4 parafermion state. We conclude in Sec. VI with a summary of the results and a discussion on future directions.

II. HAMILTONIAN

In this section, we provide analytic expressions of the FQH Hamiltonians both in the torus and the spherical geometries. Considering a recent experimental observation [63], we focus on the fully spin-polarized situation. The goal of this section is to express the electron-electron interaction Hamiltonian in the Landau level with index n (nLL) in terms of the Haldane pseudopotentials [50]. The pure Coulomb interaction can be represented by an appropriate set of the Haldane pseudopotentials.

In the torus geometry [64, 65], the unit cell has a shape of parallelogram defined by two vectors $\mathbf{L}_1$ and $\mathbf{L}_2$ with periodic

boundary condition. The area of the unit cell is set equal to $|\mathbf{L}_1 \times \mathbf{L}_2| = 2\pi l_B^2 N_\phi$, where l_B is the magnetic length and N_ϕ is the number of the flux quanta. The aspect ratio of the unit cell is defined by $r_a = |\mathbf{L}_1|/|\mathbf{L}_2|$, which is set equal to unity in this work unless stated otherwise. The nLL FQH Hamiltonian is written in terms of the torus basis states as follows:

$$H_{n\mathrm{LL}} = \frac{1}{2} \sum_{j_1,j_2,j_3,j_4} M^{(n)}_{j_1 j_2 j_3 j_4} c^\dagger_{j_1} c^\dagger_{j_2} c_{j_3} c_{j_4}, \tag{5}$$

where ${c_j}^\dagger$ and c_j are the creation and annihilation operators, respectively, acting on the j-th state with j being the linear momentum quantum number. The matrix element $M^{(n)}_{j_1 j_2 j_3 j_4}$ is given by [29, 50]

$$\begin{aligned} M^{(n)}_{j_1 j_2 j_3 j_4} =& \delta'_{j_1 - j_4, t} \delta'_{j_1 + j_2, j_3 + j_4} \\ & \times \sum_{m=0}^{\infty} \frac{2V^{(n)}_m}{N_\phi} \sum_{\mathbf{q}} e^{i q_x (X_{j_1} - X_{j_3})} e^{-\frac{q^2}{2}} L_m\left(q^2\right), \end{aligned} \tag{6}$$

where $X_j = 2\pi j/|\mathbf{L}_2|$ for $j = 1, 2, \ldots, N_\phi$ and $\mathbf{q} = s\mathbf{q}_1 + t\mathbf{q}_2$ $[s, t \in \mathbb{Z}]$ with $\mathbf{q}_1$ and $\mathbf{q}_2$ being the reciprocal vectors defined via the reciprocal relations, $\mathbf{L}_1 \cdot \mathbf{q}_1 = 2\pi$ and $\mathbf{L}_2 \cdot \mathbf{q}_2 = 2\pi$. The primed Kronecker delta is defined so that $\delta'_{i,j} = 1$ if $i = j$ modulo N_ϕ, and 0 otherwise. $L_m(x)$ is the Laguerre polynomial. Called the Haldane pseudopotential, $V^{(n)}_m$ is the potential energy of an electron pair with relative angular momentum m. For a given electron-electron interaction specified by its Fourier component $V_{\mathbf{k}}$, the Haldane pseudopotentials are given as follows [50]:

$$V^{(n)}_m = \frac{1}{2\pi} \int d^2\mathbf{k}\, V_{\mathbf{k}} L_m(k^2) L^2_n\left(\frac{k^2}{2}\right) e^{-k^2}. \tag{7}$$

Note that, in the case of the Coulomb interaction, the $\mathbf{q} = 0$ component is excluded in the $\mathbf{q}$ summation of Eq. (6) to take into account the positive background correction. It is convenient to vary the Coulomb interaction by adding the Haldane pseudopotential variations $\delta V^{(n)}_m$ to the pure Coulomb values $V^{(n)}_{\mathrm{Coul},m}$. In particular, we obtain the exact 7/3 ground state, $\Psi_{7/3}[\delta V^{(1)}_1]$, by diagonalizing the torus FQH Hamiltonian in Eq. (5) as a function of $\delta V^{(1)}_1$. Note that all eigenstates of the torus FQH Hamiltonian can be classified in terms of the pseudomomentum $\mathbf{Q} = Q_1\mathbf{q}_1 + Q_2\mathbf{q_2} \equiv (Q_1, Q_2)$, which is conserved due to the translational invariance. Here, Q_1 and Q_2 are integers between 0 and $\gcd(N, N_\phi)$ [65].

For the bilayer quantum Hall (BQH) system, the Hamiltonian is written as follows:

$$H_{\mathrm{BQH}} = \frac{1}{2} \sum_{j_1,j_2,j_3,j_4} M^{\sigma\sigma'}_{j_1 j_2 j_3 j_4} c^\dagger_{j_1\sigma} c^\dagger_{j_2\sigma'} c_{j_3\sigma'} c_{j_4\sigma}, \tag{8}$$

where the matrix element $M^{\sigma\sigma'}_{j_1 j_2 j_3 j_4}$ depends not only on the orbital momentum indices, j_k, but also on the layer indices, σ and σ'. When $\sigma = \sigma'$ and $\sigma \neq \sigma'$, $M^{\sigma\sigma'}_{j_1 j_2 j_3 j_4}$ describes the intralayer and the interlayer interactions, respectively. Similar to Eq. (6), the layer-dependent Haldane pseudopotentials $V^{\sigma\sigma'}_m$ can be related with $M^{\sigma\sigma'}_{j_1 j_2 j_3 j_4}$ as follows:

$$\begin{aligned} M^{\sigma\sigma'}_{j_1 j_2 j_3 j_4} =& \delta'_{j_1 - j_4, t} \delta'_{j_1 + j_2, j_3 + j_4} \\ & \times \sum_{m=0}^{\infty} \frac{2V^{\sigma\sigma'}_m}{N_\phi} \sum_{\mathbf{q}} e^{i q_x (X_{j_1} - X_{j_3})} e^{-\frac{q^2}{2}} L_m\left(q^2\right), \end{aligned} \tag{9}$$

where $V^{\sigma\sigma'}_m = V^{\mathrm{intra}}_m$ and V^{inter}_m if $\sigma = \sigma'$ and $\sigma \neq \sigma'$, respectively.

In the spherical geometry [43], the nLL FQH Hamiltonian is written for a given two-body interaction $V(\mathbf{r}_1, \mathbf{r}_2)$ as follows:

$$H_{n\mathrm{LL}} = \frac{1}{2} \sum_{m_1 m_2 m'_1 m'_2} \langle l m_1, l m_2 | V | l m'_1, l m'_2 \rangle c^\dagger_{m_1} c^\dagger_{m_2} c_{m'_2} c_{m'_1}, \tag{10}$$

where the orbital angular momentum l is given by $l = Q + n$ for nLL, and Q is the magnetic monopole strength. The azimuthal quantum numbers, m_1, m_2, m'_1, and m'_2, are summed over the range of $\{-l, -l+1, \ldots, l-1, l\}$. In terms of the Haldane pseudopotentials, an isotropic two-body interaction can be written as [2]

$$\begin{aligned} & \langle l m_1, l m_2 | V(r) | l m'_1, l m'_2 \rangle \\ & = \sum_{L=0}^{2l} \sum_{M=-L}^{L} \langle l m_1, l m_2 | L M \rangle V_L \langle L M | l m'_1, l m'_2 \rangle, \end{aligned} \tag{11}$$

where the spherical Haldane pseudopotential V_L is given as the potential energy of an electron pair with total angular momentum L, or equivalently with the relative angular momentum $2l - L$. Specifically, for the pure Coulomb interaction, i.e., $V(\mathbf{r}_1, \mathbf{r}_2) = 1/|\mathbf{r}_1 - \mathbf{r}_2|$,

$$\begin{aligned} & \langle l m_1, l m_2 | V | l m'_1, l m'_2 \rangle \\ & = \frac{1}{R} \sum_{l'} \sum_{m} \langle l m'_1, l' m | l m_1 \rangle \langle l m_2, l' m | l m'_2 \rangle \langle l Q, l' 0 | l Q \rangle^2, \end{aligned} \tag{12}$$

where the radius of the sphere R is determined by $4\pi R^2 B = 2Qhc/e$, or simply $R = \sqrt{Q}$ if we set the magnetic length $l_B = \sqrt{\hbar c/eB}$ equal to unity.

III. TRIAL STATES

As mentioned in Sec. I, we are interested in the following four trial states; (i) the Laughlin state, Ψ_{333}, (ii) the fermionic Haffnian state, Ψ_{Hf}, which is shown to be equivalent to $\mathcal{A}\Psi_{551}$, (iii) the antisymmetrized product state of two CF seas at 1/6 filling, $\mathcal{A}\Psi_{6\mathrm{CFS}\otimes 6\mathrm{CFS}}$, and (iv) the PH conjugate of the Z_4 parafermion state, $\mathcal{C}_{\mathrm{PH}}\Psi_{Z_4}$, where Ψ_{Z_4} is known to be equivalent to $\mathcal{A}\Psi_{330}$. Below, we provide concrete mathematical forms of the trial states and explain how to

obtain them numerically by applying the antisymmetrization and the PH conjugation operators onto the corresponding bilayer quantum Hall ground states. See Ref. [29] for details on how to perform the antisymmetrization and the PH conjugation operators on the exact states.

First, the Laughlin state Ψ_{333} is given as follows:

$$\Psi_{333} = \prod_{i<j}^{N/2} (z_i - z_j)^3 (\omega_i - \omega_j)^3 \prod_{k,l}^{N/2} (z_k - \omega_l)^3 , \quad (13)$$

which z_i and ω_j denote the coordinates of the i-th and the j-th electron in each layer. Apparently, Ψ_{333} is invariant with respect to antisymmetrization since its orbital part of the wave function is already antisymmetrized as is. Numerically, one can obtain Ψ_{333} as the exact zero-energy ground state of the 0LL FQH Hamiltonian in Eq. (5) with $V_1^{(0)}$ set equal to positive nonzero and all the other Haldane pseudopotentials zero [66]. Alternatively, one can also obtain Ψ_{333} with a good accuracy by diagonalizing the 0LL FQH Hamiltonian since the exact Coulomb ground state in the LLL is practically identical to Ψ_{333}.

Second, the Halperin (551) state Ψ_{551} [67, 68] is written as

$$\Psi_{551} = \prod_{i<j}^{N/2} (z_i - z_j)^5 (\omega_i - \omega_j)^5 \prod_{k,l}^{N/2} (z_k - \omega_l) . \quad (14)$$

On the other hand, the fermionic Haffnian state can be written as [62]

$$\Psi_{\rm Hf} = \Phi_1^3 {\rm Hf}\left(\frac{1}{(Z_i - Z_j)^2}\right) = \Phi_1^3 {\rm Det}\left(\frac{1}{Z_i - Z_j}\right), \quad (15)$$

where $\Phi_1 = \prod_{i<j}^{N}(Z_i - Z_j)$ with Z_i being the unified index defined as $Z_i = z_i$ and $Z_{i+N/2} = \omega_i$ with $i = 1, 2, \ldots, N/2$. Note that Φ_1^3 is equivalent to Ψ_{333}. Above, Hf denotes the Haffnian of a symmetric matrix, which is related with the determinant via ${\rm Hf}(M_{ij}^2) = {\rm Det}(M_{ij})$ [13].

Now, we would like to prove that the antisymmetrized projection of the Halperin (551) state, $\mathcal{A}\Psi_{551}$, is entirely equivalent to $\Psi_{\rm Hf}$ up to a normalization constant. To this end, it is convenient to rewrite $\mathcal{A}\Psi_{551}$ as follows:

$$\mathcal{A}\Psi_{551} = \Phi_1 \mathcal{S} \prod_{i<j}^{N/2} (z_i - z_j)^4 (\omega_i - \omega_j)^4 , \quad (16)$$

where $\mathcal{S}$ is the symmetrization operator. In order to express Eq. (16) in the form of a paired state, we exploit two identities for the symmetrized Jastrow-factor polynomial. The first identity is given by

$$\mathcal{S} \prod_{i<j}^{N/2} (z_i - z_j)^4 (\omega_i - \omega_j)^4 = 2^{(1-N/2)} \left[\mathcal{S} \prod_{i<j}^{N/2} (z_i - z_j)^2 (\omega_i - \omega_j)^2\right]^2 , \quad (17)$$

which is proved in Appendix A. The second identity is the well-known analytic relationship between the symmetrized Halperin (220) wave function and the bosonic MR Pfaffian wave function, which is fundamentally due to Cauchy's identity [56]:

$$\mathcal{S} \prod_{i<j}^{N/2} (z_i - z_j)^2 (\omega_i - \omega_j)^2 = C_{N/2} \Phi_1 {\rm Pf}\left(\frac{1}{Z_i - Z_j}\right), \quad (18)$$

where the constant factor $C_n = (-1)^{n(n-1)/2} n!$. Above, Pf denotes the Pfaffian of a skew-symmetric matrix. Using these two identities in Eqs. (17) and (18), one can rewrite $\mathcal{A}\Psi_{551}$ up to a normalization constant as follows:

$$\mathcal{A}\Psi_{551} = \Phi_1^3 \left[{\rm Pf}\left(\frac{1}{Z_i - Z_j}\right)\right]^2 = \Phi_1^3 {\rm Det}\left(\frac{1}{Z_i - Z_j}\right), \quad (19)$$

where it is used that $[{\rm Pf}(M_{ij})]^2 = {\rm Det} M_{ij}$. This proves that $\mathcal{A}\Psi_{551}$ is entirely equivalent to $\Psi_{\rm Hf}$ up to a normalization constant, i.e., $\Psi_{\rm Hf} = \mathcal{A}\Psi_{551}$.

It is interesting to mention that the above proof can be applied to the bosonic counterpart of Eq. (19). That is, the symmetrized product state of two Laughlin states at quarter filling is equivalent to the bosonic Haffnian state at half filing up to a normalization constant. Mathematically,

$$\mathcal{S}\Psi_{440} = \Phi_1^2 {\rm Hf}\left(\frac{1}{(Z_i - Z_j)^2}\right) = \Phi_1^2 {\rm Det}\left(\frac{1}{Z_i - Z_j}\right), \quad (20)$$

which was actually mentioned previously in Ref. [69], but no proof was provided there.

It is worthwhile to note from Eq. (19) that the antisymmetrized projection of the bilayer state is not necessarily incompressible even if the original bilayer state is so. The Halperin (551) state is incompressible, whereas the Haffnian state is known to be not [70]. This result is based on the conformal field theory with conformal blocks, which correspond to trial wave functions. It is shown that the conformal field theory becomes irrational if its conformal block corresponds to the Haffnian wave function. As a result, the number of excitation types in the Haffnian state is not finite. This property manifests itself in finite-size numerical calculations as a diverging degeneracy of the ground states in the torus geometry and that of the quasihole states in the spherical geometry as a function of electron number. Both numerical and analytical studies have shown exactly this property for the bosonic Haffnian state [71, 72]. We believe that the same phenomenon should happen for the fermionic Haffnian state.

According to Eq. (19), the fermionic Haffnian state can be obtained as the antisymmetrized projection of the Halperin (551) state, which can be implemented conveniently in the torus geometry. Similar to the Laughlin state, the Halperin (551) state can be obtained as the exact zero-energy ground state of the BQH Hamiltonian in Eq. (8) with $V_{1,3}^{\rm intra}$ and

V_0^{inter} set equal to positive nonzero and all the other Haldane pseudopotentials zero [68].

Meanwhile, in the spherical geometry, the fermionic Haffnian state can be directly obtained as the exact (non-degenerate) zero-energy ground state of the following three-body interaction Hamiltonian [73]:

$$H_{\text{Hf}} = \sum_{i\neq j\neq k} \Big[V_0 P_{ijk}\left(3N_\phi/2-3\right) + V_2 P_{ijk}\left(3N_\phi/2-5\right) + V_3 P_{ijk}\left(3N_\phi/2-6\right) \Big], \tag{21}$$

where $P_{ijk}(L)$ is the projection operator onto the three-particle state at a given total angular momentum L. The V_1 term is absent due to a symmetry reason [74]. We have checked that the ground state of H_{Hf} is indeed identical to the antisymmetrized projection of the Halperin (551) state in the spherical geometry up to machine precision.

Third, we consider the antisymmetrized product state of two CF seas at 1/6 filling, $\mathcal{A}\Psi_{^6\text{CFS}\otimes{}^6\text{CFS}}$. To this end, it is necessary to know how to obtain the CF sea state at 1/6 filling, $\Psi_{^6\text{CFS}}$. Naïvely, one may conjecture that $\Psi_{^6\text{CFS}}$ is obtained as the ground state of the Coulomb interaction at $\nu = 1/6$ in the LLL. Unfortunately, however, this conjecture is not correct since the actual ground state is likely to be the Wigner crystal of composite fermions rather than the quantum Hall liquid state [75, 76]. In the torus geometry, this fact manifests itself as the non-uniform ground state, which occurs at odd pseudomomenta. In this sense, the previously-mentioned phase diagram in Fig. 1 (a) is actually not accurate. The actual 1/3 bilayer ground state at $d/l_B \gg 1$ is predicted to be the product state of two Wigner crystals of composite fermions at 1/6 filling.

For this reason, we instead construct $\Psi_{^6\text{CFS}}$ as the ground state of the 0LL FQH Hamiltonian in Eq. (5) with $V^{(0)}_{1,3,5}$ set equal to unity and all the other Haldane pseudopotentials zero. It is important to note that $\mathcal{A}\Psi_{^6\text{CFS}\otimes{}^6\text{CFS}}$ does not necessarily describe a compressible phase even if $\Psi_{^6\text{CFS}\otimes{}^6\text{CFS}}$ is compressible. The reason is in some sense similar to why the antisymmetrized incompressible state is not necessarily incompressible, as shown in the example of $\Psi_{\text{Hf}} = \mathcal{A}\Psi_{551}$. Perhaps, a more directly relevant example is the previously-mentioned antisymmetrized product state of two CF seas at quarter filling, $\mathcal{A}\Psi_{^4\text{CFS}\otimes{}^4\text{CFS}}$, which has a significant square of overlap (over 90% when PH-symmetrized in the $N = 12$ system) with the exact $5/2$ state at the Coulomb point [29], which is known to be incompressible. It is not clear at this point whether $\mathcal{A}\Psi_{^6\text{CFS}\otimes{}^6\text{CFS}}$ is indeed incompressible. Despite this uncertainty, we believe that it is worthwhile to investigate how large overlap it can have with the exact $7/3$ state around the Coulomb point. In some sense, the overlap can provide us with a hint for the compressibility of $\mathcal{A}\Psi_{^6\text{CFS}\otimes{}^6\text{CFS}}$.

Finally, let us consider the PH conjugate of the Z_4 parafermion state, $\mathcal{C}_{\text{PH}}\Psi_{Z_4}$. As mentioned previously, the Z_4 parafermion state can be obtained as the antisymmetrized projection of the Halperin (330) state [59, 60], i.e., $\Psi_{Z_4} = \mathcal{A}\Psi_{330}$ with the Halperin (330) state defined as

$$\Psi_{330} = \prod_{i<j}^{N/2} \left(z_i - z_j\right)^3 \left(\omega_i - \omega_j\right)^3 \prod_{k,l}^{N/2} \left(z_k - \omega_l\right)^0 . \tag{22}$$

In principle, Ψ_{330} can be obtained as the exact zero-energy ground state of the BQH Hamiltonian in Eq. (8) with V_1^{intra} set equal to positive nonzero and all the other Haldane pseudopotentials zero.

Practically, we use the fact that Ψ_{330} is the direct product of two Laughlin states, each of which can be obtained in a much smaller Hilbert space. For example, for $N = 12$ in the torus geometry, the size of the Hilbert space for the 2/3 BQH system is around 3.6×10^{15}, whereas that for the 1/3 FQH system is around 2.9×10^{6}. Since there is a three-fold center-of-mass degeneracy at $\nu = 1/3$ in the torus geometry, there are nine possible ways of constructing the zero-energy ground state of Ψ_{330} by taking the product of any two of these degenerate Laughlin states. This naturally accounts for the nine-fold degeneracy of Ψ_{330} in the torus geometry [77]. Once Ψ_{330} is obtained, we apply the antisymmetrization operator to obtain $\mathcal{A}\Psi_{330} = \Psi_{Z_4}$, and then the PH conjugation operator to obtain $\mathcal{C}_{\text{PH}}\Psi_{Z_4}$.

It is worthwhile to mention that a recent numerical work by Peterson *et al.* in the spherical geometry [55] has shown that Ψ_{Z_4} has a significant overlap with exact $8/3$ state with full spin polarization. In the limit of zero Landau level mixing, the same significant overlap should be obtained between $\mathcal{C}_{\text{PH}}\Psi_{Z_4}$ and the exact 7/3 state. One of the most crucial technical differences between their work and ours is that, here, we use the torus geometry, where $\mathcal{C}_{\text{PH}}\Psi_{Z_4}$ and other competing states including Ψ_{333}, Ψ_{Hf}, and $\mathcal{A}\Psi_{^6\text{CFS}\otimes{}^6\text{CFS}}$ can be compared on an equal footing unlike the spherical geometry, where all those states occur at different flux sectors.

Besides the Z_4 parafermion state, there are several other non-Abelian candidate states in the bilayer quantum Hall system at $\nu = 2/3$, which include the intralayer Pfaffian state [78], the interlayer Pfaffian state [79, 80], the Fibonacci state [81, 82], and the Bonderson-Slingerland state [41]. We remark that it is not easy to construct these states in the torus geometry since the parent Hamiltonian generating each state as the zero-energy ground state is unknown. On the other hand, in the spherical geometry, one can obtain the second-quantized amplitudes of each state by making use of the Jack polynomial representation when the form of the wave function is known [83]. While these states could be important in some parameter regimes, here we are not interested in these states since they are not constructed via the bilayer mapping approach.

In the following section, to investigate which trial state is most relevant at $\nu = 7/3$, we compute the overlap between the exact $7/3$ state and each of the four trial states, Ψ_{333}, Ψ_{Hf}, $\mathcal{A}\Psi_{^6\text{CFS}\otimes{}^6\text{CFS}}$, and $\mathcal{C}_{\text{PH}}\Psi_{Z_4}$.

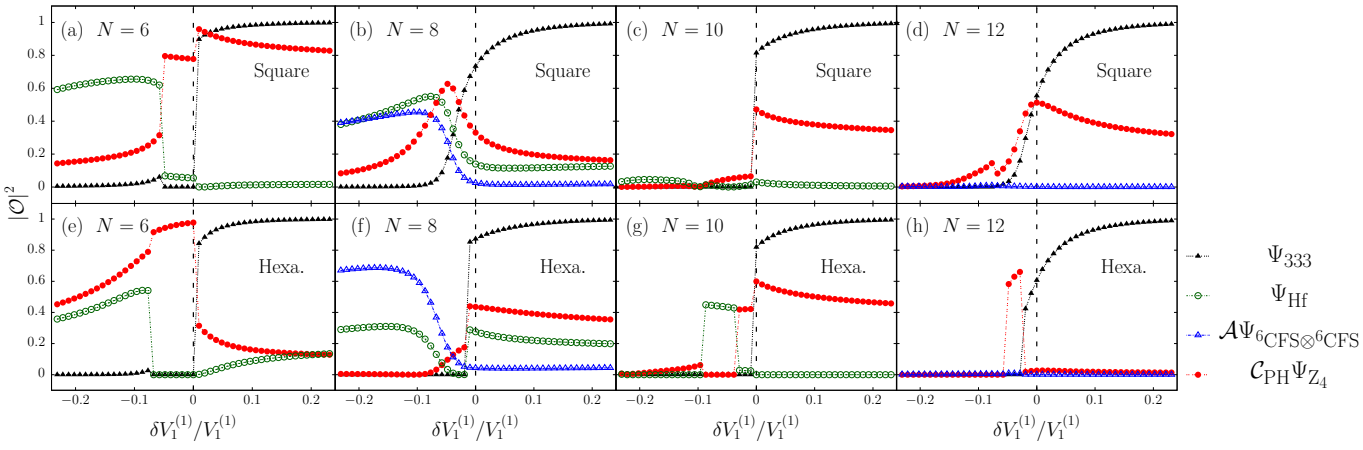

FIG. 2. (Color online) Square of overlap $|\mathcal{O}|^2$ between the exact 7/3 state, $\Psi_{7/3}[\delta V_1^{(1)}]$, and each of the four trial states, Ψ_{333}, $\Psi_{\rm Hf}$, $\mathcal{A}\Psi_{^6{\rm CFS}\otimes{}^6{\rm CFS}}$, and $\mathcal{C}_{\rm PH}\Psi_{Z_4}$, as a function of the Haldane pseudopotential variation $\delta V_1^{(1)}/V_1^{(1)}$ with $V_1^{(1)} = 0.415419$. The exact 7/3 state $\Psi_{7/3}[\delta V_1^{(1)}]$ is obtained by exactly diagonalizing the 1LL FQH Hamiltonian in Eq. (5) at pseudomomentum $\mathbf{Q} = (N/2, N/2)$, where the global ground state occurs for the entire range of $\delta V_1^{(1)}$. The shape of the unit cell is square in (a)–(d), and hexagonal in (e)–(h). The electron-electron interaction is purely Coulombic at $\delta V_1^{(1)} = 0$.

IV. OVERLAP

In this section, we compute the overlap between the exact 7/3 state and each of the four trial states, Ψ_{333}, $\Psi_{\rm Hf}$, $\mathcal{A}\Psi_{^6{\rm CFS}\otimes{}^6{\rm CFS}}$, and $\mathcal{C}_{\rm PH}\Psi_{Z_4}$ by using ED in the torus and the spherical geometries up to $N = 12$. The exact 7/3 state $\Psi_{7/3}[\delta V_1^{(1)}]$ is obtained by exactly diagonalizing the 1LL FQH Hamiltonian in Eq. (5) as a function of the Haldane pseudopotential variation $\delta V_1^{(1)}/V_1^{(1)}$ with $V_1^{(1)} = 0.415419$.

The torus geometry can accommodate different parallelogram shapes of the unit cell via the continuous variation of the angle between two lateral vectors $\mathbf{L}_1$ and $\mathbf{L}_2$, which can deform the unit cell from square to hexagon [28]. It is important to note that, while Ψ_{333} can be defined for all particle numbers, the two trial states, $\Psi_{\rm Hf}$ and $\mathcal{C}_{\rm PH}\Psi_{Z_4}$, can be defined only for even particle numbers. Also, due to a property of the antisymmetrized product state, $\mathcal{A}\Psi_{^6{\rm CFS}\otimes{}^6{\rm CFS}}$ can be defined only when the particle number is a multiple of four.

Figure 2 shows the square of overlap between $\Psi_{7/3}[\delta V_1^{(1)}]$ and each of the four trial states, Ψ_{333}, $\Psi_{\rm Hf}$, $\mathcal{A}\Psi_{^6{\rm CFS}\otimes{}^6{\rm CFS}}$, and $\mathcal{C}_{\rm PH}\Psi_{Z_4}$ as a function of the Haldane pseudopotential variation $\delta V_1^{(1)}/V_1^{(1)}$ for various particle numbers in the torus geometry. As one can see, there are somewhat wild fluctuations in the behavior of the overlap across various particle numbers. Despite these fluctuations, however, it is possible to extract the following properties.

First, the overlap between $\Psi_{7/3}[\delta V_1^{(1)}]$ and Ψ_{333} shows the most stable behavior as a function of $\delta V_1^{(1)}/V_1^{(1)}$ regardless of the particle number. Specifically, the overlap is close to unity for sufficiently large $\delta V_1^{(1)}/V_1^{(1)}$, but decreases very fast as $\delta V_1^{(1)}/V_1^{(1)}$ approaches the Coulomb point. At moderately negative $\delta V_1^{(1)}/V_1^{(1)}$, the overlap becomes negligibly small.

Second, the overlap between $\Psi_{7/3}[\delta V_1^{(1)}]$ and $\Psi_{\rm Hf}$ is sizable at moderately negative $\delta V_1^{(1)}/V_1^{(1)}$ for finite-size systems with relatively small particle numbers, say, $N = 6$ and 8. Unfortunately, however, the overlap seems to decrease fast as the particle number increases. Therefore, we conclude that $\Psi_{\rm Hf}$ has little chance for representing the exact 7/3 state around the Coulomb point in the thermodynamic limit.

It is worth mentioning that we construct $\Psi_{\rm Hf}$ as the linear combination between two degenerate ground states [which are the antisymmetrized projections of two degenerate Halperin (551) states] at $Q = (N/2, N/2)$ to generate the maximum overlap with $\Psi_{7/3}[\delta V_1^{(1)}]$. Note that the fermionic Haffnian ground state is expected to have a diverging degeneracy as a function of particle number in the torus geometry, which can be revealed in the energy spectrum of the parent Hamiltonian generating $\Psi_{\rm Hf}$ [84]. Here, we do not consider the possibility of other degenerate Haffnian states, which could be different from those obtained via the antisymmetrization of the above two degenerate Halperin (551) states. We stress, however, that such a degeneracy issue does not occur for the ground state in the spherical geometry. It is shown below (in Fig. 3) that the decreasing behavior of the overlap as a function of particle number is similarly observed in the spherical geometry.

Third, the overlap between $\Psi_{7/3}[\delta V_1^{(1)}]$ and $\mathcal{A}\Psi_{^6{\rm CFS}\otimes{}^6{\rm CFS}}$ is actually rather similar to that between $\Psi_{7/3}[\delta V_1^{(1)}]$ and $\Psi_{\rm Hf}$. That is, the overlap between $\Psi_{7/3}[\delta V_1^{(1)}]$ and $\mathcal{A}\Psi_{^6{\rm CFS}\otimes{}^6{\rm CFS}}$ is sizable at moderately negative $\delta V_1^{(1)}/V_1^{(1)}$ at $N = 8$, but almost completely collapses at $N = 12$ for the entire range of $\delta V_1^{(1)}/V_1^{(1)}$. Therefore, we also conclude that $\mathcal{A}\Psi_{^6{\rm CFS}\otimes{}^6{\rm CFS}}$ has little chance for representing the exact 7/3 state around the Coulomb point in the thermodynamic limit.

Finally, while fluctuating across various particle numbers, the overlap between $\Psi_{7/3}[\delta V_1^{(1)}]$ and $\mathcal{C}_{\rm PH}\Psi_{Z_4}$ exhibits consistently sizable values around the Coulomb point. It is in-

teresting to note that the overlap between $\Psi_{7/3}[\delta V_1^{(1)}]$ and $\mathcal{C}_{\rm PH}\Psi_{Z_4}$ seems to be peaked around the Coulomb point, where the Laughlin state loses almost entirely its overlap with the exact 7/3 state. Therefore, we conclude that $\mathcal{C}_{\rm PH}\Psi_{Z_4}$ has a reasonable chance for representing the exact 7/3 state around the Coulomb point, which is the highest among those of the four trial states.

It is important to note that there is a double degeneracy for Ψ_{Z_4} at $\mathbf{Q} = (N/2, N/2)$, which is known to be the *zero physical momentum* for even particle numbers [65]. In our calculation, we construct Ψ_{Z_4} as a linear combination between these two degenerate states to generate the maximum overlap with $\Psi_{7/3}[\delta V_1^{(1)}]$. In Sec. V, we discuss the degeneracy issue in more details since it can provide additional evidence for the validity of $\mathcal{C}_{\rm PH}\Psi_{Z_4}$.

Now, we would like to discuss what happens in the spherical geometry. In particular, we are interested in the overlap between the exact $7/3$ state and each of the following two trial states, Ψ_{333} and $\Psi_{\rm Hf}$. We do not consider $\mathcal{A}\Psi_{6\rm CFS\otimes 6CFS}$ since its overlap with the exact 7/3 state is already too low in the torus geometry. We do not think that the situation is going to be different in the spherical geometry. On the other hand, at present, it is technically difficult for us to construct $\mathcal{C}_{\rm PH}\Psi_{Z_4}$ in the spherical geometry since we do not know how to apply the PH conjugation operator in this geometry. Therefore, at present, we cannot compute the overlap between the exact 7/3 state and $\mathcal{C}_{\rm PH}\Psi_{Z_4}$ in the spherical geometry.

Before computing the overlap with the exact 7/3 state, first, we would like to check if $\Psi_{\rm Hf}$ is a worthy trial state. To this end, we compute the Coulomb interaction energies of Ψ_{333} and $\Psi_{\rm Hf}$ by performing the (variational) Monte Carlo simulations up to $N = 50$, the results of which can be extrapolated to the thermodynamic limit. To make such large-scale Monte Carlo simulations possible, it is important to write the trial states in concrete mathematical forms. Unfortunately, only the LLL representations of Ψ_{333} [Eq. (13)] and $\Psi_{\rm Hf}$ [Eq. (15)] are known concretely. To overcome this problem, we follow a trick invented in Ref. [85]. In this trick, the Coulomb interaction potential is replaced by the effective interaction potential $V_{\rm eff}(r) = \frac{1}{r} + a_1 e^{-\alpha_1 r^2} + a_2 r^2 e^{-\alpha_2 r^2}$, where a_1, a_2, α_1, and α_2 are fixed by requiring that the first four pseudopotentials of $V_{\rm eff}(r)$ in the LLL should be equal to the first four pseudopotentials of the Coulomb interaction in the SLL. As a result of using this trick, we are able to estimate that, in the thermodynamic limit, the Coulomb interaction energies of Ψ_{333} and $\Psi_{\rm Hf}$ are $-0.325(0)$ and $-0.320(9)$ in units of $e^2/\epsilon l_B$, respectively. This means that the two trial states are very competitive in the SLL. For comparison, note that, in the LLL, the Coulomb interaction energies of Ψ_{333} and $\Psi_{\rm Hf}$ are estimated to be $-0.4097(3)$ and $-0.3719(1)$ in units of $e^2/\epsilon l_B$, respectively.

It is interesting to mention that, when the "flux shift" is set equal to 5 (which corresponds to $\Psi_{\rm Hf}$), the ground state energy exhibits a clear even-odd effect, where the ground state energy oscillates depending on whether the particle number is even or odd [22]. This suggests that a certain pairing correlation exists in this state. When the flux shift is set equal to 3 (which is the Laughlin value), no such even-odd effect is observed.

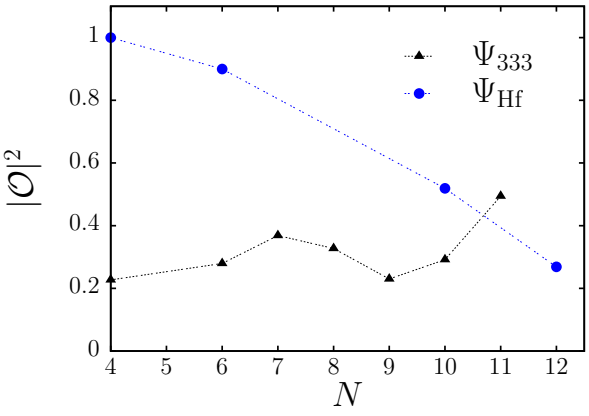

FIG. 3. (Color online) Square of overlap $|\mathcal{O}|^2$ between the exact 7/3 state at the Coulomb point and each of the two trial states, Ψ_{333} and $\Psi_{\rm Hf}$, in the spherical geometry. The two trial states occur at different flux shifts; $S = 3$ for Ψ_{333} and $S = 5$ for $\Psi_{\rm Hf}$. Note that $\Psi_{\rm Hf}$ can occur only at even particle numbers. The data points at $N = 5$ for Ψ_{333} and $N = 8$ for $\Psi_{\rm Hf}$ are missing due to the fact that the total spin quantum numbers are non-zero here.

Figure 3 shows the square of overlap between the exact 7/3 state and Ψ_{333} with the flux shift set equal to 3, and that between the exact 7/3 state and $\Psi_{\rm Hf}$ with the flux shift set equal to 5. It is important to note that the exact 7/3 states are defined as the Coulomb ground states occurring in two different flux sectors each corresponding to the flux shift of Ψ_{333} and $\Psi_{\rm Hf}$, respectively. As one can see, the overlap between the exact 7/3 state and $\Psi_{\rm Hf}$ is initially quite high, but decreases rapidly as the particle number increases. This behavior is completely consistent with what is observed in the torus geometry. Therefore, we arrive at the same conclusion that $\Psi_{\rm Hf}$ has little chance for representing the exact 7/3 state around the Coulomb point in the thermodynamic limit.

V. QUASIDEGENERACY IN THE ENERGY SPECTRUM

The substantial overlap between the exact 7/3 state and $\mathcal{C}_{\rm PH}\Psi_{Z_4}$ around the Coulomb point motivates us to investigate if there is some signature for $\mathcal{C}_{\rm PH}\Psi_{Z_4}$ in the exact energy spectrum. A particular signature that we would like to focus on here is the characteristic degeneracy of $\mathcal{C}_{\rm PH}\Psi_{Z_4}$ occurring at specific pseudomomentum channels in the torus geometry, as explained below.

The degeneracy of Ψ_{Z_4} can be evaluated by examining the structure of the conformal field theory, which says that the degenearacy should be fifteen-fold in the torus geometry [40]. Alternatively, the number of degeneracy can be determined by examining the root configurations of Ψ_{Z_4} in the thin-torus limit, which are obtained by the rule that no more than four particles can occupy six consecutive orbitals [86, 87]. Similarly, the root configurations of $\mathcal{C}_{\rm PH}\Psi_{Z_4}$ can be obtained by the rule that no more than two particles can occupy six consecutive orbitals. By examining the allowed pseudomomentum for each root configuration, one can show that degenerate

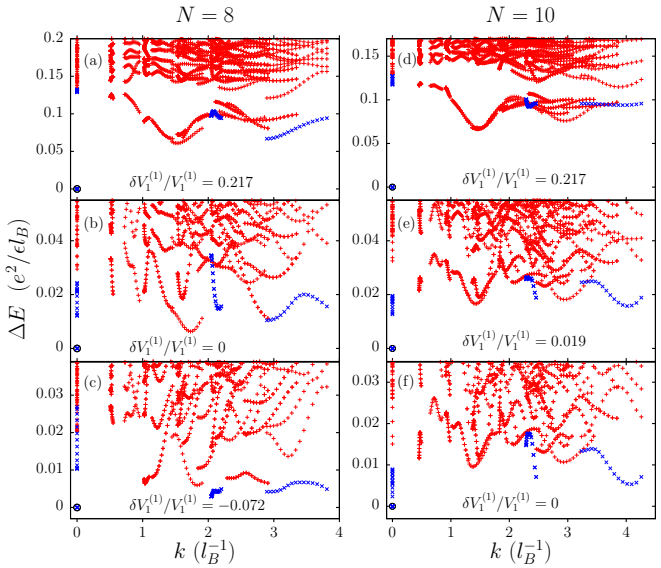

FIG. 4. (Color online) Exact energy spectra in the torus geometry for various values of the Haldane pseudopotential variation $\delta V_1^{(1)}/V_1^{(1)}$ as a function of the magnitude of the physical momentum $k = |\mathbf{k}|$ in units of $1/l_B$. The energy spectra are obtained by collecting data points from many different parallelogram shapes of the unit cell obtained via the continuous variation of the angle between two lateral vectors $\mathbf{L}_1$ and $\mathbf{L}_2$, which deforms the unit cell from square to hexagon [28]. Lowest-energy states at pseudomomenta $\mathbf{Q} = (0,0)$, $(N/2,0)$, $(0,N/2)$, and $(N/2,N/2)$ for each different shape of the unit cell are denoted as blue x's in comparison with all other states denoted as red crosses. Note that energy eigenstates at $(N/2,0)$ and $(0,N/2)$ are degenerate to each other due to the unity aspect ratio. See the text for the relationship between $\mathbf{Q}$ and $\mathbf{k}$.

ground states should occur at pseudomomenta $\mathbf{Q} = (0,0)$, $(N/2,0)$, $(0,N/2)$, and $(N/2,N/2)$ with doubly degenerate ground states at $\mathbf{Q} = (N/2,N/2)$. This makes the total number of degeneracy become 15 $(= 3 \times 5)$ because each set of the above five degenerate ground states has three center-of-mass degenerate copies of its own. As a consequence, if Ψ_{Z_4} describes the exact 7/3 state accurately, there should be degenerate, or at least quasidegenerate copies of the ground state at pseudomomenta $\mathbf{Q} = (0,0)$, $(N/2,0)$, $(0,N/2)$, $(N/2,N/2)$, and their center-of-mass-shifted counterparts.

Figure 4 shows exact energy spectra in the torus geometry for various values of the Haldane pseudopotential variation $\delta V_1^{(1)}/V_1^{(1)}$ as a function of the magnitude of the physical momentum $k = |\mathbf{k}|$ in units of $1/l_B$. The physical momentum $\mathbf{k}$ is related with $\mathbf{Q}$ via $\mathbf{k} = \mathbf{Q} - \mathbf{k}_0$, where $\mathbf{k}_0$ denotes the zero momentum, which corresponds to $\mathbf{k}_0 = (N'/2, N'/2)$ for even N' and $\mathbf{k}_0 = 0$ for odd N', where $N' = \gcd(N, N_\phi)$ [65]. Energy spectra are computed in two different finite-size systems with $N = 8$ and 10, which show similar behaviors as a function of $\delta V_1^{(1)}/V_1^{(1)}$.

At sufficiently large positive $\delta V_1^{(1)}/V_1^{(1)}$, the energy spectrum exhibits a well-developed magnetoroton structure with its minimum located at $|\mathbf{k}|/l_B \simeq 1.4$, which is the defining signature of the Laughlin state [43, 65, 88, 89]. This is consistent with the fact that the overlap between the exact 7/3 state $\Psi_{7/3}[\delta V_1^{(1)}]$ and Ψ_{333} is essentially unity at sufficiently large positive $\delta V_1^{(1)}/V_1^{(1)}$.

On the other hand, as $\delta V_1^{(1)}/V_1^{(1)}$ is lowered approaching the Coulomb point, the energy spectrum undergoes an intriguing transition from the Laughlin-type spectrum to the spectrum with a quasidegeneracy in the ground state energy that is characteristic to the PH conjugate of the Z_4 parafermion state. That is, lowest-energy states at $\mathbf{Q} = (0,0)$, $(N/2,0)$, $(0,N/2)$, and $(N/2,N/2)$ are pulled away from other excited states, and become essentially the lowest-energy excited states around the Coulomb point or at slightly negative $\delta V_1^{(1)}$. We believe that this provides a compelling piece of evidence supporting that the PH conjugate of the Z_4 parafermion state is closely related with the exact 7/3 ground state around the Coulomb point.

It is worth mentioning that the quasidegenerate excited state at $\mathbf{k} = 0$ was previously interpreted as a signature for the onset of the incompressible-to-compressible phase transition [50]. In our interpretation, this state is regarded as a degenerate copy of the Z_4 parafermion states occurring at $\mathbf{Q} = (N/2,N/2)$ along with the others at $\mathbf{Q} = (0,0)$, $(N/2,0)$, and $(0,N/2)$.

VI. CONCLUSION

In this work, we investigate the exact nature of the FQH state at $\nu = 7/3$ by using ED in the torus as well as the spherical geometries. Specifically, we compute the overlap between the exact $7/3$ ground state and various competing states including (i) the Laughlin state, Ψ_{333}, (ii) the fermionic Haffnian state, $\Psi_{\rm Hf}$, (iii) the antisymmetrized product state of two CF seas at 1/6 filling, $\mathcal{A}\Psi_{^6\rm CFS\otimes{}^6CFS}$, and (iv) the PH conjugate of the Z_4 parafermion state, $\mathcal{C}_{\rm PH}\Psi_{Z_4}$.

It is shown that, while extremely accurate at large positive Haldane pseudopotential variation $\delta V_1^{(1)}$, the Laughlin state loses almost entirely its overlap with the exact 7/3 state around the Coulomb point. At slightly negative $\delta V_1^{(1)}$, it is shown that the particle-hole (PH) conjugate of the Z_4 parafermion state has a substantial overlap with the exact 7/3 state, which is the highest among those of the above four trial states. Around the Coulomb point, the energy spectrum exhibits an intriguing change from the Laughlin-type spectrum with a well-developed magnetoroton structure to the spectrum with a quasidegeneracy in the ground state energy that is characteristic to the PH conjugate of the Z_4 parafermion state.

All the above four trial states are constructed according to a guiding principle called the bilayer mapping approach, where a trial state is obtained as the antisymmetrized projection of a bilayer quantum Hall state with interlayer distance d as a variational parameter. The bilayer mapping approach can be regarded an alternative to the Z_k parafermion approach, although the two approaches coincide in the case of the Z_4 parafermion state. That is, $\Psi_{Z_4} = \mathcal{A}\Psi_{330}$.

An interesting future direction is to investigate what happens if one applies the bilayer mapping approach to the

12/5 (13/5) problem in the 2/5-filled SLL. Natural trial states obtained in the bilayer mapping approach are the antisymmetrized projections of the Halperin (441) and (550) states, $\mathcal{A}\Psi_{441}$ and $\mathcal{A}\Psi_{550}$, at $\nu = 12/5$ and their PH conjugates at $\nu = 13/5$. In light of the substantial overlap between the exact 7/3 state and $\mathcal{C}_{\rm PH}\Psi_{Z_4}$, it would be interesting to compute the overlap between the exact 12/5 state and $\mathcal{A}\Psi_{550}$. It is noted that the antisymmetrized Halperin $(nn0)$ state, $\mathcal{A}\Psi_{nn0}$, had been proposed previously in a different context [59], while detailed properties of $\mathcal{A}\Psi_{550}$ were not studied. Meanwhile, it had been argued in various numerical studies [40, 47, 48] that the Z_3 parafermion state could provide a good trial state faithfully representing the exact 13/5 state. If so, it would also interesting to investigate if there is a connection between the Z_3 parafermion state and the PH conjugate of $\mathcal{A}\Psi_{550}$.

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Appendix A: Proof of an identify for the symmetrized Jastrow-factor polynomial

In this Appendix, we prove Eq. (17). We begin by setting $N/2 = n$, and

$$x_i = z_i,\ x_{i+n} = \omega_i, \tag{A1}$$

and rewrite Eq. (17) as

$$2^{n-1}\sum_{(I,J);a_1=1}\Delta(I)^4\Delta(J)^4 = \left[\sum_{(I,J);a_1=1}\Delta(I)^2\Delta(J)^2\right]^2, \tag{A2}$$

where

$$\Delta(\{a_1,a_2,\ldots,a_r\})^l = \prod_{1\le p<q\le r}\left(x_{a_p}-x_{a_q}\right)^l, \tag{A3}$$

and the summations are taken over

$$I=\{a_1,a_2\ldots,a_n\},\ 1=a_1<a_2<\cdots<a_n \tag{A4}$$
$$J=\{b_1,b_2\ldots,b_n\},\ b_1<b_2<\cdots<b_n \tag{A5}$$
$$\{a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n\}=\{1,2,\ldots,2n\}. \tag{A6}$$

Equation (A2) trivially holds for $n = 1$. Below, we prove it for general n by induction.

To this end, let us set

$$P_n(x_1,x_2\ldots,x_{2n}) = \sum_{(I,J);a_1=1}\Delta(I)^4\Delta(J)^4 \tag{A7}$$
$$= \sum_{(I',J')}\prod_{2\le p\le n}(x_1-x_{a_p})^4\Delta\left(I'\right)^4\prod_{2\le q\le n}(x_{b_1}-x_{b_q})^4\Delta\left(J'\right)^4, \tag{A8}$$

and

$$Q_n(x_1,x_2\ldots,x_{2n}) = \sum_{(I,J);a_1=1}\Delta(I)^2\Delta(J)^2 \tag{A9}$$
$$= \sum_{(I',J')}\prod_{2\le p\le n}(x_1-x_{a_p})^2\Delta\left(I'\right)^2\prod_{2\le q\le n}(x_{b_1}-x_{b_q})^2\Delta\left(J'\right)^2, \tag{A10}$$

where $I' = \{a_2,\ldots,a_n\}$ and $J' = \{b_2,\ldots,b_n\}$. We assume the induction hypothesis, which is written in terms of P_{n-1} and Q_{n-1} as follows:

$$2^{n-2}P_{n-1} = Q_{n-1}^2. \tag{A11}$$

First, we prove that Eq. (A11) holds for $x_1 = x_2 = x$. Since $\Pi(x_1 - x_{a_p}) = 0$ when $x_1 = x_2 = x$ and $a_2 = 2$, we have

$$P_n(x,x,x_3,\ldots,x_{2n}) = \sum_{(I',J')}\prod_{2\le p\le n}(x-x_{a_p})^4\Delta\left(I'\right)^4\prod_{2\le q\le n}(x-x_{b_q})^4\Delta\left(J'\right)^4 \tag{A12}$$
$$= \prod_{3\le k\le 2n}(x-x_k)^4\sum_{(I',J')}\Delta(I')^4\Delta(J')^4 \tag{A13}$$
$$= 2\prod_{3\le k\le 2n}(x-x_k)^4\sum_{(I',J');a_2=3}\Delta(I')^4\Delta(J')^4 \tag{A14}$$
$$= 2\prod_{3\le k\le 2n}(x-x_k)^4P_{n-1}(x_3,\ldots,x_{2n}) \tag{A15}$$

and

$$Q_n(x,x,x_3,\ldots,x_{2n}) = 2\prod_{3\le k\le 2n}(x-x_k)^2Q_{n-1}(x_3,\ldots,x_{2n}), \tag{A16}$$

which, by the help of the induction hypothesis in Eq. (A11), give rise to

$$Q_n(x,x,x_3,\ldots,x_{2n})^2 = 2^{n-1}P_n\left(x,x,x_3,\ldots,x_{2n}\right). \tag{A17}$$

This implies

$$2^{n-1}P_n - Q_n^2 = (x_1-x_2)^s\mathcal{R}, \tag{A18}$$

where the polynomial $\mathcal{R}$ is nonzero for $x_1 = x_2 = x$. In order to determine s, we prove

$$2^{n-1}\frac{\partial P_n(x,x,x_3\ldots,x_{2n})}{\partial x} = \frac{\partial Q_n(x,x,x_3\ldots,x_{2n})^2}{\partial x}. \tag{A19}$$

By Eq. (A15), we have

$$\frac{\partial P_n(x,x,x_3\ldots,x_{2n})}{\partial x} = \prod_{3\le l\le 2n}(x-x_l)^4 \sum_{3\le k\le 2n}\frac{8}{x-x_k}P_{n-1}(x_3,\ldots,x_{2n}). \tag{A20}$$

Similarly, by Eq. (A16), we have

$$\frac{\partial Q_n(x,x,x_3,\ldots,x_{2n})^2}{\partial x} = 2Q_n(x,x,x_3,\ldots,x_{2n}) \tag{A21}$$

$$\times \prod_{3\le l\le 2n}(x-x_l)^2 \sum_{3\le k\le 2n}\frac{4}{x-x_k}Q_{n-1}(x_3,\ldots,x_{2n}) \tag{A22}$$

$$= \prod_{2\le l\le 2n}(x-x_l)^4 \sum_{3\le k\le 2n}\frac{16}{x-x_k}Q_{n-1}(x_3,\ldots,x_{2n})^2, \tag{A23}$$

which, by the help of the induction hypothesis [Eq. (A11)] and Eq. (A20), leads to the proof of Eq. (A19). Thus, the partial derivative of both sides of Eq. (A18) with respect to x for $x_1 = x_2 = x$ gives rise to

$$s(x_1-x_2)^{s-1}\mathcal{R}|_{x_1=x_2=x} = 0, \tag{A24}$$

which is satisfied only for $s \ge 2$. Hence, by the symmetry, we have

$$2^{n-1}P_n - Q_n^2 = \prod_{1\le i<j\le 2n}(x_i-x_j)^2\mathcal{V} \tag{A25}$$

for some polynomial $\mathcal{V}$. However, the degree of each monomial in the left-hand side, which is $4n(n-1)$, is less than that in the right-hand side, which is at least $2n(2n-1)$. Therefore, $\mathcal{V}$ must be zero. Q.E.D.

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