Siegel families with application to class fields

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Abstract

We investigate certain families of meromorphic Siegel modular functions on which Galois groups act in a natural way. By using Shimura's reciprocity law we construct some algebraic numbers in the ray class fields of CM-fields in terms of special values of functions in these Siegel families.

1 Introduction

For a positive integer N let $\mathfrak{F}_N$ be the field of meromorphic modular functions of level N (defined on $\mathbb{H}=\{\tau\in\mathbb{C}\mid \mathrm{Im}(\tau)>0\}$) whose Fourier coefficients belong to the Nth cyclotomic field. As is well known, $\mathfrak{F}_N$ is a Galois extension of $\mathfrak{F}_1$ whose Galois group is isomorphic to $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ ([8, §6.1–6.2]). Now, let $N\geq 2$ and consider a set

$$V_N=\{\mathbf{v}\in\mathbb{Q}^2\mid N\text{ is the smallest positive integer for which }N\mathbf{v}\in\mathbb{Z}^2\}$$

as the index set. We call a family $\{f_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in V_N}$ of functions in $\mathfrak{F}_N$ a *Fricke family* of level N if each $f_{\mathbf{v}}(\tau)$ depends only on $\pm\mathbf{v}\pmod{\mathbb{Z}^2}$ and satisfies

$$f_{\mathbf{v}}(\tau)^\alpha=f_{\alpha^T\mathbf{v}}(\tau)\quad(\alpha\in\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}),$$

where α^T means the transpose of α. For example, Siegel functions of one-variable form such a Fricke family of level N ([5, Proposition 1.3 in Chapter 2]). See also [2] or [4].

Let K be an imaginary quadratic field with the ring of integers $\mathcal{O}_K$, and let $\mathfrak{f}$ be a proper nontrivial ideal of $\mathcal{O}_K$. We denote by $\mathrm{Cl}(\mathfrak{f})$ and $K_{\mathfrak{f}}$ the ray class group modulo $\mathfrak{f}$ and its corresponding ray class field modulo $\mathfrak{f}$, respectively. If $\{f_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$ is a Fricke family of level N in which every $f_{\mathbf{v}}(\tau)$ is holomorphic on $\mathbb{H}$, then we can assign to each ray class $\mathcal{C}\in\mathrm{Cl}(\mathfrak{f})$ an algebraic number $f_{\mathfrak{f}}(\mathcal{C})$ as a special value of a function in $\{f_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$. Furthermore, we attain by Shimura's reciprocity law that $f_{\mathfrak{f}}(\mathcal{C})$ belongs to $K_{\mathfrak{f}}$ and satisfies

$$f_{\mathfrak{f}}(\mathcal{C})^{\sigma_{\mathfrak{f}}(\mathcal{D})}=f_{\mathfrak{f}}(\mathcal{C}\mathcal{D})\quad(\mathcal{D}\in\mathrm{Cl}(\mathfrak{f})),$$

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where $\sigma_{\mathfrak{f}}$ is the Artin reciprocity map for $\mathfrak{f}$ ([5, Theorem 1.1 in Chapter 11]).

In this paper, we shall define a Siegel family $\{h_M(Z)\}_M$ of level N consisting of meromorphic Siegel modular functions of (higher) genus g and level N, which would be a generalization of a Fricke family of level N in case $g = 1$ (Definition 3.1). It turns out that every Siegel family of level N is induced from a meromorphic Siegel modular function for the congruence subgroup $\Gamma^1(N)$ (Theorem 3.5).

Let K be a CM-field and let $\mathfrak{f} = N\mathcal{O}_K$. Given a Siegel family $\{h_M(Z)\}_M$ of level N, we shall introduce a number $h_{\mathfrak{f}}(\mathcal{C})$ by a special value of a function in $\{h_M(Z)\}_M$ for each ray class $\mathcal{C} \in \mathrm{Cl}(\mathfrak{f})$ (Definition 4.4). Under certain assumptions on K (Assumption 4.1) we shall prove that if $h_{\mathfrak{f}}(\mathcal{C})$ is finite, then it lies in the ray class field $K_{\mathfrak{f}}$ whose Galois conjugates are of the same form (Theorem 6.2 and Corollary 6.3). To this end, we assign a principally polarized abelian variety to each nontrivial ideal of $\mathcal{O}_K$, and apply Shimura's reciprocity law to $h_{\mathfrak{f}}(\mathcal{C})$.

2 Actions on Siegel modular functions

First, we shall describe the Galois group between fields of meromorphic Siegel modular functions in a concrete way.

Let g be a positive integer, and let $\eta_g = \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}$. For every commutative ring R with unity we denote by

$$\begin{aligned} \mathrm{GSp}_{2g}(R) &= \left\{\alpha \in \mathrm{GL}_{2g}(R) \mid \alpha^T \eta_g \alpha = \nu(\alpha)\eta_g \text{ with } \nu(\alpha) \in R^\times\right\}, \\ \mathrm{Sp}_{2g}(R) &= \{\alpha \in \mathrm{GSp}_{2g}(R) \mid \nu(\alpha) = 1\}. \end{aligned}$$

Let

$$G = \mathrm{GSp}_{2g}(\mathbb{Q}),$$

and let $G_{\mathbb{A}}$ be the adelization of G, G_0 its non-archimedean part and G_∞ its archimedean part. One can extend the multiplier map $\nu : G \to \mathbb{Q}^\times$ continuously to the map $\nu : G_{\mathbb{A}} \to \mathbb{Q}_{\mathbb{A}}^\times$, and set

$$G_{\infty+} = \{\alpha \in G_\infty \mid \nu(\alpha) > 0\}, \quad G_{\mathbb{A}+} = G_0 G_{\infty+}, \quad G_+ = G \cap G_{\mathbb{A}+}.$$

Furthermore, let

$$\begin{aligned} \Delta &= \left\{ \begin{bmatrix} I_g & O_g \\ O_g & sI_g \end{bmatrix} \mid s \in \prod_p \mathbb{Z}_p^\times \right\}, \\ U_1 &= \prod_p \mathrm{GSp}_{2g}(\mathbb{Z}_p) \times G_{\infty+}, \\ U_N &= \{x \in U_1 \mid x_p \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z}_p)} \text{ for all rational primes } p\} \end{aligned}$$

for every positive integer N. Then we have

$$U_N \trianglelefteq U_1 \leq G_{\mathbb{A}+} \quad \text{and} \quad G_{\mathbb{A}+} = U_N \Delta G_+$$

([10, Lemma 8.3 (1)]).

Note that the group $G_{\infty+}$ acts on the Siegel upper half-space $\mathbb{H}_g = \{Z \in M_g(\mathbb{C}) \mid Z^T = Z,\ \mathrm{Im}(Z) \text{ is positive definite}\}$ by

$$\alpha(Z) = (AZ+B)(CZ+D)^{-1} \quad (\alpha \in G_{\infty+},\ Z \in \mathbb{H}_g),$$

where A, B, C, D are $g \times g$ block matrices of α. Let $\mathcal{F}_N$ be the field of meromorphic Siegel modular functions of genus g for the congruence subgroup

$$\Gamma(N) = \left\{\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z})}\right\}$$

of the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ whose Fourier coefficients belong to the Nth cyclotomic field $\mathbb{Q}(\zeta_N)$ with $\zeta_N = e^{2\pi i/N}$. That is, if $f \in \mathcal{F}_N$, then

$$f(Z) = \sum_h c(h) e(\mathrm{tr}(hZ)/N) \quad \text{for some } c(h) \in \mathbb{Q}(\zeta_N),$$

where h runs over all $g \times g$ positive semi-definite symmetric matrices over half integers with integral diagonal entries, and $e(w) = e^{2\pi i w}$ for $w \in \mathbb{C}$ ([3, Theorem 1 in §4]). Let

$$\mathcal{F} = \bigcup_{N=1}^{\infty} \mathcal{F}_N.$$

PROPOSITION 2.1. *There exists a homomorphism $\tau : G_{\mathbb{A}+} \to \mathrm{Aut}(\mathcal{F})$ satisfying the following properties: Let $f(Z) = \sum_h c(h)e(\mathrm{tr}(hZ)/N) \in \mathcal{F}_N$.*

(i) *If $\alpha \in G_+ = \{\alpha \in G \mid \nu(\alpha) > 0\}$, then*

$$f^{\tau(\alpha)} = f \circ \alpha.$$

(ii) *If $\beta = \begin{bmatrix} I_g & O_g \\ O_g & sI_g \end{bmatrix} \in \Delta$ and t is a positive integer such that $t \equiv s_p \pmod{N\mathbb{Z}_p}$ for all rational primes p, then*

$$f^{\tau(\beta)} = \sum_h c(h)^{\sigma} e(\mathrm{tr}(hZ)/N),$$

where σ is the automorphism of $\mathbb{Q}(\zeta_N)$ given by $\zeta_N^{\sigma} = \zeta_N^t$.

(iii) *For every positive integer N we have*

$$\mathcal{F}_N = \{f \in \mathcal{F} \mid f^{\tau(x)} = f \text{ for all } x \in U_N\}.$$

(iv) $\ker(\tau) = \mathbb{Q}^{\times} G_{\infty+}$.

PROOF. See [10, Theorem 8.10]. □

Since

$$U_N(\mathbb{Q}^\times G_{\infty+})/\mathbb{Q}^\times G_{\infty+} \simeq U_N/(U_N \cap \mathbb{Q}^\times G_{\infty+}) \simeq \begin{cases} U_1/\pm G_{\infty+} & \text{if } N=1, \\ U_N/G_{\infty+} & \text{if } N>1, \end{cases}$$

we see by Proposition 2.1 (iii) and (iv) that $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1$ with

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq U_1/\pm U_N. \tag{1}$$

PROPOSITION 2.2. *We have*

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}.$$

PROOF. Let $\alpha \in U_1$. Take a matrix A in $M_{2g}(\mathbb{Z})$ for which $A \equiv \alpha_p \pmod{N \cdot M_{2g}(\mathbb{Z}_p)}$ for all rational primes p. Define a matrix $\psi(\alpha) \in M_{2g}(\mathbb{Z}/N\mathbb{Z})$ by the image of A under the natural reduction $M_{2g}(\mathbb{Z}) \to M_{2g}(\mathbb{Z}/N\mathbb{Z})$. Then by the Chinese remainder theorem $\psi(\alpha)$ is well defined and independent of the choice of A. Furthermore, let t be an integer relatively prime to N such that $t \equiv \nu(\alpha_p) \pmod{N\mathbb{Z}_p}$ for all rational primes p. We then derive that

$$t\eta_g \equiv \nu(\alpha_p)\eta_g \equiv \alpha_p^T \eta_g \alpha_p \equiv A^T \eta_g A \equiv \psi(\alpha)^T \eta_g \psi(\alpha) \pmod{N \cdot M_{2g}(\mathbb{Z}_p)}$$

for all rational primes p, and hence $\psi(\alpha) \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Thus we obtain a group homomorphism

$$\psi : U_1 \to \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}).$$

Let $\beta \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, and take a preimage B of β under the natural reduction $M_{2g}(\mathbb{Z}) \to M_{2g}(\mathbb{Z}/N\mathbb{Z})$. Since $\nu(\beta) \in (\mathbb{Z}/N\mathbb{Z})^\times$ and

$$B^T \eta_g B \equiv \beta^T \eta_g \beta \equiv \nu(\beta)\eta_g \pmod{N \cdot M_{2g}(\mathbb{Z})},$$

B belongs to $\mathrm{GSp}_{2g}(\mathbb{Z}_p)$ for every rational prime p dividing N. Let $\alpha = (\alpha_p)_p$ be the element of $\prod_p \mathrm{GSp}_{2g}(\mathbb{Z}_p)$ given by

$$\alpha_p = \begin{cases} B & \text{if } p \mid N, \\ I_{2g} & \text{otherwise.} \end{cases}$$

We then see that $\alpha \in U_1$ and $\psi(\alpha) = \beta$. Thus ψ is surjective.

Clearly, U_N is contained in $\ker(\psi)$. Let $\gamma \in \ker(\psi)$. Since $\gamma_p \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z}_p)}$ for all rational primes p, we get $\gamma \in U_N$, and hence $\ker(\psi) = U_N$. Therefore ψ induces an isomorphism $U_1/U_N \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, from which we achieve by (1)

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq U_1/\pm U_N \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}.$$

□

REMARK 2.3. We have the decomposition

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq G_N \cdot \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\},$$

where

$$G_N = \left\{ \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \mid \nu \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$

By Proposition 2.1 one can describe the action of $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on $\mathcal{F}_N$ as follows: Let $f(Z) = \sum_h c(h)e(\mathrm{tr}(hZ)/N) \in \mathcal{F}_N$.

(i) An element $\beta = \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix}$ of G_N acts on f by

$$f^\beta = \sum_h c(h)^\sigma e(\mathrm{tr}(hZ)/N),$$

where σ is the automorphism of $\mathbb{Q}(\zeta_N)$ satisfying $\zeta_N^\sigma = \zeta_N^\nu$.

(ii) An element γ of $\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ acts on f by

$$f^\gamma = f \circ \gamma',$$

where γ' is any preimage of γ under the natural reduction $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$.

3 Siegel families of level N

By making use of the description of $\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$ in §2 we shall introduce a generalization of a Fricke family in higher dimensional cases.

Let $N \geq 2$. For $\alpha \in M_{2g}(\mathbb{Z})$ we denote by $\widetilde{\alpha}$ its reduction modulo N. Define a set

$$\mathcal{V}_N = \left\{ (1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix} \mid \alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z}) \text{ such that } \widetilde{\alpha} \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \right\}.$$

Let M be an element of $\mathcal{V}_N$ stemmed from $\alpha \in M_{2g}(\mathbb{Z})$ such that $\widetilde{\alpha} \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, and let β be an element of $M_{2g}(\mathbb{Z})$ satisfying $\widetilde{\beta} \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Then it is straightforward that $\beta^T M$ is also an element of $\mathcal{V}_N$ given by the product $\alpha\beta$.

Definition 3.1. We call a family $\{h_M(Z)\}_{M \in \mathcal{V}_N}$ a *Siegel family* of level N if it satisfies the following properties:

(S1) Each $h_M(Z)$ belongs to $\mathcal{F}_N$.

(S2) $h_M(Z)$ depends only on $\pm M \pmod{M_{2g\times g}(\mathbb{Z})}$.

(S3) $h_M(Z)^\sigma = h_{\sigma^T M}(Z)$ for all $\sigma \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq \mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$.

By $\mathcal{S}_N$ we mean the set of such Siegel families of level N.

Remark 3.2. Let $\{h_M(Z)\}_M \in \mathcal{S}_N$.

(i) The property (S3) yields a right action of the group $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on $\{h_M(Z)\}_M$.

(ii) Let $M = (1/N)\begin{bmatrix} A^T \\ B^T \end{bmatrix} \in \mathcal{V}_N$, and so there is a matrix $\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})$ such that $\widetilde{\alpha} \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Considering $\widetilde{\alpha}$ as an element of $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ we obtain

$$h_{(1/N)\left[\begin{smallmatrix} I_g \\ O_g \end{smallmatrix}\right]}(Z)^{\widetilde{\alpha}} = h_{(1/N)\alpha^T\left[\begin{smallmatrix} I_g \\ O_g \end{smallmatrix}\right]}(Z) = h_M(Z).$$

Thus the action of $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on $\{h_M(Z)\}_M$ is transitive.

Let

$$\Gamma^1(N) = \left\{ \gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix} \pmod{N \cdot M_{2g}(\mathbb{Z})} \right\},$$

and let $\mathcal{F}_N^1(\mathbb{Q})$ be the field of meromorphic Siegel modular functions for $\Gamma^1(N)$ with rational Fourier coefficients.

Lemma 3.3. *If* $\{h_M(Z)\}_M \in \mathcal{S}_N$, *then* $h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z) \in \mathcal{F}_N^1(\mathbb{Q})$.

Proof. For any $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma^1(N)$ we deduce by (S2) and (S3) that

$$h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(\gamma(Z)) = h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z)^{\widetilde{\gamma}} = h_{\gamma^T\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z) = h_{(1/N)\left[\begin{smallmatrix} A^T \\ B^T \end{smallmatrix}\right]}(Z) = h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z)$$

because $A \equiv I_g,\ B \equiv O_g \pmod{N \cdot M_g(\mathbb{Z})}$. Thus $h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z)$ is modular for $\Gamma^1(N)$.

For every $\nu \in (\mathbb{Z}/N\mathbb{Z})^\times$ we see by (S2) and (S3) that

$$h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z)^{\left[\begin{smallmatrix} I_g & O_g \\ O_g & \nu I_g \end{smallmatrix}\right]} = h_{\left[\begin{smallmatrix} I_g & O_g \\ O_g & \nu I_g \end{smallmatrix}\right]\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z) = h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z),$$

which implies that $h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z)$ has rational Fourier coefficients. This proves the lemma. $\square$

One can consider $\mathcal{S}_N$ as a field under the binary operations

$$\begin{aligned} \{h_M(Z)\}_M + \{k_M(Z)\}_M &= \{(h_M + k_M)(Z)\}_M, \\ \{h_M(Z)\}_M \cdot \{k_M(Z)\}_M &= \{(h_M k_M)(Z)\}_M. \end{aligned}$$

By Lemma 3.3 we get the ring homomorphism

$$\begin{aligned} \phi_N \ :\ \mathcal{S}_N &\to \mathcal{F}_N^1(\mathbb{Q}) \\ \{h_M(Z)\}_M &\mapsto h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z). \end{aligned}$$

Lemma 3.4. *If* $M \in \mathcal{V}_N$, *then there is* $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})$ *such that* $\widetilde{\gamma} \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ *and* $M = (1/N)\begin{bmatrix} A^T \\ B^T \end{bmatrix}$.

Proof. Let $\alpha = \begin{bmatrix} A & B \\ U & V \end{bmatrix} \in M_{2g}(\mathbb{Z})$ such that $\widetilde{\alpha} \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and $M = (1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix}$. In $M_{2g}(\mathbb{Z}/N\mathbb{Z})$, decompose $\widetilde{\alpha}$ as

$$\widetilde{\alpha} = \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix} \quad \text{with } \nu = \nu(\widetilde{\alpha}) \in (\mathbb{Z}/N\mathbb{Z})^\times$$

so that $\begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix}$ belongs to $\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Since the reduction $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ is surjective ([7]), we can take $\gamma \in M_{2g}(\mathbb{Z})$ satisfying $\widetilde{\gamma} = \begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix}$. □

Theorem 3.5. *$\mathcal{S}_N$ and $\mathcal{F}_N^1(\mathbb{Q})$ are isomorphic via ϕ_N.*

Proof. Since $\mathcal{S}_N$ and $\mathcal{F}_N^1(\mathbb{Q})$ are fields, it suffices to show that ϕ_N is surjective.

Let $h(Z) \in \mathcal{F}_N^1(\mathbb{Q})$. For each $M \in \mathcal{V}_N$, take any $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})$ such that $\widetilde{\gamma} \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and $M = (1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix}$ by using Lemma 3.4. And, set

$$h_M(Z) = h(Z)^{\widetilde{\gamma}}.$$

We claim that $h_M(Z)$ is independent of the choice of γ. Indeed, if $\gamma' = \begin{bmatrix} A & B \\ C' & D' \end{bmatrix} \in M_{2g}(\mathbb{Z})$ such that $\widetilde{\gamma'} \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, then we attain in $M_{2g}(\mathbb{Z}/N\mathbb{Z})$ that

$$\widetilde{\gamma'}\widetilde{\gamma}^{-1} = \begin{bmatrix} A & B \\ C' & D' \end{bmatrix} \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix} = \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix}$$

by the fact $\widetilde{\gamma}, \widetilde{\gamma'} \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Let δ be an element of $\mathrm{Sp}_{2g}(\mathbb{Z})$ such that $\widetilde{\delta} = \widetilde{\gamma'}\widetilde{\gamma}^{-1}$. We then achieve

$$h(Z)^{\widetilde{\gamma'}} = (h(Z)^{\widetilde{\gamma'}\widetilde{\gamma}^{-1}})^{\widetilde{\gamma}} = h(\delta(Z))^{\widetilde{\gamma}} = h(Z)^{\widetilde{\gamma}}$$

because $h(Z)$ is modular for $\Gamma^1(N)$ and $\delta \in \Gamma^1(N)$.

Now, for any $\sigma = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ with $\nu = \nu(\sigma)$ we derive that

$$\begin{aligned} h_M(Z)^\sigma &= h(Z)^{\widetilde{\gamma}\sigma} \\ &= h(Z)^{\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]\left[\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right]} \\ &= h(Z)^{\left[\begin{smallmatrix} I_g & O_g \\ O_g & \nu I_g \end{smallmatrix}\right]\left[\begin{smallmatrix} AP+BR & AQ+BS \\ \nu^{-1}(CP+DR) & \nu^{-1}(CQ+DS) \end{smallmatrix}\right]} \\ &= h(Z)^{\left[\begin{smallmatrix} AP+BR & AQ+BS \\ \nu^{-1}(CP+DR) & \nu^{-1}(CQ+DS) \end{smallmatrix}\right]} \quad \text{since } h(Z) \text{ has rational Fourier coefficients} \\ &= h_{\left[\begin{smallmatrix} (AP+BR)^T \\ (AQ+BS)^T \end{smallmatrix}\right]}(Z) \end{aligned}$$

$$
\begin{aligned}
&= h_{\left[\begin{smallmatrix} P^T & R^T \\ Q^T & S^T \end{smallmatrix}\right]\left[\begin{smallmatrix} A^T \\ B^T \end{smallmatrix}\right]}(Z) \\
&= h_{\sigma^T M}(Z).
\end{aligned}
$$

This shows that the family $\{h_M(Z)\}_M$ belongs to $\mathcal{S}_N$. Furthermore, since

$$
\phi_N(\{h_M(Z)\}_M) = h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z) = h(Z)^{\left[\begin{smallmatrix} I_g & O_g \\ O_g & I_g \end{smallmatrix}\right]} = h(Z),
$$

ϕ is surjective as desired. □

Remark 3.6. (i) By Proposition 2.2 and Remark 2.3 we obtain

$$
\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_N^1(\mathbb{Q})) \simeq G_N \cdot \left\{ \gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \mid \gamma = \pm \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix} \right\}.
$$

(ii) Let $\mathcal{F}_{1,N}(\mathbb{Q})$ be the field of meromorphic Siegel modular functions for

$$
\Gamma_1(N) = \left\{ \gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} I_g & * \\ O_g & I_g \end{bmatrix} \pmod{N \cdot M_{2g}(\mathbb{Z})} \right\}
$$

with rational Fourier coefficients. If we set

$$
\omega = \begin{bmatrix} (1/\sqrt{N})I_g & O_g \\ O_g & \sqrt{N} I_g \end{bmatrix},
$$

then we know that $\omega \in \mathrm{Sp}_{2g}(\mathbb{R})$ and

$$
\omega \begin{bmatrix} A & B \\ C & D \end{bmatrix} \omega^{-1} = \begin{bmatrix} A & (1/N)B \\ NC & D \end{bmatrix} \quad \text{for all } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R}).
$$

This implies

$$
\omega \Gamma^1(N) \omega^{-1} = \Gamma_1(N),
$$

and so $\mathcal{F}_{1,N}(\mathbb{Q})$ and $\mathcal{F}_N^1(\mathbb{Q})$ are isomorphic via

$$
\begin{aligned}
\mathcal{F}_{1,N}(\mathbb{Q}) &\rightarrow \mathcal{F}_N^1(\mathbb{Q}) \\
h(Z) &\mapsto (h \circ \omega)(Z) = h((1/N)Z).
\end{aligned}
$$

4 Special values associated with a Siegel family

As an application of a Siegel family of level N we shall construct a number associated with each ray class modulo N of a CM-field.

Let n be a positive integer, K be a CM-field with $[K : \mathbb{Q}] = 2n$ and $\{\varphi_1, \ldots, \varphi_n\}$ be a set of embeddings of K into $\mathbb{C}$ such that $(K, \{\varphi_i\}_{i=1}^n)$ is a CM-type. We fix a finite Galois extension L of $\mathbb{Q}$ containing K, and set

$$
S = \{\sigma \in \mathrm{Gal}(L/\mathbb{Q}) \mid \sigma|_K = \varphi_i \text{ for some } i \in \{1, 2, \ldots, n\}\},
$$

$$\begin{aligned} S^* &= \{\sigma^{-1} \mid \sigma \in S\}, \\ H^* &= \{\gamma \in \mathrm{Gal}(L/\mathbb{Q}) \mid \gamma S^* = S^*\}. \end{aligned}$$

Let K^* be the subfield of L corresponding to the subgroup H^* of $\mathrm{Gal}(L/\mathbb{Q})$, and let $\{\psi_1, \ldots, \psi_g\}$ be the set of all embeddings of K^* into $\mathbb{C}$ arising from the elements of S^*. Then we know that $(K^*, \{\psi_j\}_{j=1}^g)$ is a primitive CM-type and

$$K^* = \mathbb{Q}\left(\sum_{i=1}^{n} a^{\varphi_i} \mid a \in K\right)$$

([9, Proposition 28 in §8.3]). We call this CM-type $(K^*, \{\psi_j\}_{j=1}^g)$ the reflex of $(K, \{\varphi_i\}_{i=1}^n)$. Using this CM-type we define an embedding

$$\begin{aligned} \Psi : K^* &\to \mathbb{C}^g \\ a &\mapsto \begin{bmatrix} a^{\psi_1} \\ \vdots \\ a^{\psi_g} \end{bmatrix}. \end{aligned}$$

For each purely imaginary element c of K^* we associate an $\mathbb{R}$-bilinear form

$$\begin{aligned} E_c : \ \mathbb{C}^g \times \mathbb{C}^g &\to \mathbb{R} \\ (\mathbf{u}, \mathbf{v}) &\mapsto \sum_{j=1}^{g} c^{\psi_j}(u_j \overline{v}_j - \overline{u}_j v_j) \quad (\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_g \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_g \end{bmatrix}). \end{aligned}$$

Then, one can readily check that

$$E_c(\Psi(a), \Psi(b)) = \mathrm{Tr}_{K^*/\mathbb{Q}}(ca\overline{b}) \quad \text{for all } a, b \in K^* \tag{2}$$

by utilizing the fact $\overline{a^{\psi_j}} = \overline{a}^{\psi_j}$ for all $a \in K^*$ $(1 \leq j \leq g)$.

Assumption 4.1. In what follows we assume the following conditions:

(i) $(K^*)^* = K$.

(ii) There is a purely imaginary element ξ of K^* and a $\mathbb{Z}$-basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_{2g}\}$ of the lattice $\Psi(\mathcal{O}_{K^*})$ in $\mathbb{C}^g$ for which

$$\Big[E_\xi(\mathbf{a}_i, \mathbf{a}_j)\Big]_{1 \leq i,j \leq 2g} = \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}.$$

In this case, we say that the complex torus $(\mathbb{C}^g/\Psi(\mathcal{O}_{K^*}), E_\xi)$ is a principally polarized abelian variety with a symplectic basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_{2g}\}$. See [9, §6.2].

(iii) $\mathfrak{f} = N\mathcal{O}_K$ for an integer $N \geq 2$.

Remark 4.2. The Assumption 4.1 (i) is equivalent to saying that $(K, \{\varphi_i\}_{i=1}^n)$ is a primitive CM-type, namely, the abelian varieties of this CM-type are simple ([9, Proposition 26 in §8.2]).

By Assumption 4.1 (i) one can define a group homomorphism

$$\begin{array}{rccl} \mathfrak{g}: & K^\times & \to & (K^*)^\times \\ & d & \mapsto & \prod_{i=1}^n d^{\varphi_i}, \end{array}$$

and extend it continuously to the homomorphism $\mathfrak{g}: K^\times_\mathbb{A} \to (K^*)^\times_\mathbb{A}$ of idele groups. It is also known that for each fractional ideal $\mathfrak{a}$ of K there is a fractional ideal $\mathcal{G}(\mathfrak{a})$ of K^* such that

$$\mathcal{G}(\mathfrak{a})\mathcal{O}_L = \prod_{i=1}^n (\mathfrak{a}\mathcal{O}_L)^{\varphi_i}$$

([9, §8.3]). Let $\mathcal{C}$ be a given ray class in $\mathrm{Cl}(\mathfrak{f})$. Take any integral ideal $\mathfrak{c}$ in $\mathcal{C}$, and let

$$\mathcal{N}(\mathfrak{c}) = \mathcal{N}_{K/\mathbb{Q}}(\mathfrak{c}) = |\mathcal{O}_K/\mathfrak{c}|.$$

Lemma 4.3. *$(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$ is also a principally polarized abelian variety.*

Proof. It follows from (2) that

$$\begin{aligned} E_{\xi\mathcal{N}(\mathfrak{c})}(\Psi(\mathcal{G}(\mathfrak{c})^{-1}), \Psi(\mathcal{G}(\mathfrak{c})^{-1})) &= \mathrm{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c})\mathcal{G}(\mathfrak{c})^{-1}\overline{\mathcal{G}(\mathfrak{c})^{-1}}) \\ &= \mathrm{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{O}_{K^*}) \\ &= E_\xi(\Psi(\mathcal{O}_{K^*}), \Psi(\mathcal{O}_{K^*})) \\ &\subseteq \mathbb{Z} \end{aligned}$$

because E_ξ is a Riemann form on $\mathbb{C}^g/\Psi(\mathcal{O}_{K^*})$. Thus $E_{\xi\mathcal{N}(\mathfrak{c})}$ defines a Riemann form on $\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1})$.

Now, let $\{\mathbf{b}_1, \ldots, \mathbf{b}_{2g}\}$ be a symplectic basis of the abelian variety $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$ so that

$$\Psi(\mathcal{G}(\mathfrak{c})^{-1}) = \sum_{j=1}^{2g} \mathbb{Z}\mathbf{b}_j \quad \text{and} \quad \left[E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{b}_i, \mathbf{b}_j)\right]_{1\le i,j\le 2g} = \begin{bmatrix} O_g & -\mathcal{E} \\ \mathcal{E} & O_g \end{bmatrix},$$

where $\mathcal{E} = \begin{bmatrix} \varepsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_g \end{bmatrix}$ is a $g \times g$ diagonal matrix for some positive integers $\varepsilon_1, \ldots, \varepsilon_g$ satisfying $\varepsilon_1 \mid \cdots \mid \varepsilon_g$. Furthermore, let $b_1 \ldots, b_{2g}$ be elements of $\mathcal{G}(\mathfrak{c})^{-1}$ such that $\mathbf{b}_j = \Psi(b_j)$ $(1 \le j \le 2g)$. Since $\mathcal{O}_{K^*} \subseteq \mathcal{G}(\mathfrak{c})^{-1}$, we have

$$\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{2g} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_{2g} \end{bmatrix} \alpha \quad \text{for some } \alpha \in M_{2g}(\mathbb{Z}) \cap \mathrm{GL}_{2g}(\mathbb{Q}), \tag{3}$$

and hence

$$\begin{bmatrix} a_1^{\psi_1} & \cdots & a_{2g}^{\psi_1} \\ \vdots & & \vdots \\ a_1^{\psi_g} & \cdots & a_{2g}^{\psi_g} \\ \overline{a_1^{\psi_1}} & \cdots & \overline{a_{2g}^{\psi_1}} \\ \vdots & & \vdots \\ \overline{a_1^{\psi_g}} & \cdots & \overline{a_{2g}^{\psi_g}} \end{bmatrix} = \begin{bmatrix} b_1^{\psi_1} & \cdots & b_{2g}^{\psi_1} \\ \vdots & & \vdots \\ b_1^{\psi_g} & \cdots & b_{2g}^{\psi_g} \\ \overline{b_1^{\psi_1}} & \cdots & \overline{b_{2g}^{\psi_1}} \\ \vdots & & \vdots \\ \overline{b_1^{\psi_g}} & \cdots & \overline{b_{2g}^{\psi_g}} \end{bmatrix} \alpha.$$

Taking determinant and squaring gives rise to the identity

$$\Delta_{K^*/\mathbb{Q}}(a_1,\ldots,a_{2g}) = \Delta_{K^*/\mathbb{Q}}(b_1,\ldots,b_{2g})\det(\alpha)^2.$$

It then follows that

$$\begin{aligned}\det(\alpha)^2 = \frac{|\Delta_{K^*/\mathbb{Q}}(a_1,\ldots,a_{2g})|}{|\Delta_{K^*/\mathbb{Q}}(b_1,\ldots,b_{2g})|} = \frac{d_{K^*/\mathbb{Q}}(\mathcal{O}_{K^*})}{d_{K^*/\mathbb{Q}}(\mathcal{G}(\mathfrak{c})^{-1})} &= \mathcal{N}_{K^*/\mathbb{Q}}(\mathcal{G}(\mathfrak{c}))^2 \\ &= \mathcal{N}_{K^*/\mathbb{Q}}(\mathcal{G}(\mathfrak{c})\overline{\mathcal{G}(\mathfrak{c})}) \\ &= \mathcal{N}(\mathfrak{c})^{2g},\end{aligned} \tag{4}$$

where $d_{K^*/\mathbb{Q}}$ stands for the discriminant of a fractional ideal of K^* ([6, Proposition 13 in Chapter III]). And, we deduce by (3) that

$$\begin{aligned}\mathcal{N}(\mathfrak{c})\begin{bmatrix} O_g & -I_g \\ I_g & O_g\end{bmatrix} &= \Big[\mathcal{N}(\mathfrak{c})E_\xi(\mathbf{a}_i,\mathbf{a}_j)\Big]_{1\le i,j\le 2g} \\ &= \Big[E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{a}_i,\mathbf{a}_j)\Big]_{1\le i,j\le 2g} \\ &= \alpha^T\Big[E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{b}_i,\mathbf{b}_j)\Big]_{1\le i,j\le 2g}\,\alpha \\ &= \alpha^T\begin{bmatrix} O_g & -\mathcal{E} \\ \mathcal{E} & O_g\end{bmatrix}\alpha.\end{aligned}$$

By taking determinant we get

$$\mathcal{N}(\mathfrak{c})^{2g} = \det(\alpha)^2(\varepsilon_1\cdots\varepsilon_g)^2,$$

which yields by (4) that $\varepsilon_1 = \cdots = \varepsilon_g = 1$, and so $\mathcal{E} = I_g$. Therefore, $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$ becomes a principally polarized abelian variety. □

As in the proof of Lemma 4.3 we take a symplectic basis $\{\mathbf{b}_1,\ldots,\mathbf{b}_{2g}\}$ of the principally polarized abelian variety $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$, and let $b_1,\ldots,b_{2g}$ be elements of $\mathcal{G}(\mathfrak{c})^{-1}$ such that $\mathbf{b}_j = \Psi(b_j)$ $(1\le j\le 2g)$. We then have

$$\begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_{2g}\end{bmatrix} = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_{2g}\end{bmatrix}\alpha \quad \text{for some } \alpha = \begin{bmatrix} A & B \\ C & D\end{bmatrix} \in M_{2g}(\mathbb{Z})\cap \mathrm{GSp}_{2g}(\mathbb{Q}). \tag{5}$$

Since $\nu(\alpha) = \mathcal{N}(\mathfrak{c})$ is relatively prime to N, the reduction $\widetilde{\alpha}$ of α modulo N belongs to $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Let $Z_\mathfrak{c}^*$ be the CM-point associated with the symplectic basis $\{\mathbf{b}_1,\ldots,\mathbf{b}_{2g}\}$, namely

$$Z_\mathfrak{c}^* = \begin{bmatrix}\mathbf{b}_{g+1} & \cdots & \mathbf{b}_{2g}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_g\end{bmatrix}$$

which belongs to $\mathbb{H}_g$ ([1, Proposition 8.1.1]).

Definition 4.4. Let $\{h_M(Z)\}_M \in \mathcal{S}_N$. For a given ray class $\mathcal{C}\in \mathrm{Cl}(\mathfrak{f})$ we define

$$h_\mathfrak{f}(\mathcal{C}) = h_{(1/N)\left[\begin{smallmatrix} B \\ D\end{smallmatrix}\right]}(Z_\mathfrak{c}^*).$$

REMARK 4.5. Here, the index matrix $(1/N)\begin{bmatrix} B \\ D \end{bmatrix}$ is obtained by the fact

$$(\begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}\alpha)^T = \begin{bmatrix} B^T & D^T \\ -A^T & -C^T \end{bmatrix}.$$

5 Well-definedness of $h_{\mathfrak{f}}(\mathcal{C})$

In this section we shall show that the value $h_{\mathfrak{f}}(\mathcal{C})$ given in Definition 4.4 depends only on the ray class $\mathcal{C}$, and hence it is independent of the choice of a symplectic basis and an integral ideal in $\mathcal{C}$.

PROPOSITION 5.1. *$h_{\mathfrak{f}}(\mathcal{C})$ does not depend on the choice of a symplectic basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_{2g}\}$ of $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$.*

PROOF. Let $\{\widehat{\mathbf{b}}_1, \ldots, \widehat{\mathbf{b}}_{2g}\}$ be another symplectic basis of $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$, and so

$$\begin{bmatrix} \widehat{\mathbf{b}}_1 & \cdots & \widehat{\mathbf{b}}_{2g} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_{2g} \end{bmatrix}\beta \quad \text{for some } \beta = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \mathrm{GL}_{2g}(\mathbb{Z}). \tag{6}$$

We then derive that

$$\begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} = \begin{bmatrix} E_{\xi\mathcal{N}(\mathfrak{c})}(\widehat{\mathbf{b}}_i, \widehat{\mathbf{b}}_j) \end{bmatrix}_{1\leq i,j\leq 2g} = \beta^T \begin{bmatrix} E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{b}_i, \mathbf{b}_j) \end{bmatrix}_{1\leq i,j\leq 2g}\beta = \beta^T \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}\beta,$$

which shows that $\beta \in \mathrm{Sp}_{2g}(\mathbb{Z})$. Since

$$\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{2g} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_{2g} \end{bmatrix}\alpha = \begin{bmatrix} \widehat{\mathbf{b}}_1 & \cdots & \widehat{\mathbf{b}}_{2g} \end{bmatrix}\beta^{-1}\alpha$$

by (5) and (6), the special value obtained by $\{\widehat{\mathbf{b}}_1, \ldots, \widehat{\mathbf{b}}_{2g}\}$ is

$$h_{(1/N)\beta^{-1}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(\widehat{Z}_{\mathfrak{c}}^*),$$

where $\widehat{Z}_{\mathfrak{c}}^*$ is the CM-point corresponding to $\{\widehat{\mathbf{b}}_1, \ldots, \widehat{\mathbf{b}}_{2g}\}$.

On the other hand, we attain that

$$\begin{aligned}
\widehat{Z}_{\mathfrak{c}}^* &= \begin{bmatrix} \widehat{\mathbf{b}}_{g+1} & \cdots & \widehat{\mathbf{b}}_{2g} \end{bmatrix}^{-1} \begin{bmatrix} \widehat{\mathbf{b}}_1 & \cdots & \widehat{\mathbf{b}}_g \end{bmatrix} \\
&= \left(\begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_g \end{bmatrix} Q + \begin{bmatrix} \mathbf{b}_{g+1} & \cdots & \mathbf{b}_{2g} \end{bmatrix} S\right)^{-1} \\
&\quad \left(\begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_g \end{bmatrix} P + \begin{bmatrix} \mathbf{b}_{g+1} & \cdots & \mathbf{b}_{2g} \end{bmatrix} R\right) \quad \text{by (6)} \\
&= \left(P^T \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_g \end{bmatrix}^T + R^T \begin{bmatrix} \mathbf{b}_{g+1} & \cdots & \mathbf{b}_{2g} \end{bmatrix}^T\right) \\
&\quad \left(Q^T \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_g \end{bmatrix}^T + S^T \begin{bmatrix} \mathbf{b}_{g+1} & \cdots & \mathbf{b}_{2g} \end{bmatrix}^T\right)^{-1} \quad \text{since } (\widehat{Z}_{\mathfrak{c}}^*)^T = \widehat{Z}_{\mathfrak{c}}^* \\
&= \left(P^T \left(\begin{bmatrix} \mathbf{b}_{g+1} & \cdots & \mathbf{b}_{2g} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_g \end{bmatrix}\right)^T + R^T\right)
\end{aligned}$$

$$
\begin{aligned}
&\left(Q^T\left(\begin{bmatrix}\mathbf{b}_{g+1} & \cdots & \mathbf{b}_{2g}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_g\end{bmatrix}\right)^T + S^T\right)^{-1}\\
= \;& (P^T(Z_\mathfrak{c}^*)^T + R^T)(Q^T(Z_\mathfrak{c}^*)^T + S^T)^{-1}\\
= \;& (P^T Z_\mathfrak{c}^* + R^T)(Q^T Z_\mathfrak{c}^* + S^T)^{-1} \quad \text{because } (Z_\mathfrak{c}^*)^T = Z_\mathfrak{c}^*\\
= \;& \beta^T(Z_\mathfrak{c}^*). \qquad (7)
\end{aligned}
$$

Thus we deduce that

$$
\begin{aligned}
h_{(1/N)\beta^{-1}\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(\widehat{Z}_\mathfrak{c}^*) &= h_{(1/N)\beta^{-1}\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(\beta^T(Z_\mathfrak{c}^*)) \quad \text{by (7)}\\
&= (h_{(1/N)\beta^{-1}\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(Z))^{\beta^T}|_{Z=Z_\mathfrak{c}^*}\\
&= h_{(1/N)(\beta^T)^T\beta^{-1}\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(Z_\mathfrak{c}^*) \quad \text{by the property (S3) of } \{h_M(Z)\}_M\\
&= h_{(1/N)\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(Z_\mathfrak{c}^*).
\end{aligned}
$$

This proves that $h_\mathfrak{f}(\mathcal{C})$ is independent of the choice of a symplectic basis of $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$. □

REMARK 5.2. In like manner one can readily show that $h_\mathfrak{f}(\mathcal{C})$ does not depend on the choice of a symplectic basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_{2g}\}$ of $(\mathbb{C}^g/\Psi(\mathcal{O}_K), E_\xi)$.

PROPOSITION 5.3. *$h_\mathfrak{f}(\mathcal{C})$ does not depend on the choice of an integral ideal $\mathfrak{c}$ in $\mathcal{C}$.*

PROOF. Let $\mathfrak{c}'$ be another integral ideal in the class $\mathcal{C}$, and hence

$$
\mathfrak{c}'\mathfrak{c}^{-1} = (1+a)\mathcal{O}_K \quad \text{for some } a \in \mathfrak{f}\mathfrak{a}^{-1}, \qquad (8)
$$

where $\mathfrak{a}$ is an integral ideal of K relatively prime to $\mathfrak{f}$. Since $1 \in \mathfrak{c}^{-1}$ and $(1+a) \in \mathfrak{c}'\mathfrak{c}^{-1} \subseteq \mathfrak{c}^{-1}$, we get $a \in \mathfrak{c}^{-1}$. Thus we derive that

$$
\begin{aligned}
a\mathfrak{a}\mathfrak{c} &\subseteq \mathfrak{f}\mathfrak{c} \cap \mathfrak{a} \quad \text{by the facts } a \in \mathfrak{f}\mathfrak{a}^{-1} \text{ and } a \in \mathfrak{c}^{-1}\\
&\subseteq \mathfrak{f} \cap \mathfrak{a}\\
&= \mathfrak{f}\mathfrak{a} \quad \text{because } \mathfrak{f} \text{ and } \mathfrak{a} \text{ are relatively prime,}
\end{aligned}
$$

from which it follows that $a \in \mathfrak{f}\mathfrak{c}^{-1}$. We then achieve by the fact $\mathfrak{f} = N\mathcal{O}_K$ that

$$
\mathfrak{g}(1+a) = \prod_{i=1}^n (1+a)^{\varphi_i} \in K^* \cap \prod_{i=1}^n (1 + N(\mathfrak{c}^{-1}\mathcal{O}_L)^{\varphi_i}) \subseteq K^* \cap (1 + N\mathcal{G}(\mathfrak{c})^{-1}\mathcal{O}_L) = 1 + N\mathcal{G}(\mathfrak{c})^{-1}. \quad (9)
$$

Let

$$
b_j' = \mathfrak{g}(1+a)^{-1} b_j \quad \text{and} \quad \mathbf{b}_j' = \Psi(b_j') \quad (1 \le j \le 2g). \qquad (10)
$$

We know that $\{\mathbf{b}_1', \ldots, \mathbf{b}_{2g}'\}$ is a $\mathbb{Z}$-basis of the lattice $\Psi(\mathcal{G}(\mathfrak{c}')^{-1})$ in $\mathbb{C}^g$ and

$$
\mathbf{b}_j' = T\mathbf{b}_j \quad \text{with } T = \begin{bmatrix} (\mathfrak{g}(1+a)^{-1})^{\psi_1} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & (\mathfrak{g}(1+a)^{-1})^{\psi_g}\end{bmatrix}. \qquad (11)
$$

Furthermore, we get that

$$\begin{aligned}
\left[E_{\xi\mathcal{N}(\mathfrak{c}')}(\mathbf{b}'_i, \mathbf{b}'_j)\right]_{1\le i,j\le 2g} &= \left[\mathrm{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c}')b'_i\overline{b'_j})\right]_{1\le i,j\le 2g} \quad \text{by (2)}\\
&= \left[\mathrm{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c}')\mathfrak{g}(1+a)^{-1}b_i\overline{\mathfrak{g}(1+a)^{-1}b_j})\right]_{1\le i,j\le 2g} \quad \text{by (10)}\\
&= \left[\mathrm{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c}')\mathrm{N}_{K/\mathbb{Q}}(1+a)^{-1}b_i\overline{b_j})\right]_{1\le i,j\le 2g}\\
&= \left[\mathrm{Tr}_{K/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c})b_i\overline{b_j})\right]_{1\le i,j\le 2g}\\
&\quad\ \text{by (8) and the fact } \mathrm{N}_{K/\mathbb{Q}}(1+a)>0\\
&= \left[E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{b}_i, \mathbf{b}_j)\right]_{1\le i,j\le 2g} \quad \text{by (2)}\\
&= \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}.
\end{aligned}$$

Thus $\{\mathbf{b}'_1, \ldots, \mathbf{b}'_{2g}\}$ is a symplectic basis of $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c}')^{-1}), E_{\xi\mathcal{N}(\mathfrak{c}')})$, and its associated CM-point $Z^*_{\mathfrak{c}'}$ is given by

$$\begin{aligned}
Z^*_{\mathfrak{c}'} &= \begin{bmatrix}\mathbf{b}'_{g+1} & \cdots & \mathbf{b}'_{2g}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{b}'_1 & \cdots & \mathbf{b}'_g\end{bmatrix}\\
&= \begin{bmatrix}T\mathbf{b}_{g+1} & \cdots & T\mathbf{b}_{2g}\end{bmatrix}^{-1}\begin{bmatrix}T\mathbf{b}_1 & \cdots & T\mathbf{b}_g\end{bmatrix} \quad \text{by (11)}\\
&= Z^*_{\mathfrak{c}}.
\end{aligned} \tag{12}$$

Let $\alpha = \begin{bmatrix}a_{ij}\end{bmatrix}$, $\alpha' = \begin{bmatrix}a'_{ij}\end{bmatrix} \in M_{2g}(\mathbb{Z})$ such that

$$\begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_{2g}\end{bmatrix} = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_{2g}\end{bmatrix}\alpha = \begin{bmatrix}\mathbf{b}'_1 & \cdots & \mathbf{b}'_{2g}\end{bmatrix}\alpha'. \tag{13}$$

For each $1 \le i \le 2g$ we obtain that

$$\begin{aligned}
\sum_{j=1}^{2g} a'_{ji}b_j &= \mathfrak{g}(1+a)\sum_{j=1}^{2g} a'_{ji}b'_j \quad \text{by (10)}\\
&= a_i\mathfrak{g}(1+a) \quad \text{by (13)}\\
&\in a_i(1+N\mathcal{G}(\mathfrak{c})^{-1}) \quad \text{by (9)}\\
&\subseteq a_i + N\mathcal{G}(\mathfrak{c})^{-1} \quad \text{because } a_i \in \mathcal{O}_K\\
&= \sum_{j=1}^{2g} a_{ji}b_j + N\sum_{j=1}^{2g}\mathbb{Z}b_j \quad \text{by (13)}.
\end{aligned}$$

This yields $\alpha \equiv \alpha' \pmod{N \cdot M_{2g}(\mathbb{Z})}$, and hence

$$(1/N)\alpha \equiv (1/N)\alpha' \pmod{M_{2g}(\mathbb{Z})}. \tag{14}$$

Now, the result follows from (12), (14) and the property (S2) of $\{h_M(Z)\}_M$. □

6 Galois actions on $h_{\mathfrak{f}}(\mathcal{C})$

Finally we shall show that if $h_{\mathfrak{f}}(\mathcal{C})$ is finite, then it lies in the ray class field $K_{\mathfrak{f}}$ and satisfies the natural transformation formula under the Artin reciprocity map for $\mathfrak{f}$.

Let $r : K^* \to M_{2g}(\mathbb{Q})$ be the regular representation with respect to the ordered basis $\{a_1, \ldots, a_{2g}\}$ of K^* over $\mathbb{Q}$ given by

$$a \begin{bmatrix} a_1 \\ \vdots \\ a_{2g} \end{bmatrix} = r(a) \begin{bmatrix} a_1 \\ \vdots \\ a_{2g} \end{bmatrix} \quad (a \in K^*). \tag{15}$$

Then it can be extended to the map $r : (K^*)_{\mathbb{A}} \to M_{2g}(\mathbb{Q}_{\mathbb{A}})$ of adele rings.

LEMMA 6.1 (Shimura's Reciprocity Law). *Let f be an element of $\mathcal{F}$ which is finite at $Z_{\mathfrak{c}}^*$.*

(i) *The special value $f(Z_{\mathfrak{c}}^*)$ lies in K_{ab}.*

(ii) *For every $s \in K_{\mathbb{A}}^{\times}$ we have $r(\mathfrak{g}(s)) \in G_{\mathbb{A}+}$ and*

$$f(Z_{\mathfrak{c}}^*)^{[s,K]} = f^{\tau(r(\mathfrak{g}(s)^{-1}))}(Z_{\mathfrak{c}}^*).$$

PROOF. See [10, Lemma 9.5 and Theorem 9.6]. □

THEOREM 6.2. *If $h_{\mathfrak{f}}(\mathcal{C})$ is finite, then it belongs to $K_{\mathfrak{f}}$. And it satisfies*

$$h_{\mathfrak{f}}(\mathcal{C})^{\sigma_{\mathfrak{f}}(\mathcal{D})} = h_{\mathfrak{f}}(\mathcal{C}\mathcal{D}) \quad \textit{for every } \mathcal{D} \in \mathrm{Cl}(\mathfrak{f}),$$

where $\sigma_{\mathfrak{f}}$ is the Artin reciprocity map for $\mathfrak{f}$.

PROOF. Since $h_{\mathfrak{f}}(\mathcal{C})$ belongs to K_{ab} by Lemma 6.1 (i), there is a sufficiently large positive integer M so that $N \mid M$ and $h_{\mathfrak{f}}(\mathcal{C}) \in K_{\mathfrak{m}}$ with $\mathfrak{m} = M\mathcal{O}_K$. Take an integral ideal $\mathfrak{d}$ in $\mathcal{D}$ relatively prime to $\mathfrak{m}$ by using the surjectivity of the natural map $\mathrm{Cl}(\mathfrak{m}) \to \mathrm{Cl}(\mathfrak{f})$. Let $\{\mathbf{d}_1, \ldots, \mathbf{d}_{2g}\}$ be a symplectic basis of the principally polarized abelian variety $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c}\mathfrak{d})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c}\mathfrak{d})})$, and let $d_1, \ldots, d_{2g}$ be elements of $\mathcal{G}(\mathfrak{c}\mathfrak{d})^{-1}$ such that $\mathbf{d}_j = \Psi(d_j)$ $(1 \le j \le 2g)$. Since $\mathcal{G}(\mathfrak{c})^{-1} \subseteq \mathcal{G}(\mathfrak{c}\mathfrak{d})^{-1}$, we get

$$\begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_{2g} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 & \cdots & \mathbf{d}_{2g} \end{bmatrix} \delta \quad \text{for some } \delta \in M_{2g}(\mathbb{Z}) \cap \mathrm{GL}_{2g}(\mathbb{Q}). \tag{16}$$

We then have that

$$\begin{aligned} \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} &= \begin{bmatrix} E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{b}_i, \mathbf{b}_j) \end{bmatrix}_{1 \le i,j \le 2g} \\ &= \delta^T \begin{bmatrix} E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{d}_i, \mathbf{d}_j) \end{bmatrix}_{1 \le i,j \le 2g} \delta \quad \text{by (16)} \\ &= \delta^T \begin{bmatrix} \mathcal{N}(\mathfrak{c})\mathcal{N}(\mathfrak{c}\mathfrak{d})^{-1} E_{\xi\mathcal{N}(\mathfrak{c}\mathfrak{d})}(\mathbf{d}_i, \mathbf{d}_j) \end{bmatrix}_{1 \le i,j \le 2g} \delta \\ &= \mathcal{N}(\mathfrak{d})^{-1} \delta^T \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} \delta. \end{aligned}$$

This claims that

$$\delta \in M_{2g}(\mathbb{Z}) \cap G_+ \text{ with } \nu(\delta) = \mathcal{N}(\mathfrak{d}). \tag{17}$$

Furthermore, if we let $Z^*_{\mathfrak{c}\mathfrak{d}}$ be the CM-point associated with $\{\mathbf{d}_1, \ldots, \mathbf{d}_{2g}\}$, then we obtain

$$Z^*_{\mathfrak{c}\mathfrak{d}} = (\delta^{-1})^T (Z^*_{\mathfrak{c}}) \tag{18}$$

in a similar way to the argument in the proof of Proposition 5.1.

Let $s = (s_p)_p$ be an idele of K such that

$$\begin{cases} s_p = 1 & \text{if } p \,|\, M, \\ s_p(\mathcal{O}_K)_p = \mathfrak{d}_p & \text{if } p \nmid M. \end{cases} \tag{19}$$

If we set $\widetilde{\mathcal{D}}$ to be the ray class in $\mathrm{Cl}(\mathfrak{m})$ containing $\mathfrak{d}$, then we attain by (19)

$$[s, K]|_{K_{\mathfrak{m}}} = \sigma_{\mathfrak{m}}(\widetilde{\mathcal{D}}), \tag{20}$$

$$\mathfrak{g}(s)_p^{-1}(\mathcal{O}_{K^*})_p = \mathcal{G}(\mathfrak{d})_p^{-1} \quad \text{for all rational primes } p. \tag{21}$$

It then follows from (15)$\sim$(21) that for every rational prime p, the entries of each of the vectors

$$r(\mathfrak{g}(s)^{-1})_p \begin{bmatrix} b_1 \\ \vdots \\ b_{2g} \end{bmatrix} \quad \text{and} \quad (\delta^{-1})^T \begin{bmatrix} b_1 \\ \vdots \\ b_{2g} \end{bmatrix}$$

form a basis of $\mathcal{G}(\mathfrak{c}\mathfrak{d})_p^{-1} = \mathcal{G}(\mathfrak{c})^{-1}\mathcal{G}(\mathfrak{d})_p^{-1}$. So, there is a matrix $u = (u_p)_p \in \prod_p \mathrm{GL}_{2g}(\mathbb{Z}_p)$ satisfying

$$r(\mathfrak{g}(s)^{-1}) = u(\delta^{-1})^T. \tag{22}$$

Since δ^T and $\begin{bmatrix} I_g & O_g \\ O_g & \mathcal{N}(\delta)I_g \end{bmatrix}$ can be viewed as elements of $\mathrm{GSp}_{2g}(Z/M\mathbb{Z})$ by (17), there exists a matrix $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that

$$\delta^T \equiv \begin{bmatrix} I_g & O_g \\ O_g & \mathcal{N}(\delta)I_g \end{bmatrix} \gamma \pmod{M \cdot M_{2g}(\mathbb{Z})} \tag{23}$$

owing to the surjectivity of the reduction $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/M\mathbb{Z})$. Since $r(\mathfrak{g}(s)^{-1})_p = I_{2g}$ for all $p \,|\, M$ by (19), we get $u_p = \delta^T$ for all $p \,|\, M$ by (22). Hence we deduce by (23) that

$$u_p\gamma^{-1} \equiv \begin{bmatrix} I_g & O_g \\ O_g & \mathcal{N}(\delta)I_g \end{bmatrix} \pmod{M \cdot M_{2g}(\mathbb{Z}_p)} \quad \text{for all rational primes } p. \tag{24}$$

On the other hand, we have by (5) and (16) that

$$\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{2g} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_{2g} \end{bmatrix} \alpha = (\begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_{2g} \end{bmatrix} \delta^{-1})(\delta\alpha) = \begin{bmatrix} \mathbf{d}_1 & \cdots & \mathbf{d}_{2g} \end{bmatrix} (\delta\alpha). \tag{25}$$

Letting $\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ we induce that

$$h_{\mathfrak{f}}(\mathcal{C})^{\sigma_{\mathfrak{m}}(\widetilde{\mathcal{D}})} = h_{\mathfrak{f}}(\mathcal{C})^{[s,K]} \quad \text{by (20)}$$

$$
\begin{aligned}
&= h_{(1/N)\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(Z_{\mathfrak{c}}^*)^{[s,K]} \quad \text{by Definition 4.4}\\
&= h_{(1/N)\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(Z)^{\tau(r(\mathfrak{g}(s)^{-1}))}|_{Z=Z_{\mathfrak{c}}^*} \quad \text{by Lemma 6.1 (ii)}\\
&= h_{(1/N)\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(Z)^{\tau(u(\delta^{-1})^T)}|_{Z=Z_{\mathfrak{c}}^*} \quad \text{by (22)}\\
&= h_{(1/N)\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(Z)^{\tau(u\gamma^{-1})\tau(\gamma)\tau((\delta^{-1})^T)}|_{Z=Z_{\mathfrak{c}}^*}\\
&= h_{(1/N)\left[\begin{smallmatrix}I_g & O_g\\O_g & \mathcal{N}(\delta)I_g\end{smallmatrix}\right]\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(Z)^{\tau(\gamma)\tau((\delta^{-1})^T)}|_{Z=Z_{\mathfrak{c}}^*} \quad \text{by (24) and (S3)}\\
&= h_{(1/N)\gamma^T\left[\begin{smallmatrix}I_g & O_g\\O_g & \mathcal{N}(\delta)I_g\end{smallmatrix}\right]\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(Z)^{\tau((\delta^{-1})^T)}|_{Z=Z_{\mathfrak{c}}^*} \quad \text{by (S3)}\\
&= h_{(1/N)\delta\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(Z)^{\tau((\delta^{-1})^T)}|_{Z=Z_{\mathfrak{c}}^*} \quad \text{by (23) and (S2)}\\
&= h_{(1/N)\delta\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}((\delta^{-1})^T(Z_{\mathfrak{c}}^*)) \quad \text{due to the fact } \delta\in G_+ \text{ and (A1)}\\
&= h_{\mathfrak{f}}(\mathcal{CD}) \quad \text{by (18), (25) and Definition 4.4.}
\end{aligned}
$$

In particular, suppose that $\mathfrak{d} = d\mathcal{O}_K$ for some $d \in \mathcal{O}_K$ such that $d \equiv 1 \pmod{\mathfrak{f}}$. Then $\mathcal{D}$ is the identity class of $\mathrm{Cl}(\mathfrak{f})$, and so the above observation implies that $\sigma_{\mathfrak{m}}(\widetilde{\mathcal{D}})$ leaves $h_{\mathfrak{f}}(\mathcal{C})$ fixed. Therefore, we conclude that $h_{\mathfrak{f}}(\mathcal{C})$ lies in $K_{\mathfrak{f}}$. □

Corollary 6.3. *Let H be a subgroup of $\mathrm{Cl}(\mathfrak{f})$ defined by*

$$
\begin{aligned}
H &= \langle \mathcal{D}\in\mathrm{Cl}(\mathfrak{f}) \mid \mathcal{D} \text{ contains an integral ideal } \mathfrak{d} \text{ of } K \text{ for which}\\
&\qquad \mathcal{G}(\mathfrak{d}) = \mathfrak{g}(d)\mathcal{O}_{K^*} \text{ for some } d\in\mathcal{O}_K \text{ such that } \mathfrak{g}(d)\equiv 1 \pmod{N\mathcal{O}_{K^*}}\rangle,
\end{aligned}
$$

and let $K_{\mathfrak{f}}^H$ be the fixed field of H. If $h_{\mathfrak{f}}(\mathcal{C})$ is finite, then it belongs to $K_{\mathfrak{f}}^H$.

Proof. Let $\mathcal{C}_0$ be the identity class of $\mathrm{Cl}(\mathfrak{f})$. Since $h_{\mathfrak{f}}(\mathcal{C}_0) \in K_{\mathfrak{f}}$ by Theorem 6.2, $K(h_{\mathfrak{f}}(\mathcal{C}_0))$ is a Galois extension of K as a subfield of $K_{\mathfrak{f}}$. Furthermore, since

$$h_{\mathfrak{f}}(\mathcal{C}_0)^{\sigma_{\mathfrak{f}}(\mathcal{C})} = h_{\mathfrak{f}}(\mathcal{C}_0\mathcal{C}) = h_{\mathfrak{f}}(\mathcal{C})$$

by Theorem 6.2, $K(h_{\mathfrak{f}}(\mathcal{C}_0))$ contains $h_{\mathfrak{f}}(\mathcal{C})$. Thus it suffices to show that $h_{\mathfrak{f}}(\mathcal{C}_0)$ belongs to $K_{\mathfrak{f}}^H$.

To this end, let $\mathcal{D}$ be an element of $\mathrm{Cl}(\mathfrak{f})$ containing an integral ideal $\mathfrak{d}$ of K for which

$$\mathcal{G}(\mathfrak{d}) = \mathfrak{g}(d)\mathcal{O}_{K^*} \quad \text{for some } d\in\mathcal{O}_K \text{ such that } \mathfrak{g}(d)\equiv 1 \pmod{N\mathcal{O}_{K^*}}.$$

Now that

$$(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{d})^{-1}), E_{\xi\mathcal{N}(\mathfrak{d})}) = (\mathbb{C}^g/\Psi(\mathfrak{g}(d)^{-1}\mathcal{O}_{K^*}), E_{\xi\mathcal{N}(d\mathcal{O}_K)}),$$

we obtain

$$h_{\mathfrak{f}}(\mathcal{C}_0)^{\sigma_{\mathfrak{f}}(\mathcal{D})} = h_{\mathfrak{f}}(\mathcal{D}) = h_{\mathfrak{f}}([d\mathcal{O}_K]),$$

where $[\mathfrak{a}]$ is the ray class containing $\mathfrak{a}$ for a fractional ideal $\mathfrak{a}$ of K. Moreover, since $\mathfrak{g}(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}}$, we achieve

$$h_{\mathfrak{f}}([d\mathcal{O}_K]) = h_{\mathfrak{f}}([\mathcal{O}_K]) = h_{\mathfrak{f}}(\mathcal{C}_0)$$

in like manner as in the proof of Proposition 5.3. This proves that $h_{\mathfrak{f}}(\mathcal{C}_0)$ belongs to $K_{\mathfrak{f}}^H$. □

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