Representing Strategic Games and Their Equilibria in Many-Valued Logics*

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Abstract

We introduce the notion of logical $\boldsymbol{A}$-games for a fairly general class of algebras $\boldsymbol{A}$ of real truth-values. This concept generalizes the Boolean games of Harrenstein et al. as well as the recently defined Łukasiewicz games of Marchioni and Wooldridge. We demonstrate that a wide range of strategic n-player games can be represented as logical $\boldsymbol{A}$-games. Moreover we show how to construct, under rather general conditions, propositional formulas in the language of $\boldsymbol{A}$ that correspond to pure and mixed Nash equilibria of logical $\boldsymbol{A}$-games.

Keywords: strategic games, many-valued logics, Nash equilibria, Łukasiewicz games

1 Introduction

Various types of connections between logic and game theory increasingly receive attention in the literature. (We refer to [27] for a recent monograph devoted to several aspects of this topic.) This paper is a contribution to a special line of research that has been initiated by the introduction of the concept of a Boolean game in [14]. Originally, Boolean games have been introduced as two-person zero-sum extensive-form games. However, here we follow the bulk of literature that views Boolean games as special strategic n-player games, where each player's payoff function is expressed by a classical propositional (i.e., Boolean) formula and her strategies consist in truth-value assignments to a subset of the propositional variables occurring in the payoff functions. The focus on classical formulas severely limits the scope of strategic games that can be represented in this format. In particular, one is often interested in finite games with more than just two possible payoff values, which entails that the payoff functions cannot be identified with Boolean formulas. For example, nearly all of the well-known strategic games that are usually represented by a 2×2 payoff matrix, such as the Prisoner's Dilemma, Chicken, the Coordination Game, etc. (see, e.g., [10, 19, 24, 26]), fall into this category. This fact has motivated Marchioni and Wooldridge [17, 18] to generalize Boolean games to so-called Łukasiewicz games, where the payoff

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functions are represented by formulas of an appropriate (finite or infinite) Łukasiewicz logic and the strategies are assignments of the logic's truth values to relevant variables. A number of well-known strategic games, or at least modified variants of them, are representable as Łukasiewicz games. Moreover, it is shown in [17] that for any given finite Łukasiewicz game $\mathcal{G}$ there is a propositional formula $\varphi_{\mathcal{G}}$ which is satisfiable in the corresponding Łukasiewicz logic iff $\mathcal{G}$ has a pure Nash equilibrium. In [18] this result is generalized to infinite-valued Łukasiewicz logic. However, instead of directly expressing Nash equilibria by propositional formulas, a detour via classical first-order theories of corresponding algebras is employed.

The main aim of this paper is to generalize and expand the approach of Marchioni and Wooldridge in at least three different aspects:

(1) We show that the restriction to Łukasiewicz logics as the underlying formalism for the representation of games is neither necessary nor convenient. In fact, rather than just proposing additional many-valued logic as possible representation formalisms, we aim at identifying *general conditions* that are sufficient for representing wide classes of games as well as expressing their Nash equilibria.

(2) We remove yet another, quite different limitation of Łukasiewicz games: Marchioni and Wooldridge [17, 18] identify the set of strategies of a given player with the set of *all* assignments of truth values to the variables controlled by that player. While this makes sense for Boolean games, it amounts to an unnecessary, and in fact rather obstructive, restriction in the many-valued setting. As we will demonstrate, by simply using *subsets* of all possible assignments to represent a player's strategies, not only a wider class of games, but in particular *all finite* strategic games become representable as logical games.

(3) So far, only the characterization of the existence of *pure* Nash equilibria by logical formulas has been considered in the literature. We will show that, for sufficiently expressive logics, also *mixed* Nash equilibria can be characterized by propositional formulas. The emphasis here is not on their existence, which in all cases relevant here is guaranteed by Nash's Theorem, but on the fact that we may use propositional variables to represent probability distributions and thus obtain a one-to-one correspondence between the assignments satisfying a particular formula and the mixed equilibria of the games in question.

Overall, we demonstrate that many-valued logics provide an adequate setting for the formal representation of large classes of strategic games, in particular of all finite strategic games. This includes the reduction of questions about pure as well as mixed Nash equilibria into questions about the satisfiability of appropriate propositional formulas. While our results immediately suggest straightforward algorithms for checking (the existence of) equilibria, it remains to be seen which further benefits can be reaped from our logical approach to the representation of strategic n-person games.

Let us mention some further features that distinguish our approach. While we talk about "logical games", the central reference is actually to a wide class of so-called standard algebras, i.e., algebras over (subsets of) the real unit interval $[0, 1]$. For our purpose, the distinction between formulas and terms over an algebra is in fact immaterial. As already mentioned, in [18] the existence of pure Nash equilibria is not expressed directly by propositional formulas, but only indirectly via classical first-order theories of particular algebras. We will show that this detour is unnecessary. A further item that deserves to be emphasized right away concerns the very concept of representing a given strategic game as a logical game (with respect to a given algebra). Note that the notion of representability is only implicit in [17, 18] as well as in the literature on Boolean games. ([11] discusses succinct representability of Boolean games, but does not refer to general strategic games.) By making representability explicit, we disambiguate this somewhat vague notion and are able to formally characterize the scope of representable games. Moreover, this move supports the identification of different conditions on expressibility that are sufficient to express Nash equilibria at various levels of succinctness.

We emphasize that the aim of this paper is to demonstrate that many-valued logics provide a versatile and very general tool for the formalization of strategic games. Once appropriate notions and conditions are identified, checking that the corresponding logical representations are indeed adequate is routine and consequently left to the reader in most cases.

The paper is organized as follows. In Section 2 we present the basic concepts and terminology used in later sections: Subsection 2.1 fixes some notions regarding algebras and logics over (subsets of) the real unit interval $[0,1]$. Subsection 2.2 reviews basic game-theoretic notions and illustrates these by presenting a number of concrete examples that are taken up in later sections. Section 3 introduces the concept of a logical game with respect to an arbitrary standard algebra. We demonstrate that a wide class of ordinary strategic games can be represented as logical games and provide corresponding examples. In Section 4.1 we show how (the existence of) pure Nash equilibria in logical games can be expressed by propositional formulas under rather weak conditions. Section 4.2 is devoted to the construction of propositional formulas that correspond to mixed Nash equilibria for suitable classes of logical games. We conclude with Section 5 containing a short summary and some remarks on possible directions of future research, with emphasis on dealing with infinite games.

2 Preliminary notions

2.1 Logics and algebras

We will work with logics expressed in various propositional languages. A (propositional) language $\mathcal{L}$ is understood as a collection of connectives equipped with arities. The corresponding set $Fm_{\mathcal{L}}$ of propositional formulas is defined over a countably infinite set of propositional variables as usual:

- $Fm_{\mathcal{L}}$ contains all propositional variables.
- $\circ(\varphi_1, \ldots, \varphi_n) \in Fm_{\mathcal{L}}$ if $\varphi_1, \ldots, \varphi_n \in Fm_{\mathcal{L}}$ and $\circ$ is an n-ary connective in $\mathcal{L}$. (We will use infix notation for familiar binary connectives. Nullary connectives are also called *truth constants.*)
- Nothing else is in $Fm_{\mathcal{L}}$.

In this paper we consider special many-valued logics, each of which is determined by a single particular algebra of truth degrees; proof systems will play no role here. (The interested reader can find information about deductive aspects of the kinds of many-valued logics treated here in the handbook chapter [21].) We will always assume that each language contains at least three binary connectives $\wedge$, $\vee$, and $\to$. We will identify propositional languages with algebraic types, connectives with operation symbols, and formulas with terms of the corresponding algebra.

Definition 2.1. A *standard algebra* $\boldsymbol{A}$ (of truth degrees) in a language $\mathcal{L}_{\boldsymbol{A}}$ is a tuple $\langle A, \langle \circ^{\boldsymbol{A}} \rangle_{\circ \in \mathcal{L}_{\boldsymbol{A}}} \rangle$, where:

- The domain A is a subset of the real interval $[0,1]$ such that $1 \in A$.
- For each n-ary connective $\circ \in \mathcal{L}_{\boldsymbol{A}}$, its interpretation $\circ^{\boldsymbol{A}}$ in $\boldsymbol{A}$ is an n-ary operation on A (or an element of A if $n = 0$).
- For every $x, y \in A$:
  - $x \wedge^{\boldsymbol{A}} y = 1$ iff $x = 1$ and $y = 1$.
  - $x \vee^{\boldsymbol{A}} y = 1$ iff $x = 1$ or $y = 1$.
  - $x \to^{\boldsymbol{A}} y = 1$ iff $x \leq y$.

Note that the realization $\wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}$ of the connectives $\wedge, \vee$ in the algebra $\boldsymbol{A}$ need not be the minimum and maximum (under the usual order of reals). Even if this will be the case in typical standard algebras, the only conditions required of $\wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}$ are those of Definition 2.1, as they already ensure the validity of all theorems given below. Restricting the interpretation of $\wedge, \vee$ in $\boldsymbol{A}$ to the lattice operations would thus impose an unnecessary limitation on the class of admissible logics and on the generality of the results.

Example 2.2. Let us list several prominent algebras that can be seen as standard algebras in the sense of Definition 2.1:

- The *two-valued Boolean algebra* $\boldsymbol{2}$ in the language $\wedge, \vee, \to, \neg, \overline{0}, \overline{1}$ (where $x \to y$ is defined as $\neg x \vee y$; we will not mention arities of well-known connectives).
- The *standard* G-*algebra* $[0,1]_{\mathrm{G}} = \langle [0,1], \wedge, \vee, \to, \overline{0}, \overline{1} \rangle$ (G for Gödel), where $\langle [0,1], \wedge, \vee, \overline{0}, \overline{1} \rangle$ is the lattice $[0,1]$ with the usual order, and $x \to y = 1$ if $x \leq y$ and $x \to y = y$ otherwise. (For G-algebras see, e.g., [3, 4].)
- The $(n+1)$-*valued* G-*algebra* $\boldsymbol{G}_n$, i.e., the subalgebra of $[0,1]_{\mathrm{G}}$ with the domain $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$. (See, e.g., [3, 4].)
- The *standard* MV-*algebra* $[0,1]_{\text{Ł}} = \langle [0,1], \&, \to, \wedge, \vee, \overline{0}, \overline{1} \rangle$, where $\langle [0,1], \wedge, \vee, \overline{0}, \overline{1} \rangle$ is the lattice $[0,1]$ with the usual order of reals, $x \& y = \max(x+y-1, 0)$, and $x \to y = \min(1-x+y, 1)$. In MV-algebras (and their expansions), it is customary to introduce the defined connectives $\neg x = x \to \overline{0}$; $x \oplus y = \neg x \to y$; and $x \ominus y = x \& \neg y$. In (the subalgebras of) the standard MV-algebra, they are realized as $1-x$; $\min(x+y, 1)$; and $\max(x-y, 0)$ respectively. Let us remark that MV-algebras are often introduced in the language $\oplus, \neg, \overline{0}$, in which case $\&, \to, \wedge, \vee, \overline{1}$ are defined connectives; the two definitions are term-wise equivalent. (For MV-algebras see, e.g., [8].)
- The $(n+1)$-*valued* MV-*algebra* $\boldsymbol{Ł}_n$ ($\boldsymbol{Ł}$ for Łukasiewicz), i.e., the subalgebra of $[0,1]_{\text{Ł}}$ with the domain $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$. (See, e.g., [8].)
- If the truth constants for all elements of $\boldsymbol{Ł}_n$ (i.e., nullary connectives $\overline{0}, \overline{\frac{1}{n}}, \ldots, \overline{\frac{n-1}{n}}, \overline{1}$ interpreted by the corresponding elements of $\boldsymbol{Ł}_n$) are added to the language, we denote the resulting expansion of $\boldsymbol{Ł}_n$ by $\boldsymbol{Ł}_n^c$ (see, e.g., [6]). Analogously we define the expansion $\boldsymbol{G}_n^c$ of $\boldsymbol{G}_n$ by truth constants.
- The *standard* PŁ-*algebra* $[0,1]_{\text{PŁ}}$, which is an expansion of $[0,1]_{\text{Ł}}$ by a binary connective $\odot$, interpreted as the usual algebraic product of reals. (PŁ-algebras are also known as PMV-algebras; see, e.g., [12, Sect. 5].)
- The *standard* ŁΠ-*algebra* $[0,1]_{\text{ŁΠ}}$, which is an expansion of $[0,1]_{\text{PŁ}}$ by a binary connective $\to_\Pi$, interpreted as $x \to_\Pi y = 1$ if $x \leq y$ and $x \to_\Pi y = \frac{y}{x}$ otherwise. (See, e.g., [12, Sect. 5].)
- The expansions of $[0,1]_{\text{Ł}}$, $[0,1]_{\mathrm{G}}$, $[0,1]_{\text{PŁ}}$, and $[0,1]_{\text{ŁΠ}}$ with nullary operations (i.e., constants) $\bar{r}$ for all $r \in [0,1] \cap \mathbb{Q}$, where each constant $\bar{r}$ is interpreted by the rational number r. We denote these algebras, respectively, as $[0,1]_{\mathbb{Q}\text{Ł}}$, $[0,1]_{\mathbb{Q}\mathrm{G}}$, $[0,1]_{\mathbb{Q}\text{PŁ}}$, and $[0,1]_{\text{ŁΠ}\frac{1}{2}}$. (The traditional symbol for the last mentioned algebra is due to the fact that in $[0,1]_{\text{ŁΠ}\frac{1}{2}}$, all rational constants are definable from the constant for $\frac{1}{2}$. For the expansions of the standard MV-, G-, PŁ-, and ŁΠ-algebra by rational constants see, e.g., [12].)
- The expansion of any standard algebra $\boldsymbol{A}$ such that $0 \in A$ by the unary operation $\triangle$ interpreted as $\triangle x = 1$ if $x = 1$ and $\triangle x = 0$ otherwise; we denote this algebra by $\boldsymbol{A}^{\triangle}$. (The operation $\triangle$ is definable in $\boldsymbol{Ł}_n$, $\boldsymbol{Ł}_n^c$, $[0,1]_{\text{ŁΠ}}$, and $[0,1]_{\text{ŁΠ}\frac{1}{2}}$, so $\boldsymbol{Ł}_n^{\triangle} = \boldsymbol{Ł}_n$ modulo term-wise equivalence, and similarly for $\boldsymbol{Ł}_n^{c\triangle}$, $[0,1]_{\text{ŁΠ}}^{\triangle}$, and $[0,1]_{\text{ŁΠ}\frac{1}{2}}^{\triangle}$. For expansions by $\triangle$ see, e.g., [13, Ch. 2.4] or [4, Sect. 2.2.1].)

Algebra	Functions	Domains	Pieces
$[0,1]_{Ł}$	continuous	linear	linear functions with integer coefficients
$[0,1]_{Ł}^{\triangle}$	all	linear	linear functions with integer coefficients
$[0,1]_{\mathbb{Q}Ł}$	continuous	linear	linear functions with integer coefficients and a rational shift
$[0,1]_{\mathbb{Q}Ł}^{\triangle}$	all	linear	linear functions with integer coefficients and a rational shift
$[0,1]_{PŁ}^{\triangle}$	all	all	polynomials with integer coefficients
$[0,1]_{\mathbb{Q}PŁ}^{\triangle}$	all	all	polynomials with rational coefficients
$[0,1]_{Ł\Pi\frac{1}{2}}$	all	all	fractions of polynomials with integer coefficients
$[0,1]_{Ł\Pi}$	all	all	functions f expressible in $[0,1]_{Ł\Pi\frac{1}{2}}$ such that $f[\{0,1\}^n] \subseteq \{0,1\}$

Table 1: Characterization of functions expressible in prominent standard algebras.

Let us recall several standard notions of the algebraic semantics of many-valued logics. (For a detailed modern exposition see, e.g., [9].)

Definition 2.3. Let $\boldsymbol{A}$ be a standard algebra. An $\boldsymbol{A}$-*evaluation* is a mapping e assigning an element of A to each propositional variable. Every $\boldsymbol{A}$-evaluation can be uniquely extended to a mapping from $Fm_{\mathcal{L}_{\boldsymbol{A}}}$ into A, by setting $e(\circ(\varphi_1,\dots,\varphi_n)) = \circ^{\boldsymbol{A}}(e(\varphi_1),\dots,e(\varphi_n))$ for each n-ary connective $\circ \in \mathcal{L}_{\boldsymbol{A}}$ and formulas $\varphi_1,\dots,\varphi_n$.

An $\mathcal{L}_{\boldsymbol{A}}$-formula φ is *satisfied* by an $\boldsymbol{A}$-evaluation e if $e(\varphi) = 1$. A formula φ is $\boldsymbol{A}$-*satisfiable* if it is satisfied by some $\boldsymbol{A}$-evaluation.

The *logic of* $\boldsymbol{A}$ is identified with the consequence relation $\models_{\boldsymbol{A}}$, defined as follows:

$$\Gamma \models_{\boldsymbol{A}} \varphi \text{ if and only if for each } \boldsymbol{A}\text{-evaluation } e\text{: if } e[\Gamma] \subseteq \{1\}, \text{ then } e(\varphi) = 1.$$

A trivial, but important observation is that the value of a formula φ in an $\boldsymbol{A}$-evaluation depends only on the variables occurring in φ. Let $\boldsymbol{v}$ be a sequence $v_1,\dots,v_n$ of pairwise different variables; we shall write $\varphi(v_1,\dots,v_n)$, or just $\varphi(\boldsymbol{v})$, to denote that all variables occurring in φ are among those in $\boldsymbol{v}$. Given a formula $\varphi(v_1,\dots,v_n)$ and a sequence of formulas $\psi_1,\dots,\psi_n$, we shall write $\varphi(\psi_1,\dots,\psi_n)$ to denote the formula where each variable v_i is replaced by the formula ψ_i.

Definition 2.4. Given a formula $\varphi(v_1,\dots,v_n)$, we define the mapping $\varphi^{\boldsymbol{A}}\colon A^n \to A$ by setting:

$$\varphi^{\boldsymbol{A}}(a_1,\dots,a_n) = e(\varphi),$$

where e is any $\boldsymbol{A}$-evaluation such that $e(v_i) = a_i$.

For any given standard algebra $\boldsymbol{A}$, it is an interesting question how to describe the class of all functions $\varphi^{\boldsymbol{A}}$. The classes of functions expressible in prominent standard algebras are described in Table 1. These delimitations are of a "piecewise" character, i.e., based on a decomposition of the corresponding hypercube $[0,1]^n$ into domains; in particular, each row of the table specifies: (i) whether the functions are all those which satisfy the other constraints listed on the row, or just the continuous ones; (ii) whether the possible domains are all Q-semialgebraic sets,[1] or just the linear ones; and (iii) how the functions restricted to these domains are characterized.[2] For further details see [20, 22] or [1, Sect. 4.1], where also the (more complicated) result for Gödel algebras is presented. Let us furthermore remark that the case of $[0,1]_{PŁ}$ is a long-standing open problem related to the so-called Pierce–Birkhoff conjecture [5, 16].

[1] A set $S \subseteq [0,1]^n$ is *(linear)* Q-*semialgebraic* if it is a Boolean combination of sets of the form $\{\langle x_1,\dots,x_n\rangle \in [0,1]^n \mid P(x_1,\dots,x_n) > 0\}$ for (linear) polynomials P with integer coefficients.

[2] In the table, a *shift* means an absolute coefficient of the polynomial.

2.2 Strategic Games

We present some basic notions and results concerning strategic games with finitely many players. In particular, we review the most fundamental solution concept for such games, namely that of a *Nash equilibrium*, both for pure and mixed strategies. The examples in this subsection are intended to illustrate these concepts and are taken up in later sections to demonstrate that many well known games can be represented as logical games (in the sense of Definition 3.1). We use standard game-theoretic notation and terminology; see, e.g., [19, Ch. 4–5] or [25].

Definition 2.5. A strategic game $\mathcal{G}$ is an ordered triple

$$\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\}\rangle, \text{ where:}$$

1. $N = \{1, \ldots, n\}$ is a finite set of *players.*
2. Each $S_i \neq \varnothing$ is a *strategy set* of player $i \in N$.
3. Putting $S = S_1 \times \cdots \times S_n$, each function $f_i \colon S \to \mathbb{R}$ is called the *payoff function* (or: *utility function*) of player i.

Note that we do not restrict the cardinality of the strategy sets $S_i \neq \varnothing$ at this point. A game $\mathcal{G}$ is called *finite* if each S_i is finite. An ordered n-tuple of strategies $\boldsymbol{s} = \langle s_1, \ldots, s_n\rangle \in S$ is called a *strategy profile.*

Throughout the paper we will adhere to the following conventions:

- Whenever a symbol is related to a particular player i, then we use the corresponding subscript. Thus, e.g., the strategies of player i will typically be denoted by s_i, s_i', etc. If the subscripted symbol is itself a tuple, then the second index will be written as a superscript (e.g., s_i^j and v_i^j).
- For every player $i \in N$ and a strategy profile $\boldsymbol{s} = \langle s_1, \ldots, s_n\rangle \in S$, by $\boldsymbol{s}_{-i}$ we denote the ordered $(n-1)$-tuple $\langle s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n\rangle$. By $\langle s_i', \boldsymbol{s}_{-i}\rangle$ we abbreviate the strategy profile $\langle s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n\rangle$. The utility of player $i \in N$ under the strategy profile $\langle s_i', \boldsymbol{s}_{-i}\rangle$ is written as $f_i(s_i', \boldsymbol{s}_{-i})$.

The solution concept of a Nash equilibrium captures the idea of stability in the given game. When all players choose their strategies according to a Nash equilibrium, then neither player can profit from unilaterally deviating to an alternative strategy.

Definition 2.6. Let $\mathcal{G}$ be a strategic game. A strategy profile $\boldsymbol{s}^* = \langle s_1^*, \ldots, s_n^*\rangle \in S$ is a *pure Nash equilibrium* of $\mathcal{G}$ if

$$f_i(s_i, \boldsymbol{s}_{-i}^*) \leq f_i(\boldsymbol{s}^*),$$

for every player $i \in N$ and every strategy $s_i \in S_i$.

Since one of the main aims of the paper is to show that a very wide range of strategic games, in particular all finite games, can be represented as logical games, we provide several different examples here that will be taken up again in later sections. We focus on examples that cannot be directly modeled as either Boolean or Łukasiewicz games.

Example 2.7 (New Technology)**.** Suppose that there are three firms sharing a market. In face of a new technology each firm has to decide whether to adopt it or else to stay put. We assume that the total value of the market remains unchanged; only the relative competitiveness of the firms may change in accordance to their decisions. If only one firm decides to adopt the new technology, it will gain a certain competitive advantage $c > 0$ and each of the other two firms looses $c/2$, accordingly. If two firms decide to adopt then they split the competitive gain, receiving $c/2$ each, and the third firm has to bear the full loss c. If either none or all firms adopt the new technology, no firm will gain or loose anything.

The payoff vectors of the resulting 3-player zero-sum game are as follows:

Firm 3: adopt

Firm 1	**Firm 2** adopt	**Firm 2** stay put
adopt	$(0, 0, 0)$	$(c/2, -c, c/2)$
stay put	$(-c, c/2, c/2)$	$(-c/2, -c/2, c)$

Firm 3: stay put

Firm 1	**Firm 2** adopt	**Firm 2** stay put
adopt	$(c/2, c/2, -c)$	$(c, -c/2, -c/2)$
stay put	$(-c/2, c, -c/2)$	$(0, 0, 0)$

It is not hard to see that the only pure Nash equilibrium of the game arises if every firm adopts the new technology. (In fact, adopting is a dominating strategy for each firm.)

Examples of games with infinite strategy spaces arise very naturally in many applications; however, they are highly non-trivial to analyze, in general.

Example 2.8 (Vickrey Auction)**.** As pointed out, e.g., in [24, 25] many types of auctions can be modeled as strategic games under certain assumptions. Of particular interest is the second-price sealed-bid auction with perfect information, also called *Vickrey auction,* since, in contrast to more familiar types of auctions, bidders have an incentive to bid their true value.

Our strategic game representing a Vickrey auction has n players (bidders). Independently from the others, each player $i \in N = \{1, \dots, n\}$ associates a rational value $p_i \geq 0$ to the object sold in the auction. This value will be used to define the payoff function and thus is assumed to be known to all players, as required in all strategic games. The strategy set S_i, i.e., the set of possible bids of player i, is identified with the interval $[0, t]$, where t is some fixed rational maximal bidding value, greater than $\max_{i \in N} p_i$. Each player only knows her own bid $b_i \in [0, t]$. The assumption that values and bids are capped at some fixed t is not restrictive, as real-life bidders always have finite bankrolls.) The object is assigned to the player with the highest bid. If more than one player has chosen the same highest bid then the object is assigned to the player with the lowest index among those sharing the same highest bid. The price to be paid for the object by that player is the highest bid made by any other player. This amounts to the following payoff function:

$$f_i(b_1, \dots, b_n) = \begin{cases} p_i - \max_{j \neq i} b_j & \text{if } i = \min\{j \in N \mid b_j = \max_{k \in N} b_k\} \\ 0 & \text{otherwise.} \end{cases}$$

A Nash equilibrium for this game is $\langle b_1, \dots, b_n \rangle = \langle p_1, \dots, p_n \rangle$, which means that all players bid their true value. However there are also other Nash equilibria; for example, if $p_1 > p_2 > \cdots > p_n$, then $\langle b_1, \dots, b_n \rangle = \langle p_1, 0, \dots, 0 \rangle$ and $\langle b_1, \dots, b_n \rangle = \langle p_2, p_1, 0, \dots, 0 \rangle$ are Nash equilibria, too.

Example 2.9 (Electoral Competition)**.** The following simplified model of electoral competition between two candidates appears in [15]. Let $N = \{1, 2\}$ and $S_1 = S_2 = [0, 1]$. Put:

$$f_1(s_1, s_2) = \begin{cases} 0 & \text{if } s_1, s_2 \in \{0, 1\} \\ 1/2 & \text{if } s_1 = s_2 = 1/2 \\ 1 & \text{if } s_1 = 1/2, s_2 \in \{0, 1\} \end{cases}$$

and define the values $f_1(s_1, s_2)$ elsewhere by a linear interpolation. Put $f_2 = 1 - f_1$.

This game captures the following simplified version of Hotelling's electoral competition model [24]. Assume that players 1 and 2 are candidates choosing policies $s_1, s_2 \in [0, 1]$ in order to win the elections. Each citizen has preferences over policies and votes for either player 1 or 2. In the latest poll, the preferences show that player 1

- cannot attract extreme right or extreme left voters at all,
- is preferred by centrist voters whenever player 2 chooses any of the two extreme policies, and

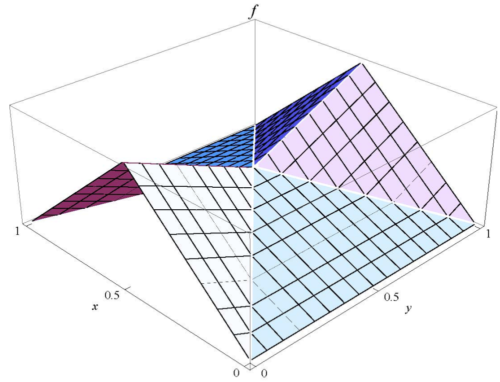

Figure 1: Payoff function $f_1(s_1, s_2) = f(x, y)$ for player 1 in the simplified electoral competition model.

- ties with player 2 if both adopt the centrist policy.

The values of f_i express the ratio of votes for player i so that $f_1 + f_2 = 1$. The unique pure Nash equilibrium in this game is the point $\langle 1/2, 1/2 \rangle$, representing the simultaneous choice of the centrist policies.

The next example shows that even very simple games may not admit pure Nash equilibria.

Example 2.10 (Matching Pennies)**.** *Matching Pennies* is a game in which each player $i \in N = \{1, 2\}$ secretly selects one of the sides of a coin. We may put $S_1 = S_2 = \{h, t\}$, with h for "head" and t for "tail". After their choices are made public, the payoffs of players 1 and 2 are given by the following table:

	h	t
h	$(1, -1)$	$(-1, 1)$
t	$(-1, 1)$	$(1, -1)$

It is easy to see the game of *Matching Pennies* has no pure Nash equilibrium.

The previous example motivates the introduction of mixed strategies over the sets S_i. In order to avoid technicalities we confine our attention to mixed strategies in *finite* strategic games (cf. Remark 2.15).

Definition 2.11. Let $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\} \rangle$ be a finite strategic game. A probability distribution p_i on the strategy set S_i of player $i \in N$ is called a *mixed strategy* of player i. More precisely, p_i is a function $S_i \to [0, 1]$ such that $\sum_{s_i \in S_i} p_i(s_i) = 1$. By Δ_i we denote the set of all mixed strategies of player i.

For any *mixed strategy profile* $\boldsymbol{p} = \langle p_1, \ldots, p_n \rangle \in \Delta = \Delta_1 \times \cdots \times \Delta_n$ we set:

$$\mathsf{E}_i(\boldsymbol{p}) = \sum_{\boldsymbol{s} \in S} \Big(f_i(\boldsymbol{s}) \cdot \prod_{i \in N} p_i(s_i) \Big).$$

The function $\mathsf{E}_i \colon \Delta \to \mathbb{R}$ is the *expected payoff (utility)* of player $i \in N$; its dependence on the payoff function f_i is tacitly understood. A mixed strategy profile $\boldsymbol{p}^* = \langle p_1^*, \ldots, p_n^* \rangle \in \Delta$ is a *mixed Nash equilibrium* of $\mathcal{G}$ if

$$\mathsf{E}_i(p_i, \boldsymbol{p}_{-i}^*) \leq \mathsf{E}_i(\boldsymbol{p}^*), \tag{1}$$

for every player $i \in N$ and every mixed strategy $p_i \in \Delta_i$.

Theorem 2.12 (Nash)**.** *Every finite strategic game $\mathcal{G}$ has a mixed Nash equilibrium.*

Note that the mixed strategy spaces Δ_i are uncountably infinite and each of them contains the original pure strategy space S_i via the embedding $s_i \in S_i \mapsto \delta_{s_i} \in \Delta_i$, where δ_{s_i} is the Dirac probability distribution at s_i:

$$\delta_{s_i}(t_i) = \begin{cases} 1 & \text{if } t_i = s_i \\ 0 & \text{if } t_i \neq s_i \end{cases} \quad \text{for each } t_i \in S_i.$$

For every pure strategy $s_i \in S_i$ let $\langle s_i, \boldsymbol{p}_{-i} \rangle$ denote the mixed strategy profile in which the mixed strategy of player i is the Dirac distribution δ_{s_i} concentrated at s_i. It can be shown that condition (1) of Definition 2.11 only needs to be checked against the pure strategies of each player:

Proposition 2.13 ([19, Corollary 5.8])**.** *For any finite strategic game $\mathcal{G}$ the following are equivalent:*

1. *$\boldsymbol{p}^*$ is a mixed Nash equilibrium of $\mathcal{G}$.*
2. *For every player $i \in N$ and every pure strategy $s_i \in S_i$,*

$$\mathsf{E}_i(s_i, \boldsymbol{p}^*_{-i}) \leq \mathsf{E}_i(\boldsymbol{p}^*).$$

Example 2.14 (Love and Hate)**.** The countable game called *Love and Hate* was analyzed in [7]. Here we present its finite variant $\mathcal{LH}$. It is played by an even number $n = 2k$ of players. Let m be an even positive integer. Each strategy space S_i is equal to the set $\left\{0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, 1\right\}$. Let $h(x, y) = 2 \cdot \min(|x - y|, 1 - |x - y|)$, for every $x, y \in S_i$. The payoff functions are defined as follows, for every $j = 1, \ldots, k$ and every strategy profile $\langle s_1, \ldots, s_n \rangle \in S$:

$$f_{2j-1}(s_1, \ldots, s_n) = h(s_{2j-1}, s_{2j}), \quad f_{2j}(s_1, \ldots, s_n) = 1 - h(s_{2j}, s_{2j+1}).$$

It can be shown (and later will be verified in Example 4.8) that a mixed Nash equilibrium in this game is the n-tuple of mixed strategies $\langle p_1, \ldots, p_n \rangle$ defined for $t_1, r_1, t_3, r_3, \ldots, t_{n-1}, r_{n-1} \in S_1$ by

$$\begin{aligned} p_1(x) = p_2(x) &= \tfrac{1}{2} \cdot (\delta_{t_1}(x) + \delta_{r_1}(x)), \\ &\cdots \\ p_{n-1}(x) = p_n(x) &= \tfrac{1}{2} \cdot (\delta_{t_{n-1}}(x) + \delta_{r_{n-1}}(x)), \end{aligned}$$

where $|t_{2j-1} - r_{2j-1}| = \frac{1}{2}$ for each $j \leq k$.

Remark 2.15. The assumption of finiteness of strategy spaces in $\mathcal{G}$ makes the ensuing theory much more understandable and technically easier. While it is possible to relax this assumption and define the mixed equilibria as general probability measures for games with infinite strategy sets, many additional assumptions are needed and the existence of a Nash equilibrium in mixed strategies is no longer guaranteed in general—see, e.g., [10]. Moreover, probability measures over an infinite universe are not directly amenable to a logical treatment since they do not admit any finite representation, in general. The problem of determining and computing the mixed strategies over $[0, 1]$ in case of games expressible by formulas in Łukasiewicz logic is studied in [15].

It is trivial to observe that any bijective re-labeling of the strategies in a finite game preserves pure Nash equilibria. Moreover, it holds as well for mixed equilibria, as stated by the next lemma.

Lemma 2.16. *Let $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\} \rangle$ and $\mathcal{G}' = \langle N, \{S'_i \mid i \in N\}, \{f'_i \mid i \in N\} \rangle$ be finite strategic games such that, for each $i \in N$, there is a bijection $c_i \colon S_i \to S'_i$ and $f'_i(\boldsymbol{c}(\boldsymbol{s})) = f_i(\boldsymbol{s})$, where $\boldsymbol{c}(\boldsymbol{s}) = \boldsymbol{c}(s_1, \ldots, s_n) = \langle c_1(s_1), \ldots, c_n(s_n) \rangle$, for every $\boldsymbol{s} \in S_1 \times \cdots \times S_n$. Then the following are equivalent for any mixed strategy profile $\boldsymbol{p} = \langle p_1, \ldots, p_n \rangle$ in game $\mathcal{G}$:*

1. *$\boldsymbol{p}$ is a mixed Nash equilibrium in $\mathcal{G}$.*
2. *$\boldsymbol{c}(\boldsymbol{p}) = \langle p_1 \circ c_1^{-1}, \ldots, p_n \circ c_n^{-1} \rangle$ is a mixed Nash equilibrium in $\mathcal{G}'$.*

It can be easily shown that pure Nash equilibria are invariant with respect to order-preserving maps:

Lemma 2.17. *Let $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\}\rangle$ be a strategic game. For every player $i \in N$, let g_i be a real non-decreasing function defined on the range of f_i. Then every pure equilibrium in $\mathcal{G}$ is also a pure equilibrium in the game $\hat{\mathcal{G}} = \langle N, \{S_i \mid i \in N\}, \{g_i \circ f_i \mid i \in N\}\rangle$.*

Mixed Nash equilibria are invariant with respect to positive affine transformations of the payoff functions—see [19, Theorem 5.35], for example.

Lemma 2.18. *Let $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\}\rangle$ be a finite strategic game. For every player $i \in N$, let $a_i > 0$, $b_i \in \mathbb{R}$, and $g_i = a_i f_i + b_i$. Let $\hat{\mathcal{G}} = \langle N, \{S_i \mid i \in N\}, \{g_i \mid i \in N\}\rangle$. Then every mixed Nash equilibrium of $\mathcal{G}$ is also a mixed Nash equilibrium of $\hat{\mathcal{G}}$.*

3 Logical games—representing strategic games

In this section, we will first (in Subsection 3.1) formally introduce the notion of a logical game, followed by some useful notational conventions. In Subsection 3.2 we define and then illustrate by various examples the concept of representing a given strategic game as a logical game. Subsection 3.3 discusses expressibility issues. Finally, Subsection 3.4 provides a series of general propositions that demonstrate how (wide classes of) finite strategic games can be represented as logical games at various levels of expressiveness of the underlying algebra.

3.1 Basic definitions

We introduce a special kind of strategic games—so-called logical $\boldsymbol{A}$-games. The standard algebra $\boldsymbol{A}$ plays two related roles in the definition of such games:

- Each player 'controls' a set of propositional variables: her strategies are assignments of values from A to those variables.
- Each payoff function is expressible by a formula in the language $\mathcal{L}_{\boldsymbol{A}}$ built from variables controlled by the players; thus each strategy profile provides the full information needed to evaluate any such 'payoff formula' and the possible payoffs are elements of A as well.

Definition 3.1. A *logical $\boldsymbol{A}$-game,* where $\boldsymbol{A}$ is a standard algebra, is an ordered tuple

$$\mathcal{G} = \langle N, V, \{V_i \mid i \in N\}, \{S_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle, \text{ where:}$$

1. $N = \{1, \ldots, n\}$ is a finite set of *players.*
2. V is a finite set of propositional variables.
3. $V_1, \ldots, V_n$ are sets of propositional variables forming a partition of V.
4. $S_i \subseteq A^{V_i}$ is the *strategy set* of player $i \in N$; we assume that S_i is non-empty for each $i \in N$.
5. The formula φ_i over variables from V in the language $\mathcal{L}_{\boldsymbol{A}}$ represents the *payoff function* of player $i \in N$; i.e., her payoff in the strategy profile $\boldsymbol{s} = \langle \boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\rangle \in S = S_1 \times \cdots \times S_n$ is $e(\varphi_i)$, in any $\boldsymbol{A}$-evaluation e such that $e(v) = \boldsymbol{s}_j(v)$ for each $j \in N$ and $v \in V_j$.

We say that $\mathcal{G}$ is:

- *Basic* if V_i is a singleton for each $i \in N$.
- *Finite* if S_i is finite for each $i \in N$.
- *Full* if $S_i = A^{V_i}$ for each $i \in N$.

Finally, we say that an element $a \in A$ is *$\mathcal{G}$-relevant* if $a = \boldsymbol{s}_i(v)$ for some $i \in N$, $v \in V_i$, and $\boldsymbol{s}_i \in S_i$; we denote the set of all $\mathcal{G}$-relevant elements of the carrier set A of $\boldsymbol{A}$ by $\boldsymbol{A}{\restriction}\mathcal{G}$.

Before introducing the notional conventions and commenting on the definition, we present an example of a logical game that is closely related to the New Technology game of Example 2.7:

Example 3.2. Let $\mathcal{NT}_{\mathbf{Ł}_4^c}$ be a 3-player logical $\mathbf{Ł}_4^c$-game such that:

- $N = \{1, 2, 3\}$.
- $V = \{v_1, v_2, v_3\}$.
- $V_i = \{v_i\}$ for each $i \in N$.
- $S_i = \{\{\langle v_i, a\rangle\} \mid a \in \{0, 1\}\}$ for each $i \in N$ (i.e., the players can only assign the values 0 or 1 to the variable they control).
- The $\mathcal{L}_{\mathbf{Ł}_4^c}$-formulas representing payoffs are as follows (see Example 2.2 for the definitions of the connectives in the algebra $\mathbf{Ł}_4^c$):

$$\varphi_1(v_1, v_2, v_3) = \big(\overline{1/2} \oplus \big(\overline{1/2} \wedge v_1\big)\big) \ominus \big(\big(\overline{1/4} \wedge v_2\big) \oplus \big(\overline{1/4} \wedge v_3\big)\big)$$
$$\varphi_2(v_1, v_2, v_3) = \big(\overline{1/2} \oplus \big(\overline{1/2} \wedge v_2\big)\big) \ominus \big(\big(\overline{1/4} \wedge v_1\big) \oplus \big(\overline{1/4} \wedge v_3\big)\big)$$
$$\varphi_3(v_1, v_2, v_3) = \big(\overline{1/2} \oplus \big(\overline{1/2} \wedge v_3\big)\big) \ominus \big(\big(\overline{1/4} \wedge v_1\big) \oplus \big(\overline{1/4} \wedge v_2\big)\big)$$

Clearly, the game is *basic* (as each player i only controls the single variable v_i), *finite* (as each player i has only two strategies, namely, $v_i \mapsto 0$ and $v_i \mapsto 1$), but *not full* (as the strategies do not exhaust $\{0, 1/4, 1/2, 3/4, 1\}^{\{v_i\}}$). Only the elements $0, 1$ are $\mathcal{NT}_{\mathbf{Ł}_4^c}$-relevant, as they are the only elements of $\mathbf{Ł}_4^c$ that can be assigned by the players to the variables they control; thus, $\mathbf{Ł}_4^c \lfloor \mathcal{NT}_{\mathbf{Ł}_4^c} = \{0, 1\}$.

The logical game $\mathcal{NT}_{\mathbf{Ł}_4^c}$ is a $\mathbf{Ł}_4^c$-representation of the New Technology game of Example 2.7, where each player's strategy $v_i \mapsto 1$ represents "adopt" and $v_i \mapsto 0$ the strategy "stay put". The details of the representation (especially how the payoff formulas correspond to the payoff functions of the strategic game) will be clarified by Definition 3.7 and Example 3.9 below.

Let us now introduce several notational conventions and identifications that will simplify the further presentation of logical games and formulation of results:

- Recall that we reserve subscripts for the index of the relevant player; a second index, if needed, is written as a superscript.
- For any $i \in N$, let us enumerate the propositional variables in V_i as $V_i = \{v_i^1, \ldots, v_i^{|V_i|}\}$. The tuple $\langle v_i^1, \ldots, v_i^{|V_i|}\rangle$ will be denoted by $\boldsymbol{v}_i$.
- By Definition 3.1, the strategies of player $i \in N$, or the elements of S_i, are (some) *mappings* from the player's set of controlled variables V_i to A. Thanks to the fixed enumeration of V_i, player i's strategies $\boldsymbol{s}_i \in S_i$ can be identified with *tuples* $\langle s_i^1, \ldots, s_i^{|V_i|}\rangle$ of elements of A, where $s_i^j = \boldsymbol{s}_i(v_i^j)$, for each $i \in N, j \in \{1, \ldots, |V_i|\}$. Most of the time we will view i's strategies as tuples rather than mappings, but freely switch between both meanings. (Formally, we just identify $A^{|V_i|}$ and A^{V_i}.)
- Since each player's strategies can be regarded as tuples of elements of A, a strategy profile $\boldsymbol{s} = \langle \boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\rangle$ can be viewed as the tuple of tuples $\langle\langle s_1^1, \ldots, s_1^{|V_1|}\rangle, \ldots, \langle s_n^1, \ldots, s_n^{|V_n|}\rangle\rangle$, which can in turn be identified with the concatenation of the inner tuples:

$$\boldsymbol{s} = \langle s_1^1, \ldots, s_1^{|V_1|}, \ldots, s_n^1, \ldots, s_n^{|V_n|}\rangle,$$

  i.e., a $|V|$-tuple of elements of A. Simultaneously, the strategy profile can be identified with a mapping from V to A, assigning to each $v_i^j \in V$ the element $s_i^j \in A$. That is, each strategy profile can also be regarded as an *evaluation* (a truth-value assignment) of all propositional variables in V. Again, we will freely switch between these representations of a strategy profile.

- Similarly, the set V can be regarded as a $|V|$-tuple

$$\boldsymbol{v} = \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle = \langle v_1^1, \ldots, v_1^{|V_1|}, \ldots, v_n^1, \ldots, v_n^{|V_n|} \rangle.$$

  This will allow us to write $\varphi_i(\boldsymbol{v})$ to signify that the variables occurring in φ_i are among those in $\boldsymbol{v}$ and to write $\varphi_i^{\boldsymbol{A}}(\boldsymbol{s})$ for the value of φ_i in the evaluation determined by the strategy profile $\boldsymbol{s}$.

- Recall from Lemma 2.16 that we write $\boldsymbol{c}(\boldsymbol{s})$ for $\langle c_1(s_1^1), \ldots, c_1(s_1^{|V_1|}), c_2(s_2^1), \ldots, c_n(s_n^{|V_n|}) \rangle$. Similarly, we write $\boldsymbol{e}(\boldsymbol{v})$ for $\langle e(v_1^1), \ldots, e(v_n^{|V_n|}) \rangle$.

Example 3.3. Recall the logical $\boldsymbol{Ł}_4^c$-game $\mathcal{NT}_{\boldsymbol{Ł}_4^c}$ of Example 3.2. By our conventions, the set V can be regarded as the triple $\boldsymbol{v} = \langle v_1^1, v_2^1, v_3^1 \rangle$, or simply $\langle v_1, v_2, v_3 \rangle$ since the game is basic. Similarly, each strategy $\boldsymbol{s}_i \in S_i$ can be identified with the 1-tuple (i.e., an element) $s_i^1 \in [0, 1]$; thus, due to the limited choice of elements by each player and the fact that the game is basic, we can identify S_i with the set $\{0, 1\} \subseteq A$. Each strategic profile $\boldsymbol{s} \in S$ can thus be viewed as a triple $\langle s_1, s_2, s_3 \rangle \in \{0, 1\}^3 \subseteq A^3$, or as an $\boldsymbol{Ł}_4^c$-evaluation of the propositional variables v_1, v_2, v_3 (by values 0 or 1).

Let us now discuss our definition of a logical $\boldsymbol{A}$-game. Definition 3.1 generalizes three classes of formalized models of strategic games appearing in the literature: the well-known Boolean games in strategic form of Harrenstein et al. [14] are full $\boldsymbol{2}$-games; the finite Łukasiewicz games of [17, 18] are full $\boldsymbol{Ł}_n$- or $\boldsymbol{Ł}_n^c$-games;[3] and the infinite Łukasiewicz games introduced in [18] are full $[0, 1]_{\mathbb{Q}Ł}$-games.

Our approach, however, is versatile enough to encompass also other types of strategic games, which are not directly captured by the two mentioned subclasses. In particular, note that whenever the payoff functions in the strategic game are of a type included in Table 1 in Section 2.1, then we can represent the game as a logical $\boldsymbol{A}$-game, where $\boldsymbol{A}$ is the corresponding algebra specified in the first column of the table.

Furthermore note that if a game $\mathcal{G}$ is full, then $\boldsymbol{A}\lfloor\mathcal{G} = A$. However, it should be stressed that in many games $A \neq \boldsymbol{A}\lfloor\mathcal{G}$: i.e., not all elements of A are available for the players as evaluations for the variables they control. In particular a logical $\boldsymbol{A}$-game $\mathcal{G}$ can be finite even if the algebra $\boldsymbol{A}$ itself is infinite. While this may look unintuitive at the first glance, it is actually not too different from the classical case, where we could also assume that strategy sets of the players may be coded as *proper* subsets of real numbers. The main role of $\boldsymbol{A}$ is to provide a way to express payoffs as formulas. The flexibility in modeling strategy sets yields two major advantages over the previous approaches:

- A much wider class of strategic games can be represented in our framework compared to the previous approaches (for details see Section 3.2).

- We can extend the algebra $\boldsymbol{A}$ by adding more elements to its domain or adding more operations to express further properties of games (see Section 4.2). Moreover, if we keep the strategy sets and payoff formulas unchanged, the resulting game will remain essentially the same (as the payoff formulas of the original game are also formulas of the larger game and we have only extended the codomain of their evaluations, but not the evaluations themselves).

The following proposition formalizes the second claim using the notion of a *subreduct*: $\boldsymbol{A}$ is a subreduct of $\boldsymbol{B}$ if $A \subseteq B$, $\mathcal{L}_{\boldsymbol{A}} \subseteq \mathcal{L}_{\boldsymbol{B}}$, and the operations of $\boldsymbol{A}$ are the restrictions of those in $\boldsymbol{B}$ to A. (If $\mathcal{L}_{\boldsymbol{A}} = \mathcal{L}_{\boldsymbol{B}}$, we speak of a *subalgebra* $\boldsymbol{A}$ of $\boldsymbol{B}$; if $A = B$, then $\boldsymbol{A}$ is called a *fragment* of $\boldsymbol{B}$.)

[3] Marchioni and Wooldridge formulate their results for what in our terminology are full $\boldsymbol{Ł}_n^c$-games. However, they also show that characteristic functions for all the elements of the domains of these algebras can be expressed by $\boldsymbol{Ł}_n$-formulas (see Section 3.3 and Definition 3.15 below). In this sense also full $\boldsymbol{Ł}_n$-games are covered in [17, 18].

Proposition 3.4. *Let $\boldsymbol{A}$ be a subreduct of $\boldsymbol{B}$. Then every logical $\boldsymbol{A}$-game is also a logical $\boldsymbol{B}$-game.*

Proposition 3.4 allows us to view all $\boldsymbol{A}$-games, for all algebras $\boldsymbol{A}$ from Example 2.2, as logical $[0,1]_{\text{ŁΠ}\frac{1}{2}}$-games and use all the expressive power of this logic (see Table 1) to analyze these games (see Section 4.2, where it will allow us to expressed mixed Nash equilibria of all finite logical games).

The following trivial proposition shows that restricting the set of strategies in $\boldsymbol{A}$-games preserves pure Nash equilibria, provided they remain available in the restricted game.

Proposition 3.5. *Let $\mathcal{G} = \langle N, V, \{V_i \mid i \in N\}, \{S_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ and $\mathcal{G}' = \langle N, V, \{V_i \mid i \in N\}, \{S'_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ be logical $\boldsymbol{A}$-games such that $S_i \subseteq S'_i$ for all $i \in N$. If the strategy profile $\boldsymbol{s} \in S_1 \times \cdots \times S_n$ in $\mathcal{G}$ is a pure Nash equilibrium of $\mathcal{G}'$, then it is a pure Nash equilibrium of $\mathcal{G}$ as well.*

Sometimes we can reverse the implication:

Example 3.6. Let $\mathcal{G} = \langle N, V, \{V_i \mid i \in N\}, \{S_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ and $\mathcal{G}' = \langle N, V, \{V_i \mid i \in N\}, \{S'_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$, where $V = \{v_1, \dots, v_n\}$ and for each $i \leq n$: $V_i = V'_i = \{v_i\}$ (i.e., both $\mathcal{G}$ and $\mathcal{G}'$ are basic), $S_i = [0,1] \cap \mathbb{Q}$, and $S'_i = [0,1]$; i.e., each player i selects the value for the variable v_i from $[0,1]$ in $\mathcal{G}'$ and from $[0,1] \cap \mathbb{Q}$ in $\mathcal{G}$ (thus $\mathcal{G}'$ is full, while $\mathcal{G}$ is not). Then every pure Nash equilibrium of $\mathcal{G}$ is a pure Nash equilibrium of $\mathcal{G}'$ as well: see [15, Prop. 3.6].

3.2 Representing strategic games—examples

As indicated above, logical $\boldsymbol{A}$-games can be viewed as special strategic games. Thus the notions of pure and mixed Nash equilibria are defined for logical games exactly for strategic games, using the payoff functions $\varphi_i^{\boldsymbol{A}}(\boldsymbol{s})$. We have also seen, at the end of Section 2.2, that strategic games related to one another by simple transformations share (pure and/or mixed) Nash equilibria. The following definition spells out what it means to represent a given strategic game by a logical $\boldsymbol{A}$-game and adapts the corresponding classes of transformations between games to our setting.

Definition 3.7. Let $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\}\rangle$ be a strategic game and let $\boldsymbol{A}$ be a standard algebra. We say that $\mathcal{G}$ is *represented* by a logical $\boldsymbol{A}$-game $\hat{\mathcal{G}} = \langle N, V, \{V_i \mid i \in N\}, \{S'_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ via g and $\boldsymbol{c} = \langle c_i \rangle_{i \in N}$ if:

1. $g\colon [0,1] \to \mathbb{R}$ is a strictly increasing function.
2. $c_i\colon S_i \to S'_i$ is a bijection for each $i \in N$.
3. $f_i(\boldsymbol{s}) = g(\varphi_i^{\boldsymbol{A}}(\boldsymbol{c}(\boldsymbol{s})))$ for each $\boldsymbol{s} \in S_1 \times \cdots \times S_n$.

The representation is called *affine* if g is an affine function.

Using Lemmas 2.16–2.18 we obtain the following basic fact, which shows how the Nash equilibria of a strategic game and its logical representation are related. For a mixed strategy $\boldsymbol{p}$ in $\mathcal{G}$ and bijections c_i as above, $\boldsymbol{c}(\boldsymbol{p})$ denotes the corresponding image of $\boldsymbol{p}$ in $\hat{\mathcal{G}}$, just as in Lemma 2.16.

Lemma 3.8. *Let $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\}\rangle$ be a strategic game and let $\hat{\mathcal{G}} = \langle N, V, \{V_i \mid i \in N\}, \{S'_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ be a logical $\boldsymbol{A}$-game representing $\mathcal{G}$ via g and $\boldsymbol{c}$. Then:*

1. *A strategy profile $\boldsymbol{s}^*$ is a pure Nash equilibrium in $\mathcal{G}$ iff $\boldsymbol{c}(\boldsymbol{s}^*)$ is a pure Nash equilibrium in $\hat{\mathcal{G}}$.*
2. *If $\mathcal{G}$ is finite and the representation is affine, then $\boldsymbol{p}^*$ is a mixed Nash equilibrium in $\mathcal{G}$ iff $\boldsymbol{c}(\boldsymbol{p}^*)$ is a mixed Nash equilibrium in $\hat{\mathcal{G}}$.*

In Section 3.4 we state a series of propositions, demonstrating that wide classes of finite games can be (affinely) represented by appropriate logical games of various levels of complexity. Ultimately, Proposition 3.26 shows that for a sufficiently expressive algebra $\boldsymbol{A}$, *all* finite games can be represented (though not necessarily affinely) as logical $\boldsymbol{A}$-games. We will see that many of those representations are nevertheless affine and thus preserve even mixed equilibria. However, the representations that are directly obtained from those propositions are using rather complex formulas, in general. In the following, we revisit the examples from Section 2.2 to indicate that many prominent strategic games can in fact be logically represented in a more compact and natural fashion.

Example 3.9. Recall the strategic game *New Technology* of Example 2.7 (let us denote it by $\mathcal{NT}$) and the logical $\boldsymbol{Ł}_4^c$-game $\mathcal{NT}_{\boldsymbol{Ł}_4^c}$ of Example 3.2. Definition 3.7 makes precise in which sense $\mathcal{NT}_{\boldsymbol{Ł}_4^c}$ represents $\mathcal{NT}$:

- There are just two strategies ("adopt" and "stay put") for each player i in $\mathcal{NT}$. As hinted in Example 3.2, these are encoded by assigning the values 1 and 0, respectively, to the variable v_i the player controls in $\mathcal{NT}_{\boldsymbol{Ł}_4^c}$. This provides the bijection c_i between the two-element strategy sets of player i in both games.

- Recall from Example 3.3 that strategy profiles in $\mathcal{NT}_{\boldsymbol{Ł}_4^c}$ can be viewed as triples $\langle a_1, a_2, a_3\rangle \in \{0,1\}^3$ evaluating the variables v_1, v_2, v_3. By the interpretation of the connectives in $\boldsymbol{Ł}_4^c$ (see Example 2.2), the payoff formula φ_1 of $\mathcal{NT}_{\boldsymbol{Ł}_4^c}$ (defined in Example 3.2) evaluates as

$$\varphi_1^{\boldsymbol{Ł}_4^c}(a_1,a_2,a_3) = \left(\frac{1}{2} + \frac{a_1}{2}\right) - \left(\frac{a_2}{4} + \frac{a_3}{4}\right)$$

  for each $a_1, a_2, a_3 \in \{0,1\}$, and similarly for φ_2 and φ_3. Observe that the resulting values $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ of the payoff formulas φ_i in $\boldsymbol{Ł}_4^c$ can be transformed to the corresponding payoff values

$$f_i(\boldsymbol{s}) \in \{-c, -{}^{c}\!/_{2}, 0, {}^{c}\!/_{2}, c\}$$

  (as defined for each corresponding strategy profile $\boldsymbol{s}$ of $\mathcal{NT}$ in Example 2.7) by the strictly increasing function $g\colon x \mapsto 2c(x - \frac{1}{2})$ from $[0,1]$ to $\mathbb{R}$. The representation is affine, as g is clearly an affine function.

Thus the logical game $\mathcal{NT}_{\boldsymbol{Ł}_4^c}$ represents the finite strategic game *New Technology* affinely, therefore both games have the same pure and mixed Nash equilibria (modulo the transformation by $\boldsymbol{c}$ and g).

Example 3.10. The finite variant $\mathcal{LH}$ of *Love and Hate* from Example 2.14 can be affinely represented as a (full and basic) logical $\boldsymbol{Ł}_m$-game $\mathcal{LH}_{\boldsymbol{Ł}_m}$. First observe that each strategy space $S_i = \left\{0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}, 1\right\}$ is just the universe of the finite Łukasiewicz chain $\boldsymbol{Ł}_m$. We can thus take $V_i = \{v_i\}$ and c_i as identity for each $i \in N$.

Using the interpretation of connectives in $\boldsymbol{Ł}_m$ (see Example 2.2), one can easily verify that $h(x,y) = \eta^{\boldsymbol{Ł}_m}(x,y)$ for the $\mathcal{L}_{\boldsymbol{Ł}_m}$-formula η defined as

$$\eta(x,y) = \big(\theta(x,y) \wedge \neg\theta(x,y)\big) \oplus \big(\theta(x,y) \wedge \neg\theta(x,y)\big),$$

where $\theta(x,y) = \neg(x \to y) \vee \neg(y \to x)$. (Observe that $\theta^{\boldsymbol{Ł}_m}(x,y) = |x-y|$ for every $x, y \in S_i$.) The payoff functions are thus directly expressible by $\mathcal{L}_{\boldsymbol{Ł}_m}$-formulas $\varphi_{2j-1} = \eta(v_{2i-1}, v_{2i})$ for odd players and $\varphi_{2j} = \neg\eta(v_{2j}, v_{2j+1})$ for even players. The representation is affine, as the transformation g is the identity function.

Example 3.11. Recall the *Vickrey Auction* game from Example 2.8. This game of n players is determined by values each player associates to the object sold in the auction. Recall that we assume that the values $p_1, \dots, p_n$ are non-negative rational numbers and that bids $b_1, \dots, b_n$ of all players are from the interval $[0,t]$ for some rational number $t > \max_{i \le n} p_i$.

Let us consider a $[0,1]^{\triangle}_{\mathbb{Q}Ł}$-game $\mathcal{VA}^{\triangle}_{\mathbb{Q}Ł}$, where $N = \{1, 2, \ldots, n\}$, $V = \{v_1, \ldots, v_n\}$, and for each $i \in N$, we put $V_i = \{v_i\}$, $S_i = [\frac{1}{2}, 1]$, and the payoff formulas $\varphi_i(v_1, \ldots, v_n)$ are constructed as follows.

Let $\overline{r_i} = \overline{\frac{t+p_i}{2t}}$ be the truth constant corresponding to $\frac{t+p_i}{2t}$ (recall that we assume t and all p_i to be rational). Let $\boldsymbol{v} = \langle v_1, \ldots, v_n\rangle$ and define:

$$\kappa_i(\boldsymbol{v}) = \bigvee_{j \neq i} v_j$$
$$\iota_i(\boldsymbol{v}) = \triangle\Big(\big(\bigvee_j v_j\big) \to v_i\Big) \wedge \neg\triangle\Big(v_i \to \bigvee_{j<i} v_j\Big)$$

(if $i = 1$, then the empty disjunction $\bigvee_{j<i} v_j$ is understood as 0). The payoff formula φ_i is defined as follows (for the definition of connectives in the algebra $[0,1]^{\triangle}_{\mathbb{Q}Ł}$ see Example 2.2):

$$\varphi_i(\boldsymbol{v}) = \Big(\overline{\tfrac{1}{2}} \oplus \big(\iota_i(\boldsymbol{v}) \wedge (\overline{r_i} \ominus \kappa_i(\boldsymbol{v}))\big)\Big) \ominus \Big(\iota_i(\boldsymbol{v}) \wedge \big(\kappa_i(\boldsymbol{v}) \ominus \overline{r_i}\big)\Big).$$

We can show that $\mathcal{VA}^{\triangle}_{\mathbb{Q}Ł}$ affinely represents the Vickrey auction game via $g(x) = 2t(x - \frac{1}{2})$ and $c_i(x) = \frac{t+x}{2t}$ for each $i \in N$.

First observe that $\iota_i^{[0,1]^{\triangle}_{\mathbb{Q}Ł}}(\boldsymbol{v})$ indicates the player who wins the auction. Indeed, for each $b_1, \ldots, b_n \in [0, t]$,

$$\iota_i^{[0,1]^{\triangle}_{\mathbb{Q}Ł}}(\boldsymbol{c}(\boldsymbol{b})) = \begin{cases} 1 & \text{if } b_i \geq \max_{j\in N} b_j \text{ and } b_i > \max_{j<i} b_j, \\ 0 & \text{otherwise,} \end{cases}$$

where $\boldsymbol{c}(\boldsymbol{b})$ denotes the tuple $\langle c_1(b_1), \ldots, c_n(b_n)\rangle$. Furthermore notice that if i wins the auction, then κ_i represents the second highest bid, i.e., the price to be paid for the auctioned object:

$$\kappa_i^{[0,1]^{\triangle}_{\mathbb{Q}Ł}}(\boldsymbol{c}(\boldsymbol{b})) = \max_{j\neq i} c_j(b_j) = \tfrac{1}{2} + \tfrac{\max_{j\neq i} b_j}{2t}.$$

Consequently, by the semantics of the connectives in $[0,1]^{\triangle}_{\mathbb{Q}Ł}$ (see Example 2.2), for each $b_1, \ldots, b_n \in [0, t]$:

$$\varphi_i^{[0,1]^{\triangle}_{\mathbb{Q}Ł}}(\boldsymbol{c}(\boldsymbol{b})) = \begin{cases} \frac{1}{2} + \frac{p_i - \max_{j\neq i} b_j}{2t} & \text{if } i = \min\{j \in N \mid b_j = \max_{k\in N} b_k\}; \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Thus indeed, $f_i(b_1, \ldots, b_n) = g\big(\varphi_i^{[0,1]^{\triangle}_{\mathbb{Q}Ł}}(\boldsymbol{c}(\boldsymbol{b}))\big)$.

Remark 3.12. Marchioni and Wooldridge [18] aim to show that (a variant of) the second-price sealed-bid auction with perfect information can be formalized as a Łukasiewicz game. However, to achieve that goal they explicitly impose some highly problematic restrictions: (1) all players are assumed to assign the same value to the object in question; (2) the players can only submit bids that are smaller or equal to the assigned value; (3) the payoff is set to 0 for all players if there is a tie at the highest bid. Since the value assigned to the object is common knowledge (as required in any strategic game), each of these assumptions trivializes the game. Jointly these restrictions amount to a game that hardly reflects any essential feature of the Vickrey auction. In any case, our logical game $\mathcal{VA}^{\triangle}_{\mathbb{Q}Ł}$ does not impose any of the three mentioned restrictions.

Example 3.13. Recall the *Electoral Competition* model introduced in Example 2.9. It follows directly from the definitions of the two payoffs functions f_1 and f_2 that both are continuous and piecewise linear (affine), where each of the finitely many linear pieces has only integer coefficients. We have seen in Table 3.8 that these functions are represented two-variable formulas in infinite-valued Łukasiewicz logic (see [20, 8] for details). This means that the electoral model is representable as a logical $\boldsymbol{A}$-game $\mathcal{EC}_{Ł}$, where $\boldsymbol{A}$ is the standard MV-algebra (see Example 2.2).

	h	t
h	$(1,0)$	$(0,1)$
t	$(0,1)$	$(1,0)$

Table 2: Matching Pennies with transformed payoffs

Example 3.14. Recall the game of *Matching Pennies* from Example 2.10. This game is represented by the Boolean game $\mathcal{MP}_{\mathbf{2}}$ specified in Table 2, resulting from the original payoff table by applying the affine transformation $x \mapsto \frac{1}{2}x+\frac{1}{2}$. Thus, $A = \{0,1\}$ and we may identify h with 0 and t with 1. The payoff of player 1 is then determined by the Boolean function $f_1(v_1, v_2) = 1-|v_1-v_2|$, expressible by the classical propositional formula $\varphi_1 = (v_1 \wedge \neg v_2) \vee (\neg v_1 \wedge v_2)$; f_2 can be represented either analogously or simply as $\neg\varphi_1$.

3.3 Expressible games

In Sections 4.1 and 4.2 we will study the expressibility of pure and mixed equilibria, respectively, by formulas of the logics in question. For that purpose we will need an additional condition on logical games.

Definition 3.15. Let $\boldsymbol{A}$ be a standard algebra and $a \in A$. We say that $\boldsymbol{A}$ has:

- a *(definable) truth constant* for a if there is an $\mathcal{L}_{\boldsymbol{A}}$-formula $\bar{a}$ such that $e(\bar{a}) = a$ for every $\boldsymbol{A}$-evaluation e;
- a *pseudo-characteristic formula* for a if there is an $\mathcal{L}_{\boldsymbol{A}}$-formula χ_a over a single variable such that for every $x \in A$ we have $\chi_a^{\boldsymbol{A}}(x) = 1$ iff $x = a$;
- a *characteristic formula* for a if there is an $\mathcal{L}_{\boldsymbol{A}}$-formula δ_a over a single variable such that for every $x \in A$ we have $\delta_a^{\boldsymbol{A}}(x) = 1$ if $x = a$ and $\delta_a^{\boldsymbol{A}}(x) = 0$ otherwise.

We say that a logical $\boldsymbol{A}$-game $\mathcal{G}$ is:

- *weakly expressible* if there is a pseudo-characteristic formula for each $a \in \boldsymbol{A}{\restriction}\mathcal{G}$ in $\boldsymbol{A}$;
- *expressible* if there is a truth constant for each $a \in \boldsymbol{A}{\restriction}\mathcal{G}$ in $\boldsymbol{A}$.

Clearly if there is a truth constant $\bar{a}$ for a in $\boldsymbol{A}$, then $\chi_a(p) = (p \to \bar{a}) \wedge (\bar{a} \to p)$ is a pseudo-characteristic formula for a. Thus all expressible games are weakly expressible. Also note that if $\mathcal{L}_{\boldsymbol{A}}$ contains the connective $\triangle$ (see Example 2.2) and χ_a is a pseudo-characteristic formula for a in $\boldsymbol{A}$, then $\triangle\chi_a$ is a characteristic formula for a.

Since the Boolean algebra $\boldsymbol{2}$ has truth constants for both of its elements 0 and 1, all Boolean games are expressible. Note that this includes, e.g., the game $\mathcal{MP}_{\mathbf{2}}$ of Example 3.14. The analogous claim is no longer true for (finite-valued) Łukasiewicz games, but well-known results tell us the following:

Lemma 3.16.

1. *Let $a = \frac{m}{n}$, where m and n are relatively prime positive integers with $m \leq n$. Then there is an $\mathcal{L}_{[0,1]_Ł}$-formula $\xi_{m,n}(v)$ such that $\xi_{m,n}^{[0,1]_Ł}(a) = \frac{1}{n}$ and $\xi_{m,n}^{[0,1]_Ł}(x) < \frac{1}{n}$ if $x \neq a$.*
2. *For each rational $a \in [0,1]$ there is a pseudo-characteristic $\mathcal{L}_{[0,1]_Ł}$-formula.*
3. *There is neither a characteristic formula nor a definable truth constant in $[0,1]_Ł$ for any rational $a \notin \{0,1\}$.*

4. *There is no pseudo-characteristic formula (so,* a fortiori, *no characteristic formula nor a definable truth constant) in* $[0,1]_Ł$ *for any irrational* $a \in [0,1]$.

5. *Let n be a positive integer and* $m \leq n$. *Then there is a characteristic formula for* $\frac{m}{n}$ *in* $\boldsymbol{Ł}_n$.

Proof. Claim 1 follows, e.g., from the theory of Farey–Schauder hats as developed in [8, Section 3]. To obtain claim 2 for any rational $a = \frac{m}{n}$ (where m, n are relatively prime), it is sufficient to take the formula $\bigoplus_{i=1}^{n} \xi_{m,n}$, where $\xi_{m,n}$ is as in claim 1. Claims 3 and 4 are direct consequences of functional representation of Łukasiewicz infinite-valued logic (see Table 1). Finally, in order to obtain claim 5 it suffices to take the formula $\&_{i=1}^{n} \chi_{\frac{m}{n}}$, where $\chi_{\frac{m}{n}}$ is a pseudo-characteristic formula for $\frac{m}{n}$ in $[0,1]_Ł$ (a more involved proof of this fact is provided in [17, Lemma 3] and in [18, Lemma 7.2]). □

Thus all logical $\boldsymbol{Ł}_n$-games (including, e.g., the game $\mathcal{LH}_{\boldsymbol{Ł}_m}$ of Example 3.10) are weakly expressible and so are all logical $[0,1]_Ł$-games $\mathcal{G}$ such and only such that $[0,1]_Ł\lfloor\mathcal{G} \subseteq \mathbb{Q}$. Observe that, consequently, the game $\mathcal{EC}_Ł$ of Example 3.13 is not even weakly expressible (although it would become weakly expressible if restricted to rational payoffs).

Furthermore, recall that the algebras $\boldsymbol{Ł}_n^c$ and $\boldsymbol{G}_n^c$ introduced in Example 2.2 contain truth constants for all elements of their domains; consequently, all $\boldsymbol{Ł}_n^c$- and $\boldsymbol{G}_n^c$-games are expressible (including, e.g., the game $\mathcal{NT}_{\boldsymbol{Ł}_4^c}$ of Example 3.2). Example 2.2 also introduced several standard algebras containing truth constants $\bar{r}$ for all $r \in [0,1] \cap \mathbb{Q}$: clearly for any such algebra $\boldsymbol{A}$ and any $\boldsymbol{A}$-game $\mathcal{G}$, if $\boldsymbol{A}\lfloor\mathcal{G} \subseteq \mathbb{Q}$ then $\mathcal{G}$ is expressible. (Thus, e.g., the game $\mathcal{VA}_{\mathbb{Q}Ł}^{\triangle}$ of Example 3.11 is expressible.)

Here we can also illustrate the usefulness of Proposition 3.4: Consider a $[0,1]_Ł$-game $\mathcal{G}$ such that $\{0,1\} \subsetneq [0,1]_Ł\lfloor\mathcal{G} \subseteq \mathbb{Q}$. We know that $\mathcal{G}$ is weakly expressible, but not expressible. However, by Proposition 3.4, we can see $\mathcal{G}$ as a $[0,1]_{\mathbb{Q}Ł}$-game which has the same pure and mixed equilibria and is clearly expressible.

3.4 Representing strategic games—general results

In the next three propositions we investigate a particularly simple type of games: finite strategic games with at most two possible payoff values. Note that all finite win/loose games fall into this category. It is easy to show that all such games (modulo certain encodings of strategies) can be presented as Boolean games (logical $\boldsymbol{2}$-games in our terminology). The only possible complication arises from the fact that the sets of strategies need not be limited to a binary choice. This implies that each player may have to control more than one variable, in general, so that all of her possible choices can be represented. This result is implicit already in [14]; however, we make it explicit and prove it, as it provides a convenient preparation for more complex cases elaborated later.

Convention 3.17. By Lemma 2.16 one may identify without loss of generality any given set of strategies of a finite game with an initial segment of the set of natural numbers. For sake of conciseness we will do so in the rest of the paper.

Proposition 3.18. *Let* $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\}\rangle$ *be a finite strategic game such that the union of ranges of all* f_i *is* $\{a, b\}$, *where* $a < b$. *For any* $i \in N$, *let* n_i *denote the least natural number such that* $|S_i| \leq 2^{n_i}$. *Moreover, for any* $s_i \in S_i$, *let* $\langle s_i^1, \dots, s_i^{n_i}\rangle \in 2^{n_i}$ *denote the binary representation of* s_i. *(Recall that by Convention 3.17 we assume that* s_i *is a natural number).*

Then the game $\mathcal{G}$ *is affinely represented by an expressible logical* $\boldsymbol{2}$*-game* $\hat{\mathcal{G}} = \langle N, V, \{V_i \mid i \in N\}, \{S'_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ *via* g *and* $\boldsymbol{c} = \langle c_i\rangle_{i\in N}$, *where for each* $i \in N$:

- $V_i = \{v_i^1, \dots, v_i^{n_i}\}$ *(and* $V = \bigcup_{i\in N} V_i$*).*
- $S'_i = \{\langle s_i^1, \dots, s_i^{n_i}\rangle \mid s_i \in S_i\}$. *(Recall that by our conventions, strategies can be identified with tuples of elements of the algebra; see beginning of Section 3.1.)*

- $\varphi_i = \bigvee_{\substack{\boldsymbol{s} \in S_1 \times \cdots \times S_n \\ f_i(\boldsymbol{s}) = b}} \bigwedge_{k \in N} \bigwedge_{j \leq n_k} \delta_{s_k^j}(v_k^j)$,

  where $\delta_x(v)$ is a characteristic formula of $x \in \{0,1\}$ in $\boldsymbol{2}$ (e.g., $\delta_0(v) = \neg v$ and $\delta_1(v) = v$).

- $c_i(s_i) = \langle s_i^1, \ldots, s_i^{n_i} \rangle$ *for each* $s_i \in S_i$.

- $g(x) = (b - a)x + a$.

Proof. First observe that $\hat{\mathcal{G}}$, as specified above, is indeed a logical $\boldsymbol{2}$-game. Moreover, $\boldsymbol{2}\lfloor\hat{\mathcal{G}} = \{0,1\}$, therefore $\hat{\mathcal{G}}$ is expressible (recall that the requisite truth constants $\overline{0}$ and $\overline{1}$ are part of $\mathcal{L}_{\boldsymbol{2}}$). Clearly each c_i is bijective and g is an affine monotone injection. It remains to check that for each strategy profile $\boldsymbol{s}$,

$$f_i(\boldsymbol{s}) = g(\varphi_i^{\boldsymbol{2}}(\boldsymbol{c}(\boldsymbol{s}))).$$

It is easy to see that for any strategy profile $\boldsymbol{s}$ we have:

$$e\Big(\bigwedge_{k \in N} \bigwedge_{j \leq n_k} \delta_{s_k^j}(v_k^j)\Big) = \begin{cases} 1 & \text{if } \langle e(v_k^1), \ldots, e(v_k^{n_k})\rangle = \langle s_k^1, \ldots, s_k^{n_k}\rangle = c_k(s_k) \\ & \qquad \text{for each } k \in N \text{ (i.e., } e(\boldsymbol{v}) = \boldsymbol{c}(\boldsymbol{s})); \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $e(\varphi_i(\boldsymbol{v})) = 1 = \varphi_i^{\boldsymbol{2}}(\boldsymbol{c}(\boldsymbol{s}))$ iff $f_i(\boldsymbol{s}) = b$. □

Next we show that, instead of working with $\sum_{i \in N} n_i$ 'binary' variables, we could represent such games with just one variable for each player, but at the price of using a logic with more truth values.

Proposition 3.19. *Let $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\}\rangle$ be a finite strategic game such that the union of ranges of all f_i is $\{a, b\}$, where $a < b$, and let $m = \max_{i \in N} |S_i| - 1$. Then $\mathcal{G}$ is affinely represented by a* basic *weakly expressible logical $\boldsymbol{Ł}_m$-game $\hat{\mathcal{G}} = \langle N, V, \{V_i \mid i \in N\}, \{S_i' \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ via g and $\boldsymbol{c} = \langle c_i \rangle_{i \in N}$, where for each $i \in N$:*

- $V_i = \{v_i\}$ *(thus $V = \{v_1, \ldots, v_n\}$).*

- $S_i' = \left\{\frac{s_i}{m} \mid s_i \in S_i\right\}$.

- $\varphi_i = \bigvee_{\substack{\boldsymbol{s} \in S_1 \times \cdots \times S_n \\ f_i(\boldsymbol{s}) = b}} \bigwedge_{k \in N} \delta_{\frac{s_k}{m}}(v_k)$,

  where $\delta_x(v)$ is a characteristic formula of x in $\boldsymbol{Ł}_m$. (Recall from Lemma 3.16 that there is a characteristic formula for each element of $\boldsymbol{Ł}_m$.)

- $c_i(s_i) = \frac{s_i}{m}$ *for each* $s_i \in S_i$.

- $g(x) = (b - a)x + a$.

Proof. The proof is analogous to that of Proposition 3.18. Besides small structural differences—the current game is only weakly expressible and basic—we only alter (actually simplify) the key observation for any strategy profile $\boldsymbol{s}$:

$$e\Big(\bigwedge_{k \in N} \delta_{\frac{s_k}{m}}(v_k)\Big) = \begin{cases} 1 & \text{if } \langle e(v_1), \ldots, e(v_n)\rangle = \left\langle \frac{s_1}{m}, \ldots, \frac{s_n}{m}\right\rangle = \boldsymbol{c}(\boldsymbol{s}); \\ 0 & \text{otherwise.} \end{cases}$$

Thus we still have $e(\varphi_i(\boldsymbol{v})) = 1 = \varphi_i^{\boldsymbol{Ł}_m}(\boldsymbol{c}(\boldsymbol{s}))$ iff $f_i(\boldsymbol{s}) = b$. □

Of course, one can combine the previous two approaches and work with any number of truth values between 2 and $\max_{i \in N} |S_i|$ at the price of using more variables, and we can render it in different logics. Let us now formalize these ideas in the next proposition (note that Propositions 3.18 and 3.19 are its corollaries).

Proposition 3.20. *Let $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\}\rangle$ be a finite strategic game such that the union of ranges of all f_i is $\{a, b\}$, where $a < b$. Let $m \geq 1$ and n_i be the least integer such that $|S_i| \leq (m+1)^{n_i}$ and let us, for a given $s_i \in S_i$ denote the $(m+1)$-ary representation of s_i by $\langle s_i^1, \ldots, s_i^{n_i}\rangle \in \{0, \ldots, m\}^{n_i}$.*

Moreover let $\boldsymbol{A}$ be a standard algebra with distinct elements $x_0, \ldots, x_m \in A$ such that for each $i \leq m$ there is a characteristic formula $\delta_i(v)$ for x_i in $\boldsymbol{A}$.

Then $\mathcal{G}$ is affinely represented by a weakly expressible logical $\boldsymbol{A}$-game $\hat{\mathcal{G}} = \langle N, V, \{V_i \mid i \in N\}, \{S'_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ via g and $\boldsymbol{c} = \langle c_i\rangle_{i\in N}$, where for each $i \in N$:

- $V_i = \{\langle v_i^1, \ldots, v_i^{n_i}\rangle\}$ *(thus $V = \bigcup_{i\in N} V_i$).*
- $S'_i = \left\{\left\langle x_{s_i^1}, \ldots, x_{s_i^{n_i}}\right\rangle \mid s_i \in S_i\right\}$.
- $\varphi_i = \displaystyle\bigvee_{\substack{\boldsymbol{s}\in S_1\times\cdots\times S_n\\ f_i(\boldsymbol{s})=b}} \bigwedge_{k\in N} \bigwedge_{j\leq n_k} \delta_{s_k^j}(v_k^j)$.
- $c_i(s_i) = \left\langle x_{s_i^1}, \ldots, x_{s_i^{n_i}}\right\rangle$ *for each $s_i \in S_i$.*
- $g(x) = (b-a)x + a$.

Furthermore, the representing game is basic iff $m \geq \max_{i\in N} |S_i| - 1$.

Proof. A straightforward combination of the proofs of Propositions 3.18 and 3.19. □

Now we leave games with binary payoffs and deal with the strategic games with any finite number r of possible payoff values. To achieve a more digestible presentation, we first formulate a result for a fixed infinitely-valued logic and basic games; only then a general variant is presented. On the other hand, the less general version provides an *affine* representation, which cannot be guaranteed in the general case. For simplicity we use the logic given by the algebra $[0,1]^{\triangle}_{\mathbb{Q}\mathrm{G}}$ (see Example 2.2); indeed in this algebra we have both the corresponding truth constant $\bar{a}$ and a characteristic formula δ_a for each rational a and thus each logical $[0,1]^{\triangle}_{\mathbb{Q}\mathrm{G}}$-game $\mathcal{G}$ where $\boldsymbol{A}\lfloor\mathcal{G} \subseteq \mathbb{Q}$ is expressible. Notice that by Proposition 3.4, we can as well use the logic of any algebra expanding $[0,1]^{\triangle}_{\mathbb{Q}\mathrm{G}}$, for example $[0,1]^{\triangle}_{\mathbb{Q}Ł}$, $[0,1]^{\triangle}_{\mathbb{Q}\mathrm{P}Ł}$, or $[0,1]_{Ł\Pi\frac{1}{2}}$ (as the connectives of $[0,1]_{\mathrm{G}}$ are definable in all these algebras).

Proposition 3.21. *Let $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\}\rangle$ be a finite strategic game such that the union of the ranges of all f_i is a set of rational numbers $\{o_1, \ldots, o_r\}$, where $o_1 < o_2 < \cdots < o_r$, and let $m = \max_{i\in N} |S_i| - 1$.*

Then $\mathcal{G}$ is affinely represented by a basic *expressible logical $[0,1]^{\triangle}_{\mathbb{Q}\mathrm{G}}$-game $\hat{\mathcal{G}} = \langle N, V, \{V_i \mid i \in N\}, \{S'_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ via g and $\boldsymbol{c} = \langle c_i\rangle_{i\in N}$, where for each $i \in N$:*

- $V_i = \{v_i\}$ *(thus $V = \{v_1, \ldots, v_n\}$).*
- $S'_i = \{\frac{s_i}{m} \mid s_i \in S_i\}$.
- $\varphi_i = \displaystyle\bigvee_{\boldsymbol{s}\in S_1\times\cdots\times S_n} \Big(\overline{g^{-1}(f_i(\boldsymbol{s}))} \wedge \bigwedge_{k\leq n} \delta_{\frac{s_k}{m}}(v_k)\Big)$,

  where $\delta_a(v)$ is a characteristic formula of a in $[0,1]^{\triangle}_{\mathbb{Q}\mathrm{G}}$.
- $c_i(s_i) = \frac{s_i}{m}$ *for each $s_i \in S_i$.*
- $g(x) = (o_r - o_1)x + o_1$.

Proof. First observe that $\hat{\mathcal{G}}$ is indeed a logical $[0,1]^{\triangle}_{\mathbb{Q}\mathrm{G}}$-game: notice that $g^{-1}(f_i(\boldsymbol{s}))$ is a rational number, as $f_i(\boldsymbol{s})$ is rational and $g\colon [0,1] \to \mathbb{R}$ is an affine function with rational coefficients; thus there is a corresponding truth constant and φ_i is indeed an $\mathcal{L}_{[0,1]^{\triangle}_{\mathbb{Q}\mathrm{G}}}$-formula. Moreover,

$[0,1]^{\triangle}_{\mathbb{Q}G}{\restriction}\hat{\mathcal{G}} \subseteq [0,1] \cap \mathbb{Q}$ and therefore $\hat{\mathcal{G}}$ is expressible (recall that all rational truth constants are present in the language $\mathcal{L}_{[0,1]^{\triangle}_{\mathbb{Q}G}}$). Clearly, each c_i is a bijection and g is an affine strictly increasing function with rational coefficients. It remains to check that for each strategy profile $\boldsymbol{s}$,

$$f_i(\boldsymbol{s}) = g\big(\varphi_i^{[0,1]^{\triangle}_{\mathbb{Q}G}}(\boldsymbol{c}(\boldsymbol{s}))\big).$$

Observe that for any strategy profile $\boldsymbol{s}$ we have:

$$e\Big(\overline{g^{-1}(f_i(\boldsymbol{s}))} \wedge \bigwedge_{k\leq n} \delta_{\frac{s_k}{m}}(v_k)\Big) = \begin{cases} g^{-1}(f_i(\boldsymbol{s})) & \text{if } \langle e(v_1),\dots,e(v_n)\rangle = \langle \frac{s_1}{m},\dots,\frac{s_n}{m}\rangle = \boldsymbol{c}(\boldsymbol{s}); \\ 0 & \text{otherwise.}\end{cases}$$

Therefore we have $\varphi_i^{[0,1]^{\triangle}_{\mathbb{Q}G}}(\boldsymbol{c}(\boldsymbol{s})) = g^{-1}(f_i(\boldsymbol{s}))$, as required. □

Furthermore, finite strategic games with rational payoffs can also be affinely represented by logical $\boldsymbol{A}$-games for sufficiently expressive *finite* algebras $\boldsymbol{A}$. For simplicity, the following proposition is formulated for finite standard G-algebras with truth constants and $\triangle$ (see Example 2.2); i.e., $\boldsymbol{A} = \boldsymbol{G}^{c\triangle}_m$ for sufficiently large m. However, Proposition 3.4 again ensures that all expansions of $\boldsymbol{G}^{c\triangle}_m$ can be used as well (notice that in particular, $\boldsymbol{Ł}^c_m$ falls within this class, since all connectives of $\boldsymbol{G}^{c\triangle}_m$ are definable in $\boldsymbol{Ł}^c_m$).

Proposition 3.22. *Let $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\}\rangle$ be a finite strategic game such that the union of the ranges of all f_i is a set of rational numbers $\{\frac{p_1}{q},\dots,\frac{p_r}{q}\}$, where $q, p_1,\dots p_r$ are integers and $p_1 < p_2 < \cdots < p_r$. Let m be a natural number such that $m \geq \max\{p_r - p_1, |S_1|,\dots,|S_n|\} - 1$.*

Then $\mathcal{G}$ is affinely represented by a basic expressible logical $\boldsymbol{G}^{c\triangle}_m$-game $\hat{\mathcal{G}} = \langle N, V, \{V_i \mid i \in N\}, \{S'_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ via g and $\boldsymbol{c} = \langle c_i\rangle_{i\in N}$, where for each $i \in N$:

- *V_i, S'_i, φ_i, and c_i are defined as in Proposition 3.21, using $\delta_a(v) = \triangle((v \to \bar{a}) \wedge (\bar{a} \to v))$.*
- *$g(x) = (mx + p_1)/q$.*

The proof of Proposition 3.22 is essentially the same as the proof of Proposition 3.21, therefore we omit it. As shown by the next Proposition 3.24, the presence of truth constants can be avoided in standard MV-chains $\boldsymbol{Ł}_m$ of suitable lengths, at the price of having a slightly larger algebra and only *weak* expressibility of the representing game. The formalizability of the payoff function in $\boldsymbol{Ł}_m$ is based on the following lemma:

Lemma 3.23. *Let m be a prime number, $a, b \in \boldsymbol{Ł}_m$ and $a \notin \{0,1\}$. Then there is an $\mathcal{L}_{\boldsymbol{Ł}_m}$-formula $\zeta_{m,a,b}(v)$ such that $\zeta^{\boldsymbol{Ł}_m}_{m,a,b}(a) = b$.*

Proof. Let $a = \frac{p}{m}$ and $b = \frac{q}{m}$. Since m is prime, by Lemma 3.16(1) there is a formula $\xi_{p,m}(v)$ such that $\xi^{[0,1]_Ł}_{p,m}\big(\frac{p}{m}\big) = \frac{1}{m}$. Since $\boldsymbol{Ł}_m$ is a subalgebra of $[0,1]_Ł$, we obtain $\xi^{\boldsymbol{Ł}_m}_{p,m}\big(\frac{p}{m}\big) = \frac{1}{m}$ as well; thus it is sufficient to take $\bigoplus_{i=1}^{q} \xi_{p,m}$ for $\zeta_{m,a,b}$. □

Proposition 3.24. *Let $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\}\rangle$ be a finite strategic game such that the union of the ranges of all f_i is a set of rational numbers $\{\frac{p_1}{q},\dots,\frac{p_r}{q}\}$, where $q, p_1,\dots p_r$ are integers and $p_1 < p_2 < \cdots < p_r$. Let m be a prime number such that $m \geq \max\big(p_r - p_1, |S_1|+1,\dots,|S_n|+1\big)$.*

Then $\mathcal{G}$ is affinely represented by a basic weakly expressible logical $\boldsymbol{Ł}_m$-game $\hat{\mathcal{G}} = \langle N, V, \{V_i \mid i \in N\}, \{S'_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ via g and $\boldsymbol{c} = \langle c_i\rangle_{i\in N}$, where for each $i \in N$:

- *$V_i = \{v_i\}$ (thus $V = \{v_1,\dots,v_n\}$).*
- *$S'_i = \{\frac{s_i+1}{m} \mid s_i \in S_i\}$.*
- *$\varphi_i = \displaystyle\bigvee_{\boldsymbol{s}\in S_1\times\cdots\times S_n} \Big(\zeta_{m,\frac{s_i+1}{m},g^{-1}(f_i(\boldsymbol{s}))}(v_i) \wedge \bigwedge_{k\leq n} \delta_{\frac{s_k+1}{m}}(v_k)\Big)$,*

  where $\zeta_{m,a,b}(v)$ is the formula from Lemma 3.23 and $\delta_a(v)$ is a characteristic formula of a in $\boldsymbol{Ł}_m$.

- $c_i(s_i) = \frac{s_i+1}{m}$ *for each* $s_i \in S_i$.
- $g(x) = (mx + p_1)/q$.

The proof of Proposition 3.24 is analogous to that of Proposition 3.21 (just observe that by Lemma 3.23, $\zeta^{\text{Ł}_m}_{m,\frac{s_i+1}{m},g^{-1}(f_i(\boldsymbol{s}))}(v_i) = g^{-1}(f_i(\boldsymbol{s})))$.

Example 3.25. By Propositions 3.21–3.24, the strategic game $\mathcal{NT}$ of Example 2.7 can be affinely represented not only as the logical Ł_4^c-game $\mathcal{NT}_{\text{Ł}_4^c}$ of Examples 3.2 and 3.9, but also, e.g., as a logical $[0,1]^{\triangle}_{\mathbb{Q}\mathrm{G}}$-game (by Proposition 3.21), a logical $\boldsymbol{G}_4^{c\triangle}$-game (by Proposition 3.22), or a logical Ł_7-game (by Proposition 3.24). Proposition 3.4 and the variability of m in Propositions 3.22 and 3.24 admit further algebras for logical representation of $\mathcal{NT}$, including, e.g., $[0,1]_{\mathbb{Q}\text{Ł}}$, $[0,1]_{\text{Ł}\Pi\frac{1}{2}}$, Ł_5^c, Ł_6^c, Ł_{11}, Ł_{13}, etc. In all these cases, the representation is affine and the representing games are basic and expressible (or weakly expressible in the case of Ł_m-games).

Notice that while the payoff formulas produced by Propositions 3.21–3.24 are rather large, a much more compact logical representation of $\mathcal{NT}$ exists in algebras that contain Ł_4^c as a subreduct: see the formulas φ_i in Example 3.2. Notice also that despite the representability as an Ł_4^c- or Ł_7-game, $\mathcal{NT}$ cannot be represented as a finite Łukasiewicz game of Marchioni and Wooldridge, since the sets of strategies and payoff values have different cardinalities.

As already mentioned above, the following general version comes at the price of loosing the affinity of representations.

Proposition 3.26. *Let* $\mathcal{G} = \langle N, \{S_i \mid i \in N\}, \{f_i \mid i \in N\}\rangle$ *be a finite game such that the union of ranges of all* f_i *is* $\{o_1, \ldots, o_r\}$, *where* $o_1 < o_2 < \cdots < o_r$, *and let* $m = \max_{i\in N} |S_i| - 1$. *Let* $\boldsymbol{A}$ *be an arbitrary standard algebra with distinct elements* $a_0, \ldots, a_{m-1}$ *and distinct elements* $b_1 < \cdots < b_r$ *(the two sets can overlap, though) and such that there are characteristic formulas* δ_i *in* $\boldsymbol{A}$ *for each* a_i, $i < m$, *and (definable) truth constants* $\overline{b}_i$ *for each* b_i, $i \le r$.

Then $\mathcal{G}$ *is represented by a* basic *weakly expressible* $\boldsymbol{A}$*-logical game* $\hat{\mathcal{G}} = \langle N, V, \{V_i \mid i \in N\}, \{S'_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ *via* g *and* $\boldsymbol{c} = \langle c_i\rangle_{i\in N}$, *where for each* $i \in N$:

- $V_i = \{v_i\}$ *(thus* $V = \{v_1, \ldots, v_n\}$*).*
- $S'_i = \{a_{s_i} \mid s_i \in S_i\}$.
- $\varphi_i = \displaystyle\bigvee_{\boldsymbol{s}\in S_1\times\cdots\times S_n} \Big(\overline{g^{-1}(f_i(\boldsymbol{s}))} \wedge \bigwedge_{k\le n} \delta_{s_k}(v_k)\Big)$.
- $c_i(s_i) = a_{s_i}$ *for each* $s_i \in S_i$.
- $g\colon [0,1] \to \mathbb{R}$ *is a strictly increasing function with* $g(b_j) = o_j$ *for every* $j \le r$.

Proof. Observe that $\hat{\mathcal{G}}$ is a logical $\boldsymbol{A}$-game; moreover, $\boldsymbol{A}\lfloor\hat{\mathcal{G}} = \{a_1, \ldots, a_m\}$ and thus $\hat{\mathcal{G}}$ is weakly expressible (by the theorem's assumptions). Clearly each c_i is a bijection and thus it remains to check that for each strategy profile $\boldsymbol{s}$ we have

$$f_i(\boldsymbol{s}) = g(\varphi_i(\boldsymbol{c}(\boldsymbol{s}))).$$

As before, it suffices to observe that for any strategy profile $\boldsymbol{s}$ we have

$$e\Big(\overline{g^{-1}(f_i(\boldsymbol{s}))} \wedge \bigwedge_{k\le n} \delta_{s_k}(v_k)\Big) = \begin{cases} b_{f_i(\boldsymbol{s})} & \text{if } \langle e(v_1), \ldots, e(v_n)\rangle = \langle a_{s_1}, \ldots, a_{s_n}\rangle = \boldsymbol{c}(\boldsymbol{s}); \\ 0 & \text{otherwise.} \end{cases}$$

We obtain $\varphi_i^{\boldsymbol{A}}(\boldsymbol{c}(\boldsymbol{s})) = b_{f_i(\boldsymbol{s})}$ and so $g(\varphi_i^{\boldsymbol{A}}(\boldsymbol{c}(\boldsymbol{s}))) = g(b_{f_i(\boldsymbol{s})}) = f_i(\boldsymbol{s})$. □

Just like for the combination of Proposition 3.18 and 3.19 into Proposition 3.20, we could also put together Proposition 3.21 and 3.26, but refrain from doing so here.

4 Expressing equilibria of logical games

In this section we show how pure and mixed Nash equilibria of logical games can be expressed by propositional formulas, under particular conditions.

Recall from Lemma 3.8 that whenever a logical game represents a strategic game, it has the same pure equilibria (modulo the representation); and if the game is finite and the representation is affine, then even mixed equilibria are preserved by the representation. Consequently the formulas derived below characterize equilibria not only in logical games themselves, but also in the strategic games they may represent.

Throughout this section we use the notation $\mathcal{G} = \langle N, V, \{V_i \mid i \in N\}, \{S_i \mid i \in N\}, \{\varphi_i \mid i \in N\}\rangle$ for any finite logical $\boldsymbol{A}$-game, where $\boldsymbol{A}$ is a standard algebra. Furthermore, let $S = S_1 \times \cdots \times S_n$.

4.1 Pure Nash equilibria

A crucial observation is the fact that in (weakly) expressible games each player's choice of a strategy can be encoded by $\mathcal{L}_{\boldsymbol{A}}$-formulas. In this section we show how this fact can be employed to express by an $\mathcal{L}_{\boldsymbol{A}}$-formula that a certain strategy profile is a pure Nash equilibrium. Consequently, as we will show, it can also be expressed that such an equilibrium exists. For simplicity we start with expressible games and deal with the more complicated case of weakly expressible ones later. Recall that in these games we have a truth constant $\bar{a}$ for each $a \in \boldsymbol{A}\lfloor\mathcal{G}$.

Let us consider auxiliary variables $\boldsymbol{w} = \langle w^1, w^2, \dots, w^{\max_{i\in N}|V_i|}\rangle$ different from those in V and define formulas $\gamma_i(\boldsymbol{v}, \boldsymbol{w})$, for each $i \in N$, and $\gamma_{\mathcal{G}}(\boldsymbol{v})$ as follows:

$$\gamma_i = \varphi_i(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i-1}, w^1, \dots, w^{|V_i|}, \boldsymbol{v}_{i+1}, \dots, \boldsymbol{v}_n) \to \varphi_i(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$$
$$\gamma_{\mathcal{G}} = \bigwedge_{i\in N} \bigwedge_{\boldsymbol{s}_i \in S_i} \gamma_i(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i-1}, \bar{s}_i^1, \dots, \bar{s}_i^{|V_i|}, \boldsymbol{v}_{i+1}, \dots, \boldsymbol{v}_n).$$

Lemma 4.1. *Let $\mathcal{G}$ be an expressible finite $\boldsymbol{A}$-game. Then the following are equivalent for each strategy profile $\boldsymbol{s}^*$:*

1. *$\boldsymbol{s}^*$ is a pure Nash equilibrium of $\mathcal{G}$.*
2. *$\boldsymbol{s}^*$ satisfies $\gamma_{\mathcal{G}}(\boldsymbol{v})$.*

Proof. The statement is a straightforward consequence of the definition of a pure Nash equilibrium and the properties of the connectives $\wedge$ and $\to$ ensured by Definition 2.1 for all standard algebras of truth degrees. □

Theorem 4.2. *Let $\mathcal{G}$ be an expressible finite logical $\boldsymbol{A}$-game. Then the following are equivalent:*

1. *$\mathcal{G}$ allows for a pure Nash equilibrium.*
2. *The following formula is satisfiable:*

$$\Big(\bigvee_{\boldsymbol{s}\in S} \bigwedge_{i\in N} \bigwedge_{j\le|V_i|} (\chi_{s_i^j}(v_i^j))\Big) \wedge \gamma_{\mathcal{G}}(\boldsymbol{v}). \tag{2}$$

Moreover, if the game $\mathcal{G}$ is full, then (2) can be replaced just by $\gamma_{\mathcal{G}}(\boldsymbol{v})$.

Proof. The statement follows from Lemma 4.1. It suffices to observe that an evaluation e satisfies the left conjunct in (2) if and only if $\boldsymbol{e}(\boldsymbol{v})$ is a strategy profile. □

Recall that all Boolean games are expressible. Thus Lemma 4.1 and Theorem 4.2 apply to Boolean games [14] in particular. Likewise, finite Łukasiewicz games are covered. More precisely, Theorem 2 of [17] amounts to a variant of a particular subcase of Theorem 4.2, where the underlying algebra is $\boldsymbol{Ł}_n^c$, only *full* games are considered, and a somewhat more complex variant of our $\gamma_{\mathcal{G}}(\boldsymbol{v})$

is used. Full $\boldsymbol{Ł}_n$-games are treated only indirectly in [17, 18], by showing that they are (in our terminology) weakly expressible. This case is covered by Lemma 4.3 and Theorem 4.4, below.

If the game is just weakly expressible, we have to use a more complex formula instead of $\gamma_{\mathcal{G}}$ to formulate and prove analogues of Lemma 4.1 and Theorem 4.2. In fact, we can keep the definition of the formulas γ_i, but we have to modify $\gamma_{\mathcal{G}}$ to include additional auxiliary variables that correspond to all the elementary strategies and will play the role of truth constants.

Formally, we introduce new variables $\{q_a \mid a \in \boldsymbol{A}{\restriction}\mathcal{G}\}$, different from those in V and $\boldsymbol{w}$. Note that the set $\boldsymbol{A}{\restriction}\mathcal{G}$ is a subset of A and thus is naturally ordered. Therefore we can use $\boldsymbol{q}$ to unambiguously denote the sequence of those variables. Now we can define $\gamma'_{\mathcal{G}}$ as a formula over the variables $\boldsymbol{v}$ and $\boldsymbol{q}$:

$$\gamma'_{\mathcal{G}} = \Big(\bigwedge_{a\in \boldsymbol{A}\restriction\mathcal{G}} \chi_a(q_a)\Big) \wedge \Big(\bigwedge_{i\in N}\bigwedge_{\boldsymbol{s}_i\in S_i} \gamma_i(\boldsymbol{v}_1,\dots,\boldsymbol{v}_{i-1}, q_{s_i^1},\dots q_{s_i^{|V_i|}}, \boldsymbol{v}_{i+1},\dots,\boldsymbol{v}_k)\Big).$$

Note that the formula $\bigwedge_{a\in \boldsymbol{A}\restriction\mathcal{G}} \chi_a(q_a)$ is satisfiable only by those evaluations that map each variable q_a to a. In this manner we obtain the promised generalizations of Lemma 4.1 and Theorem 4.2, which are then applicable, e.g., to all $\boldsymbol{Ł}_m$-logical games and all $[0,1]_{Ł}$-logical games with finite sets of (assignment of) rationals as strategies.

Lemma 4.3. *Let $\mathcal{G}$ be a weakly expressible finite $\boldsymbol{A}$-game. Then the following are equivalent for each strategy profile $\boldsymbol{s}^*$:*

1. *$\boldsymbol{s}^*$ is a pure Nash equilibrium of $\mathcal{G}$.*
2. *The $\boldsymbol{A}$-evaluation $\boldsymbol{s}^*, a_1, \dots, a_{|\boldsymbol{A}\restriction\mathcal{G}|}$ satisfies $\gamma'_{\mathcal{G}}(\boldsymbol{v},\boldsymbol{q})$.*

Theorem 4.4. *Let $\mathcal{G}$ be a weakly expressible finite $\boldsymbol{A}$-game. Then the following are equivalent:*

1. *$\mathcal{G}$ allows for a pure Nash equilibrium.*
2. *The following formula is satisfiable:*

$$\Big(\bigvee_{\boldsymbol{s}\in S}\bigwedge_{i\in N}\bigwedge_{j\le |V_i|} (\chi_{s_i^j}(v_i^j))\Big) \wedge \gamma'_{\mathcal{G}}(\boldsymbol{v},\boldsymbol{q}). \tag{3}$$

Moreover, if the game $\mathcal{G}$ is full, then (3) *can be replaced just by the formula $\gamma'_{\mathcal{G}}(\boldsymbol{v})$.*

4.2 Mixed Nash equilibria

In order to characterize mixed strategy profiles we have to express probability distributions and corresponding expected payoffs in a propositional language. On the algebraic side this means that the additive as well as multiplicative structure of the real unit interval $[0,1]$ must be employed. Interestingly enough, there are numerous natural examples of many-valued logics that provide such a rich semantics over $[0,1]$: in particular, the logics of algebras expanding the standard PŁ-algebra $[0,1]_{PŁ}$ (including $[0,1]^{\triangle}_{PŁ}$, $[0,1]_{\mathbb{Q}PŁ}$, $[0,1]^{\triangle}_{\mathbb{Q}PŁ}$ $[0,1]_{ŁΠ}$, or $[0,1]_{ŁΠ\frac{1}{2}}$, see Example 2.2) fall within this class.

We proceed with a simple lemma, crucial for expressing probability distributions. The underlying idea is based on MV-algebraic partitions of unity; see, e.g., [23]. We present its proof for the readers convenience.

Lemma 4.5. *Let $\boldsymbol{A}$ expand the standard* MV*-algebra $[0,1]_{Ł}$. For every $n \ge 2$ there is an $\mathcal{L}_{\boldsymbol{A}}$-formula $\delta(p_1,\dots,p_n)$ such that an $\boldsymbol{A}$-evaluation $\boldsymbol{a}\in[0,1]^n$ satisfies δ iff $\sum_{i\le n} a_i = 1$.*

Proof. We define

$$\delta = \Big(\bigoplus_{i\le n} p_i\Big) \wedge \bigwedge_{i\le n}\Big(\Big(\bigoplus_{\substack{j\le n\\ j\ne i}} p_j\Big) \to \neg p_i\Big).$$

Clearly the satisfiability of the first conjunct implies $\sum_{i\leq n} a_i \geq 1$. Therefore $a_i > 0$ for at least one $i \leq n$. Moreover the satisfiability of the second conjunct implies

$$\bigoplus_{\substack{j\leq n\\ j\neq i}}^{\boldsymbol{A}} a_j \leq 1 - a_i < 1.$$

This yields the inequality

$$\sum_{\substack{j\leq n\\ j\neq i}} a_j = \bigoplus_{\substack{j\leq n\\ j\neq i}}^{\boldsymbol{A}} a_j \leq 1 - a_i,$$

which entails $\sum_{j\leq n} a_j \leq 1$. The converse direction is trivial. □

For each $i \in N$ and each strategy $\boldsymbol{s}_i \in S_i$ let us introduce a variable $p_i^{\boldsymbol{s}_i}$. Moreover, let $\boldsymbol{p}_i$ denote the tuple $\langle p_i^{\boldsymbol{s}_i} \mid \boldsymbol{s}_i \in S_i\rangle$. (The tuple is unique with respect to the lexicographic order on $A^{|V_i|}$.) For every $i \in N$, any $\boldsymbol{A}$-evaluation of the variables $\boldsymbol{p}_i$ can be thought of as a mapping $S_i \to [0,1]$. This enables us to formulate a particular instance of Lemma 4.5 and thus to obtain:

Lemma 4.6. *Let $\boldsymbol{A}$ expand the standard* PL*-algebra $[0,1]_{\mathrm{PL}}$ and $\mathcal{G}$ be a finite expressible logical $\boldsymbol{A}$-game. Then for each $i \leq n$ there is a formula $\mathtt{ProbDistr}_i(\boldsymbol{p}_i)$ such that an evaluation $\mathbf{pr}_i \in [0,1]^{S_i}$ satisfies $\mathtt{ProbDistr}_i$ iff $\mathbf{pr}_i$ is a probability distribution over S_i.*

As a consequence, an element $\mathbf{pr} = \langle \mathbf{pr}_1, \ldots, \mathbf{pr}_n\rangle \in [0,1]^S$, where each $\mathbf{pr}_i$ satisfies the formula $\mathtt{ProbDistr}_i$, can be seen as a mixed strategy profile in the game $\mathcal{G}$. This enables us to define the expected payoff for a player i in a finite expressible logical $\boldsymbol{A}$-game $\mathcal{G}$ as follows:

$$\mathsf{E}_i(\boldsymbol{p}) = \mathsf{E}_i(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) = \bigoplus_{\mathbf{s}\in S}\Big(\varphi_i(\bar{s}_1^1, \ldots, \bar{s}_n^{|V_n|}) \odot \bigodot_{j\leq n} p_j^{\boldsymbol{s}_j}\Big).$$

Recall that by $\boldsymbol{p}_{-i}$ we denote the sequence of variables $\boldsymbol{p}$ where the subsequence $\boldsymbol{p}_i$ removed; for every pure strategy $\boldsymbol{a}_i \in S_i$, by $(\boldsymbol{a}_i, \boldsymbol{p}_{-i})$ we denote the mixed strategy profile in which the mixed strategy of player i is the Dirac distribution $\delta_{\boldsymbol{a}_i}$ concentrated at $\boldsymbol{a}_i$. Moreover we define the expected payoff for a player i in a mixed strategy profile $(\boldsymbol{a}_i, \boldsymbol{p}_{-i})$ as follows:

$$\mathsf{E}_i(\boldsymbol{a}, \boldsymbol{p}_{-i}) = \bigoplus_{\substack{\boldsymbol{s}\in S\\ \boldsymbol{s}_i=\boldsymbol{a}}}\Big(\varphi_i(\bar{s}_1^1, \ldots, \bar{s}_n^{|V_n|}) \odot \bigodot_{\substack{j\leq n\\ j\neq i}} p_j^{\boldsymbol{s}_j}\Big).$$

The above definitions and conventions allow us to formulate the following theorem, which provides the announced logical characterization of mixed Nash equilibria; its proof is a straightforward consequence of Proposition 2.13.

Theorem 4.7. *Let $\boldsymbol{A}$ be an algebra expanding $[0,1]_{\mathrm{PŁ}}$. Let $\mathcal{G}$ be a finite expressible logical $\boldsymbol{A}$-game and let $\mathbf{pr}^* \in [0,1]^S$ be a mixed strategy profile in $\mathcal{G}$. Then the following are equivalent:*

1. *$\mathbf{pr}^*$ is a mixed Nash equilibrium.*
2. *$\mathbf{pr}^*$ satisfies the following formula:*

$$\bigwedge_{i\leq n}\Big(\mathtt{ProbDistr}_i(\boldsymbol{p}_i) \wedge \bigwedge_{\boldsymbol{a}_i\in S_i}\big(\mathsf{E}_i(\boldsymbol{a}_i, \boldsymbol{p}_{-i}) \to \mathsf{E}_i(\boldsymbol{p})\big)\Big). \tag{4}$$

Example 4.8. Recall the finite variant $\mathcal{LH}$ of *Love and Hate* from Example 2.14. In Example 3.10 it was represented as a logical $\boldsymbol{Ł}_m$-game $\mathcal{LH}_{\boldsymbol{Ł}_m}$ with the same pure and mixed equilibria (since the transformation was affine via the identity functions g and $\boldsymbol{c}$). In virtue of Proposition 3.4, $\mathcal{LH}_{\boldsymbol{Ł}_m}$ can as well be regarded as a $[0,1]_{\mathrm{PŁ}}$-game, which again has the same pure and mixed equilibria. A routine calculation shows that the n-tuple $\mathbf{pr}^*$ of the mixed strategies $\langle p_1, \ldots, p_n\rangle$ defined in Example 2.14 satisfies the formula (4) in $[0,1]_{\mathrm{PŁ}}$, and thus $\mathbf{pr}^*$ is indeed a mixed Nash equilibrium in $\mathcal{LH}$ and $\mathcal{LH}_{\boldsymbol{Ł}_m}$.

5 Conclusion

We have taken up a line of research initiated by the introduction of Boolean games in [14] and generalized in [17, 18] to Łukasiewicz games. These are particular types of strategic games, where the payoff function for each player is specified by a propositional formula (of classical logic or some Łukasiewicz logic, respectively) and where each strategy assigns truth values (Boolean or many-valued) to the propositional variables under control of the player in question. The scope of general strategic games that can be directly represented as Boolean or Łukasiewicz games is limited by the specific type of the formulas that represent the payoff functions. Motivated by this observation, we present a more general approach referring to a wide class of algebras of truth values in the real unit interval. For any such algebra $\boldsymbol{A}$ there is a corresponding notion of logical $\boldsymbol{A}$-game, where propositional formulas over the corresponding language $\mathcal{L}_{\boldsymbol{A}}$ express the players' payoff functions. Based on a formal definition of the representability of a general strategic game as a logical $\boldsymbol{A}$-game, we have shown for several quite general classes of finite strategic games how they can be represented as logical $\boldsymbol{A}$-games. Furthermore we have shown that, for sufficiently expressible algebras $\boldsymbol{A}$, the existence of a pure Nash equilibrium in a logical $\boldsymbol{A}$-game can be expressed by an $\boldsymbol{A}$-formula. This is due to the observation that strategy profiles of logical games can be identified with evaluations (truth-value assignments) and the fact that the equilibrium conditions can be expressed by corresponding propositional formulas. As is well known, finite strategic games always admit a Nash equilibrium in terms of mixed strategies. However, it has been left open so far whether such mixed equilibria can be characterized by propositional formulas. We have taken up this challenge and proved that, for sufficiently rich algebras $\boldsymbol{A}$, one can encode probability distributions over strategies (i.e., evaluations) and find an $\mathcal{L}_{\boldsymbol{A}}$-formula that is satisfied by an interpretation if and only if that interpretation encodes a mixed Nash equilibrium.

Several directions for future research seem natural. For example, we did not consider complexity issues in this paper. Moreover, from a logical point of view, a particularly interesting question arises for mixed equilibria in infinite games: The logical machinery developed here is quite obviously insufficiently expressive to deal with probability distributions over infinite sets of strategies. However, we conjecture that our results can be generalized also to that case by employing quantified propositional logics (see, e.g., [2]).

Another option would be to deepen the research on the game-theoretic side with the goal of characterizing the special structure of Nash equilibria associated with certain classes of infinite games in which the payoffs are continuous and the strategy space of each player is isomorphic to $[0,1]$. The motivation comes from known results for separable (polynomial) games [10, Chapter 11]: Every continuous game that falls within this class has at least one mixed Nash equilibrium whose components are probability measures with finite supports. Specifically, let us suppose that each player $i \in N$ has $[0,1]$ as her strategy space. The payoff functions $f_i\colon [0,1]^n \to [0,1]$ are assumed to be polynomials in n variables. Although the set of all mixed strategies in this game is the set of all Borel probability measures over $[0,1]$, which is far beyond the scope of any straightforward logic-based treatment, it is known that every polynomial game has a Nash equilibrium consisting of finite mixed strategies p_i only. Namely for every $i \in N$ there exist strategies $s_i^1, \dots, s_i^m \in [0,1]$ and coefficients $\alpha_i^1, \dots, \alpha_i^m \geq 0$ such that $\sum_{j=1}^m \alpha_i^j = 1$ and

$$p_i = \sum_{j=1}^{m} \alpha_i^j \cdot \delta_{s_i^j}, \tag{5}$$

where each $\delta_{s_i^j}$ is the Dirac probability distribution at s_i^j. In [15] a sufficient condition for the existence of finite mixed equilibria in constant-sum games given by McNaughton functions has been given. This means that the scope of our logical analysis of mixed equilibria (Section 4.2) can possibly be expanded further, capturing also the existence of equilibria of the type (5) directly in a sufficiently strong propositional language $\mathcal{L}_{\boldsymbol{A}}$.

Finally let us draw attention to the fact, that when Harrenstein *et al.* introduced Boolean games in [14], they addressed several topics that go beyond the mere representability of certain games by logical formulas. In particular they also considered operations on games, a form of relativized validity and satisfiability motivated by their games, and a calculus for deriving winning strategies. It would be certainly interesting to see to what extent these and related topics can be developed also in the considerably more general many-valued setting presented here.

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