The half plane UIPT is recurrent

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January 2016

Abstract

We prove that the half plane version of the uniform infinite planar triangulation (UIPT) is recurrent.

The key ingredients of the proof are a construction of a new full plane extension of the half plane UIPT, based on a natural decomposition of the half plane UIPT into independent layers, and an extension of previous methods for proving recurrence of weak local limits (still using circle packings).

1 Introduction

The **half plane uniform infinite planar triangulation**, abbreviated as the **HUIPT** below, is a random planar triangulation, closely related to the well-known and extensively studied **uniform infinite planar triangulations** (**UIPT**), but with the topology of the half plane. The HUIPT is an interesting object in its own right, and in some ways is nicer than the UIPT. For example, it possesses a simpler form of the domain Markov property (detailed definition are provided in Section 2). The problem of establishing the recurrence of the UIPT had been open for many years. This was a motivation for the seminal work of Benjamini and Schramm [14], and was resolved in recent work of Gurel-Gurevich and Nachmias [19]. However, recurrence of the HUIPT does not follow from their work, as it is not known if it is possible to realize the HUIPT as a subgraph of the UIPT (indeed, there are possible arguments that it is not possible to have such a coupling). In this article we establish the recurrence of the half-plane UIPT.

Theorem 1. *The simple random walk on the half plane uniform infinite planar triangulation is almost-surely recurrent.*

While our proof incorporates some ideas from [14, 19], new methods are also needed. A crucial ingredient in those works is that the graphs under consideration are weak local limits of finite planar graphs, with a root that is chosen uniformly among all vertices. After embedding the graphs in carefully chosen manner in the plane, this leads to a fundamental Lemma on geometry of arbitrary point sets in the plane [14, Lemma 4.2]. A quantitative version of this lemma [19, Lemma 3.4] was exploited to prove the recurrence of the UIPT, and is used in this work as well (see Lemma 6.1 below).

Crucially, the methods of [14, 19] do not apply directly, since the HUIPT is not a weak local limit of finite planar graphs. The original construction of the HUIPT is as a weak limit of uniform triangulations with boundaries where the root is restricted to the boundary. In particular, the root is not a uniform vertex. The main novelty of this work lies in the technique used to overcome this obstacle. Along the way we obtain a certain random full plane map M which we call the **layered UIPT**. The layered UIPT contains the HUIPT as a subgraph. We believe M to be of independent interest and to have further applications. We prove that M is recurrent which implies that the HUIPT is recurrent.

Another difficulty stems from the fact that (unlike the UIPT), the HUIPT is not stationary for the simple random walk. Indeed, viewed from the random walker, the HUIPT should converge in distribution w.r.t. the local topology to the UIPT, as the walker will typically be far from the boundary. The map M we introduce is not stationary itself, but there is a certain local modification of M which is stationary, and even reversible. Thus in a certain sense, the map M can be seen as a stationary reversible version of the HUIPT. (A random rooted graph (G, ρ) is stationary reversible with respect to simple random walk $\{\rho, X_1, \dots\}$ if the law of the doubly marked graph (G, ρ, X_1) is the same as the law of (G, X_1, ρ) see [10]; reversibility and the related property of unimodularity has been exploited in the past to great advantage [10, 12, 7, 11, 6, 5, 13].)

Finally, a central tool we use is a decomposition of the HUIPT and of the layered UIPT into independent layers (see Section 3). An analogous decomposition was used by Krikun [24, 23] for the UIPQ. However, the domain Markov property of HUIPT gives this decomposition a particularly elegant structure. Such a decomposition has great potential for the study of random maps. A forthcoming recent work of Curien and Le Gall [18] analyzes first passage percolation and other perturbations of the metric structure of the UIPT via such a decomposition. A continuum version of this decomposition has been introduces in recent work of Miller and Sheffield as part of a characterization of the Brownian map [26].

1.1 Outline of proof

A naive approach to proving recurrence of the HUIPT is to use the result of Gurel-Gurevich and Nachmias in [19]. Let B_r be the hull of the combinatorial ball of radius r around the root (the hull is obtained by adding the finite components of the complement of the ball). These are finite planar graphs with exponential tail on the degrees, so their limit is almost surely recurrent. If the root is near the boundary, then the limit is the HUIPT. However, the root is unlikely to be near the boundary, and the limit is the full plane UIPT. If we can show that the limit contains H as a subgraph, then we would be done. However, the limit is the UIPT, and inclusion of the HUIPT is an open problem.

A more refined approach is to find some subset S of the vertices of the ball such that if we pick uniformly a root uniformly from S we obtain a limit which contains H as a subgraph. One natural choice is to set S to be ∂B_r, so that the limit is the HUIPT. However, since $|\partial B_r| \approx r^2$, this set is much smaller than the volume of B_r. Thus the limit is not absolutely continuous with respect to the weak local limit of B_r, and we are still short of a proof.

An improvement would be to take S to be the union of the boundaries ∂B_j, for $1 \leq j \leq r$. This set is still much smaller than the volume of B_r. However, the situation can be salvaged: This set S disconnects the balls into small components (the blocks below); Understanding the structure of S gives some control over the structure of the resulting limit. One can circle pack the limiting graph, and the circles corresponding to the set S will have no accumulation points in the plane. Moreover, Lemma 6.1 gives us control over the number of vertices of S in a Euclidean ball. In practice, it is more convenient to replace B_r by a different subgraph of the HUIPT, which is done below.

In order to complete the proof, we also need some new estimates on the volume of balls in the HUIPT under a certain modified metric, as well as estimates on vertex degrees. With these in place, we can push through the proof of [19].

We comment that there are also natural measures on half planar quadrangulations, and more general 'uniform' half planar maps. There seems to be no crucial obstacle to extending our results to such more general classes of maps. We restrict here to triangulations where the the layer decomposition is particularly nice. As noted, a similar decomposition was used by Krikun for quadrangulations, and with care it seems the layered structure as well as the rest of our argument can be carried over to such more general maps.

Organization. In Section 2 we include some background material which we use, concerning the weak local topology, planar maps, the UIPT and HUIPT, and circle packings. Readers familiar with these topics may wish to skip to Section 3 where we describe the layer decomposition of the HUIPT, and describe the full plane map M containing the HUIPT. We also prove there estimates on the volume growth and vertex degrees in M. In Section 4 we show that a certain sequence of finite maps with suitable distribution for the root converge to M. Finally, in Section 6 we combine all ingredients and prove Theorem 1. We end with some comments on possible extensions and open questions in Section 7.

2 Background

2.1 Planar maps: The UIPT and relatives

Recall that a **planar map** is a proper embedding in the plane of a connected (multi) graph in the plane, considered up to orientation preserving homeomorphisms. Components of the complement of the map are called faces, and are assumed to be simple discs. All our maps are **rooted**, meaning there is a marked directed edge, called the root. Equivalently, a planar map is a graph together with a cyclic order on the edges at each vertex, such that the graph can be embedded with the edges leaving the vertex in order.

Our maps will have a distinguished face which we shall call the external face. The edges and vertices incident to the external face will be called the **boundary** of the map. When a map has a boundary, we shall often assume the root is one of the boundary edges. The boundary throughout this paper will be either a simple cycle or a simple doubly infinite path. In the latter case, the map may be embedded in the half plane with the boundary along a line. Such a map is referred to as a half plane map.

The local topology on the space of rooted graphs is generated by the following metric: for rooted graphs G, H, we define

$$d(G,H) = e^{-R} \qquad \text{where} \qquad R = \sup\{r : B_r(G) \cong B_r(H)\}.$$

Here B_r denotes the ball of radius r around the corresponding roots in the graph distance, and $\cong$ denotes isomorphism of rooted maps. For maps, we require the equivalence relation to preserve the cyclic order on edges at vertices.

This topology on graphs or maps induces a weak topology on the space of measures on graphs (resp. maps). A finite, possibly random, graph yields a measure on rooted graphs by taking the root to be a uniform directed edge (or vertex). The **weak local limit** (or **Benjamini-Schramm limit**) of a sequence of finite graphs is the weak limit of the induced measures. The starting point of our work is the following result of Gurel-Gurevich and Nachmias (and of Benjamini and Schramm with a bounded degree assumption).

Theorem 2.1 ([14, 19]). *Let G_n be finite planar graphs such that the degree of a uniform vertex has uniformly exponential tail. Then $\lim G_n$ is almost surely recurrent.*

It has been known for some time [9, 3, 8] that the uniform measures on finite planar triangulations with boundary converge in the weak local topology as the area of the map and the boundary length tend to infinity.

Theorem 2.2. *[9, 8] If T_n is a uniform rooted triangulation (map with all faces triangles) with n vertices, then the limit $T = \lim T_n$ exists. If $T_{n,m}$ is a uniform boundary rooted triangulation with m boundary vertices and n internal vertices, then we have the limit*

$$T_{n,m} \xrightarrow[m,n/m\to\infty]{d} H.$$

The limits T and H are the **UIPT** and **half plane UIPT**. We denote the law of H by $\mathbb{H}$.

The map H also enjoys **translation invariance** with respect to the root. This means that the law of the map remains invariant if we translate the root along the boundary. See [8] for a detailed definition.

The distribution of a neighbourhood of the root in the HUIPT has a simple and explicit formula which can be taken as an alternative direct definition of HUIPT.

Lemma 2.3 ([8]). *Let Q be a simply connected triangulation with a simple boundary, with some marked connected segment of ∂Q containing the root, and let H be the HUIPT. Consider the event A_Q that Q is a sub-map of H*

with the roots coinciding and the marked segment being the intersection of Q with ∂H. Then

$$\mathbb{H}(A_Q) = 6^{\#V_i(Q)} 9^{-\#F(Q)}$$

where $\#V_i(Q)$ is the number of vertices of Q not in ∂H and $\#F(Q)$ is the number of faces of Q. Moreover, conditioned on A_Q, the complement $H \setminus Q$ also has law $\mathbb{H}$.

The final claim of this lemma is referred to as the **domain Markov property** of the HUIPT (see [8]).

2.2 Peeling

One of the main tools we are going to use is known as **peeling** which was introduced by Watabiki [27] and given its present form by Angel [3]. This technique can be applied to more general class of maps, we focus primarily on HUIPT. The central idea is to explore (or "peel") a map face by face. There can be many possible algorithms to do it, and generally an algorithm is chosen depending on the purpose. The domain Markov property in the HUIPT gives the peeling process a rather simple form. For further applications of this powerful tool see e.g. [4, 16, 25, 8, 11].

Consider the unique triangle incident to the root edge of the half plane UIPT H. One of the following two events must occur: With probability $2/3$, the triangle can be incident to an internal vertex. Otherwise the triangle incident to the root edge is attached to a vertex on the boundary which is at a distance i to the left (resp. right) of the root edge along the boundary. Let p_i be the probability of this event. Moreover, let $p_{i,k}$ be the event that the finite face enclosed by such a triangle has k vertices. Let $\phi_{k,i}$ denote the number of triangulations of an i-gon with k internal vertices. The following were derived in [3].

$$\begin{aligned} p_{i,k} &= \phi_{k,i+1} \left(\frac{1}{9}\right)^i \left(\frac{2}{27}\right)^k \\ p_i &= \sum_k p_{i,k} = \frac{2}{4^i} \frac{(2i-2)!}{(i-1)!(i+1)!} \sim \frac{1}{2\sqrt{\pi}} i^{-5/2} \end{aligned} \tag{2.1}$$

The **Boltzmann triangulation** of an m-gon with weight $q \leq \frac{2}{27}$, is the probability measure on that assigns weight $q^n / Z_m(q)$ to each rooted triangulation of the m-gon having n internal vertices, where

$$Z_m(q) = \sum_n \phi_{n,m} q^n.$$

The partition function Z_m can be computed explicitly, and is finite for $|q| \leq 2/27$. When peeling a face, on the event that the face connects to a vertex at distance i, the resulting component with boundary $i+1$ is filled with a Boltzmann triangulation with weight $2/27$.

Having revealed the triangle incident to the root edge and the finite component of its complement (if any), the unrevealed map is another half plane map having law $\mathbb{H}$ by the domain Markov property. This enables us to peel the HUIPT via a succession of i.i.d. peeling steps. Note that the probabilities $p_{i,k}$ do not depend on the edge we choose to peel, by translation invariance.

2.3 Circle Packings

As in some prior works [14, 19, 6], circle packings play a central role for us. We state here the two key results needed. We refer the reader to [22] and the above papers for further information.

A **circle packing** of a graph G is a collection of circles in the plane with disjoint interiors, one corresponding to each vertex, such that two circles are tangent if and only if the corresponding vertices are adjacent. The Kobe-Andreev-Thurston Circle Packing Theorem states that every finite planar graph has a circle packing. There are extensions to infinite planar triangulations, which we do not need at present.

In order to control the geometry of graphs in terms of circle packings, it is useful to control the ratio of radii of circles. This is done by the so called Ring Lemma, which states that in a circle packing of a triangulation, the ratio of radii of adjacent circles is bounded by some constant depending only on the maximal degree of the graphs (for non-boundary vertices).

3 Layer decomposition

Given a half planar map H, we define its layer decomposition as follows. For each i, we will have a half plane map H_i. These will form a decreasing family of sub-maps of H, and each is a half plane map. The boundary of H_i is

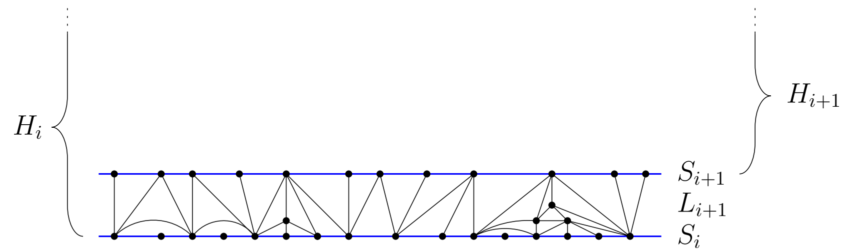

Figure 1: Construction of layer L_i from H_i. L_i is the hull of all faces incident to the boundary S_i. The entire HUIPT is H_0.

denoted S_i, and is a doubly infinite simple path in H. The vertices in S_i are called **skeleton vertices**.

Inductively, we start with $H_0 = H$ and its boundary $S_0 = \partial H_0$. Having defined H_i and S_i, define the **layer** L_{i+1} to be the hull (relative to H_i) of the set of faces of H_i incident to the boundary S_i. Thus L_{i+1} forms a layer near the boundary of H_i. We then define the next sub-map $H_{i+1} = H_i \setminus L_{i+1}$, and S_{i+1} to be its boundary (see Figure 1). For each i we have that S_i is a simple infinite path which separates L_i from L_{i+1}. Conversely, the boundary of L_i is $S_i \cup S_{i+1}$. Note also that by construction the sets S_i are disjoint.

Note that we have not yet determined a root for the maps H_i. A root can be chosen for each H_i in various manners, and we will do that below. However, the construction above is independent of the choice of root.

Let e be some edge in S_{i+1} for some $i \geq 0$. Then there is unique face in L_{i+1} containing e, and the third vertex of that face must be in S_i, since otherwise that face would not have been included in L_i. For two adjacent edges $e, e' \in S_{i+1}$, the corresponding triangles of L_i that contain e and e' split L_i into two infinite and one finite component. We refer to the finite components arising in this way as **holes**, since the sub-map induced by skeleton vertices is missing all vertices in such holes. Note that it is possible that the two triangles share a common edge, in which case the hole degenerates to that single edge. It is also possible that the two triangles share two vertices, one in S_i and one in S_{i+1}, but the edges are distinct, and in that case the hole is a 2-gon. Both occur in the lower layer in Figure 2. This observation implies that L_i can be decomposed as an alternating sequence of (possibly degenerate) holes and faces containing the edges of S_i. We can thus partition L_i to a sequence of **blocks**, where each block consists of a hole and the triangle immediately to its right. The **lower boundary** of a block in L_i is

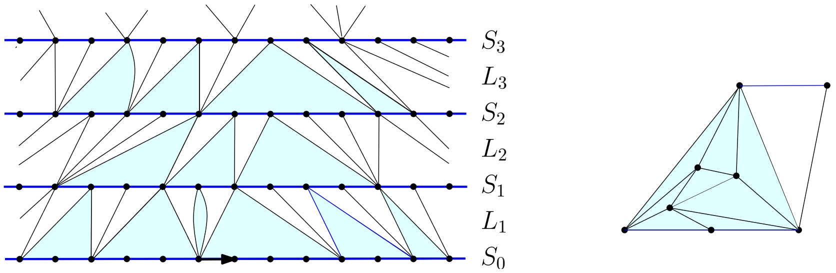

Figure 2: Left: Decomposition of layers into alternating holes and faces adjacent to top boundary edges. Holes are shaded, and vertices and edges within holes are not shown. Note that some holes degenerate. Right: A hole and face to its right form a block.

the set of edges of S_{i-1} in the block, which can be any non-negative integer. (The upper boundary always consists of a single edge.) Apart from the lower and the upper boundary, the block has two more boundary edges, where it is attached to blocks to its left and right.

3.1 Decomposition of the half plane UIPT

Up to this point we described the layer decomposition of a general half plane map. We now focus our attention on the specific case of the half-plane UIPT. While in our case, the description above is faithful, in arbitrary half-plane maps things could break down. For example, it is possible that L_1 is the entire map. Indeed, this is the case in the sub-critical half plane maps with the domain Markov property that were constructed in [8].

Lemma 3.1. *For the HUIPT, almost surely, L_1 is not the entire half plane, S_1 is a doubly infinite simple path and H_1 is also a half plane map. Moreover, if we choose a root for H_{i+1} as a function of $L_1, \dots, L_i$, then H_{i+1} has the law of the half plane UIPT, and is independent of L_1. Consequently, the layers $\{L_i\}_{i\geq 1}$ are i.i.d.*

Peeling to reveal a layer. To prove Lemma 3.1, it shall be useful to consider the following application of peeling in the half plane UIPT. An analogue of this for the UIPT was used in [3] to study the volume growth of the UIPT. In the HUIPT, the process becomes simpler. Initially, make the root edge active. At any later time, the active edges are those at the boundary of the unseen part of the map that are not on the original boundary. The active edges form a single contiguous segment, and we peel either the rightmost or leftmost active edge. Let Y_n be the number of active edges in this segment after n steps (with $Y_0 = 1$).

Let Y_n be the length of this segment after n steps, except that by convention we set $Y_0 = 1$. Define now the i.i.d. sequence ξ_n as follows. If the nth step connects the peeled edge to a new internal vertex, then $\xi_n = 1$. If it connects to a vertex at distance i towards the rest of the active segment then $\xi_n = -i$. Finally, if the face connects to a vertex to the right, $\xi_n = 0$. It is easy to see that ξ_n determines the change in Y_n. Specifically, he have

$$Y_n = (Y_{n-1} + \xi_n) \vee 1. \tag{3.1}$$

The ξ_n variables are i.i.d. with distribution

$$\mathbb{P}(\xi = i) = \begin{cases} 2/3 & i = 1, \\ 1/6 & i = 0, \\ p_i/2 & i < 0, \end{cases} \tag{3.2}$$

where p_i is given in (2.1). It follows from the computations in [3] that $\mathbb{E}(\xi) = 1/3$. Note also that every peeled face is incident to some vertex in the original boundary, and so all faces revealed in this procedure are part of L_0. Finally, the number of edges of the original boundary that are swallowed at each step are also i.i.d. with mean $1/3$.

Proof of Lemma 3.1. When peeling to reveal a layer, since $\mathbb{E}\xi = 1/3 > 0$, the strong law of large numbers implies that Y_n/n converges to $1/3$ almost surely and in particular, Y_n tends to infinity almost surely.

Start by peeling at the rightmost active edge n times. The law of large numbers ensures that the number of edges to the right of the root that are swallowed grows like $n/3$. While some of the previously active edges contributing to Y_n are swallowed at a later step, at each n there is probability $1/3$ that $Y_n = \inf_{t \geq n} Y_t$ (via Theorem 3 of [1]), and in that case, only the rightmost active edge is subsequently swallowed. In particular, the number of boundary vertices to the left of the root that are swallowed is tight.

Next, reverse direction, and peel towards the left for n additional steps. At this time we revealed some finite map P_n which contains all faces incident to edges within distance a_n along the boundary to the left and b_n along the boundary to the right with both a_n, b_n close to $n/3$ with high probability.

Thus P_n converges to the layer L_1. Moreover, the number of edges from the Y_n that are swallowed in the second stage is tight, and therefore with high probability some of them remain on the boundary as $n \to \infty$. This implies that L_1 is not the entire map.

To see that H_1 is again a half plane UIPT, and is independent of L_1, root $M_n = H \setminus P_n$ at some canonically chosen vertex ρ_n, say the first one reached in the process that is on the boundary of P_n. From the domain Markov property, (M_n, ρ_n) has the law of the half plane UIPT, and is independent of P_n. This completes the proof, since ρ_n is eventually constant, and so (M_n, ρ_n) converges to (H_1, ρ). Finally, by translation invariance of H_1, we can choose a root for H_1 as any function of L_1 and the law of H_1 will not change.

By induction, the same holds for subsequent layers. □

Proposition 3.2. *In each layer L_i we have the following.*

1. *The blocks are independent. All have the same law, except for the block containing the root edge which is biased by the size of its lower boundary. Given the block containing the root edge, the root edge is distributed uniformly among the edges in its lower boundary.*
2. *The number of edges B in the lower boundary of a block (other than the one containing the root edge) satisfies $\mathbb{E}(B) = 1$ and $\mathbb{P}(B > t) \sim ct^{-3/2}$.*
3. *Conditioned on the lower boundary length of B, the component of H within the hole is a Boltzmann map of an $(B+2)$-gon with parameter $2/27$.*

We remark that the proof yields the precise distribution of the lower boundary size of a block in terms of the partition function of triangulations, which is explicitly known. We do not need the formula for this distribution.

Proof. We enumerate the blocks $\{B_i\}_{i\in\mathbb{Z}}$ using integers with B_0 being the block containing the root edge. Consider a sequence of blocks $(B_i)_{i\in[j,k]\cap\mathbb{Z}}$ with $j \le 0 \le k$. Suppose B_i has b_i lower boundary edges and v_i vertices in its hole. Let B_0 also have a marked edge on its lower boundary. We compute the probability that these are consecutive blocks of L_1, with the marked edge of B_0 being the root edge. A block has $2v_i + b_i + 1$ faces. Joining these blocks, the total number of vertices internal to M is $V = 1 + \sum v_i + 1$, including also the upper boundary vertices. Letting $F = \sum 2v_i + b_i + 1$, by Lemma 2.3, the probability of these blocks being part of the map is $6^V 9^{-F}$.

In order for these to be blocks in L_1, it is also necessary that if we peel along the boundary to the right than no internal vertex (revealed so far) is swallowed, and that to the left the first step reveals an internal vertex, and afterwards no additional internal vertex is swallowed. These have probability 1/3 and 2/27 respectively, which are just a constant for us. Thus the probability of having the blocks B_i is

$$C \prod 6^{1+v_i} 9^{-2v_i-b_i-1} = C \prod \frac{2}{3}\left(\frac{2}{27}\right)^{v_i} 9^{-b_i}.$$

for some absolute constant C. (Careful calculation shows $C = 1$; However, we need not worry about the value of C, since it is determined by the fact that these probabilities add up to 1, and its value is canceled out in what follows.) This shows that the blocks are independent, that given b_i, and that the hole is filled with a Boltzmann triangulation with parameter 2/27. Moreover, the probability that $b_i = m$ is proportional to $\sum_n \phi_{n,m}(2/27)^n 9^{-m}$, which decays as $ct^{-5/2}$ via (2.1). Finally, for B_0 there is a marked edge on the lower boundary, so the probability of it having $b_0 = m$ is proportional to $\sum_n m\phi_{n,m}(2/27)^n 9^{-m}$, i.e. it is a biased by its lower boundary. That the root is distributed uniformly among the vertices in the lower boundary of its block follows from translation invariance. $\square$

We are going to denote by $\mathbb{B}$ the **law of a block** described in Proposition 3.2 and by $\mathbb{B}^{\text{bias}}$ the law of the block containing the root (i.e. biased by the size of its lower boundary). Let $D_{\max}$ denote the maximal degree of a vertex in a block. The following shall come as no surprise to the reader familiar with random maps.

Lemma 3.3. *Then for some $c, C > 0$ and all $r \geq 1$, $\mathbb{B}(D_{\max} > r) \leq Ce^{-cr}$, and $\mathbb{B}^{bias}(D_{\max} > r) \leq Ce^{-cr}$.*

The proof follows a fairly standard argument, and is not too difficult, however, we have not been able to locate this statement in the literature. The proof is separated into three steps. The first is known lemma about the degree of a boundary vertex in a Boltzmann triangulation.

Lemma 3.4. *There are constants c, C such that for any m, if B is a Boltzmann triangulation of an m-gon, rooted at $\rho \in \partial B$, then $\mathbb{P}(d_\rho > r) \leq Ce^{-cr}$.*

Proof. This follows from the same argument for exponential degree distribution in the UIPT from [9]. $\square$

Lemma 3.5. *There are constants c, C such that for any m, if B is a Boltzmann triangulation of an m-gon, rooted at $\rho \in \partial B$, and $D_{\max}$ the maximal degree of any internal vertex, then $\mathbb{P}(D_{\max} > r) \leq Cm^2 e^{-cr}$.*

Proof. We perform a peeling process to reveal B, each time peeling at some edge and revealing one face. However, if the revealed face separates the map into two sub-maps, we do not reveal either of them immediately, but proceed to explore one and then the other in some arbitrary order. Thus at time i we have revealed i faces, and the remainder of B is a collection of independent Boltzmann maps of some cycles (unless the process has terminated, in which case there is no complement).

Let $\mathcal{F}_i$ be the sigma algebra generated by this peeling process up to the ith step. Let A_i be the event that a new vertex is revealed at step i, and let $A_{i,r} \subset A_i$ be the event that this vertex has degree greater than r. Our goal is to bound $\mathbb{P}(\cup_i A_{i,r})$:

$$\begin{aligned}\mathbb{P}(D_{\max} > r) &\leq \sum_i \mathbb{P}(A_{i,r}) \\ &\leq \sum_i \mathbb{E}\left[\mathbb{P}(A_{i,r} | \mathcal{F}_i)\right] \leq \sum_i \mathbb{E}\left[\mathbb{1}_{A_i} \mathbb{P}(d_i > r | \mathcal{F}_i)\right]\end{aligned}$$

since $A_i \in \mathcal{F}_i$, where d_i is the degree of the vertex revealed at step i. When a new vertex is revealed, its degree is 2. Conditioned on $\mathcal{F}_i$, the component of vertex i is filled with a Boltzmann map, and so by Lemma 3.4, we have $\mathbb{P}(d_i > r | \mathcal{F}_i) \leq Ce^{-cr}$. Thus we have

$$\begin{aligned}\mathbb{P}(D_{\max} > r) &\leq \sum_i \mathbb{E}\mathbb{1}_{A_i} Ce^{-cr} \\ &= Ce^{-cr}\mathbb{E}|B| \leq Cm^2 e^{-cr},\end{aligned}$$

Where $|B|$ is the number of vertices in B, which is known to have expectation of order m^2 (see [9, Proposition 5.1]). ☐

Proof of Lemma 3.3. We know from the second item in Proposition 3.2 that the probability that the boundary of a Boltzmann map of law $\mathbb{B}$ or $\mathbb{B}^{\text{bias}}$ is larger than $e^{\varepsilon r}$ is exponentially small. The rest follows from Lemma 3.5 for $\varepsilon = c/3$, with the c from Lemma 3.5. ☐

3.2 Full plane extension of H

Given the layer decomposition of the half plane UIPT, we now construct a plane triangulation M with no boundary which contains H as a sub-map. Eventually we will also show that M is almost surely recurrent, implying Theorem 1.

To construct M start with the half plane UIPT $H = \bigcup_{i\geq 1} L_i$. We add a sequence of layers below the boundary S_0, to create a full plane map. For each $i \leq 0$, the layer L_i is composed of a doubly infinite sequence of i.i.d. blocks with law $\mathbb{B}$, attached to form a layer. Note that there is no size biased block in L_i for $i \leq 0$. Thus for $i \leq 0$, if L_i is rooted at some vertex on the top boundary, it is translation invariant in law. We then identify the top boundary of L_i with the bottom boundary of L_{i+1} for every $i \leq 0$. The full plane map is $M = \bigcup_{i\in\mathbb{Z}} L_i$. By translation invariance of the lower layers, the law of the resulting full plane map does not depend on which edge in the bottom boundary of L_{i+1} is identified with the root of L_i. The boundary between L_{i+1} and L_i is denoted S_i also for $i < 0$. Vertices of S_i for any i are also called skeleton vertices. The map M is rooted at the root ρ of H. Theorem 1 is an immediate consequence of the following.

Theorem 3.6. *The full plane extension M is almost surely recurrent.*

Let $S = \bigcup S_i$ be the skeleton of the map (M, ρ). We define a graph on S, where two vertices $x, y \in S$ are adjacent if they are both incident to some hole in some layer of M. Call this graph $\text{Skel}(M)$. Note that adjacent vertices in any S_i are adjacent in $\text{Skel}(M)$, and that neighbours in $\text{Skel}(M)$ are either both in S_i for some i or in S_i and S_{i+1} for some i. Note that $\text{Skel}(M)$ does not have a natural map structure. However, it consists of finite cliques, and the intersection graph between the cliques is planar and approximates M in some ways. (We do not need this, and do not make this precise here.) The nonempty blocks in M corresponds naturally to blocks in $\text{Skel}(M)$ with the vertices in the boundary of a block forming a clique (we will keep referring to them as blocks in $\text{Skel}(M)$) We shall use the notation $\text{Skel}(A)$ to denote the corresponding graph also for various $A \subset M$, which will be the subgraph of $\text{Skel}(M)$ induced by vertices of A.

Our immediate goal is to show that $\text{Skel}(M)$ has polynomial volume growth. Let $B_{sk}(r)$ denote the combinatorial ball of radius r around ρ of $\text{Skel}(M)$ along with all the finite components of its complement.

Proposition 3.7. *The random variables $r^{-4}|B_{sk}(r)|$ form a tight family.*

We start with a few additional definitions. For any skeleton vertex $v \in S_i$ we can associate a unique hole in the layer above it (L_{i+1}) that contains the edge of S_i to the right of v. This hole is also incident to S_{i+1} at a unique vertex. Call this vertex the **parent** $p(v)$ of v, and define $p^{(r)}$ to be the r fold composition of the operation p. Equivalently, the parent of a vertex $v \in S_i$ is the rightmost vertex of S_{i+1} that is adjacent to v in $\mathrm{Skel}(M)$.

Using Proposition 3.2, it is natural to study the maps via certain critical Galton-Watson trees derived from the block decomposition. A similar construction was used by Krikun for the full plane UIPQ in [23, 24], except that the trees there are not Galton-Watson trees. In the layered map, we define a tree as follows. The vertices are the skeleton vertices. The parent of v is $p(v)$. The set of all offspring of a vertex v form a tree, which we denote by T_v. The following is clear from this discussion and Proposition 3.2.

Lemma 3.8. *For every skeleton vertex $v \in S_0$, the tree T_v is a critical Galton-Watson tree with offspring distribution Z satisfying $\mathbb{P}(Z > k) \sim ck^{-3/2}$. Moreover, the law of the (infinite) tree rooted at ρ is the fringe of the same Galton-Watson tree.*

See Aldous [2] for the theory of fringes of trees. We do not need this theory in full generality, but use the following consequences:

- Every ancestor $p^{(r)}(\rho)$ of ρ has offspring distributed as size biased Z.
- The previous ancestor $p^{(r-1)}(\rho)$ is a uniform child of its parent.
- All other children of $p^{(r)}(\rho)$ produce independent Galton-Watson trees of offspring.

While a-priori it is not obvious that our parent definition defines a single tree and not a forest. The connectivity can be deduced from the criticality of the trees by showing that for any two vertices of $x, y \in S_i$, $p^{(r)}(x) = p^{(r)}(y)$ for r large enough. This is straightforward, but we do not need the connectivity for our purposes, so omit details. Proposition 3.7 is an immediate consequence of the following.

Lemma 3.9. *Let $\widetilde{B}(2r)$ be the hull of the ball of radius $2r$ around $p^{(r)}(\rho)$ in Skel$(M \setminus H_r)$, then $r^{-4}|\widetilde{B}(2r)|$ are tight.*

Note that $M \setminus H_r$ is the half plane map consisting of L_i for $i \leq r$, and thus we are considering a ball in the skeleton of this map, centred at a boundary vertex.

Proof of Proposition 3.7. The ball $B_{sk}(r)$ is contained in layers $L_{-r}, \dots, L_r$, since the Skel-distance from ρ to $p^{(r)}(\rho)$ is r, we have that $B_{sk}(r) \subset \widetilde{B}(2r)$. $\square$

For the proof of Lemma 3.9, we require the following standard estimate regarding survival probabilities for Galton-Watson trees. Note that the offspring distribution in T_v has infinite second moment, so the survival probabilities do not decay as c/n and the volume up to level n, conditioned on survival to that level is not quadratic. Recall that it is possible to obtain an infinite version of a critical Galton-Watson tree by conditioning it to survive up to generation n and then taking the limit as $n \to \infty$ in the local weak topology. Moreover such an infinite tree has a single infinite path – the spine – and finite trees attached to it. The tree can be described as follows. The root has an offspring distribution which is the size biased version of the original offspring distribution. A uniformly picked child v has a tree conditioned to survive below it, while all its siblings have unconditioned Galton-Watson trees of descendants. We refer to [21] for a detailed account of Galton-Watson trees conditioned to survive. The two following lemmas are standard. Proofs can be found e.g. in [15].

Lemma 3.10. *Let T be a critical Galton-Watson tree with offspring distribution satisfying $\mathbb{P}(Z > k) \sim ck^{-3/2}$. Then the probability that T survives to generation n decays as cn^{-2} for some c.*

Lemma 3.11. *Let T^* be a critical Galton-Watson tree with offspring distribution satisfying $\mathbb{P}(Z > k) \sim ck^{-3/2}$ conditioned to survive. Let W_n be the number of offspring in the nth generation of T^*. Let $Y_n = \sum_{t=1}^{n} W_n$. Then $n^{-2}W_n$ and $n^{-3}Y_n$ converge to some non-zero random variables in law.*

Proof of Lemma 3.9. We will construct inductively a growing sequence of subgraphs $\{P_i\}$ around $p^{(r)}(\rho)$ in such a way that P_i contains the hull of the ball of radius i around $p^{(r)}(\rho)$ in $\mathrm{Skel}(M \setminus H_r)$. For all i, all vertices of P_i will be in layers $S_{r-i}, \dots, S_i$ (as is the ball they bound). As a basis, we set $P_0 = \{p^{(r)}(\rho)\}$. We also define a two-sided sequence of vertices in S_r, starting with $U_0 = \{p^{(r)}(\rho)\}$. For $i > 0$, having defined P_{i-1}, and $U_{1-i}, \dots, U_{i-1}$, let U_i be the nearest vertex in S_r to the right of U_i such that the tree below U_i survives for at least i generations. Similarly, U_{-i} is the nearest vertex in S_r to the left of U_{1-i} such that the tree below U_{-i} survives for at least i generations.

We now define P_i as follows. We take all vertices of the trees at U_{-i} and at U_i, from S_r down to S_{r-i}. Since the definition of the trees is asymmetric in

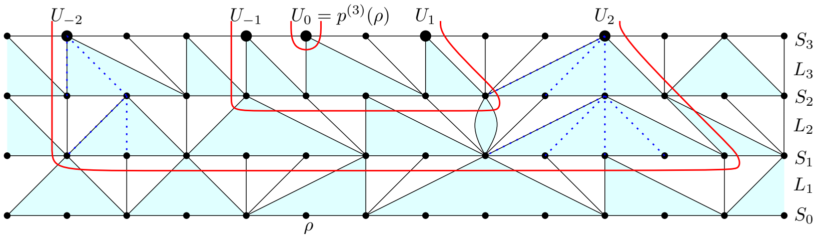

Figure 3: Construction of the maps P_0,P_1,P_2 in the proof of Lemma 3.9. The vertices $U_{-2}, \dots, U_2$ are larger. The first two generations of the trees below $U_{\pm 2}$ are dotted in blue. The boundaries of the maps are marked in red. Each boundary forms a cutset around the previous one.

that of the holes, it is convenient for the tree at U_i to also take the rightmost vertex of the rightmost hole at each level. (This vertex is in a tree further to the right). Finally, we also take in each of these levels all vertices between these two trees.

Note that since the first i levels of the tree below U_i are strictly to the right of the tree below U_i, and similarly on the left, we have that P_i indeed form an increasing sequence of subgraphs. See Figure 3 for an illustration.

The rest of the proof consists of two claims. First, that the set P_i contains the ball of radius i around $p^{(r)}(\rho)$ in the map $\operatorname{Skel}(M \setminus H_r)$. Secondly, we estimate of the size of these sets.

The first claim is proved by induction. Clearly the claim is true for P_0. For the induction step, we argue that the internal boundary of P_i (i.e. vertices of P_i connected to its complement) is completely contained in the two trees below $U_{\pm i}$ together with the segment of S_{r-i} between the two trees. In particular, the boundary is disjoint of P_{i-1}, and hence each P_i contains a ball of radius one around P_{i-1}.

A level S_j is naturally partitioned into intervals of vertices with a common parent. The lower boundary of a hole is one such interval, together with the first vertex of the next interval to the right. Edges of $\operatorname{Skel}(M)$ are either within intervals, or between adjacent intervals, or between a vertex and its parent, or between a vertex and the parent of an adjacent interval. Since every level from $S_{r-i}, \dots, S_r$ contains some vertices from the trees under U_i and U_{-i} these two trees indeed separate the rest of P_i from vertices to the right and left. Clearly only vertices in S_{r-i} can be connected to vertices further down in the map, and the first claim is proved.

Finally, we consider the size of P_{2r}, which consists of the first $2r$ generations from the trees rooted at each vertex between U_-2r and U_{2r}. The tree rooted at U_0 is special: Its first r levels are those of the tree conditioned to survive, with one vertex at each level having the size-biased offspring distribution. After level r, it transitions to a critical tree. The trees at all other vertices of S_r are critical Galton-Watson trees, except that the choice of U_i is not independent of the trees.

Fix some $\varepsilon > 0$. For the tree at U_0, by Lemma 3.11 we have for some C that the number of vertices in generations $0, \dots, r$ is at most Cr^3 and the number of vertices at generation r is at most Cr^2 with probability at least $1 - \varepsilon$. Below each of the vertices at generation r we consider the first r generation of an independent critical Galton-Watson tree. On the event that generation r is not too large, this adds in expectation at most another Cr^3 vertices, and by Markov's inequality the total contribution from the tree at U_0 is at most $(C + C/\varepsilon)r^3$ with probability at least $1 - 2\varepsilon$.

The trees at other vertices of S_r are all independent critical Galton-Watson trees, and we consider the first $2r$ levels of these trees. The expected size of each such tree is $2r+1$ (including its root). It is convenient to identify the vertices of S_r with $\mathbb{Z}$, with U_0 being 0. Between U_{i-1} and U_i we consider trees until finding one that survives to generation i, and so U_i is a stopping time. Since $U_i - U_{i-1}$ is geometric with mean of order Ci^2 (by Lemma 3.10), we have $\mathbb{E}U_{2r} \leq Cr^3$. By Wald's identity, the expected total size of the trees from U_0 to U_{2r} is at most Cr^4. By symmetry, the same holds for trees to the left of U_0, and the claimed tightness follows. □

Lemma 3.12. *Let Δ be the maximal degree in M of the vertices in $B_{sk}(r)$. There exists $C > 0$ such that $\mathbb{P}(\Delta > C \log r) \to 0$ as $r \to \infty$.*

Proof. This almost follows immediately from Proposition 3.7 and Lemma 3.3 except we need to take care of the fact that a vertex can be incident to many blocks in the layer above it. However, such a number is still Geometric, so the lemma follows. We make this rigorous below.

For every edge e in $(\bigcup_i S_i) \cap B_{sk}(r+1)$, consider the first edge to the right or left of this edge whose block has nonempty lower boundary. Since the blocks are independent and every block has a positive probability of having a non-zero lower boundary, the set of edges S_e we need to check until we find such an edge has a Geometric number of elements. Using this information and Lemma 3.3, we see that the maximal degree d_e of vertices in in all these blocks corresponding to edges in S_e has exponential tail. It is easy to see that $\Delta \le 2 \max_e d_e$ where the maximum is over all the edges in $(\cup_i S_i) \cap B_{sk}(r+1)$. Now using Proposition 3.7, we know that $\mathbb{P}(B_{sk}(r) > r^5)$ converges to 0. We can take a union bound over r^5 many edges on the event $\{B_{sk}(r) \le r^5\}$ and choose a large enough C to arrive at the desired conclusion. □

4 M as a distributional local limit

We now define a sequence of finite maps $M_n \subset H$. Each M_n will inherit the layered structure from H, and so some vertices will be skeleton vertices. M_n will have the property that if we select a root ρ_n uniformly from the skeleton vertices, then (M_n, ρ_n) converges in distribution to M.

For a skeleton vertex $v \in H$, recall the definition of the parent $p(v)$ of v from Section 3.2, and that $p^{(k)}$ is the k-fold composition of the operation p. Now we set up a coordinate system for skeleton vertices as follows. The vertices of S_k will have coordinates $\{(k,n)\}_{n\in\mathbb{Z}}$, in the order they occur in S_k. The root vertex ρ has coordinates $(0,0)$ and for any $k > 0$, the vertex $p^{(k)}(\rho)$ has coordinates $(k,0)$. Having defined these, the vertex of S_k at a distance j to the right (resp. left) of $(k,0)$ has coordinates (k,j) (resp. $(k,-j)$). See Figure 4 for an example. Note that coordinates are only defined for S_k with $k \ge 0$.

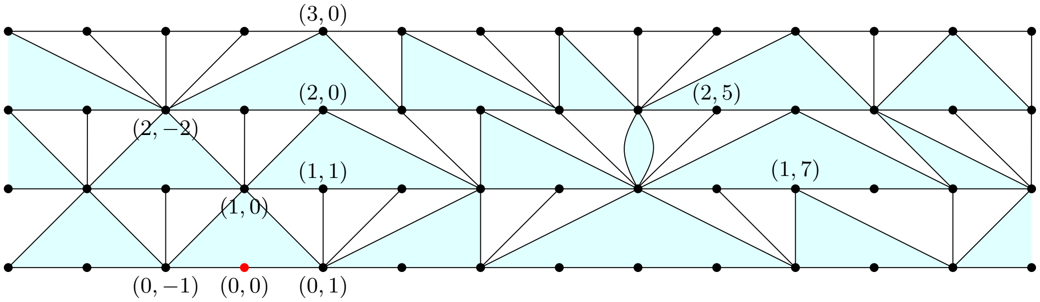

Figure 4: The coordinate system for Skel(M), with some coordinates noted.

Lemma 4.1. *Fix $k \ge 0$. Define $\ell' = \ell'(k,\ell)$ by $(k+1,\ell') = p((k,\ell))$. Then almost surely,*

$$\frac{\ell'}{\ell} \xrightarrow[\ell\to\infty]{} 1.$$

Proof. The blocks in layer L_{k+1} corresponding to vertices $(k+1,i)$ for $i>0$ form an i.i.d. sequence of blocks distributed as $\mathcal{B}$. The statement is now an immediate consequence of Proposition 3.2 (specifically that $\mathbb{E}B_i = 1$) and the Strong Law of Large Numbers which implies $\ell/\ell' \to 1$. $\square$

Now define M_n as follows. The skeleton vertices of M_n, denoted $\mathrm{Skel}(M_n)$ is the set $\{(i,j) : 0 \le i \le n, 1 \le j \le n\}$. The holes of M_n includes all holes in H all of whose skeleton vertices are contained in $\mathrm{Skel}(M_n)$. Finally take a root ρ_n for M_n, which is a uniformly selected selected skeleton vertex from M_n.

Next, we show that ρ_n is far away from the boundary of M_n with high probability.

Lemma 4.2. *We have*

$$d_{Skel}(\rho_n, \partial M_n) \xrightarrow[n,k\to\infty]{(d)} \infty$$

where d_{Skel} is graph distance in Skel$(M_n(k))$.

Proof. Consider the subgraph of the skeleton graph where $x \sim y$ if they are either at the same level and adjacent, or one is the parent of the other. Let d_p be the distance in this graph. It is easy to see that $d_p(x,y) \le 3d_{\mathrm{Skel}}(x,y)$, so it suffices to prove that for arbitrary r, with high probability $d_p(\rho_n, \partial M_n) \ge r$.

Observe that since M_n always has $n(n+1)$ skeleton vertices by definition, and the coordinates of ρ_n are independent of M_n. Let the root ρ_n have coordinates (i,j). As $n\to\infty$, with high probability $r < i < n-r$. On this event, the ball $B_p(\rho_n, r)$ does not reach levels S_0 or S_n. Fix any such i. For any $\varepsilon > 0$ there is some M so that with probability at least $1-\varepsilon$, for every vertex (x,y) with $x \in [i-r, i+r]$ and $y > M$, if (x',y') is adjacent to (x,y) in the d_p metric then $y'/y \in (e^{-\varepsilon}, e^{\varepsilon})$. Call this event B_i, and assume it holds. If $\rho_n = (i,j)$ and $j > e^{r\varepsilon}M$ then every vertex in $B_p(\rho_n, r)$ has second coordinate in $[e^{-r\varepsilon}j, e^{r\varepsilon}j]$. If n is large enough, then with high probability $e^{r\varepsilon}M < j < e^{-r\varepsilon}n$, and then the ball $B_p(\rho_n, r)$ is contained in M_n. $\square$

Proposition 4.3. *We have*

$$(M_n, \rho_n) \xrightarrow[n\to\infty]{(d)} (M, \rho)$$

in the weak local topology.

Proof. The coordinates (i_n, j_n) of a the uniform root ρ_n tend to infinity in distribution as $n \to \infty$. By Lemma 4.2 large balls around ρ_n are contained in M_n, so the weak local limit of (M_n, ρ_n) is the same as the limit of (M, ρ_n). Since i_n, j_n are independent of M, it suffices to show that for a fixed sequence $\{(i_n, j_n)\} \to \{\infty, \infty\}$, if we take $\rho_n = (i_n, j_n)$, then (M, ρ_n) converges in distribution to the full plane map (M, ρ).

Given the layers $L_1, \dots, L_i$, the half plane map above them (denoted H_i) has law $\mathbb{H}$. Thus the layers above ρ_n have the law of the HUIPT, and are independent of the i layers below ρ_n. Note that translation invariance implies that the block above ρ_n is precisely size-biased, as are subsequent blocks above it.

Since the blocks in the first i layers are independent with law $\mathbb{B}$, the layer below ρ_n has distribution similar to that of layer L_0 in M, except for one block at distance $j_n \to \infty$. The same is true for all i_n levels below ρ_n. By Lemma 4.2, the distance to these biased blocks tends to infinity, giving the result. □

5 Bounding degrees: the star-tree transformation

Following [19], we apply the so-called star-tree transformation to our maps to get maps with bounded degrees. These can then be embedded in the plane using circle packings, which are better behaved when vertices have bounded degrees.

The star-tree transform is constructed roughly as follows: starting with a map G, possibly with large degrees, take its dual, which has large faces, triangulate each face to get a triangulation, and take the dual again to get a three regular map G' which is related to the original map. The triangulation step can be done in various ways, and we will be more specific below. Each vertex of G of degree d is replaced by a 3-regular tree which connects to other trees at its leaves. Crucially for the recurrence arguments, we make all of these trees as balanced as possible, so that a vertex of degree d (star) is replaced by a tree of diameter $O(\log d)$.

To make this precise, we first cut every edge in half, so that every vertex becomes a star with d leaves. Next, each such star is replaced by a balanced tree with $d-2$ internal vertices of degree 3 and d leaves. The leaves are in

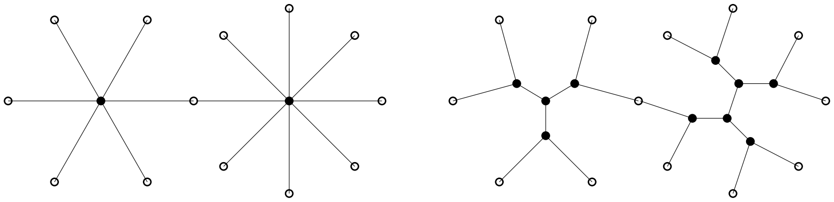

Figure 5: The star tree transform. The white circles are the vertices created when cutting edges in half. Here, vertices of degree 6 and 8 are replaced by balanced binary trees with the same number of leaves.

bijection with the leaves of the star that the tree is replacing, in cyclic order. The leaves are identified as in the original map with leaves on other trees. This creates a map with maximal degree 3. (The new map is not 3-regular, since vertices of degree 1 or 2 maintain their degree and identified leaves have degree 2.) The choices of tree for each vertex is arbitrary, except for being maximally balanced. See Figure 5 for an illustration.

When the star-tree transform is applied to a map G, we call the resulting map G'. Clearly G is a minor of G', as it can be recovered by contracting each tree back to a single vertex. A vertex of degree d in G corresponds to $(d-2)\vee 1$ vertices in G'. Edges in the map G' are now assigned conductances. All edges of a tree associated with a vertex of degree d are given conductance $w_e = d$. This allows us to use the following lemma.

Lemma 5.1 ([19])**.** *Let G be a planar map, and G' the weighted star-tree transform of G. If G' is recurrent, then so is G.*

For a rooted map, we can give the transformed map G' a root ρ' by choosing uniformly a root within the tree (including the leaves) corresponding to ρ.

Recall the rooted graph (M_n, ρ_n) from Section 4, where some vertices are the skeleton. Apply the star-tree transform to M_n to get M_n'. The skeleton vertices of M_n' are all vertices in trees associated with skeleton vertices of M_n, including the leaves.

There are two ways to choose a root for M_n'. First, we could choose a root uniformly among all skeleton vertices of M_n'. The law of the resulting rooted map is denoted ν_n. Take an arbitrary subsequential limit of ν_n and call it ν. A second way is to take the rooted (M_n, ρ_n), and take the root of M_n' to be a uniform vertex from the tree associated with ρ_n. We call the law of this rooted map μ_n. Note that the star tree transform is continuous in the local topology. Since (M_n, ρ_n) converges to (M, ρ) we have that (M'_n, ρ'_n) converges to (M', ρ'), with law $\mu = \lim \mu_n$, where ρ' is a uniform vertex in the tree associated to the root ρ of M.

Lemma 5.2. *The measure μ is absolutely continuous with respect to ν.*

Proof. Given M_n, each skeleton vertex is equally likely to be the root under ν_n. Under μ_n a skeleton vertex in the tree of a vertex $v \in M_n$ has probability proportional to $(2\deg(v) - 2)^{-1}$ of being the root, since we need to choose $\rho_n = v$ and the associated tree has $2\deg(v) - 2$ vertices. Thus the Radon-Nikodym derivative $d\mu_n/d\nu_n$ is proportional to $(2\deg(\rho_n) - 2)^{-1}$. Since every skeleton vertex has degree at least 2, $d\mu_n/d\nu_n \leq 1/2$. In the case of an identified leaf between skeleton vertices u and v, the probability is proportional to $(2\deg(u) - 2)^{-1} + (2\deg(v) - 2)^{-1} \leq 1$. Thus using dominated convergence, μ is absolutely continuous with respect to ν. $\square$

Finally in order to use circle packing, it is useful to work with triangulations. We triangulate each face of M'_n, M' to obtain a triangulation. This can be done while maintaining bounded degrees, as in [14]. By a slight abuse of notation, we also denote the resulting maps by (M'_n, ρ'_n) and (M', ρ') and their law by ν. Since adding edges can only makes a graph transient (via the Rayleigh Monotonicity Principle), we immediately deduce the following using Lemmas 5.1 and 5.2.

Corollary 5.3. *If (M', ρ') is ν-almost surely recurrent, then (M, ρ) is almost surely recurrent, as is its subgraph H.*

Finally, we shall also need a simple lemma relating adjacency in M_n and M'_n. Let $\pi : M'_n \to M_n$ be the projection mapping each vertex in the tree corresponding to a vertex v to v. A vertex arising from the splitting of an edge in two is mapped (arbitrarily) to one of the two endpoints of the edge.

Lemma 5.4. *If $u \sim v$ in M'_n, then either $\pi(u) = \pi(v)$ or else $\pi(u) \sim \pi(v)$.*

Proof. Since M_n is a triangulation, after vertices are replaced by trees, each face consists of paths from three trees corresponding to a face of M_n, and three vertices corresponding to the edges between the trees. All additional edges in M'_n connect vertices within a face. $\square$

6 Recurrence via circle packing

All the tools are in place, and we are ready to build on the methods of [14, 19] to prove our main result. Throughout this section we have maps (M'_n, ρ'_n) and (M', ρ') with law ν_n and ν respectively.

Let us recall some useful terminology. Given a set of points $\mathcal{C}$ in a metric space, the radius of isolation R_x of a point $x \in \mathcal{C}$ is the minimal distance to another point of $\mathcal{C}$. Following [14], we say that a point $x \in \mathcal{C}$ is **(δ, s)-unsupported** if all but s of the points in $B(x, \delta^{-1} R_x)$ can be covered by a ball of radius δR_x. Otherwise it is **(δ, s)-supported**. A key idea in [14], is that for small δ and large s, a finite set cannot have too many (δ, s)-supported points. We use a quantitative form of this appears in [19, Lemma 3.4]:

Lemma 6.1 ([19])**.** *There exists some A, so that for any finite $\mathcal{C} \subset \mathbb{R}^2$, for all $\delta \in (0, 1/2)$ and $s \geq 2$, the fraction of (δ, s)-supported points in $\mathcal{C}$ is at most $\frac{A \log(\delta^{-1})}{\delta^2 s}$.*

In previous work, this lemma was applied to the set of centres of a circle packing of a given graph. A key difference from previous work, is that we take the set $\mathcal{C}$ to be the set of centres of the circles corresponding to skeleton vertices, and not all vertices. Let P_n be some (arbitrarily chosen) circle packing of M'_n in $\mathbb{R}^2$ (which exists in light of the Circle Packing Theorem [22]). Since M'_n is a bounded degree triangulation (with boundary), we may take P_n so that ratios of radii of adjacent circles are bounded.

Having fixed some circle packing for M'_n, we now consider the uniform skeleton root ρ'_n. Apply a translation and dilation to P_n so that the circle corresponding to the root ρ'_n is the unit disc, and let Q be the image of $\mathcal{C}$ after this transformation, which is now defined on the same probability space as H, M_n and M'_n.

Lemma 6.2. *Let E_r be the event that all but r^3 points of $Q \cap \{|z| < r\}$ can be covered by a disc of radius r^{-1}. There exists some A, such that for all $r \geq 2, n \geq 1$ we have $\mathbb{P}(E_r) > 1 - \frac{A \log r}{r}$.*

Proof. an arbitrary M'_n, and take $\mathcal{C}$ to be the set of the centres of circles of skeleton points in M'_n. For a uniform vertex v, scale so that the circle of v is the unit circle. By the Ring Lemma, the radius of isolation R_v is in $[1, C]$ for some absolute constant C.

Now apply Lemma 6.1 with $s = r^3$ and $\delta = 1/(Cr)$. We find that if v is uniform in $\mathcal{C}$, with the claimed high probability, all but r^3 points in

$\mathcal{C} \cap \{|z| \le CrR_v\}$ can be covered by a disc of radius R_v/Cr. Since $CrR_v \ge r$ and $R_v/Cr \le 1/r$, this implies the claim. $\square$

Consider the subgraph of M_n' induces by vertices in $\{|z| < r\}$. Let $\Gamma(n,r)$ denote the connected cluster of ρ_n' in this graph, and let $\bar\Gamma(n,r)$ denote $\Gamma(n,r)$ together with all edges connecting the cluster to vertices outside $\{|z| > r\}$. A major step in our proof is to show that for some constant α, the resistance in $\bar\Gamma(n,r)$ from ρ_n' to the complement of $\{|z| > r\}$ is at least α with high probability. Of course, this is the same as the resistance in M_n' between the same vertex sets. Moreover, we shall prove all this not just for the resistance from ρ_n', but from any finite neighbourhood of ρ_n', i.e. there is some $\alpha > 0$ so that for any finite set the resistance from the set to the complement of $\{|z| > r\}$ is at least α for r large enough. Towards this, we first prove that (with high probability) the maximal conductance of any edge in $\bar\Gamma(n,r)$ is at most $C \log r$, and that if all conductances are changed to 1 then the resistance between the involved vertex sets are at least $c \log r$. (Recall that the conductance of an edge is the degree of the vertex corresponding to it before the star-tree transform.) The claim then follows by Rayleigh monotonicity with $\alpha = c/C$. In what follows, $R_{\text{eff}}(A, B; w)$ denotes the resistance from A to B with edge weights w. The graph is implicit and should be clear from the context.

Lemma 6.3. *Fix k, and let $B_k \subset M_n'$ be the ball of graph radius k around ρ_n'. For some c, for all r large enough, in $\bar\Gamma(n,r)$ we have*

$$R_{\text{eff}}(B_k, \{|z| \ge r\}; 1) \ge c_1 \log r.$$

Proof. The radius of the circle of ρ_n' is 1. By the Ring Lemma, radii of adjacent circles have bounded ratio, so every vertex of B_k is contained in $\{|z| < r'\}$ for some $r' = r'(k)$. The resistance across the annulus $\{r' < |z| < r\}$ is now seen to be at least $c \log(r/r')$ by the arguments of [20, 14] (see for example [19, Corollary 3.3]). $\square$

Lemma 6.4. *Let $w_{\max}$ denote the maximal conductance of any edge in $\bar\Gamma(n,r)$. Then for some c_0 we have*

$$\lim_{r\to\infty} \limsup_{n\to\infty} \mathbb{P}(w_{\max} \ge c_0 \log r) = 0.$$

Proof. Fix $\varepsilon > 0$, and consider the event E_r of Lemma 6.2. For $r \ge r_0(\varepsilon)$ we have $\mathbb{P}(E_r) \ge 1 - \varepsilon$. Assume $r \ge 1$ and that E_{2r} holds, and let $U =$

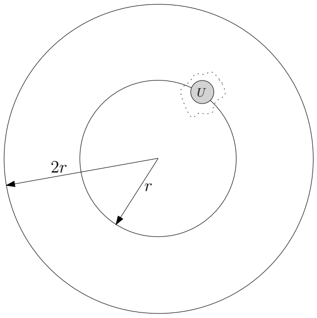

Figure 6: Illustration of the proof of Lemma 6.4. The grey disc U may include an arbitrary number of skeleton vertices, but the rest of the large ball, including the cycle around U contain at most $8r^3$ skeleton vertices. The cycle is needed if U intersects but is not surrounded by the ball of radius r.

$\{|z - z_0| < 1/2r\}$ be a disc such that $\{|z| < 2r\} \setminus U$ contains at most $(2r)^3$ skeleton vertices.

We consider several possibilities according to the location of U. If U is disjoint of $\{|z| < r\}$, then $\{|z| < r\}$ contains at most $8r^3$ skeleton vertices. Otherwise, $|z_0| < r+1/2$ (since $r \geq 1$). Suppose U contains at least 2 vertices, which therefore have circles of radius at most $1/r$. Let $U_a = \{|z-z_0| < a/r\}$. From the Ring Lemma it follows that for some a, the vertices in the annulus $U_a \setminus U$ disconnect U from the complement of U_a. In that case, since M'_n is a triangulation, there is a cycle in that annulus that surrounds U. For r large enough, this cycle lies in $\{|z| < 2r\} \setminus U$, and so it contains at most $8r^3$ skeleton vertices (and possibly more non-skeleton vertices).

Let us summarize our findings so far. For r large enough and any n, with probability at least $1 - \varepsilon$, there is a set of at most $8r^3$ skeleton vertices in M'_n that contains every skeleton vertex in $\{|z| < r\}$ except possibly those in U. If any vertices from U are missed, the set also contains all skeleton vertices from a cycle separating U from ρ'_n. That cycle need not be contained in $\{|z| < r\}$. See Figure 6.

Now, any path γ in M'_n which does not contain any boundary vertex of M'_n projects via π to a path in M_n. The restriction of $\pi(\gamma)$ to the skeleton vertices is a path in $\mathrm{Skel}(M_n)$, which visits no more skeleton vertices than γ (by Lemma 5.4). For any r, for large enough n with probability $1 - \varepsilon$, no boundary vertex of M'_n is in $\bar{\Gamma}(n, r)$ since otherwise, the skeleton distance from ρ'_n to a boundary vertex is at most $8r^3$ (Lemma 4.2). Thus for any r, for large enough n, with probability $1-2\varepsilon$, $\Gamma(n,r)$ is contained in the hull of $B_{\mathrm{Skel}}(\rho'_n, 8r^3)$. The result now follows from Lemma 3.12. $\square$

As noted, by combining Lemmas 6.3 and 6.4 we get the following with $\alpha = c_1/c_0$.

Proposition 6.5. *Fix an integer k and $\varepsilon > 0$, and let $B_k \subset M'_n$ be the ball of graph distance k around ρ'_n. For some $\alpha > 0$, for all r large enough, we have with probability at least $1-\varepsilon$ as $n \to \infty$*

$$R_{\mathrm{eff}}(B_k, \{|z| \geq r\}; w) \geq \alpha.$$

Proof of Theorem 1 and Theorem 3.6. The argument is similar to the argument of [19]. We start with the observation that an electrical network G is recurrent if and only if for some $\alpha > 0$, for every graph distance ball $B_k = B_k(\rho)$ there exists a finite vertex set S such that

$$R_{\mathrm{eff}}(B_k, G \setminus S; w) > \alpha.$$

Fix k, and $\varepsilon > 0$. By Proposition 6.5, for any large enough n, with probability $1-\varepsilon$ there is some finite S such that in M'_n we have $R_{\mathrm{eff}}(B_k, M'_n \setminus S; w) > \alpha$. Moreover, with high probability for some R, the set S is contained in the hull of a ball of radius R in M'_n. Going to the limit, we find that for n large enough, with probability at least $1-2\varepsilon$ the resistance in M' from B_k to the complement of some large finite set S is at least α. Since ε is arbitrary, this implies that M' is ν-almost surely recurrent.

By Lemma 5.2, this implies that M' is μ-almost surely recurrent, which in turn also implies recurrence of M, and of H. $\square$

7 Extensions

Resistance estimates. From the argument above we also get some explicit estimates on the growth of the resistance in M. In the annulus between Euclidean radii 2^n and 2^{n+1} the maximal degree is of order n. Since the resistance across the annulus without weight is at least C, this indicates that the resistance to distance 2^n is at least $c \log n$, i.e. the resistance to Euclidean distance R is $\log\log R$. This argument can be made precise, but we do not pursue this here. It would be interesting to get better bounds on the growth of the resistance (it is believed to grow like log).

Other classes of maps. One natural generalization is to consider uniform infinite domain Markov half plane triangulations with self-loops. Such triangulations can be obtained by taking a HUIPT and decomposing every edge into i.i.d. Geometric number of edges and attaching self-loops on one of the vertices in the 2-gons thus formed by tossing a fair coin (see [8] for detailed discussion on this.) Note that a self-loops with any finite triangulation inside it do not effect recurrence or transience so we can delete them. We can now form an equivalent network by collapsing the geometric number of multiple edges into a single edge and giving this edge a conductance which is equal to the number of edges combined to form it. Thus the equivalent network is HUIPT but with i.i.d. geometric conductances on each edge. It can be checked by the diligent reader that our analysis of the HUIPT goes through in this case also, implying recurrence of this case as well.

A more difficult problem is proving recurrence of more general half planar maps. It is easy to see that a layer decomposition is still possible for various other classes of half plane uniform infinite maps. For quadrangulations, a similar layer decomposition was introduced by Krikun in [23]. The main estimate needed is that the maximal degree in the skeleton balls grow logarithmically in the radius (an analogue of Lemma 3.12.) For maps with even larger faces, a layer decomposition is still possible but it becomes more complicated.

Hyperbolic maps. A one parameter family of hyperbolic versions of the half plane UIPT were constructed in [8]. A full plane hyperbolic version was constructed in [17] and it was shown in [7] that the half plane versions can be be realized as a sub-map of the full plane ones. One can carry out the layer decomposition and a full plane extension of the half plane maps in exactly the same way as done in this paper. Call such a full plane map M^{hyp}. The volume of the triangulation inside the holes in this situation will have exponential tail. It is not too difficult to see that the lower half of the triangulation in M^{hyp} is recurrent. In this situation, if we look at the sequence of hulls of radius r and uniformly pick a vertex, the map converges locally to some rerooted version of the lower half, and so the maps are a local limit of finite planar graphs with exponential degree distribution. Exploring the connection between M^{hyp} and the full plane map defined in [17] is also of interest.

Stationarity. It is easy to see that if we put appropriate conductances on the edges of Skel(M) and bias by the degree (in M) of the root vertex, we obtain a stationary reversible graph. A similar construction can be carried out for the hyperbolic versions to obtain Skel(M^{hyp}). For a simple random walk $Y_0, Y_1, \dots$ in Skel(M^{hyp}) or Skel(M), if we let $\ell(Y_i)$ denote the index of the layer below Y_i an application of ergodic theorem lets us conclude

$$\frac{\ell(Y_i)}{i} \to s \tag{7.1}$$

almost surely for some constant s. It follows from the results in [7] that $s > 0$ almost surely in Skel(M^{hyp}) and the recurrence result in this paper shows $s = 0$ almost surely for Skel(M). Notice that simple random walk in M^{hyp} spends a positive fraction of its time in the skeleton vertices (this is easy to see again via stationarity and exponential tail of the volume of the holes). From all this we can deduce the existence of the speed of simple random walk away from the boundary in H. This answers a question in [7] where only positive liminf speed from the boundary was established.

Acknowledgments

This work begun while GR was at UBC, and completed during a visit of OA to the Isaac Newton Institute. OA was partly supported by NSERC, the Isaac Newton Institute and the Simons Foundation. GR was supported by the Engineering and Physical Sciences Research Council under grant EP/103372X/1.

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