COMPUTING ROBUST CONTROLLED INVARIANT SETS OF LINEAR SYSTEMS

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Abstract. We consider controllable linear discrete-time systems with perturbations and present two methods to compute robust controlled invariant sets. The first method results in an (arbitrarily precise) outer approximation of the maximal robust controlled invariant set, while the second method provides an inner approximation.

1. Introduction

Before introducing the problem addressed in this paper we would like to mention that all the relevant notation is explained in the appendix. Let us consider two matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ with $m \le n$ and a nonempty set $W \subseteq \mathbb{R}^n$. Throughout this paper, we analyze linear, time-invariant, discrete-time systems with additive perturbations of the form

$$\xi(t+1) \in A\xi(t) + B\nu(t) + W, \qquad W \neq \varnothing \tag{1}$$

where $\xi(t) \in \mathbb{R}^n$ and $\nu(t) \in \mathbb{R}^m$ is the *state signal*, respectively, *input signal* and W is the set of *disturbances*. In addition to the dynamics, we consider *state constraints* and *input constraints* given by the sets

$$X \subseteq \mathbb{R}^n \quad \text{and} \quad U \subseteq \mathbb{R}^m. \tag{2}$$

We are interested in the computation of *feedback strategies* [1, Chap. VIII] (in short *feedbacks*) that non-deterministically map state histories to admissible inputs

$$\mu : \bigcup_{T\in\mathbb{Z}_{\ge 0}} (\mathbb{R}^n)^{[0;T]} \rightrightarrows U \tag{3}$$

and force the trajectories of (1) to evolve inside the state constraint set X. In the following, we use $\mathcal{F}(U)$ to denote the set of all *strict* feedback strategies of the form (3). A feedback μ is strict, if for all $\xi \in \cup_{T\in\mathbb{Z}_{\ge 0}}(\mathbb{R}^n)^{[0;T]}$ we have $\mu(\xi) \neq \varnothing$.

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A *trajectory* of (1) with *initial state* $x \in \mathbb{R}^n$ and feedback $\mu \in \mathcal{F}(U)$, is a sequence $\xi : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$ for which there exists $\nu : \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$ so that (1) and

$$\nu(t) \in \mu(\xi|_{[0;t]})$$

holds for all $t \in \mathbb{Z}_{\geq 0}$. We use $\mathcal{B}_{x,\mu} \subseteq (\mathbb{R}^n)^{\mathbb{Z}_{\geq 0}}$ to denote the set of trajectories ξ with initial state x and feedback μ.

A set $R \subseteq \mathbb{R}^n$ is called *robust controlled invariant* (w.r.t. (1) and U) if for all $x \in R$, there exists $\mu \in \mathcal{F}(U)$ so that for all $\xi \in \mathcal{B}_{x,\mu}$ and $t \in \mathbb{Z}_{\geq 0}$ we have $\xi(t) \in R$, or equivalently: for every $x \in R$ there exists $u \in U$ so that $Ax + Bu + W \subseteq R$.

It is well-known [2] that the feedbacks of interest are characterized by the *maximal robust controlled invariant set* [3, 4], also known as *infinite reachable set* [2] or *discriminating kernel* [5, 6], contained in X, i.e.,

$$R(X) = \{x \in \mathbb{R}^n \mid \exists_{\mu \in \mathcal{F}(U)} \forall_{\xi \in \mathcal{B}_{x,\mu}} \forall_{t \in \mathbb{Z}_{\geq 0}}\ \xi(t) \in X\}. \tag{4}$$

The set $R(X)$ is called maximal, since $R \subseteq X$ being robust controlled invariant, implies $R \subseteq R(X)$. Given $R(X)$, the following map characterizes all feedbacks of interest

$$C(x) = \{u \in U \mid Ax + Bu + W \subseteq R(x)\}. \tag{5}$$

Theorem 1. *Consider the system* (1) *and the constraint sets* (2). *Let* $R(X)$ *and* C *be defined in* (4) *and* (5), *respectively. Let* $x \in X$, *then the feedback* $\mu \in \mathcal{F}(U)$ *satisfies*

$$\forall_{\xi \in \mathcal{B}_{x,\mu}} \forall_{t \in \mathbb{Z}_{\geq 0}} \xi(t) \in X \text{ iff } \forall_{\xi \in \mathcal{B}_{x,\mu}} \forall_{t \in \mathbb{Z}_{\geq 0}} \mu(\xi|_{[0;t]}) \subseteq C(\xi(t)).$$

The result, which is given in [7, Thm. 1] and also appears in a slightly different form in [2, Prop. 3], shows that it is sufficient to consider *static* feedback strategies, i.e., feedbacks of the form $\mu : \mathbb{R}^n \rightrightarrows U$, to render the set X invariant. Moreover, and more importantly, it shows that it is sufficient to know $R(X)$ from which any feedback $\mu \in \mathcal{F}(U)$ that enforces the state constraints X on (1) can be derived.

Let $\mathrm{pre}(R) = \{x \in \mathbb{R}^n \mid \exists_{u \in U} Ax + Bu + W \subseteq R\}$ denote the set of states that are mapped into R by the dynamics when the input is appropriately chosen. In [2], Bertsekas introduced the iteration

$$R_0 = X, \quad R_{i+1} = \mathrm{pre}(R_i) \cap X \tag{6}$$

to compute the discriminating kernel and showed a variant of the following theorem.

Theorem 2. *Consider the system* (1) *and the constraint sets* (2). *Let* $R(X)$ *and* $(R_i)_{i \in \mathbb{N}}$ *be defined in* (4) *and* (6), *respectively. Suppose that* X *is closed and* U *is compact, then*

$$R(X) = \lim_{i \to \infty} R_i. \tag{7}$$

The theorem appears in a variety of different flavors in the literature, see e.g. [2, Prop. 4], [8, Prop. 4.8], [9, Thm. 5.1], [3, Sec. 5], [10, Cor. 2] and [4, Thm. 5.2]. In [2–4, 10] the convergence of $R(X) = \lim_{i \to \infty} R_i$ is shown with respect to the Hausdorff metric, provided that the constraint sets are compact and the sets R_i (or $R(X)$) are nonempty. In [8, Prop. 4.8], the convergence is shown for merely closed sets X, U with a slightly different set iteration $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$ in which the order of quantification of the control and the disturbance is interchanged. We provide a proof of Theorem 2, which considers the set iteration (6) with a possible unbounded state constraint set, with respect to the set convergence defined in [11, Chatper 4] in the appendix. In the proof, we use the same argument as already presented in [2], in which the compactness of U is exploited to show that the set $\lim_{i \to \infty} R_i$ is robust controlled invariant.

Theorem 2 shows that the discriminating kernel $R(X)$ can, in principle, be outer approximated by the sets $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$ with arbitrary precision. Nevertheless, even if the sets $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$ are computable, the approximation is not very useful since in general the sets $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$ are not robust controlled invariant and it is not possible to derive a feedback from any R_i that ensures that the system always evolves inside the state constraint set.

However, in some cases it is possible to determine the maximal robust controlled invariant set by the iteration (6). If there exists $i \in \mathbb{Z}_{\geq 0}$ so that two consecutive iterations in (6) result in equal sets, i.e., $R_{i+1} = R_i$, then $R_i = R(X)$. In this case, we say that $R(X)$ is *finitely determined* [12, Thm 2.3]. Depending on the dynamics (A, B) and the shape of X, U and W there exist conditions which ensure that $R(X)$ is finitely determined, see [13]. A large class of cases is covered by the following conditions. Suppose that (A, B) is controllable, then without loss of generality, we may assume that the system is in Brunovsky normal form, also known as Controller Form, see [14, Sec. 6.4.1]. In this representation, if $W = \{0\}$ and the sets X and U are given by a finite union of *hyper-rectangles*, then the maximal control invariant set is finitely determined, see [15–17].

Unfortunately, for one of the most popular settings, where (A, B) is assumed to be controllable, $W = \{0\}$ and the sets X and U are assumed to be polytopes with the origin in the interior, $R(X)$ is not finitely determined. Nevertheless, in this case, one can modify the iteration (6) and set $R_0 = \{0\}$ (instead of $R_0 = X$). As a result, each set R_i is controlled invariant and in fact R_i is the i-step null-controllable set [18, 19] and the union of the sets R_i converges to the *largest null-controllable set* $N(X)$, i.e., the set of all initial states from which the system can be forced to the origin in finite time without violating the constrains. As R_i converges to the maximal null controllable set $N(X)$ and the closure of $N(X)$ equals $R(X)$, see [19, Prop. 1], the iteration (6) with $R_0 = \{0\}$ provides an algorithm for the arbitrarily precise (inner) approximation of $R(X)$, with the considerable advantage that the approximation is robust controlled invariant. Moreover, this approach provides a so-called *anytime algorithm*, i.e., for each iteration $i \in \mathbb{Z}_{\geq 0}$ the set R_i is controlled invariant and a feedback can be derived, which enforces the trajectories of (1) with initial state in R_i to evolve inside the constraint set X. Additionally, due to the convergence of R_i, the mismatch between R_i and $R(X)$ decreases as the computation continues.

An alternative modification of the iteration (6), which also provides an invariant approximation of $R(X)$, is presented in [20] and [4, Sec. 5.2]. In contrast to the approach in [18, 19] the initial set is unchanged $R_0 = X$, but in each iteration the successor set is computed by $R_{i+1} = \mathrm{pre}(\lambda R_i) \cap X$ for some fixed contraction factor $\lambda \in]0, 1[$. The computation of $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$ terminates, once the inclusion $R_i \subseteq \hat{\lambda} R_{i+1}$ holds for $\hat{\lambda} \in]\lambda, 1[$. Given that X, U and W are polytopes with the origin in its interior, it is shown in [4, Prop. 5.9] that there exists $i \in \mathbb{Z}_{\geq 0}$ so that the termination condition is satisfied $R_i \subseteq \hat{\lambda} R_{i+1}$ and R_i is robust controlled invariant. Furthermore, if there exists a λ-contractive set in X (see [4, Def. 4.18] and [4, Thm. 4.48]) then it is guaranteed that R_i is nonempty.

In this paper, we assume that the dynamics (A, B) are controllable and the constraint sets X and U are compact. Under these assumptions, we propose two schemes for the inner and outer invariant approximation of the discriminating kernel. For the invariant outer approximation of $R(X)$, we leave the set iteration (6) untouched, but introduce a stopping criterion, similar to (5.10) in [4], by

$$R_i \subseteq R_{i+n} + \varepsilon \mathbb{B}. \tag{8}$$

We show that for every $\varepsilon \in \mathbb{R}_{>0}$ there exists an $i \in \mathbb{Z}_{\geq 0}$ so that (8) holds. Based on the set R_{i+n}, we derive a *δ-relaxed robust control invariant set* R, i.e., $R(X) \subseteq R \subseteq X + \delta \mathbb{B}$ and R is robust controlled invariant w.r.t. (1) and $U + \delta \mathbb{B}$. Here $\delta = c\varepsilon$, where $c \in \mathbb{R}_{\geq 0}$ is a constant that is known a-priori and the relaxation of the constraints can be made arbitrarily small by choosing an appropriate $\varepsilon \in \mathbb{R}_{>0}$. Moreover, we show that the set R converges to $R(X)$ as ε decreases to zero. Note that this approach can also be used in an anytime scheme. In that situation, at each iteration $i \geq n$, we determine $\varepsilon \in \mathbb{R}_{\geq 0}$ so that (8) holds. If the constraint relaxation δ is tolerable, we stop the computation, otherwise, we continue with R_{i+1}.

For the inner invariant approximation of $R(X)$, we modify the iteration (6) to

$$R_0 = X, \quad R_{i+1} = \mathrm{pre}_\rho(R_i) \cap X \tag{9}$$

where the map pre_ρ is defined for $\rho \in \mathbb{R}_{\geq 0}$ by

$$\mathrm{pre}_\rho(R) = \{x \in \mathbb{R}^n \mid \exists_{u \in U} : Ax + Bu + W + \rho \mathbb{B} \subseteq R\}. \tag{10}$$

This approach is very much in spirit of the scheme presented in [20], in which the set sequence is constructed by $R_{i+1} = \mathrm{pre}(\lambda R_i) \cap X$. Given $\rho \in \mathbb{R}_{\geq 0}$, we show that there exists $i \in \mathbb{Z}_{\geq 0}$ so that $R_i \subseteq R_{i+1} + \rho\mathbb{B}$ holds and that R_{i+1} is robust controlled invariant.

Compared to existing approaches, we do not impose any restrictions on the shape of the constraint sets [15–17], nor do we assume that they contain the origin in its interior [18–20], but simply consider compact constraint sets. Specifically, we allow sets given by finite unions of polytopes, i.e., the sets $X_i \subseteq \mathbb{R}^n$, $U_j \subseteq \mathbb{R}^m$, $W_k \subseteq \mathbb{R}^n$ with $i \in [1; I]$, $j \in [1; J]$, $k \in [1; K]$ and $I, J, K \in \mathbb{Z}_{\geq 1}$ are polytopes and

$$X = \bigcup_{i \in [1;I]} X_i, \quad U = \bigcup_{j \in [1;J]} U_j, \quad W = \bigcup_{k \in [1;K]} W_k. \tag{11}$$

In this case, the sets $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$ are computable [10, Sec. III.B] and the proposed scheme for the outer invariant approximation is δ-complete [21]: Let $\delta \in \mathbb{R}_{>0}$, (A, B) be controllable and X, U, $W \neq \varnothing$ be defined in (11), then the proposed algorithm either returns an empty set R_{i+n}, in which case the problem has no solution, i.e., $R(X) = \varnothing$, or we obtain a δ-relaxed robust controlled invariant set R.

Constrains sets in the form of (11) arise in a variety of different situations, see e.g. [22], and are particularly important in the synthesis problems with respect to safe linear temporal logic specifications [17].

2. Outer Invariant Approximation

We begin with a lemma which shows that the stopping criterion (8) is valid.

Lemma 1. *Consider the system* (1) *and the constraint sets X and U given in* (2)*. Let $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$ be defined according to* (4)*. Suppose that X and U are compact, then for any $\varepsilon \in \mathbb{R}_{>0}$ there exists $i \in \mathbb{Z}_{\geq 0}$ so that* (8) *holds.*

Proof. Let $\varepsilon \in \mathbb{R}_{>0}$. From Theorem 2 and the boundedness of $R(X)$ and $(R_i)_{i \in \mathbb{N}}$ we obtain that $\lim_{i \to \infty} d_H(R(X), R_i) = 0$, see [11, pp. 117]. Hence, we can pick $i^* \in \mathbb{Z}_{\geq 0}$ so that $d_H(R(X), R_i) \leq \varepsilon/2$ holds for all $i \geq i^*$ and we obtain the inequality $d_H(R_{i^*+n}, R_{i^*}) \leq d_H(R_{i^*+n}, R(X)) + d_H(R(X), R_{i^*}) \leq \varepsilon$ which implies that (8) holds. □

In the following, we make use of *δ-constraint i-step null-controllable sets* $N_i^\delta \subseteq \mathbb{R}^n$, i.e., the set of initial states from which the unperturbed system $\xi(t+1) = A\xi(t) + B\nu(t)$ can be forced to the origin while satisfying the input and state constraints $U = \delta\mathbb{B}$ and $X = \delta\mathbb{B}$. Let $\delta \in \mathbb{R}_{>0}$, then we define the sequence of sets $(N_i^\delta)_{i \in \mathbb{Z}_{\geq 0}}$ recursively by

$$\begin{aligned} N_0^\delta &= \{0\}, \\ N_{i+1}^\delta &= \{x \in \mathbb{R}^n \mid \exists_{u \in \delta\mathbb{B}}\ Ax + Bu \in N_i^\delta\} \cap \delta\mathbb{B}. \end{aligned} \tag{12}$$

Note that for a fixed $\delta \in \mathbb{R}_{>0}$ it is straightforward to compute the sets (N_i^δ) by polyhedral projection and intersection [4]. We use the following technical lemma about δ-constraint i-step null-controllable sets.

Lemma 2. *Consider the system* (1) *with $W = \{0\}$. Let N_n^δ be defined according to* (12)*. Suppose that (A, B) is controllable, then*

$$\exists_{c \in \mathbb{R}_{>0}}\ \forall_{\varepsilon \in \mathbb{R}_{>0}} : \quad \varepsilon\mathbb{B} \subseteq N_n^\delta \quad \text{with } \delta = c\varepsilon. \tag{13}$$

Proof. We show that there exists $c \in \mathbb{R}_{>0}$ such that for every $x \in \mathbb{R}^n$ there exists $\nu : [0; n[\to \mathbb{R}^m$ so that the trajectory of $\xi(t+1) = A\xi(t) + B\nu(t)$ with $\xi(0) = x$ satisfies $\xi(n) = 0$, and for all $t \in [0; n[$ we have $|\xi(t)| \leq c|x|$ and $|\nu(t)| \leq c|x|$. This implies the assertion of the lemma, since it is easy to see that $\xi(t) \in N_{n-t}^\delta$ with $\delta \geq c|x|$ holds for all $t \in [0; n]$. The trajectory at time n is given by $\xi(n) = A^n x + \mathcal{C}V$, where $\mathcal{C}$ is the controllability matrix $[B,\ AB \ldots A^{n-1}B]$ and V is a vector in $\mathbb{R}^{mn}$ with $V = [\nu(n-1)^\top,\ \ldots, \nu(0)^\top]^\top$. Let $\mathcal{C}' \in \mathbb{R}^{n \times n}$ denote a matrix containing n linearly independent columns of $\mathcal{C}$. Such a matrix always exists, since (A, B) is controllable and hence $\mathcal{C}$ hast full rank. Given $x \in \mathbb{R}^n$, we determine the input sequence V by setting the entries V' of V associated with $\mathcal{C}'$

to $V' = -(\mathcal{C}')^{-1}A^n x$ and the remaining entries of V to zero. It follows that $\xi(n) = A^n x + \mathcal{C}V = 0$. Moreover, $|V'| \leq c'|x|$ with $c' = |(\mathcal{C}')^{-1}A^n|$ holds and $|\nu(t)| \leq c'|x|$ for all $t \in [0; n[$ follows. From $\xi(t) = A^t + \sum_{s=0}^{t-1} A^{t-(s+1)}B\nu(s)$ follows that $|\xi(t)| \leq (|A^t| + \sum_{s=0}^{t-1} |A^{t-(s+1)}B|c')|x|$ holds and the assertion follows. □

Corollary 1. *Let $z_j \in \mathbb{R}^n$, $j \in [1; 2^n]$ denote the vertices of the unit cube $\mathbb{B}$. A constant $c \in \mathbb{R}_{>0}$ that satisfies* (13) *is given by $c = \max_{j\in[1;2^n]} c_j$ where c_j is obtained by solving the linear program*

$$\begin{aligned} & \min_{c_j, u_0, \dots, u_{n-1}} c_j \\ s.t. \quad & A^n z_j + \sum_{k=0}^{n-1} A^{n-k-1} B u_k = 0 \\ \forall_{i\in[0;n-1]} \quad & |u_i| \leq c_j \\ \forall_{i\in[1;n-1]} \quad & \left| A^i z_j + \sum_{k=0}^{i-1} A^{i-k-1} B u_k \right| \leq c_j. \end{aligned} \tag{14}$$

Note that $|x|$ denotes the infinite norm of $x \in \mathbb{R}^n$ and the corollary follows simply by the linearity of the trajectories of $\xi(t+1) = A\xi(t) + B\nu(t)$.

We proceed with the main result related with the outer invariant approximation.

Theorem 3. *Consider the system* (1) *and constraint sets* (2). *Let (A, B) be controllable and consider the sequences of sets $(R_i)_{i\in\mathbb{Z}_{\geq 0}}$ and $(N_i^\delta)_{i\in\mathbb{Z}_{\geq 0}}$ given according to* (4), *respectively* (12), *with $\varepsilon \in \mathbb{R}_{>0}$, $\delta = c\varepsilon$ and c satisfying* (13). *Let $i^* \in \mathbb{Z}_{\geq 0}$ be the smallest index, so that* (8) *holds. The set*

$$R := \bigcup_{j\in[1;n]} R_{i^*+j} + N_j^\delta \tag{15}$$

is a subset of $X + \delta\mathbb{B}$ and is robust controlled invariant w.r.t. (1) *and $U + \delta\mathbb{B}$.*

Proof. Consider the set R defined in (15). Due to the choice of $\delta = c\varepsilon$ with c satisfying (13) we have $\varepsilon\mathbb{B} \subseteq \cup_{j\in[1;n]} N_j^\delta \subseteq \delta\mathbb{B}$, which together with $R_i \subseteq X$ implies that $R \subseteq X + \delta\mathbb{B}$. Moreover, (8) and (13) imply $R_{i^*} \subseteq R$. We show that for every $x \in R$ there exists $u \in U + \delta\mathbb{B}$ so that $Ax + Bu + W \subseteq R$ which implies that R is robust controlled invariant, see [15, Prop. 2]. Let $x \in R$, then there exists $j \in [1; n]$ so that $x \in R_{i^*+j} + N_j^\delta$. Let $x = x_r + x_n$ so that $x_r \in R_{i^*+j}$ and $x_n \in N_j^\delta$. Then there exists $u_r \in U$ and $u_n \in \delta\mathbb{B}$ so that $Ax_r + Bu_r + W \subseteq R_{i^*+j-1}$ and $Ax_n + Bu_n \in N_{j-1}^\delta$ and it follows that $Ax+Bu+W \subseteq R_{i^*+j-1}+N_{j-1}$ where $u = u_r+u_n \in U+\delta\mathbb{B}$. If $j \geq 2$, it follows from the definition of R that $Ax+Bu+W \subseteq R$. If $j = 1$, we use (8) and (13) to get $Ax + Bu + W \subseteq R_{i^*} \subseteq R_{i^*+n} + \varepsilon\mathbb{B} \subseteq R$. □

Due to the construction of R it is straightforward to show that by decreasing the stopping parameter $\varepsilon \in \mathbb{R}_{>0}$ the set R defined in (15) converges to $R(X)$.

Corollary 2. *Consider the hypothesis of Theorem 3 and suppose that X and U are compact. Let R_ε denote the set R defined in* (15) *for parameter $\varepsilon \in \mathbb{R}_{>0}$. For any sequence $(\varepsilon_j)_{j\in\mathbb{Z}_{\geq 0}}$ in $\mathbb{R}_{>0}$ with limit 0 we have $R(X) = \lim_{j\to\infty} R_\varepsilon$.*

Proof. Consider the sequence $(R_i)_{i\in\mathbb{Z}_{\geq 0}}$ according to (4). Let $i^*(\varepsilon)$ denote the smallest $i^* \in \mathbb{Z}_{\geq 0}$ such that (8) holds for a fixed $\varepsilon \in \mathbb{R}_{>0}$. Consider a sequence $(\varepsilon_j)_{j\in\mathbb{Z}_{\geq 0}}$ in $\mathbb{R}_{>0}$ that converges to zero. Due to the choice of $\delta_j = c\varepsilon_j$ in Theorem 3, we see that $(\delta_j)_{j\in\mathbb{Z}_{\geq 0}}$ converges to zero and hence, $\lim_{j\to\infty} d_H(R_{i^*(\varepsilon_j)}, R_{\varepsilon_j}) = 0$. Since d_H satisfies the triangular inequality for compact subsets of $\mathbb{R}^n$, it suffices to show $\lim_{j\to\infty} d_H(R(X), R_{i^*(\varepsilon_j)}) = 0$. Let us first point out that $\varepsilon_{j'} < \varepsilon_j$ implies $i^*(\varepsilon_{j'}) \geq i^*(\varepsilon_j)$. In case that $i^*(\varepsilon_j) \to \infty$ as $j \to \infty$ we use Theorem 2 to conclude $\lim_{j\to\infty} d_H(R(X), R_{i^*(\varepsilon_j)}) = 0$. Suppose that $i^*(\varepsilon_j) \to i$ with $i \in \mathbb{Z}_{\geq 0}$, so there exists $j' \in \mathbb{Z}_{\geq 0}$ such that $i^*(\varepsilon_j) = i$ for all $j \geq j'$. Therefore $d_H(R_i, R_{i+n}) \leq \varepsilon_j$ for all ε_j with $j \geq j'$, which implies $R_i = R(X)$. □

Remark 1. *Consider the system* (1) *and compact constraint sets* (2)*. Let* (A, B) *be controllable and fix* $\varepsilon \in \mathbb{R}_{>0}$*. Suppose that we have an algorithm to iteratively compute* R_i *and check the inclusion* (8)*. It follows from Lemma 1 that there exists* $i \in \mathbb{Z}_{\geq 0}$ *so that* (8) *holds. If* $R_{i+n} = \varnothing$*, then there does not exist a feedback to enforce the constraints* X *and* U*, in particular* $R(X) = \varnothing$*. If* $R_{i+n} \neq \varnothing$*, due to the controllability of* (A, B) *we can solve the linear program* (14) *and compute the sets* $(N^{\delta}_{i+j})_{j\in[1;n]}$ *with which we construct the set* R *according to* (15)*. Then it follows from Theorem 3 that* R *is robust controlled invariant and a static feedback to enforce the constraints* $X + \delta\mathbb{B}$ *and* $U + \delta\mathbb{B}$ *is derived from the map*

$$K(x) = \{u \in U + \delta\mathbb{B} \mid Ax + Bu + W \subseteq R\}.$$

Since $R(X) \subseteq R$ *it is straightforward to see that the map defined in* (5) *satisfies* $C(x) \subseteq K(x)$ *for all* $x \in R$*.*

3. Inner Invariant Approximation

For the inner approximation of $R(X)$ we fix $\rho \in \mathbb{R}_{\geq 0}$ and analyze the sequence

$$R^{\rho}_0 = X, \quad R^{\rho}_{i+1} = \text{pre}_{\rho}(R^{\rho}_i) \cap X \tag{16}$$

where pre_{ρ} is defined in (10). The stopping criterion, as proposed in (5.10) in [4], is given by

$$R^{\rho}_i \subseteq R^{\rho}_{i+1} + \rho\mathbb{B}. \tag{17}$$

Theorem 4. *Consider the system* (1) *and compact constraint sets* (2)*. Let* $(R^{\rho}_i)_{i\in\mathbb{Z}_{\geq 0}}$ *be defined in* (16)*. For every* $\rho \in \mathbb{R}_{>0}$ *there exists an index* $i \in \mathbb{Z}_{\geq 0}$ *such that* (17) *holds and* R^{ρ}_{i+1} *is robust controlled invariant w.r.t.* (1) *and* U*.*

Proof. The proof of the existence of $i \in \mathbb{Z}_{\geq 0}$ so that (17) holds, follows by the same arguments as the proof of Lemma 1 and is omitted here.

Let $x \in R^{\rho}_{i+1} = \text{pre}_{\rho}(R^{\rho}_i) \cap X$. There exists $u \in U$ such that $Ax + Bu + W + \rho\mathbb{B} \subseteq R^{\rho}_i \subseteq R^{\rho}_{i+1} + \rho\mathbb{B}$ which implies that $Ax + Bu + W \subseteq K^{\rho}_{i+1}$ and it follows form [15, Prop. 2] that R_{i+1} is robust controlled invariant. □

In the following theorem we show that if the discriminating kernel is robust with respect to the strengthened constraint sets

$$\begin{aligned} \bar{X}_{\varepsilon} &= \{x \in \mathbb{R}^n \mid x + \varepsilon\mathbb{B} \subseteq X\} \\ \bar{U}_{\varepsilon} &= \{u \in \mathbb{R}^m \mid u + \varepsilon\mathbb{B} \subseteq U\} \end{aligned} \tag{18}$$

with $\varepsilon \in \mathbb{R}_{>0}$, then there exists a parameter $\rho \in \mathbb{R}_{>0}$ so that the discriminating kernel associated with $\bar{X}_{\varepsilon}$ and $\bar{U}_{\varepsilon}$ is contained in R^{ρ}_{i+1}.

Theorem 5. *Consider the system* (1)*,* (A, B) *being controllable and compact constraint sets* (2)*. Let* $(R^{\rho}_i)_{i\in\mathbb{Z}_{\geq 0}}$ *be defined in* (16)*. Given* $\varepsilon \in \mathbb{R}_{>0}$*, we define*

$$\bar{R}_{\varepsilon} = \{x \in \mathbb{R}^n \mid \exists_{\mu\in\mathcal{F}(\bar{U}_{\varepsilon})}\forall_{\xi\in\mathcal{B}_{x,\mu}}\forall_{t\in\mathbb{Z}_{\geq 0}}\ \xi(t) \in \bar{X}_{\varepsilon}\}$$

with $\bar{X}_{\varepsilon}$ *and* $\bar{U}_{\varepsilon}$ *given in* (18)*. For every* $\varepsilon \in \mathbb{R}_{>0}$*, there exists* $\rho \in \mathbb{R}_{>0}$ *so that* $\bar{R}_{\varepsilon} \subseteq R^{\rho}_{i+1}$ *holds, where* $i \in \mathbb{Z}_{\geq 0}$ *satisfies* (17)*.*

Proof. Let us consider the system

$$\xi(t + 1) = A\xi(t) + B\nu(t) + W + \rho\mathbb{B}. \tag{19}$$

Due to the definition of pre_{ρ}, it follows that discriminating kernel $R^{\rho}(X)$ defined w.r.t. (19) and constraints X and U, satisfies $R^{\rho}(X) \subseteq R^{\rho}_i$ for every $i \in \mathbb{Z}_{\geq 0}$. Let $\varepsilon \in \mathbb{R}_{>0}$. In the following, we show that there exists a set K which contains $\bar{R}_{\varepsilon}$ and $\rho \in \mathbb{R}_{>0}$ so that $K \subseteq X$ is robust controlled invariant w.r.t. (19) and U, which implies $\bar{R}_{\varepsilon} \subseteq K \subseteq R^{\rho}(X)$ and the assertion follows.

Given $\varepsilon \in \mathbb{R}_{>0}$, let $\delta = \varepsilon/n$ and $\rho \in \mathbb{R}_{>0}$ so that $c\rho = \delta$, where the constant c is chosen according to Lemma 2 (which is applicable, since (A, B) is controllable). Consider N^{δ}_i, $i \in [0; n]$ defined according to (12). Note that (13) implies that $\rho\mathbb{B} \subseteq N_n^\delta$. We define the set $K := \bar{R}_\varepsilon + \sum_{i=1}^n N_i^\delta$. Note that $N_i^\delta \subseteq \delta\mathbb{B}$ holds for every $i \in [1;n]$, which together with $\bar{R}_\varepsilon + \varepsilon\mathbb{B} \subseteq X$ and $\delta = \varepsilon/n$, implies $K \subseteq X$. We show that K is robust controlled invariant w.r.t. (19) and U. Let $x \in K$, then there exists $x_r \in \bar{R}_\varepsilon$ and $x_i \in \mathbb{N}_i^\delta$, $i \in [1;n]$ so that $x = x_r + \sum_{i=1}^n x_i$. Since $\bar{R}_\varepsilon$ is robust controlled invariant, we can pick $u_r \in \bar{U}_\varepsilon$ so that $Ax_r + Bu_r + W \subseteq \bar{R}_\varepsilon$ (see [15, Prop. 2]), which implies that $Ax_r + Bu_r + W + \rho\mathbb{B} \subseteq \bar{R}_\varepsilon + N_n^\delta$. Moreover, for $x_i \in N_i^\delta$, we pick $u_i \in \delta\mathbb{B}$ so that $Ax_i + Bu_i \in N_{i-1}^\delta$. Let $u = u_r + \sum_{i=1}^n u_i$. As $u_r \in \bar{U}_\varepsilon$ and $\delta \leq \varepsilon/n$ we have $u \in U$. Additionally, it is easy to see that $Ax + Bu + W + \rho\mathbb{B} \subseteq K$, which shows that K is robust controlled invariant w.r.t. (19) and U. □

4. An Illustrative Example

We proceed with a simple example taken from [15] to illustrate our results. We consider the system (1) with parameters

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; W = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \alpha \in \mathbb{R}^2 \;\middle|\; \alpha \in [-1, 1] \right\}.$$

The constraint sets are given by $U = [-100, 100]$ and $X = \{x \in \mathbb{R}^2 \mid Hx \leq h_0\}$ with

$$H = \begin{bmatrix} 1 & 1 \\ -3 & 1 \\ 0 & -1 \end{bmatrix}, \quad h_0 = \begin{bmatrix} 100 \\ -50 \\ -26 \end{bmatrix}.$$

For this particular example we are able to analytically compute the set iterations $(R_i)_{i\in\mathbb{Z}_{\geq 0}}$ defined in (6). Specifically, the sets $(R_i)_{i\in\mathbb{Z}_{\geq 0}}$ and W are polytopes and we follow the approach in [12, Sec. 3.3] to compute $\mathrm{pre}(R_i)$ in terms of the Pontryagin set difference $R_i \sim W = \{x \in R_i \mid x + W \subseteq R_i\}$, i.e.,

$$\mathrm{pre}(R_i) = \{x \in \mathbb{R}^2 \mid \exists_{u\in U} Ax + Bu \in (R_i \sim W)\}.$$

For $R_0 = X$, we apply [23, Thm. 2.4], and obtain the set difference by $R_0 \sim W = \{x \in \mathbb{R}^2 \mid Hx \leq h_0'\}$ with

$$h_0' = [98,\; -52,\; -27]^\top$$

and $\mathrm{pre}(R_0)$ follows simply by projecting the polytope

$$\left\{ (x, u) \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} HA & HB \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} h_0' \\ 100 \\ 100 \end{bmatrix} \right\}$$

onto its first two coordinates. After the intersection of $\mathrm{pre}(R_0)$ with R_0 we obtain $R_1 = \{x \in \mathbb{R}^2 \mid Hx \leq h_1\}$ with

$$h_1 = \left[100,\; -50,\; -26 - \tfrac{1}{3}\right]^\top.$$

We repeat this computation and obtain the sequence of sets by $R_i = \{x \in \mathbb{R}^2 \mid Hx \leq h_i\}$ with

$$h_i = \left[100,\; -50,\; -25 - \sum_{j=0}^{i} \tfrac{1}{3^i}\right]^\top$$

whose limit is given by $R(X) = \{x \in \mathbb{R}^2 \mid Hx \leq h\}$ with

$$h = [100,\; -50,\; -26.5]^\top.$$

The boundary of the maximal robust controlled invariant set $R(X)$ is illustrated in Figure 2 and 3 by the dotted line.

Note that $R(X)$ is not finitely determined, nor does X contain the origin in its interior. Hence, it is not possible to apply any of the methods in [4, 15–20], to invariantly approximate the maximal robust controlled invariant set. In the following we apply the results from Sections 2 and 3 to compute outer and inner invariant approximations of $R(X)$.

Outer approximation. We start by solving the linear program (14) to determine the constant $c = 2$ which satisfies (13). The δ-constraint i-step null controllable sets N_j^δ for $j \in [1;2]$ are illustrated in Figure 1. From the previous consideration it is straightforward to see that $R_i \subseteq R_{i+2} + \frac{4}{3^{i+2}}\mathbb{B}$ holds

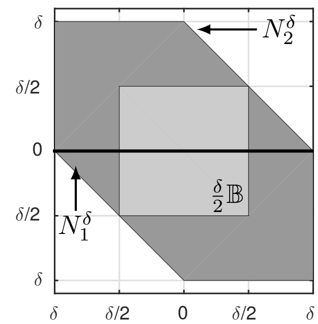

FIGURE 1. The δ-constraint 1-step (thick black bar) and 2-step (dark gray polytope) null controllable sets N_j^δ containing the ball $\frac{\delta}{2}\mathbb{B}$ (light gray box).

for all $i \in \mathbb{Z}_{\geq 0}$. Hence, in each iteration the stopping parameter is given by $\varepsilon = 4/3^{i+2}$. We illustrate the robust controlled invariant set defined in (15) for $i = 0$ and $i = 3$ relative to $R(X)$ in Figure 2. For $i = 3$, $\delta = 8/243$ and R in Figure 2 is indistinguishable form $R(X)$.

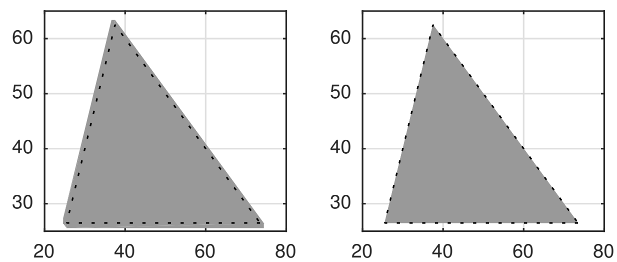

FIGURE 2. Invariant outer approximations of $R(X)$ given according to (15) for $i = 0$ (left) and $i = 3$ (right). The dotted line indicates $R(X)$.

Inner approximation. In order to obtain an inner approximation of $R(X)$, we compute the sequence of sets $(R_i^\rho)_{i \in \mathbb{Z}_{\geq 0}}$ defined in (16). Similar as before, we compute $\text{pre}_\rho(R_i^\rho)$ by using the Pontryagin set difference, i.e.,

$$\text{pre}_\rho(R_i^\rho) = \{x \in \mathbb{R}^2 \mid \exists_{u \in U} Ax + Bu \in (R_i \sim (W + \rho\mathbb{B}))\}.$$

We apply again [23, Thm. 2.4] to compute $R_i \sim (W + \rho\mathbb{B})$. Two invariant inner approximations of $R(X)$ with parameters $\rho = 1$ and $\rho = 1/10$ are illustrated in Figure 3.

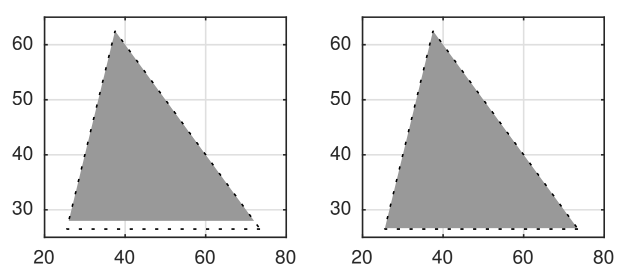

FIGURE 3. Invariant inner approximations of $R(X)$ with parameters $\rho = 1$ (left) and $\rho = 1/10$ (right). The dotted line indicates $R(X)$.

All the computations are conducted with MATLAB using the freely available `Multi-Parametric Toolbox` `http://people.ee.ethz.ch/~mpt/3/`.

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Appendix

Notation and Terminology. We use $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ to denote the set of natural numbers, integers and real numbers, respectively. We annotate those symbols with subscripts to restrict those sets in the obvious way, e.g. $\mathbb{R}_{>0}$ denotes the positive real numbers and $\mathbb{N} = \mathbb{Z}_{\geq 1}$. Given a set X and $n \in \mathbb{N}$ we use X^n to denote the n-fold Cartesian product of X with itself, i.e., $X^n = X \times \cdots \times X$. We use $\mathbb{R}^{n\times m}$, with $n, m \in \mathbb{N}$, to denote the vector space of real matrices with n rows and m columns.

For $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a \leq b$, we denote the closed, open and half-open intervals in $\mathbb{R} \cup \{\pm\infty\}$ by $[a, b]$, $]a, b[$, $[a, b[$, and $]a, b]$, respectively. For $a, b \in \mathbb{N} \cup \{\pm\infty\}$ and $a \leq b$, we use $[a; b]$, $]a; b[$, $[a; b[$, and

$]a;b]$ to denote the corresponding intervals in $\mathbb{N} \cup \{\pm\infty\}$. In $\mathbb{R}^n$, the relations $<, \leq, \geq, >$ are defined component-wise, e.g. $a < b$ iff $a_i < b_i$ for all $i \in [1;n]$.

$f\colon X \rightrightarrows Y$ denotes a *set-valued map* of X into Y, whereas $f\colon X \to Y$ denotes an ordinary map; see [11]. If f is set-valued, then f is *strict* if $f(x) \neq \varnothing$ for every $x \in X$. Given $f : X \rightrightarrows Y$ or $f : X \to Y$, the restriction of f to a subset $M \subseteq X$ is denoted $f|_M$. The set of maps $X \to Y$ is denoted Y^X, e.g. the set of functions $\xi : [0;t[\to X$, for fixed $t \in \mathbb{N}$, is denoted by $X^{[0;t[}$.

Given two sets $Q, P \subseteq \mathbb{R}^n$, we define the Minkowski set addition by $Q+P = \{y \in \mathbb{R}^n \mid \exists_{q\in Q}, \exists_{p\in P}\ y = q + p\}$. If $Q = \{q\}$, we slightly abuse notation and use $q + P = \{q\} + P$. For $\lambda \in \mathbb{R}_{\geq 0}$ we define $\lambda P = \{x \in \mathbb{R}^n \mid \exists_{p\in P}\ x = \lambda p\}$.

We use $|x|$ to denote the infinite norm of $x \in \mathbb{R}^n$ and $\mathbb{B}^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ denotes the unit ball in $\mathbb{R}^n$ centered at the origin. We drop the superscript, if the dimension is clear from the context. The Hausdorff distance between two sets $Q, P \subseteq \mathbb{R}^n$ is defined by $d_H(Q,P) = \inf\{\eta \in \mathbb{R}_{\geq 0} \mid P \subseteq Q + \eta\mathbb{B} \wedge Q \subseteq P + \eta\mathbb{B}\}$.

A *polyhedron* P is given by a matrix $H \in \mathbb{R}^{p\times n}$ and vector $h \in \mathbb{R}^p$ with $P = \{x \in \mathbb{R}^n \mid Hx \leq h\}$. A bounded polyhedron is called *polytope*.

Let $(R_i)_{i\in\mathbb{Z}_{\geq 0}}$ be a sequence of sets in $\mathbb{R}^n$. The *outer* limit and the *inner* limit are given by the sets

$$\limsup_{i\to\infty} R_i = \{x \in \mathbb{R}^n \mid \liminf_{i\to\infty} d(x, R_i) = 0\}$$
$$\liminf_{i\to\infty} R_i = \{x \in \mathbb{R}^n \mid \limsup_{i\to\infty} d(x, R_i) = 0\}.$$

If the outer and inner limits are equal, we say the *limit* exists and $\lim_{i\to\infty} R_i := \limsup_{i\to\infty} R_i = \liminf_{i\to\infty} R_i$. See [11, Ex. 4.2].

Consider $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ with $m \leq n$. We say that (A, B) is *controllable* if the *controllability matrix* $\mathcal{C} = [B,\ AB \dots A^{n-1}B]$ has full rank, see e.g. [14].

Proof of Theorem 2. We use the following lemma which is derived in [12] with the identity (2.15).

Lemma 3. *Consider the system* (1) *and the feedbacks* $\mathcal{F}(U)$ *for some* $U \subseteq \mathbb{R}^m$. *Let* $(R_i)_{i\in\mathbb{Z}_{\geq 0}}$ *be defined in* (6) *for some* $X \subseteq \mathbb{R}^n$. *Then* $x \in R_i$ *iff* $\exists_{\mu\in\mathcal{F}(U)}\forall_{\xi\in\mathcal{B}_{x,\mu}}\forall_{t\in[0;i]}\ \xi(t) \in X$.

Proof of Theorem 2. Let us first show $R(X) \subseteq \lim_{i\to\infty} R_i$. Let $x \in R(X)$. Since there exists $\mu \in \mathcal{F}(U)$ so that for all $\xi \in \mathcal{B}_{x,\mu}$ and $t \in \mathbb{Z}_{\geq 0}$ we have $\xi(t) \in X$, we see that $x \in R_i$ for all $i \in \mathbb{Z}_{\geq 0}$ and hence, $x \in \lim_{i\to\infty} R_i$.

We proceed to show $R^* := \lim_{i\to\infty} R_i \subseteq R(X) \subseteq X$. In particular, we show that for every $x \in R^*$ there exists $u \in U$ so that $Ax + Bu + W \subseteq R^*$ holds. Then we can easily derive a feedback μ so that for all $x \in R^*$, $\xi \in \mathcal{B}_{x,\mu}$ and $t \in \mathbb{Z}_{\geq 0}$ we have $\xi(t) \in X$ (see [10, Prop. 1, ii)]) which implies $R^* \subseteq R(X)$.

Let $x \in R^*$. By the definition of the limit, see [11, Def. 4.1] there exists a sequence $(x_i)_{i\in\mathbb{Z}_{\geq 0}}$ in $\mathbb{R}^n$ that converges to x with $x_i \in R_i \subseteq X$ for all $i \geq i'$ with $i' \in \mathbb{Z}_{\geq 0}$ sufficiently large. Since X is closed, it is clear that $x \in X$ and hence $R^* \subseteq X$. Let $(u_i)_{i\in\mathbb{Z}_{\geq 0}}$ be a sequence in U so that $Ax_i + Bu_i + W \subseteq R_{i-1}$ for all $i \geq i'$. Since U is compact we can assume w.l.o.g. that $(u_i)_{i\in\mathbb{Z}_{\geq 0}}$ converges to some $u \in U$, otherwise we restrict our analysis to a convergent subsequence. We are going to show that $Ax + Bu + W \subseteq R^*$. Let $x' \in Ax + Bu + W$. Since $(Ax_i + Bu_i)_{i\in\mathbb{Z}_{\geq 0}}$ converges to $Ax + Bu$, we see that there exists a sequence $x'_i \in Ax_i + Bu_i + W \subseteq R_{i-1}$ that converges to x'. It follows that there exists a (sub)sequence $(x'_i)_{i\in\mathbb{Z}_{\geq 0}}$ that converges to x' with $x'_i \in R_{i-1}$ for all $i \geq i'$. Again using the definition of the limit, we see that $x' \in R^*$, which shows $Ax + Bu + W \subseteq R^*$ and thereby, completes the proof.