

# NON-ARCHIMEDEAN VALUATIONS OF EIGENVALUES OF MATRIX POLYNOMIALS

MARIANNE AKIAN, RAVINDRA BAPAT, AND STÉPHANE GAUBERT

ABSTRACT. We establish general weak majorization inequalities, relating the leading exponents of the eigenvalues of matrices or matrix polynomials over the field of Puiseux series with the tropical analogues of eigenvalues. We also show that these inequalities become equalities under genericity conditions, and that the leading coefficients of the eigenvalues are determined as the eigenvalues of auxiliary matrix polynomials.

## 1. INTRODUCTION

**1.1. Non-archimedean valuations and tropical geometry.** A non-archimedean valuation  $\nu$  on a field  $\mathbb{K}$  is a map  $\mathbb{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  such that

$$(1a) \quad \nu(a) = +\infty \iff a = 0$$

$$(1b) \quad \nu(a + b) \geq \min(\nu(a), \nu(b))$$

$$(1c) \quad \nu(ab) = \nu(a) + \nu(b) .$$

These properties imply that  $\nu(a + b) = \min(\nu(a), \nu(b))$  for  $a, b \in \mathbb{K}$  such that  $\nu(a) \neq \nu(b)$ . Therefore, the map  $\nu$  is almost a morphism from  $\mathbb{K}$  to the *min-plus* or *tropical* semifield  $\mathbb{R}_{\min}$ , which is the set  $\mathbb{R} \cup \{+\infty\}$ , equipped with the addition  $(a, b) \mapsto \min(a, b)$  and the multiplication  $(a, b) \mapsto a + b$ . A basic example of field with a non-archimedean valuation is the field of complex Puiseux series, with the valuation which takes the leading (smallest) exponent of a series. The images by a non-archimedean valuation of algebraic subsets of  $\mathbb{K}^n$  are known as *non-archimedean amoebas*. The latter have a combinatorial structure which is studied in tropical geometry [IMS07, MS15]. For instance, Kapranov's theorem shows that the closure of the image by a non-archimedean valuation of an algebraic hypersurface over an algebraically closed field is a tropical hypersurface, i.e., the non-differentiability locus of a convex polyhedral function, see [EKL06]. This generalizes the characterization of the leading exponents of the different branches of an algebraic curve in terms of the slopes of the Newton polygon, which is part of the classical Newton-Puiseux theorem.

**1.2. Main results.** In the present paper, we consider the eigenproblem over the field of complex Puiseux series and related fields of functions. Our aim is to relate the images of the eigenvalues by the non-archimedean valuation with certain easily computable combinatorial objects called *tropical eigenvalues*.

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The first main result of the present paper, Theorem 4.4, shows that the sequence of valuations of the eigenvalues of a matrix  $\mathcal{A} \in \mathbb{K}^{n \times n}$  is weakly (super) majorized by the sequence of (algebraic) tropical eigenvalues of the matrix obtained by applying the valuation to the entries of  $\mathcal{A}$ . Next, we show that the same majorization inequality holds under more general circumstances. In particular, we consider in Theorem 5.2 and Corollary 5.3 a relaxed definition of the valuation, in the spirit of large deviations theory, assuming that the entries of the matrix are functions of a small parameter  $\epsilon$ . We do not require these functions to have a Puiseux series type expansion, but assume that they have some mild form of first order asymptotics. Moreover, the results apply to a lower bound of the valuation of the entries of  $\mathcal{A}$ .

Then, in Section 7, we assume that the entries of  $\mathcal{A}$  satisfy

$$\mathcal{A}_{ij} = a_{ij}\epsilon^{A_{ij}} + o(\epsilon^{A_{ij}}) ,$$

for some scalars  $a_{ij} \in \mathbb{C}$  and  $A_{ij} \in \mathbb{R} \cup \{+\infty\}$ , as  $\epsilon$  tends to 0. When  $a_{ij} = 0$ , this reduces to  $\mathcal{A}_{ij} = o(\epsilon^{A_{ij}})$ , so that valuations are partially known: only a lower bound is known. Applying Corollary 5.3, majorization inequalities are derived in Theorem 7.1. The assumption of Section 7 is satisfied of course if the entries of  $\mathcal{A}$  are absolutely converging Puiseux series, or more generally, if these entries belong to a polynomially bounded o-minimal structure [vdD99, Ale13]. We show in Theorem 7.4 that the majorization inequalities of Theorem 7.1 become equalities for generic values of the entries  $a_{ij}$ . The proof of the latter theorem relies on some variations of the Newton-Puiseux theorem, which we state as Theorems 6.1 and 6.2. The latter results only require a partial information on the asymptotics of the coefficients of the polynomial. The particular case where this partial information contains at least the first order asymptotics of all the coefficients of the polynomial was considered in [Die68]. However, here we show that a partial information on the first order asymptotics, giving an outer approximation of a Newton polytope, allow one to derive a partial information on the roots. The latter idea goes back at least to [Mon34] in the context of archimedean valuations.

The valuation only gives an information on the *leading exponent* of Puiseux series. Our aim in Section 8 is to refine this information, by characterizing also the coefficients  $\lambda_i \in \mathbb{C}$  of the leading monomials of the asymptotic expansions of the different eigenvalues  $\mathcal{L}^i$  of the matrix  $\mathcal{A}$ ,

$$\mathcal{L}^i \sim \lambda_i \epsilon^{\Lambda_i}, \quad 1 \leq i \leq n .$$

As a byproduct, we shall end up with an explicit form, easily checkable, of the genericity conditions under which the majorization inequalities become equalities. To this end, it is necessary to embed the standard eigenproblem in the wider class of matrix polynomial eigenproblems. Theorems 8.2 and 8.3 show in particular that the coefficients  $\lambda_i$  are the eigenvalues of certain auxiliary matrix polynomials which are determined only by the leading exponents and leading coefficients of the entries of  $\mathcal{A}$ . These polynomials are constructed from the optimal dual variables of an optimal assignment problem, arising from the evaluation of the tropical analogue of the characteristic polynomial.

### 1.3. Application to perturbation theory and discussion of related work.

The present results apply to perturbation theory [Kat95, Bau85], and specially, to the singular case in which a matrix with multiple eigenvalues is perturbed. The latter situation is the object of the theory developed by Višik and Ljusternik [VL60]

and completed by Lidskiĭ [Lid65], see [MBO97] for a survey. The goal of this theory is to give a direct characterization of the exponents, without computing the Newton polytope of the characteristic polynomial. The theorem of [Lid65] solves this problem under some genericity assumptions, requiring the non-vanishing of certain Schur complements. The question of solving degenerate instance of Lidskiĭ's theorem has been considered in particular, by Ma and Edelman [ME98] and Najman [Naj99], and also by Moro, Burke and Overton in [MBO97]. Theorems 8.2 and 8.3 generalize the theorem of Lidskiĭ, as they allow one to solve degenerate instances in which the Schur complements needed in Lidskiĭ's construction are no longer defined.

The present train of thoughts originates from a work of Friedland [Fri86], who showed that a certain deformation of the Perron root of a nonnegative matrix, in terms of Hadamard powers, converges to the maximal circuit mean of the matrix, a.k.a., the maximal tropical eigenvalue. Then, in [ABG98], we showed that the limiting Perron eigenvector, along the same deformation, can also be characterized by tropical means, under a nondegeneracy condition. An early version of the present Theorems 7.1 and 7.4 appeared as Theorem 3.8 of the authors' preprint [ABG04a]. There, we also gave a generalization of the theorem of Lidskiĭ, in which the exponents of the first order asymptotics of the eigenvalues are given by the tropical eigenvalues of certain tropical Schur complements and their coefficients are given by some associated usual Schur complements like in the true Lidskiĭ theorem. However, some singular cases remained, see for instance Example 8.4, motivating the introduction of matrix polynomial eigenproblems in further works. The results of Theorems 8.2 and 8.3 were announced without proof in the note [ABG04b]. Therefore, the present article is, for some part, a survey of results which have not appeared previously in the form of a journal article. It also provides a general presentation of eigenvalues in terms of valuation theory, with several new results or refinements, like the general majorization inequality for the eigenvalues of matrix polynomials, Theorem 8.1. The interest of this presentation is that it explains better the relation between the results obtained here, or in eigenvalues perturbation theory, for the non-archimedean valuations of matrix entries and eigenvalues, with their analogues for archimedean valuations, like the modulus map, or for some generalization of the notion of archimedean valuation which includes in particular matrix norms.

Indeed, the latter works have motivated a more recent work by Akian, Gaubert and Marchesini, who showed in [AGM14], that one form of the theorem of Friedland concerning the Perron or dominant root carries over to all eigenvalues: the sequence of moduli of all eigenvalues of a matrix is weakly log-majorized by the sequence of tropical eigenvalues, up to certain combinatorial coefficients. Therefore, the result there can be thought of as an analogue of Theorem 4.4 or 7.1 for the modulus archimedean valuation. Also, the results of [ABG04a, ABG04b] have been at the origin of the application of tropical methods by Gaubert and Sharify to the numerical computation and estimation of eigenvalues [GS09, Sha11], based on the norms of matrix polynomial coefficients. This is a subject of current interest, with work by Akian, Bini, Gaubert, Hammarling, Noferini, Sharify, and Tisseur [BNS13, HMT13, NST14, AGS13].

The present results provide a further illustration of the role of tropical algebra in asymptotic analysis, which was recognized by Maslov [Mas73, Ch. VIII].

He observed that WKB-type or large deviation type asymptotics lead to limiting equations, like Hamilton-Jacobi equations, of a tropical nature. This observation is at the origin of idempotent analysis [MS92, DKM92, KM97, LMS01]. The same deformation has been identified by Viro [Vir01], in relation with the patchworking method he developed for real algebraic curves.

Note that all the perturbation results described or recalled above study sufficient conditions for the possible computation of first order asymptotics of some roots when an information on the first order asymptotics of the data is only available. However, when all the Puiseux series expansion of the data is known, Puiseux theorem allows one to compute without any condition all the Puiseux series expansion. In the context of matrix polynomials, Murota [Mur90], gave an algorithm to compute the Puiseux series expansions of the eigenvalues of a matrix polynomial depending polynomially in the parameter  $\epsilon$ , avoiding the explicit computation of the characteristic polynomial. As for Theorems 8.2 and 8.3, his algorithm relies on a parametric optimal assignment problem.

The present work builds on tropical spectral theory. It has been inspired by the analogy with nonnegative matrix theory, of which Hans Schneider was a master. We gratefully acknowledge our debt to him.

## 2. MIN-PLUS POLYNOMIALS AND NEWTON POLYGONS

We first recall some elementary facts concerning formal polynomials and polynomial functions over the min-plus semifield, and their relation with Newton polygons.

The *min-plus semifield*,  $\mathbb{R}_{\min}$ , is the set  $\mathbb{R} \cup \{+\infty\}$  equipped with the addition  $(a, b) \mapsto \min(a, b)$ , denoted  $a \oplus b$ , and the multiplication  $(a, b) \mapsto a + b$ , denoted  $a \otimes b$  or  $ab$ . We shall denote by  $\mathbb{0} = +\infty$  and  $\mathbb{1} = 0$  the zero and unit elements of  $\mathbb{R}_{\min}$ , respectively. The familiar algebraic constructions and conventions carry out to the min-plus context with obvious changes. For instance, if  $A, B$  are matrices of compatible dimensions with entries in  $\mathbb{R}_{\min}$ , we shall denote by  $AB$  the matrix product with entries  $(AB)_{ij} = \bigoplus_k A_{ik}B_{kj} = \min_k(A_{ik} + B_{kj})$ , we denote by  $A^k$  the  $k$ th min-plus matrix power of  $A$ , etc. Moreover, if  $x \in \mathbb{R}_{\min} \setminus \{\mathbb{0}\}$ , then we will denote by  $x^{-1}$  the inverse of  $x$  for the  $\otimes$  law, which is nothing but  $-x$ , with the conventional notation. The reader seeking information on the min-plus semifield may consult [CG79, MS92, BCOQ92, KM97, ABG13, But10].

We denote by  $\mathbb{R}_{\min}[Y]$  the semiring of formal polynomials with coefficients in  $\mathbb{R}_{\min}$  in the indeterminate  $Y$ : a *formal polynomial*  $P \in \mathbb{R}_{\min}[Y]$  is nothing but a sequence  $(P_k)_{k \in \mathbb{N}} \in \mathbb{R}_{\min}^{\mathbb{N}}$  such that  $P_k = \mathbb{0}$  for all but finitely many values of  $k$ . Formal polynomials are equipped with the entry-wise sum,  $(P \oplus Q)_k = P_k \oplus Q_k$ , and the Cauchy product,  $(PQ)_k = \bigoplus_{0 \leq i \leq k} P_i Q_{k-i}$ . As usual, we denote a formal polynomial  $P$  as a formal sum,  $P = \bigoplus_{k=0}^{\infty} P_k Y^k$ . We also define the *degree* and *valuation* of  $P$ :  $\deg P = \sup\{k \in \mathbb{N} \mid P_k \neq \mathbb{0}\}$ ,  $\text{val } P = \inf\{k \in \mathbb{N} \mid P_k \neq \mathbb{0}\}$  ( $\deg P = -\infty$  and  $\text{val } P = +\infty$  if  $P = \mathbb{0}$ ). To any  $P \in \mathbb{R}_{\min}[Y]$ , we associate the *polynomial function*  $\widehat{P} : \mathbb{R}_{\min} \rightarrow \mathbb{R}_{\min}$ ,  $y \mapsto \widehat{P}(y) = \bigoplus_{k=0}^{\infty} P_k y^k$ , that is, with the usual notation:

$$(2) \quad \widehat{P}(y) = \min_{k \in \mathbb{N}} (P_k + ky) .$$

Thus,  $\widehat{P}$  is concave, piecewise affine with integer slopes. We denote by  $\mathbb{R}_{\min}\{Y\}$  the semiring of polynomial functions  $\widehat{P}$ . The morphism  $\mathbb{R}_{\min}[Y] \rightarrow \mathbb{R}_{\min}\{Y\}$ ,  $P \mapsto \widehat{P}$

is not injective, as it is essentially a specialization of the classical Fenchel transform over  $\mathbb{R}$ , which reads:

$$\mathcal{F} : \overline{\mathbb{R}}^{\mathbb{R}} \rightarrow \overline{\mathbb{R}}^{\mathbb{R}}, \mathcal{F}(g)(y) = \sup_{x \in \mathbb{R}} (xy - g(x)) ,$$

where  $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ . Indeed, for all  $y \in \mathbb{R}$ ,  $\widehat{P}(y) = -\mathcal{F}(P)(-y)$ , where the function  $k \mapsto P_k$  from  $\mathbb{N}$  to  $\mathbb{R}_{\min}$  is extended to a function

$$(3) \quad P : \mathbb{R} \rightarrow \mathbb{R}_{\min}, x \mapsto P(x), \text{ with } P(x) = \begin{cases} P_k & \text{if } x = k \in \mathbb{N} , \\ +\infty & \text{otherwise} \end{cases}$$

The following result of Cuninghame-Green and Meijer gives a min-plus analogue of the fundamental theorem of algebra.

**Theorem 2.1** ([CGM80]). *Any polynomial function  $\widehat{P} \in \mathbb{R}_{\min}\{\mathbb{Y}\}$  can be factored in a unique way as*

$$(4) \quad \widehat{P}(y) = P_n(y \oplus c_1) \cdots (y \oplus c_n) ,$$

with  $c_1 \leq \cdots \leq c_n$ .

The  $c_i$  will be called the *roots* of  $\widehat{P}$ . The *multiplicity* of the root  $c$  is the cardinality of the set  $\{j \in \{1, \dots, n\} \mid c_j = c\}$ . We shall denote by  $\mathbf{R}(\widehat{P})$  the sequence of roots:  $\mathbf{R}(\widehat{P}) = (c_1, \dots, c_n)$ . By extension, if  $P \in \mathbb{R}_{\min}[\mathbb{Y}]$  is a formal polynomial, we will call *roots of  $P$*  the roots of  $\widehat{P}$ , so  $\mathbf{R}(P) := \mathbf{R}(\widehat{P})$ . The next properties also follow from [CGM80]; they show that the definition of the roots in Theorem 2.1 is a special case of the notion of a tropical hypersurface defined as the nondifferentiability locus of a tropical polynomial [IMS07].

**Proposition 2.2** (See [CGM80]). *The roots  $c \in \mathbb{R}$  of a formal polynomial  $P \in \mathbb{R}_{\min}[\mathbb{Y}]$  are exactly the points at which the function  $\widehat{P}$  is not differentiable. The multiplicity of a root  $c \in \mathbb{R}$  is equal to the variation of slope of  $\widehat{P}$  at  $c$ ,  $\widehat{P}'(c^-) - \widehat{P}'(c^+)$ . Moreover,  $0$  is a root of  $P$  if, and only if,  $\widehat{P}'(0^-) := \lim_{c \rightarrow +\infty} \widehat{P}'(c) \neq 0$ . In that case  $\widehat{P}'(0^-)$  is the multiplicity of  $0$ , and it coincides with  $\text{val } P$ .  $\square$*

Legendre-Fenchel duality allows one to relate the tropical roots to the slopes of Newton polygons. To see this, denote by  $\text{vex } f$  the convex hull of a map  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ , and denote by  $\overline{P}$  the formal polynomial whose sequence of coefficients is obtained by restricting to  $\mathbb{N}$  the map  $\text{vex } P : k \mapsto \overline{P}_k := (\text{vex } P)(k)$ , for  $k \in \mathbb{N}$ . The function  $\text{vex } P$  is finite on the interval  $[\text{val } P, \text{deg } P]$ . Also, it should be noted that the graph of  $\text{vex } P$  is the standard *Newton polygon* associated to the sequence of points  $(k, P_k)$ ,  $k \in [\text{val } P, \text{deg } P]$ .

**Theorem 2.3** ([BCOQ92, Th. 3.43, 1 and 2]). *A formal polynomial of degree  $n$ ,  $P \in \mathbb{R}_{\min}[\mathbb{Y}]$ , satisfies  $P = \overline{P}$  if, and only if, there exist  $c_1 \leq \cdots \leq c_n \in \mathbb{R}_{\min}$  such that*

$$P = P_n(\mathbb{Y} \oplus c_1) \cdots (\mathbb{Y} \oplus c_n) .$$

The  $c_i$  are unique and given, by:

$$(5) \quad c_i = \begin{cases} P_{n-i}(P_{n-i+1})^{-1} & \text{if } P_{n-i+1} \neq 0 \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } i = 1, \dots, n .$$

The following standard observation relates the tropical roots with the Newton polygon.

**Proposition 2.4.** *The roots  $c \in \mathbb{R}$  of a formal polynomial  $P \in \mathbb{R}_{\min}[\mathbb{Y}]$  coincide with the opposites of the slopes of the affine parts of  $\text{vex } P : [\text{val } P, \text{deg } P] \rightarrow \mathbb{R}$ . The multiplicity of a root  $c \in \mathbb{R}$  coincides with the length of the interval in which  $\text{vex } P$  has slope  $-c$ .  $\square$*

*Remark 2.5.* The duality between tropical roots and slopes of the Newton polygon in Proposition 2.4 is a special case of the Legendre–Fenchel duality formula for subdifferentials:  $-c \in \partial(\text{vex } P)(x) \Leftrightarrow x \in \partial\mathcal{F}(P)(-c) \Leftrightarrow x \in \partial^+\widehat{P}(c)$  where  $\partial$  and  $\partial^+$  denote the subdifferential and superdifferential, respectively [Roc70, Th. 23.5].

The above notions are illustrated in Figure 1, where we consider the formal min-plus polynomial  $P = \mathbb{Y}^3 \oplus 5\mathbb{Y}^2 \oplus 6\mathbb{Y} \oplus 13$ . The map  $j \mapsto P_j$ , together with the map  $\text{vex } P$ , are depicted at the left of the figure, whereas the polynomial function  $\widehat{P}$  is depicted at the right of the figure. We have  $\bar{P} = \mathbb{Y}^3 \oplus 3\mathbb{Y}^2 \oplus 6\mathbb{Y} \oplus 13 = (\mathbb{Y} \oplus 3)^2(\mathbb{Y} \oplus 7)$ . Thus, the roots of  $P$  are 3 and 7, with respective multiplicities 2 and 1. The roots are visualized at the right of the figure, or alternatively, as the opposite of the slopes of the two line segments at the left of the figure. The multiplicities can be read either on the map  $\widehat{P}$  at the right of the figure (the variation of slope of  $\widehat{P}$  at points 3 and 7 is 2 and 1, respectively), or on the map  $\text{vex } P$  at the left of the figure (as the respective horizontal widths of the two segments). We conclude this section by

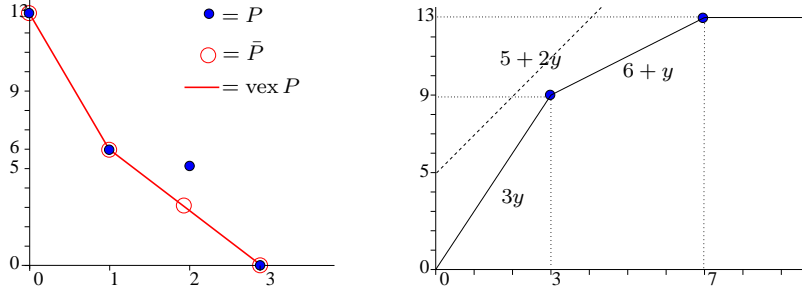


FIGURE 1. The Newton polygon of the formal min-plus polynomial  $P = \mathbb{Y}^3 \oplus 5\mathbb{Y}^2 \oplus 6\mathbb{Y} \oplus 13$  (left) and the associated polynomial function  $\widehat{P}$  (right).

two technical results.

**Lemma 2.6.** *Let  $P = \bigoplus_{i=0}^n P_i \mathbb{Y}^i \in \mathbb{R}_{\min}[\mathbb{Y}]$  be a formal polynomial of degree  $n$ . Then,  $\mathbf{R}(P) = (c_1 \leq \dots \leq c_n)$  if, and only if,  $P \geq P_n(\mathbb{Y} \oplus c_1) \cdots (\mathbb{Y} \oplus c_n)$  and*

$$(6) \quad P_{n-i} = P_n c_1 \cdots c_i \quad \text{for all } i \in \{0, n\} \cup \{i \in \{1, \dots, n-1\} \mid c_i < c_{i+1}\} .$$

*In particular,  $P_{n-i} = \bar{P}_{n-i}$  holds for all  $i$  as in (6), and  $\bar{P} = P_n(\mathbb{Y} \oplus c_1) \cdots (\mathbb{Y} \oplus c_n)$ .*

*Proof.* We first prove the “only if” part. If  $\mathbf{R}(P) = (c_1 \leq \dots \leq c_n)$ , then  $\bar{P} = \bar{P}_n(\mathbb{Y} \oplus c_1) \cdots (\mathbb{Y} \oplus c_n)$  and, by Theorem 2.3,  $\bar{P}_{n-i} = \bar{P}_n c_1 \cdots c_i$  for all  $i = 1, \dots, n$ . Recall that  $P$  defines a map  $x \mapsto P(x)$  by (3). By definition of  $\text{vex } P$ , the epigraph of  $\text{vex } P$ ,  $\text{epi } \text{vex } P$ , is the convex hull of the epigraph of  $P$ ,  $\text{epi } P$ . By a classical result [Roc70, Cor 18.3.1], if  $S$  is a set with convex hull  $C$ , any extreme point of  $C$  belongs to  $S$ . Let us apply this to  $S = \text{epi } P$  and  $C = \text{epi } \text{vex } P$ . Since  $c_i = \bar{P}_{n-i}(\bar{P}_{n-i+1})^{-1}$ , the piecewise affine map  $\text{vex } P$  changes its slope at any

point  $n-i$  such that  $c_i < c_{i+1}$ . Thus, any point  $(n-i, \text{vex } P(n-i))$  with  $c_i < c_{i+1}$  is an extreme point of  $\text{epi } \text{vex } P$ , which implies that  $(n-i, \text{vex } P(n-i)) \in \text{epi } P$ , i.e.,  $P_{n-i} \leq \text{vex } P(n-i) = \overline{P}_{n-i}$ . Since the other inequality is trivial by definition of the convex hull, we have  $P_{n-i} = \overline{P}_{n-i}$ . Obviously,  $P$  and  $\overline{P}$  have the same degree, which is equal to  $n$ , and they have the same valuation,  $k$ . Then,  $(n, \text{vex } P(n))$  and  $(k, \text{vex } P(k))$  are extreme points of  $\text{epi } \text{vex } P$ , and by the preceding argument,  $P_n = \overline{P}_n$ , and  $P_k = \overline{P}_k$ . Hence,  $P_0 = \overline{P}_0$ , if  $k = 0$ , and  $P_0 = \overline{P}_0 = +\infty$ , if  $k > 0$ . We have shown (6), together with the last statement of the lemma. Since  $\overline{P}_n = P_n$  and  $P \geq \overline{P}$ , we also obtain  $P \geq P_n(\mathbf{Y} \oplus c_1) \cdots (\mathbf{Y} \oplus c_n)$ .

For the “if” part, assume that  $P \geq P_n(\mathbf{Y} \oplus c_1) \cdots (\mathbf{Y} \oplus c_n)$  and that (6) holds. Since  $Q = P_n(\mathbf{Y} \oplus c_1) \cdots (\mathbf{Y} \oplus c_n)$  is convex, and the convex hull map  $P \mapsto \overline{P}$  is monotone, we must have  $\overline{P} \geq \overline{Q} = Q$ . Hence,  $P \geq \overline{P} \geq Q$  and since  $P_{n-i} = Q_{n-i}$  for all  $i$  as in (6), we must have  $P_{n-i} = \overline{P}_{n-i} = Q_{n-i}$ , thus  $\text{vex } P(n-i) = Q(n-i)$  at these  $i$ . Since  $\text{vex } P$  and  $Q$  are convex,  $Q$  is piecewise affine and  $Q(j) = \text{vex } P(j)$  for  $j$  at the boundary of the domain of  $Q$  and at all the  $j$  where  $Q$  changes of slope, we must have  $\text{vex } P = Q$ . Hence  $\overline{P} = Q$  and  $\mathbf{R}(P) = \mathbf{R}(\overline{P}) = \mathbf{R}(Q) = (c_1, \dots, c_n)$ .  $\square$

**Corollary 2.7.** *Let  $P = \bigoplus_{i=0}^n P_i \mathbf{Y}^i \in \mathbb{R}_{\min}[\mathbf{Y}]$  be a formal polynomial of degree  $n$ . Let  $c \in \mathbb{R}$  be a finite root of  $P$  with multiplicity  $m$ , and denote by  $m'$  the sum of the multiplicities of all the roots of  $P$  greater than  $c$  ( $+\infty$  comprised). Then,  $P_i = \overline{P}_i$  for both  $i = m'$  and  $i = m + m'$ ,  $\widehat{P}(c) = P_{m'} c^{m'} = P_{m+m'} c^{m+m'}$  and  $\widehat{P}(c) < P_i c^i$  for all  $1 \leq i < m'$  and  $m + m' < i \leq n$ .*

*Proof.* Let us denote  $\mathbf{R}(P) = (c_1 \leq \dots \leq c_n)$ . By definition of  $c$ ,  $m$  and  $m'$  we have  $m \geq 1$ ,  $m' \geq 0$ ,  $m + m' \leq n$ ,  $c = c_{n-m'-m+1} = \dots = c_{n-m'}$ ,  $c_{n-m'-m} < c$  if  $n - m' - m > 0$ , and  $c < c_{n-m'+1}$  if  $n - m' < n$ . By Lemma 6, this implies that for both  $i = m'$  and  $i = m + m'$ ,  $P_i = \overline{P}_i = P_n c_1 \cdots c_{n-i}$ . We also have  $\overline{P}_i = (\text{vex } P)(i) \leq P_i$ , we have  $P_i \geq (\text{vex } P)(i) = \overline{P}_i = P_n c_1 \cdots c_{n-i}$  for all  $i = 0, \dots, n$ . Moreover, by Theorem 2.1, we have  $\widehat{P}(c) = P_n(c \oplus c_1) \cdots (c \oplus c_n) = P_n c_1 \cdots c_{n-m'-m} c^{m+m'} = P_n c_1 \cdots c_{n-m'} c^{m'}$ . Hence,  $\widehat{P}(c) = P_{m+m'} c^{m+m'} = P_{m'} c^{m'}$ , and  $\widehat{P}(c) < P_n c_1 \cdots c_{n-i} c^i \leq P_i c^i$  for  $i < m'$  and for  $i > m + m'$ .  $\square$

### 3. TROPICAL EIGENVALUES

We now recall some classical results on tropical eigenvalues and characteristic polynomials.

The *permanent* of a matrix with coefficients in an arbitrary semiring  $(S, \oplus, \otimes)$  is defined by

$$\text{per}(A) = \bigoplus_{\sigma \in \mathfrak{S}_n} \bigotimes_{i=1}^n A_{i\sigma(i)} ,$$

where  $\mathfrak{S}_n$  is the set of permutations of  $[n] := \{1, \dots, n\}$ . In particular, for any matrix  $A \in \mathbb{R}_{\min}^{n \times n}$ ,

$$\text{per}(A) = \min_{\sigma \in \mathfrak{S}_n} |\sigma|_A ,$$

where for any permutation  $\sigma \in \mathfrak{S}_n$ , we define the *weight* of  $\sigma$  with respect to  $A$  as  $|\sigma|_A := A_{1\sigma(1)} + \dots + A_{n\sigma(n)}$ .

To any min-plus  $n \times n$  matrix  $A$ , we associate the (*directed*) graph  $G(A)$ , which has set of nodes  $[n]$  and an arc  $(i, j)$  if  $A_{ij} \neq 0$ , and the weight function which

associates the weight  $A_{ij}$  to the arc  $(i, j)$  of  $G(A)$ . In the sequel, we shall omit the word “directed” as all graphs will be directed. Then,  $\text{per}(A)$  is the value of an optimal assignment in this weighted graph. It can be computed in  $O(n^3)$  time using the Hungarian algorithm [Sch03, § 17]. We refer the reader to [BR97, § 2.4] or [Sch03, § 17] for more background on the optimal assignment problem and a discussion of alternative algorithms.

We define the *formal characteristic polynomial* of  $A$ ,

$$P_A := \text{per}(YI \oplus A) = \bigoplus_{\sigma \in \mathfrak{S}_n} \bigotimes_{i=1}^n (Y\delta_{i\sigma(i)} \oplus A_{i\sigma(i)}) \in \mathbb{R}_{\min}[Y] ,$$

where  $I$  is the identity matrix, and  $\delta_{ij} = \mathbb{1}$  if  $i = j$  and  $\delta_{ij} = 0$  otherwise. The formal polynomial  $P_A$  has degree  $n$  and its coefficients are given by  $(P_A)_k = \text{tr}_{n-k}^{\min}(A)$ , for  $k = 0, \dots, n-1$  and  $(P_A)_n = \mathbb{1}$ , where  $\text{tr}_k^{\min}(A)$  is the min-plus  $k$ -th trace of  $A$ :

$$(7) \quad \text{tr}_k^{\min}(A) := \bigoplus_{J \subset \{1, \dots, n\}, \#J=k} \left( \bigoplus_{\sigma \in \mathfrak{S}_J} \bigotimes_{j \in J} A_{j\sigma(j)} \right) ,$$

where  $\mathfrak{S}_J$  is the set of permutations of  $J$ . The associated min-plus polynomial function will be called the *characteristic polynomial function* of  $A$ , and its roots will be called the *(algebraic) eigenvalues* of  $A$ .

The algebraic eigenvalues of  $A$  (and so, its characteristic polynomial function) can be computed in  $O(n^4)$  time by the method of Burkard and Butkovič [BB03]. Gassner and Klinz [GK10] showed that this can be reduced to a  $O(n^3)$  time, using parametric optimal assignment techniques. However, it is not known whether the sequence of coefficients of the *formal characteristic polynomial*  $P_A$  can be computed in polynomial time.

The term *algebraic eigenvalue* is used here since unlike for matrices with real or complex coefficients, a root  $\lambda \in \mathbb{R}_{\min}$  of the characteristic polynomial of a  $n \times n$  min-plus matrix  $A$  may not satisfy  $Au = \lambda u$  for some  $u \in \mathbb{R}_{\min}^n$ . To avoid any confusion, we shall call a scalar  $\lambda$  with the latter property a *geometric eigenvalue*. The following statement and remarks collect some results in tropical spectral theory, which have been developed by several authors [CG79, Vor67, Rom67, GM77, CDQV83, MS92, BSvdD95, AGW05, BCGG09]. We refer the reader to [BCOQ92, But10] for more information. We say that a matrix  $A$  is *irreducible* if  $G(A)$  is strongly connected.

**Theorem 3.1** (See e.g. [But10]). *The minimal algebraic eigenvalue of a matrix  $A \in \mathbb{R}_{\min}^{n \times n}$  is given by*

$$(8) \quad \rho_{\min}(A) = \bigoplus_{k=1}^n \bigoplus_{i_1, \dots, i_k} (A_{i_1 i_2} \cdots A_{i_k i_1})^{\frac{1}{k}} ,$$

or equivalently, by the following expression called minimal circuit mean,

$$(9) \quad \min_{c \text{ circuit in } G(A)} \frac{|c|_A}{|c|} ,$$

where for all paths  $p = (i_0, i_1, \dots, i_k)$  in  $G(A)$ , we denote by  $|p|_A = A_{i_0 i_1} + \cdots + A_{i_{k-1} i_k}$  the weight of  $p$ , and by  $|p| = k$  its length, and the minimum is taken over all elementary circuits of  $G(A)$ .  $\square$



An important notion to be used in the sequel is the one of *critical circuit*, i.e., of circuit  $c = (i_1, i_2, \dots, i_k, i_1)$  of  $G(A)$  attaining the minimum in (9). The *critical graph* of  $A$  is the union of the critical circuits, that is the graph whose nodes and arcs belong to critical circuits. It is known that  $\rho_{\min}(A)$  is the minimal geometric eigenvalue of  $A$  and that the multiplicity of  $\rho_{\min}(A)$  as a geometric eigenvalue (i.e., the “dimension” of the associated eigenspace) coincides with the number of strongly connected components of the critical graph. Note also that, if  $A$  is irreducible,  $\rho_{\min}(A)$  is the unique geometric eigenvalue of  $A$ .

*Remark 3.2.* The multiplicity of  $\rho_{\min}(A)$ , as an algebraic eigenvalue, coincides with the term rank (i.e., the maximal number of nodes of a disjoint union of circuits) of the critical graph of  $A$ . This follows from the arguments of proof of Theorem 4.7 in [ABG04a].

#### 4. MAJORIZATION INEQUALITIES FOR VALUATIONS OF EIGENVALUES

The inequalities that we shall establish involve the notion of *weak majorization*, see [MO79] for background.

**Definition 4.1.** Let  $u, v \in \mathbb{R}_{\min}^n$ . Let  $u_{(1)} \leq \dots \leq u_{(n)}$  (resp.  $v_{(1)} \leq \dots \leq v_{(n)}$ ) denote the components of  $u$  (resp.  $v$ ) in increasing order. We say that  $u$  is *weakly (super) majorized* by  $v$ , and we write  $u \prec^w v$ , if the following conditions hold:

$$u_{(1)} \cdots u_{(k)} \geq v_{(1)} \cdots v_{(k)} \quad \forall k = 1, \dots, n .$$

The weak majorization relation is only defined in [MO79] for vectors of  $\mathbb{R}^n$ . Here, it is convenient to define this notion for vectors with infinite entries. We used the min-plus notation for homogeneity with the rest of the paper. The following lemma states a useful monotonicity property of the map which associates to a formal min-plus polynomial  $P$  its sequence of roots,  $R(P)$ .

**Lemma 4.2.** *Let  $P, Q \in \mathbb{R}_{\min}[X]$  be two formal polynomials of degree  $n$ . Then,*

$$(10) \quad P \geq Q \text{ and } P_n = Q_n \implies R(P) \prec^w R(Q) .$$

*Proof.* From  $P \geq Q$ , we deduce  $\bar{P} \geq \bar{Q}$ . Let  $R(P) = (c_1(P) \leq \dots \leq c_n(P))$  and  $R(Q) = (c_1(Q) \leq \dots \leq c_n(Q))$  denote the sequence of roots of  $P$  and  $Q$ , respectively. Using  $\bar{P} \geq \bar{Q}$ ,  $\bar{P}_n = P_n = Q_n = \bar{Q}_n$  and (5), we get  $c_1(P) \cdots c_k(P) = \bar{P}_{n-k}(\bar{P}_n)^{-1} \geq \bar{Q}_{n-k}(\bar{Q}_n)^{-1} = c_1(Q) \cdots c_k(Q)$ , for all  $k = 1, \dots, n$ , that is  $R(P) \prec^w R(Q)$ .  $\square$

Let  $\nu$  be a (non-archimedean) valuation on a field  $\mathbb{K}$ , i.e., a map  $\nu : \mathbb{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  satisfying the conditions (1) recalled in the introduction. We shall think of the images of  $\nu$  as elements of the min-plus semifield, writing  $\nu(ab) = \nu(a) \nu(b)$ . The main example of valuation considered here is obtained by considering the field of complex Puiseux series, with the valuation which takes the smallest exponent of a series. Recall that this field consists of the series of the form  $\sum_{k=K}^{\infty} a_k \epsilon^{k/s}$  with  $a_k \in \mathbb{C}$ ,  $K \in \mathbb{Z}$  and  $s \in \mathbb{N} \setminus \{0\}$ , in which case the smallest exponent is equal to  $K/s$  as soon as  $a_K \neq 0$ . The results of the present paper apply as well to formal series or to series that are absolutely convergent for a sufficiently small positive  $\epsilon$ .

The following proposition formulates in terms of tropical roots a well known property usually stated in terms of Newton polygons, see for instance [Bou89, Exer. VI.4.11]. It is a special case of a result proved in [EKL06] for non-archimedean

amoebas of hypersurfaces. We include a proof relying on Lemma 2.6 for the convenience of the reader, since we shall use the same argument in the sequel.

**Proposition 4.3** (See [EKL06, Th. 2.1.1]). *Let  $\mathbb{K}$  be an algebraically closed field with a (non-archimedean) valuation  $\nu$  and let  $\mathcal{P} = \sum_{k=0}^n \mathcal{P}_k \mathcal{Y}^k \in \mathbb{K}[\mathcal{Y}]$ , with  $\mathcal{P}_n = 1$ . Then, the valuations of the roots of  $\mathcal{P}$  (counted with multiplicities) coincide with the roots of the min-plus polynomial  $\nu(\mathcal{P}) := \bigoplus_{k=0}^n \nu(\mathcal{P}_k) \mathcal{Y}^k$ .*

*Proof.* Let  $\mathcal{Y}_1, \dots, \mathcal{Y}_n$  denote the roots of  $\mathcal{P}$ , ordered by nondecreasing valuation,  $c_i := \nu(\mathcal{Y}_i)$ , so that  $c_1 \leq \dots \leq c_n$ ,  $Q := \bigoplus_{k=0}^n c_1 \dots c_k \mathcal{Y}^{n-k}$ , and  $P := \nu(\mathcal{P})$ . Observe that  $P_n = Q_n = \mathbb{1}$ , and  $Q = (\mathcal{Y} \oplus c_1) \dots (\mathcal{Y} \oplus c_n)$ . Since  $\mathcal{P}_{n-k} = (-1)^k \sum_{i_1 < \dots < i_k} \mathcal{Y}_{i_1} \dots \mathcal{Y}_{i_k}$ , we get  $\nu(\mathcal{P}_{n-k}) \geq \nu(\mathcal{Y}_1 \dots \mathcal{Y}_k) = c_1 \dots c_k$ , and so  $P \geq Q$ , and  $P_n = Q_n$ . Moreover, if  $c_k < c_{k+1}$  or  $k = n$ ,  $\mathcal{Y}_1 \dots \mathcal{Y}_k$  is the only term in the sum  $(-1)^k \sum_{i_1 < \dots < i_k} \mathcal{Y}_{i_1} \dots \mathcal{Y}_{i_k}$ , having a minimal valuation, and so,  $P_{n-k} = \nu(\mathcal{P}_{n-k}) = c_1 \dots c_k = P_n c_1 \dots c_k$ . Then, it follows from Lemma 2.6 that  $R(P) = (c_1 \leq \dots \leq c_n) = (\nu(\mathcal{Y}_1), \dots, \nu(\mathcal{Y}_n))$ .  $\square$

We now establish majorization inequalities for the valuations of the eigenvalues of matrices.

**Theorem 4.4.** *Let  $\mathbb{K}$  be an algebraically closed field with a (non-archimedean) valuation  $\nu$ . Let  $\mathcal{A} = (\mathcal{A}_{ij}) \in \mathbb{K}^{n \times n}$ . Then, the sequence of valuations of the eigenvalues of  $\mathcal{A}$  (counted with multiplicities) is weakly majorized by the sequence of (algebraic) eigenvalues of the matrix  $A := (\nu(\mathcal{A}_{ij})) \in \mathbb{R}_{\min}^{n \times n}$ .*

*Proof.* Let  $\mathcal{Q} := \det(\mathcal{Y}I - \mathcal{A}) \in \mathbb{K}[\mathcal{Y}]$  be the characteristic polynomial of  $\mathcal{A}$ , and let  $P := \text{per}(\mathcal{Y}I \oplus \mathcal{A}) \in \mathbb{R}_{\min}[\mathcal{Y}]$  be the min-plus characteristic polynomial of  $\mathcal{A}$ . Let  $Q := \nu(\mathcal{Q})$ . Observe that the coefficients of  $\mathcal{Q}$  are given by  $\mathcal{Q}_k = (-1)^{n-k} \text{tr}_{n-k}(\mathcal{A})$ , for  $k = 0, \dots, n-1$  and  $\mathcal{Q}_n = 1$ , where  $\text{tr}_k(\mathcal{A})$  is the  $k$ -th trace of  $\mathcal{A}$ :

$$(11) \quad \text{tr}_k(\mathcal{A}) := \sum_{J \subset \{1, \dots, n\}, \#J=k} \left( \sum_{\sigma \in \mathfrak{S}_J} \text{sgn}(\sigma) \prod_{j \in J} \mathcal{A}_{j\sigma(j)} \right).$$

Similarly, the coefficients of  $P$  are given by  $P_k = \text{tr}_{n-k}^{\min}(A)$ , for  $k = 0, \dots, n-1$  and  $P_n = \mathbb{1}$ , where  $\text{tr}_k^{\min}(A)$  is the min-plus  $k$ -th trace of  $A$  (7). It follows that  $Q = \nu(\mathcal{Q}) \geq P$ , and  $Q_n = P_n = 0$ . Hence, by Lemma 4.2,  $R(Q) \prec^w R(P)$ . By Proposition 4.3,  $R(Q)$  coincides with the sequence of valuations of the eigenvalues of  $\mathcal{A}$ , which establishes the result.  $\square$

If the minimum in every expression (7) is attained by only one product, we have  $\nu(\mathcal{Q}) = P$  in the previous proof, and so, the majorization inequality becomes an equality. However, this condition is quite restrictive (it requires each of a family of a combinatorial optimization problem to have a unique solution). We shall see in the next section that the same conclusion holds under milder assumptions.

## 5. LARGE DEVIATION TYPE ASYMPTOTICS AND QUASIVALUATIONS

The results of the previous section apply to the field of complex Puiseux series. However, in some problems of asymptotic analysis, we need to deal with complex functions  $f$  of a small positive parameter  $\epsilon$  which may not have Puiseux series

expansions, but which only have a “large deviation” type asymptotics, meaning that the limit

$$(12) \quad \lim_{\epsilon \rightarrow 0} \frac{\log |f(\epsilon)|}{\log \epsilon} \in \mathbb{R} \cup \{+\infty\}$$

exists. Since the set of such functions is not a ring (it is not stable by sum), we introduce the larger set  $\mathcal{C}$  of continuous functions  $f$  defined on some interval  $(0, \epsilon_0)$  to  $\mathbb{C}$  with  $\epsilon_0 > 0$ , such that  $|f(\epsilon)| \leq \epsilon^{-k}$  on  $(0, \epsilon_0)$ , for some positive constant  $k$ . Since all the properties that we will prove in the sequel will hold on some neighborhoods of 0, we shall rather use the ring of *germs* at 0 of elements of  $\mathcal{C}$ , which is obtained by quotienting  $\mathcal{C}$  by the equivalence relation that identifies functions which coincide on a neighborhood of 0. This ring of germs will be also denoted by  $\mathcal{C}$ . For any germ  $f \in \mathcal{C}$ , we shall abusively denote by  $f(\epsilon)$  or  $f_\epsilon$  the value at  $\epsilon$  of any representative of the germ  $f$ . We shall make a similar abuse for vectors, matrices, polynomials whose coefficients are germs. We call *exponent* of  $f \in \mathcal{C}$ :

$$(13) \quad \mathbf{e}(f) \stackrel{\text{def}}{=} \liminf_{\epsilon \rightarrow 0} \frac{\log |f(\epsilon)|}{\log \epsilon} \in \mathbb{R} \cup \{+\infty\} ,$$

and denote by  $\mathcal{C}^\dagger$  the subset of elements  $f$  of  $\mathcal{C}$  having a large deviation type asymptotics, that is such that the liminf in the definition of  $\mathbf{e}(f)$  is a limit. A convenient setting in tropical geometry, along the lines of Alessandrini [Ale13], is to work with functions that are definable in a o-minimal model with a polynomial growth. Then, standard model theory arguments show that such functions have automatically large deviations type asymptotics, so that the present results apply in particular to this setting.

We have, for all  $f, g \in \mathcal{C}$ ,

$$(14a) \quad \mathbf{e}(f + g) \geq \min(\mathbf{e}(f), \mathbf{e}(g)) ,$$

$$(14b) \quad \mathbf{e}(fg) \geq \mathbf{e}(f) + \mathbf{e}(g) ,$$

with

$$(15) \quad \mathbf{e}(f + g) = \min(\mathbf{e}(f), \mathbf{e}(g)) \quad \text{if } \mathbf{e}(f) \neq \mathbf{e}(g) ,$$

and equality in (14b) if  $f$  or  $g$  belongs to  $\mathcal{C}^\dagger$ . An element  $f \in \mathcal{C}$  is invertible if, and only if, there exists a positive constant  $k$  such that  $|f(\epsilon)| \geq \epsilon^k$ . Then,  $\mathbf{e}(f) \neq 0$ , and the inverse of  $f$  is the map  $f^{-1} : \epsilon \mapsto f(\epsilon)^{-1}$ . Moreover, we have  $\mathbf{e}(f^{-1}) \leq -\mathbf{e}(f)$  with equality if, and only if,  $f \in \mathcal{C}^\dagger$ . Thus,  $f \mapsto \mathbf{e}(f)$  is “almost” a valuation on the ring  $\mathcal{C}$  (and thus almost a morphism  $\mathcal{C} \rightarrow \mathbb{R}_{\min}$ ).

In the sequel, we shall say that a map  $\mathbf{e}$  from a ring  $\mathcal{R}$  to  $\mathbb{R} \cup \{+\infty\}$  is a *quasi-valuation* if it satisfies (14), for all  $f, g \in \mathcal{R}$ , together with

$$(16) \quad \mathbf{e}(-1) = 0 .$$

For any quasi-valuation, we define the set:

$$(17) \quad \mathcal{R}^\dagger := \{f \in \mathcal{R} \mid \mathbf{e}(fg) = \mathbf{e}(f) + \mathbf{e}(g) \forall g \in \mathcal{R}\} .$$

From the above remarks, the map  $\mathbf{e}$  of (18) is a quasi-valuation over the ring  $\mathcal{R} = \mathcal{C}$ , and one can easily show in that case that the subset  $\mathcal{R}^\dagger$  coincides with the set  $\mathcal{C}^\dagger$  defined above. Another example of a quasi-valuation is the map

$$(18) \quad \mathbf{e}(f) \stackrel{\text{def}}{=} \liminf_{\epsilon \rightarrow 0} \frac{\log \|f(\epsilon)\|}{\log \epsilon} \in \mathbb{R} \cup \{+\infty\} ,$$

on the ring  $\mathcal{R} = \mathcal{C}^{n \times n}$  of  $n \times n$  matrices with entries in  $\mathcal{C}$ , where  $\|\cdot\|$  is any matrix norm on  $\mathcal{C}^{n \times n}$ . Indeed, there exists a constant  $C$  such that  $\|AB\| \leq C\|A\|\|B\|$ , for all  $A, B \in \mathcal{C}^{n \times n}$ . This property together with the sup-additivity of a norm imply (14). The identity matrix is the unit of  $\mathcal{R}$  and since any constant matrix  $A \in \mathcal{C}^{n \times n}$  satisfies  $\mathbf{e}(A) = 0$ , we get (16). However, the ring  $\mathcal{C}^{n \times n}$  is not commutative, so that the results of the end of the present section cannot be applied directly.

Most of the properties of the map  $\mathbf{e}$  of (18) can be transposed to the case of a general quasi-valuation, as follows. Let us denote by  $\mathcal{R}^*$  the set of invertible elements of  $\mathcal{R}$ . It is easy to see that if a map  $\mathbf{e}$  from  $\mathcal{R}$  to  $\mathbb{R} \cup \{+\infty\}$  is not identically  $+\infty$  and satisfies (14b), then  $\mathbf{e}(1) \leq 0$ , and thus  $f \in \mathcal{R}^* \Rightarrow \mathbf{e}(f) \neq +\infty$ . The condition (16) is equivalent to the condition that  $-1 \in \mathcal{R}^\dagger$  and it implies that  $\mathbf{e}(1) = 0$ ,  $1 \in \mathcal{R}^\dagger$ , and  $\mathbf{e}(-g) = \mathbf{e}(g)$  for all  $g \in \mathcal{R}$ . Then, from the latter property, a quasi-valuation  $\mathbf{e}$  satisfies necessarily (15). Moreover, the set  $\mathcal{R}^\dagger$  is necessarily a multiplicative submonoid of  $\mathcal{R}$ , the set  $\mathcal{R}^\dagger \cap \mathcal{R}^*$  is the subgroup of  $\mathcal{R}^*$  composed of the invertible elements  $f$  of  $\mathcal{R}$  such that  $\mathbf{e}(f^{-1}) = -\mathbf{e}(f)$ , and  $\mathbf{e}$  is a multiplicative group morphism on it.

For any formal polynomial with coefficients in a ring  $\mathcal{R}$  with a quasi-valuation  $\mathbf{e}$ ,  $\mathcal{P} = \sum_{j=0}^n \mathcal{P}_j \mathcal{Y}^j \in \mathcal{R}[\mathcal{Y}]$ , we define its quasivaluation similarly to its valuation, see Proposition 4.3:

$$\mathbf{e}(\mathcal{P}) \stackrel{\text{def}}{=} \bigoplus_{j=0}^n \mathbf{e}(\mathcal{P}_j) \mathcal{Y}^j \in \mathbb{R}_{\min}[\mathcal{Y}] .$$

Using the same proof, while replacing the valuation by a quasi-valuation, we extend Proposition 4.3 and Theorem 4.4 as follows. These results hold in particular for the exponent application  $\mathbf{e}$  defined on  $\mathcal{C}$ , in which case Proposition 5.1 says that “the leading exponents of the roots of a polynomial are the min-plus roots of the polynomial of leading exponents”.

**Proposition 5.1.** *Let  $\mathcal{R}$  be a commutative ring with a quasi-valuation  $\mathbf{e}$ , and let  $\mathcal{R}^\dagger$  be defined by (17). Let  $\mathcal{P} = \sum_{k=0}^n \mathcal{P}_k \mathcal{Y}^k \in \mathcal{R}[\mathcal{Y}]$ , with  $\mathcal{P}_n = 1$ . Assume that  $\mathcal{P}$  has  $n$  roots (counted with multiplicities) and that they all belong to  $\mathcal{R}^\dagger$ . Then, the images by  $\mathbf{e}$  of the roots of  $\mathcal{P}$  (counted with multiplicities) coincide with the roots of the min-plus polynomial  $\mathbf{e}(\mathcal{P})$ .  $\square$*

**Theorem 5.2.** *Let  $\mathcal{R}$  be a commutative ring with a quasi-valuation  $\mathbf{e}$ , and let  $\mathcal{R}^\dagger$  be defined by (17). Let  $\mathcal{A} = (\mathcal{A}_{ij}) \in \mathcal{R}^{n \times n}$ . Assume that  $\mathcal{A}$  has  $n$  algebraic eigenvalues (counted with multiplicities) and that they all belong to  $\mathcal{R}^\dagger$ . Denote by  $\Lambda = (\Lambda_1 \leq \dots \leq \Lambda_n)$  the sequence of their images by  $\mathbf{e}$  (counted with multiplicities). Let  $\Gamma = (\gamma_1 \leq \dots \leq \gamma_n)$  be the sequence of min-plus algebraic eigenvalues of  $\mathbf{e}(\mathcal{A}) := (\mathbf{e}(\mathcal{A}_{ij})) \in \mathbb{R}_{\min}^{n \times n}$ . Then,  $\Lambda$  is weakly majorized by  $\Gamma$ .  $\square$*

The following corollary will be used in Section 7.

**Corollary 5.3.** *Let  $\mathcal{R}$  be a commutative ring with a quasi-valuation  $\mathbf{e}$ , and let  $\mathcal{R}^\dagger$  be defined by (17). Let  $\mathcal{A} = (\mathcal{A}_{ij}) \in \mathcal{R}^{n \times n}$ . Assume that  $\mathcal{A}$  has  $n$  algebraic eigenvalues (counted with multiplicities) and that they all belong to  $\mathcal{R}^\dagger$ . Denote by  $\Lambda = (\Lambda_1 \leq \dots \leq \Lambda_n)$  the sequence of their images by  $\mathbf{e}$  (counted with multiplicities). Let  $A \in \mathbb{R}_{\min}^{n \times n}$  be such that  $\mathbf{e}(\mathcal{A}_{ij}) \geq A_{ij}$ , for all  $i, j \in [n]$ , and let  $\Gamma = (\gamma_1 \leq \dots \leq \gamma_n)$  be the sequence of min-plus algebraic eigenvalues of  $A$ . Then,  $\Lambda$  is weakly majorized by  $\Gamma$ .*

*Proof.* Theorem 5.2 implies that  $\Lambda \prec^w \Gamma'$ , where  $\Gamma'$  is the sequence of min-plus algebraic eigenvalues of  $e(\mathcal{A})$ . Define the min-plus polynomials  $P := \text{per } e(\mathcal{A})$  and  $Q := \text{per } A$ . Then, by definition of min-plus eigenvalues,  $\Gamma'$  is the sequence  $R(P)$  of min-plus roots of  $P$ , and  $\Gamma$  is the sequence  $R(Q)$  of min-plus roots of  $Q$ . Since  $e(\mathcal{A})_{ij} = e(\mathcal{A}_{ij}) \geq A_{ij}$ , for all  $i, j \in [n]$ , we get that  $P \geq Q$ , and since  $P_n = Q_n$ , Lemma 4.2 shows that  $R(P) \prec^w R(Q)$ . Hence  $\lambda \prec^w \Gamma' \prec^w \Gamma$ , which finishes the proof.  $\square$

## 6. A PRELIMINARY: NEWTON-PUISEUX THEOREM WITH PARTIAL INFORMATION ON VALUATIONS

If  $\mathcal{A}$  is a matrix with entries in the field of Puiseux series, the knowledge of the valuations of the entries of  $\mathcal{A}$  is not enough to determine the valuations of the coefficients of the characteristic polynomial of  $\mathcal{A}$ , owing to potential cancellation. Therefore, in order to find conditions under which the majorization inequality in Theorem 5.2 becomes an equality, we need to state a variant of the classical Newton Puiseux theorem, in which only a partial information on the valuations of the coefficients of a polynomial, or equivalently, an “external approximation” of the Newton polytope, is available. The idea that such an information on the polytope is enough to infer a partial information on roots is classical: in the case of archimedean valuations, it already appeared for instance in the work of Montel [Mon34].

We shall say that  $f \in \mathcal{C}$  has a *first order asymptotics* if

$$(19) \quad f(\epsilon) \sim a\epsilon^A, \quad \text{when } \epsilon \rightarrow 0^+,$$

with either  $A \in \mathbb{R}$  and  $a \in \mathbb{C} \setminus \{0\}$ , or  $A = +\infty$  and  $a \in \mathbb{C}$ . In the first case, (19) means that  $\lim_{\epsilon \rightarrow 0} \epsilon^{-A} f(\epsilon) = a$ , in the second case, (19) means that  $f = 0$  (in a neighborhood of 0). Such asymptotic behaviors arise when considering precise large deviations. We have:

$$(20) \quad f(\epsilon) \sim a\epsilon^A \implies e(f) = A \text{ and } f \in \mathcal{C}^\dagger.$$

We shall also need a relation slightly weaker than  $\sim$ . If  $f \in \mathcal{C}$ ,  $a \in \mathbb{C}$  and  $A \in \mathbb{R}_{\min}$ , we write

$$(21) \quad f(\epsilon) \simeq a\epsilon^A$$

if  $f(\epsilon) = a\epsilon^A + o(\epsilon^A)$ . If  $A \in \mathbb{R}$ , this means that  $\lim_{\epsilon \rightarrow 0} \epsilon^{-A} f(\epsilon) = a$ . If  $A = +\infty$ , this means by convention that  $f = 0$ . If  $a \neq 0$  or  $A = +\infty$ , then  $f(\epsilon) \simeq a\epsilon^A$  if, and only if,  $f(\epsilon) \sim a\epsilon^A$  and in that case  $e(f) = A$ . In general,

$$(22) \quad f(\epsilon) \simeq a\epsilon^A \implies e(f) \geq A.$$

Conversely,  $e(f) > A \implies f(\epsilon) \simeq 0\epsilon^A$ . Of course, in (21),  $a\epsilon^A$  must be viewed as a formal expression, for the relation to be meaningful when  $a = 0$  and  $A \in \mathbb{R}$ . In (20), however,  $a\epsilon^A$  can be viewed either as a formal expression or as an element of  $\mathcal{C}$ .

The following results give conditions under which some or all roots of a polynomial with coefficients in  $\mathcal{C}$  have first order asymptotics, hence are elements of  $\mathcal{C}^\dagger$ , which allow one in particular to apply the results of Section 5. Although stated for polynomials with coefficients in  $\mathcal{C}$ , they are already useful in the case of polynomials with coefficients in the set of Puiseux series, for which some of the valuations are not known. Its We shall use in particular these results in the case of characteristic polynomials.

**Theorem 6.1** (Newton-Puiseux theorem with partial information on the valuations). *Let  $\mathcal{P} = \sum_{j=0}^n \mathcal{P}_j \mathcal{Y}^j \in \mathcal{C}[\mathcal{Y}]$ . Assume that there exist  $p = \sum_{j=0}^n p_j \mathcal{Y}^j \in \mathbb{C}[\mathcal{Y}]$  and  $P = \bigoplus_{j=0}^n P_j \mathcal{Y}^j \in \mathbb{R}_{\min}[\mathcal{Y}]$  satisfying  $\mathcal{P}_j(\epsilon) \simeq p_j \epsilon^{P_j}$ ,  $j = 0, \dots, n$ . Let  $c \in \mathbb{R}$  be a finite root of  $P$  with multiplicity  $m$  and assume that the polynomial*

$$(23) \quad p^{(c)} = \sum_{\substack{0 \leq j \leq n \\ \widehat{P}(c) = P_j c^j}} p_j \mathcal{Y}^j \in \mathbb{C}[\mathcal{Y}]$$

*is not identically zero. Let  $y_1, \dots, y_\ell$  denote its non-zero roots (counted with multiplicities). Then,  $\ell \leq m$ , and there exist  $\ell$  roots of  $\mathcal{P}$  (counted with multiplicities),  $\mathcal{Y}_1, \dots, \mathcal{Y}_\ell \in \mathcal{C}$ , having first order asymptotics of the form  $\mathcal{Y}_i \sim y_i \epsilon^c$ . Moreover if  $v = \text{val} p^{(c)} > 0$ , then  $v \geq m'$ , where  $m'$  is the sum of the multiplicities of all the roots of  $P$  greater than  $c$  ( $+\infty$  comprised), and there exist  $v$  roots of  $\mathcal{P}$  (counted with multiplicities)  $\mathcal{Y}_{1+\ell}, \dots, \mathcal{Y}_{v+\ell} \in \mathcal{C}$ , such that  $\mathcal{Y}_i \simeq 0 \epsilon^c$ , for  $1 \leq i - \ell \leq v$ . Finally, the remaining  $n - v - \ell$  roots  $\mathcal{Y}$  of  $\mathcal{P}$  are such that  $\mathcal{Y} \epsilon^{-c}$  tends to infinity when  $\epsilon$  goes to 0, and their number satisfies  $n - v - \ell \geq n - m - m'$ .*

Recall that to  $P$  is associated the polynomial function  $\widehat{P}$  and the convex formal polynomial  $\overline{P}$ , as in Section 2.

**Theorem 6.2** (Newton-Puiseux theorem with partial information on the valuations, continued). *Let  $\mathcal{P} = \sum_{j=0}^n \mathcal{P}_j \mathcal{Y}^j \in \mathcal{C}[\mathcal{Y}]$ , such that  $\mathcal{P}_n = 1$ . The following assertions are equivalent:*

- (1) *There exist  $\mathcal{Y}_1, \dots, \mathcal{Y}_n \in \mathcal{C}$  such that  $\mathcal{Y}_1(\epsilon), \dots, \mathcal{Y}_n(\epsilon)$  are the roots of  $\mathcal{P}(\epsilon) = \sum_{j=0}^n \mathcal{P}_j(\epsilon) \mathcal{Y}^j$  counted with multiplicities, and  $\mathcal{Y}_1, \dots, \mathcal{Y}_n$  have first order asymptotics,  $\mathcal{Y}_j(\epsilon) \sim y_j \epsilon^{Y_j}$  with  $Y_1 \leq \dots \leq Y_n$ ;*
- (2) *There exist  $p = \sum_{j=0}^n p_j \mathcal{Y}^j \in \mathbb{C}[\mathcal{Y}]$  and  $P = \bigoplus_{j=0}^n P_j \mathcal{Y}^j \in \mathbb{R}_{\min}[\mathcal{Y}]$  satisfying  $\mathcal{P}_j(\epsilon) \simeq p_j \epsilon^{P_j}$ ,  $j = 0, \dots, n$ , with  $p_n = 1$ ,  $P_n = \mathbb{1}$ ,  $p_0 \neq 0$  or  $P_0 = \mathbb{0}$ , and  $p_{n-i} \neq 0$  for all  $i \in \{1, \dots, n-1\}$  such that  $c_i < c_{i+1}$ , where  $(c_1 \leq \dots \leq c_n) = R(P)$ .*

*When these assertions hold, we have  $\mathbf{e}(\mathcal{P}) \geq P$ ,  $\overline{\mathbf{e}(\mathcal{P})} = \overline{P}$ , and  $R(\mathbf{e}(\mathcal{P})) = R(P) = (c_1 \leq \dots \leq c_n) = (Y_1 \leq \dots \leq Y_n)$ . Moreover, if  $c \in \mathbb{R}$  is a root of  $P$  with multiplicity  $m$  and  $c_{i+1} = \dots = c_{i+m} = c$ , then  $y_{i+1}, \dots, y_{i+m}$  are precisely the non-zero roots of the polynomial  $p^{(c)}$  of (23), counted with multiplicities.*

Theorem 6.2 is a “precise large deviation” version of the Newton-Puiseux theorem: we assume only the existence of asymptotic equivalents for the coefficients of  $\mathcal{P}$ , and derive the existence of asymptotic equivalents for the branches of  $\mathcal{P}(\epsilon, y) = 0$ . The Newton-Puiseux algorithm is sometimes presented for asymptotic expansions, as in [Die68]. The interest of the statements of Theorems 6.1 and 6.2, is to show that if some coefficients are known to be negligible, the asymptotics of the roots is determined only from the asymptotics of those coefficients  $\mathcal{P}_i$  such that  $(i, P_i)$  is an exposed point of the epigraph of  $\overline{P}$ .

*Example 6.3.* Consider  $\mathcal{P} = \mathcal{Y}^3 + \epsilon^5 \mathcal{Y}^2 - \epsilon^6 \mathcal{Y} + \epsilon^{13}$ . Then,  $\mathcal{P}$  is a polynomial over the field  $\mathbb{K}$  of complex Puiseux series, hence the roots of  $\mathcal{P}$  are elements of  $\mathbb{K}$ . Moreover, the min-plus polynomial  $P = \nu(\mathcal{P}) = \mathcal{Y}^3 \oplus 5\mathcal{Y}^2 \oplus 6\mathcal{Y} \oplus 13$  is the one of Figure 1, hence its roots are  $c_1 = c_2 = 3$  and  $c_3 = 7$ . In that case, Proposition 4.3 says that the valuations of roots of  $\mathcal{P}$  coincide with the roots of  $P$ .

If now  $\mathcal{P} = Y^3 + \frac{\epsilon^3}{\log \epsilon} Y^2 - \epsilon^6 Y + \epsilon^{13}$ , the coefficient  $\mathcal{P}_2$  of  $\mathcal{P}$  is not in  $\mathbb{K}$ . Moreover, although all coefficients of  $\mathcal{P}$  belong to  $\mathcal{C}$  and even  $\mathcal{C}^\dagger$ ,  $\mathcal{P}_2$  does not have a first order asymptotics. However,  $\mathcal{P}_j(\epsilon) \simeq p_j \epsilon^{P_j}$ ,  $j = 0, \dots, 3$ , with  $P = Y^3 \oplus 3Y^2 \oplus 6Y \oplus 13$  and  $p = Y^3 - Y + 1 \in \mathbb{C}[Y]$ . The min-plus polynomial  $P$  has same roots as the one defined above:  $c_1 = c_2 = 3 < c_3 = 7$  and since  $p_0 \neq 0$ ,  $p_1 \neq 0$ ,  $p_3 = 1$  and  $P_3 = \mathbf{1}$ , Assertion (2) of Theorem 6.2 holds. Hence, by Theorem 6.2, the roots of  $\mathcal{P}$  form 3 continuous branches around 0:  $\mathcal{Y}_1, \dots, \mathcal{Y}_3 \in \mathcal{C}$  with first order asymptotics, and respective exponents  $c_1 = c_2 = 3 < c_3 = 7$ .

In both cases above, Theorem 6.2 gives the additional information that the roots  $\mathcal{Y}_1, \dots, \mathcal{Y}_3$  of  $\mathcal{P}$  satisfy  $\mathcal{Y}_j(\epsilon) \sim y_j \epsilon^{c_j}$ ,  $j = 1, \dots, 3$ , where  $y_1, y_2$  are the non-zero roots of  $p^{(3)} = Y^3 - Y$ , and  $y_3$  is the non-zero root of  $p^{(7)} = -Y + 1$ . This gives for instance  $y_1 = 1$ ,  $y_2 = -1$  and  $y_3 = 1$ , so that  $\mathcal{Y}_1 \sim \epsilon^3$ ,  $\mathcal{Y}_2 \sim -\epsilon^3$  and  $\mathcal{Y}_3 \sim \epsilon^7$ .

In order to prove Theorems 6.1 and 6.2, we need the following standard result.

**Lemma 6.4.** *Let  $\mathcal{Q}(\epsilon, Y) = \sum_{i=0}^n \mathcal{Q}_i(\epsilon) Y^i$ , where the  $\mathcal{Q}_i$  are continuous functions of  $\epsilon \in [0, \epsilon_0)$ , assume  $\mathcal{Q}(0, \cdot) \neq 0$  and let  $d = \deg \mathcal{Q}(0, \cdot) \geq 0$ . Then, for any open ball  $B$  containing the roots of  $\mathcal{Q}(0, \cdot)$ , there are  $d$  continuous branches  $\mathcal{Z}_1, \dots, \mathcal{Z}_d$  defined in some interval  $[0, \epsilon_1)$ , with  $0 < \epsilon_1 \leq \epsilon_0$ , such that  $\mathcal{Z}_1(\epsilon), \dots, \mathcal{Z}_d(\epsilon)$  are exactly the roots of  $\mathcal{Q}(\epsilon, \cdot)$  in  $B$  counted with multiplicities. Moreover, the roots of  $\mathcal{Q}(\epsilon, \cdot)$  that are outside  $B$  tend to infinity when  $\epsilon$  goes to 0.*

*Proof.* We only sketch the proof, which is classical. By the Cauchy index theorem, if  $\gamma$  is any circle in  $\mathbb{C}$  containing no roots of  $\mathcal{Q}(\epsilon, \cdot)$ , the number of roots of  $\mathcal{Q}(\epsilon, \cdot)$  inside  $\gamma$  is  $(2\pi i)^{-1} \int_{\gamma} \partial_z \mathcal{Q}(\epsilon, z) (\mathcal{Q}(\epsilon, z))^{-1} dz$ . By continuity of  $\epsilon \mapsto \mathcal{Q}(\epsilon, \cdot)$ , the number of roots of  $\mathcal{Q}(\epsilon', \cdot)$  inside  $\gamma$  (counted with multiplicities) is constant for  $\epsilon'$  in some neighborhood of  $\epsilon$ . Taking  $B$  as in the lemma,  $\gamma = \partial B$ , and  $\epsilon = 0$ , we get exactly  $d$  roots of  $\mathcal{Q}(\epsilon', \cdot)$  in  $B$  for  $\epsilon'$  in some interval  $[0, \epsilon_1)$ . Consider now a ball  $B_R \supset B$  of radius  $R$ . For  $\epsilon'$  small enough, the number of roots of  $\mathcal{Q}(\epsilon', \cdot)$  in either  $B_R$  or  $B$  is equal to  $d$ , hence any root of  $\mathcal{Q}(\epsilon', \cdot)$  outside  $B$  must be outside  $B_R$ . This shows that the roots of  $\mathcal{Q}(\epsilon', \cdot)$  that do not belong to  $B$  go to infinity, when  $\epsilon' \rightarrow 0$ . Finally, by taking small balls around each root of  $\mathcal{Q}(\epsilon, \cdot)$ , with  $0 \leq \epsilon < \epsilon_1$ , we see that the map which sends  $\epsilon$  to the unordered  $d$ -tuple of roots of  $\mathcal{Q}(\epsilon, \cdot)$  that belong to  $B$ , is continuous on  $[0, \epsilon_1)$ . By a selection theorem for unordered  $d$ -tuples depending continuously on a real parameter (see for instance [Kat95, Ch. II, Section 5, 2]), we derive the existence of the  $d$  continuous branches  $\mathcal{Z}_1, \dots, \mathcal{Z}_d$ .  $\square$

*Proof of Theorem 6.1.* This is obtained by applying the first step of the Puiseux algorithm (which is part of the proof of the classical Newton-Puiseux theorem). Indeed, applying the change of variable  $y = z\epsilon^c$ , and the division of  $\mathcal{P}$  by  $\epsilon^{\widehat{P}(c)}$ , transforms the equation  $\mathcal{P}(\epsilon, y) := \sum_{j=0}^n \mathcal{P}_j(\epsilon) y^j = 0$  into an equation  $\mathcal{Q}(\epsilon, z) = 0$ , where  $\mathcal{Q}(\cdot, z)$  extends continuously to 0 with  $\mathcal{Q}(0, z) = p^{(c)}(z)$ . Since  $p^{(c)}$  is not identically zero, Lemma 6.4 implies that for  $d = \deg p^{(c)}$ , there exist  $\mathcal{Z}_1, \dots, \mathcal{Z}_d \in \mathcal{C}$  such that for all  $\epsilon \geq 0$  small enough,  $\mathcal{Z}_1(\epsilon), \dots, \mathcal{Z}_d(\epsilon)$  are exactly the roots (counted with multiplicities) of  $\mathcal{Q}(\epsilon, \cdot)$  in some ball around 0, and that the roots of  $\mathcal{Q}(\epsilon, \cdot)$  outside this ball tend to infinity when  $\epsilon$  goes to 0.

In particular  $\mathcal{Z}_1(0), \dots, \mathcal{Z}_d(0)$  are the roots of  $p^{(c)}$ , then if  $v = \text{val } p^{(c)}$ ,  $\ell = d - v$  is the number of non-zero roots and one can assume that  $\mathcal{Z}_i(0) = y_i$  for  $i \leq \ell$ . Making the reverse change of variable, we obtain that  $\mathcal{Y}_i(\epsilon) = \mathcal{Z}_i(\epsilon)\epsilon^c$ ,  $i = 1, \dots, d$  satisfy the conditions of the theorem.

It remains to show that  $\ell \leq m$ ,  $v \geq m'$ , and  $n - v - \ell \geq n - m - m'$ , where  $m'$  is the sum of the multiplicities of all the roots of  $P$  greater than  $c$ . By Corollary 2.7, we have  $\widehat{P}(c) < P_j c^j$  when  $j < m'$  or  $j > m + m'$ , from which we deduce that  $v = \text{val} p^{(c)} \geq m'$  and  $d = \text{deg} p^{(c)} \leq m' + m$ , so that  $\ell = d - v \leq m$ , and  $n - v - \ell = n - d \geq n - m - m'$ , which finishes the proof.  $\square$

*Proof of Theorem 6.2.* We first prove (1)  $\implies$  (2). Let  $Q = (\mathsf{Y} \oplus Y_1) \cdots (\mathsf{Y} \oplus Y_n)$ . Then,  $Q = \overline{Q}$ ,  $\mathsf{R}(Q) = (Y_1 \leq \cdots \leq Y_n)$  and  $Q_{n-i} = Y_1 \cdots Y_i$  for all  $i = 1, \dots, n$ . Since  $\mathcal{Y}_1(\epsilon), \dots, \mathcal{Y}_n(\epsilon)$  are the roots of  $\mathcal{P}(\epsilon, y) = 0$  counted with multiplicities, and  $\mathcal{P}_n = 1$ , it follows that  $\mathcal{P}(\epsilon, \mathsf{Y}) = \prod_{i=1}^n (\mathsf{Y} - \mathcal{Y}_i(\epsilon))$ . Hence,  $(-1)^i \mathcal{P}_{n-i}$  is the sum of all products  $\mathcal{Y}_{j_1} \cdots \mathcal{Y}_{j_i}$ , where  $j_1, \dots, j_i$  are pairwise distinct elements of  $\{1, \dots, n\}$ . By the properties of “ $\simeq$ ” (stability by addition and multiplication), and since  $\bigoplus_{j_1, \dots, j_i} Y_{j_1} \cdots Y_{j_i} = Y_1 \cdots Y_i = Q_{n-i}$ , we obtain that there exist  $p_0, \dots, p_{n-1} \in \mathbb{C}$  such that  $\mathcal{P}_j \simeq p_j \epsilon^{Q_j}$  for all  $j = 0, \dots, n-1$ . Putting  $p_n = 1$ , we also get  $\mathcal{P}_n = 1 \simeq p_n \epsilon^{Q_n}$  since  $Q_n = \mathbb{1}$ . When  $i = 1, \dots, n-1$  is such that  $Y_i < Y_{i+1}$ ,  $\mathcal{Y}_1 \cdots \mathcal{Y}_i$  is the only leading term in the sum of all  $\mathcal{Y}_{j_1} \cdots \mathcal{Y}_{j_i}$ , and then  $p_{n-i} = (-1)^i y_1 \cdots y_i \neq 0$ . Moreover, for  $i = n$ , either  $Y_n \neq 0$ , which implies that  $p_0 = (-1)^n y_1 \cdots y_n \neq 0$ , or  $Y_n = 0$ , which implies that  $\mathcal{Y}_n = 0$ ,  $\mathcal{P}_0 = 0$  and  $Q_0 = 0$ . This shows that  $(c_1, \dots, c_n) = (Y_1, \dots, Y_n)$  and  $P = Q$  are as in Point (2).

The implication (2)  $\implies$  (1) and the remaining part of the theorem will follow from several applications of Theorem 6.1. We now only assume that the  $\mathcal{P}_j \in \mathcal{C}$  satisfy Point (2) and consider  $(c_1 \leq \cdots \leq c_n) = \mathsf{R}(P)$ . From Lemma 2.6 applied to  $P$ , we get that  $P \geq P_n(\mathsf{Y} \oplus c_1) \cdots (\mathsf{Y} \oplus c_n)$ ,  $P_{n-i} = P_n c_1 \cdots c_i$  for all  $i \in \{0, n\} \cup \{i \in \{1, \dots, n-1\} \mid c_i < c_{i+1}\}$ , and  $\overline{P} = P_n(\mathsf{Y} \oplus c_1) \cdots (\mathsf{Y} \oplus c_n)$ . It follows from (14,22), that  $\mathsf{e}(P) \geq P \geq P_n(\mathsf{Y} \oplus c_1) \cdots (\mathsf{Y} \oplus c_n)$ , and from Point (2) and (20), we get that  $\mathsf{e}(P)_{n-i} = P_{n-i}$  for all  $i \in \{0, n\} \cup \{i \in \{1, \dots, n-1\} \mid c_i < c_{i+1}\}$ . Then,  $\mathsf{e}(P)$  satisfies the conditions of Lemma 2.6, which yields  $\mathsf{R}(\mathsf{e}(P)) = \mathsf{R}(P)$  and  $\overline{\mathsf{e}(P)} = \overline{P}$ .

Let  $c \neq 0$  be a root of  $P$  with multiplicity  $m$ . Then, there exists  $0 \leq i < n$  such that  $c = c_{i+1} = \cdots = c_{i+m}$ . Using the same arguments as in the proof of Theorem 6.1, we have that  $i = n - m' - m$ , where  $m'$  is the sum of the multiplicities of all the roots of  $P$  greater than  $c$ ,  $\text{val} p^{(c)} \geq m' = n - m - i$  and  $d = \text{deg} p^{(c)} \leq m' + m = n - i$ . Moreover, since either  $i = 0$  or  $c_i < c_{i+1}$ , we get, by Point (2), that  $p_{n-i} \neq 0$ , hence  $\text{deg} p^{(c)} = n - i$ . Similarly, we have either  $i + m = n$  or  $c_{i+m} < c_{i+m+1}$ . In the second case, we get  $p_{n-i-m} \neq 0$ , thus  $\text{val} p^{(c)} = n - m - i$ . In the first case,  $n - m - i = 0$  and either  $p_0 \neq 0$  or  $P_0 = 0$ . Since  $P_0 = 0$  implies  $c = c_n = 0$ , which contradicts our assumption, we must have  $p_0 \neq 0$ , hence again  $\text{val} p^{(c)} = n - m - i$ . This shows that  $\text{deg} p^{(c)} - \text{val} p^{(c)} = m$ , so that  $p^{(c)}$  has  $\ell = m$  non-zero roots and applying Theorem 6.1, we get that if  $y_{i+1}, \dots, y_{i+m}$  denote its non-zero roots, then there exist  $m$  roots of  $\mathcal{P}$  (counted with multiplicities),  $\mathcal{Y}_{i+1}, \dots, \mathcal{Y}_{i+m} \in \mathcal{C}$ , having first order asymptotics of the form  $\mathcal{Y}_j \sim y_j \epsilon^c$ ,  $i+1 \leq j \leq i+m$ .

Finally, if  $c = 0$  is a root of  $P$  with multiplicity  $m$ , then  $c_{n-m} < c_{n-m+1} = \cdots = c_n = 0$ ,  $\text{val} P = m$ , and  $P$  has  $n - m$  roots  $\neq 0$ . This implies that  $\text{val} \mathcal{P} \geq m$ , so that 0 is a root of  $\mathcal{P}$  with multiplicity  $\geq m$ . Moreover, we have shown above that there exist  $n - m$  roots of  $\mathcal{P}$  with first order asymptotics with respective exponents  $c_1 \leq \cdots \leq c_{n-m} < +\infty$ . Hence,  $\mathcal{P}$  cannot have more than  $m$  zero roots, so that  $\mathcal{P}$  has exactly  $m$  roots with first order asymptotics of the form  $Y_j \sim 0$  (and  $\text{val} \mathcal{P} = m$ ). We thus have shown Point (1), which finishes the proof of the theorem.  $\square$



## 7. GENERICITY FOR ASYMPTOTICS OF EIGENVALUES

We consider a matrix  $\mathcal{A} \in \mathcal{C}^{n \times n}$  and we shall assume that the entries  $(\mathcal{A}_\epsilon)_{ij}$  of  $\mathcal{A}_\epsilon$  have asymptotics of the form:

$$(24) \quad \begin{aligned} (\mathcal{A}_\epsilon)_{ij} &\simeq a_{ij} \epsilon^{A_{ij}}, \text{ for some matrices} \\ a &= (a_{ij}) \in \mathbb{C}^{n \times n}, \text{ and } A = (A_{ij}) \in \mathbb{R}_{\min}^{n \times n}. \end{aligned}$$

Applying Corollary 5.3, we obtain the following majorization inequality.

**Theorem 7.1.** *Let  $\mathcal{A} \in \mathcal{C}^{n \times n}$  satisfy (24) and let  $\Gamma = (\gamma_1 \leq \dots \leq \gamma_n)$  be the sequence of min-plus algebraic eigenvalues of  $A$ . Assume that the eigenvalues  $\mathcal{L}_\epsilon^1, \dots, \mathcal{L}_\epsilon^n$  of  $\mathcal{A}_\epsilon$  (counted with multiplicities) have first order asymptotics,*

$$\mathcal{L}_\epsilon^i \sim \lambda_i \epsilon^{\Lambda_i},$$

and let  $\Lambda = (\Lambda_1 \leq \dots \leq \Lambda_n)$ . Then,  $\Lambda \prec^w \Gamma$ .

*Proof.* From (20), the eigenvalues of  $\mathcal{A}_\epsilon$  are elements of  $\mathcal{C}^\dagger$  and their images by  $\mathfrak{e}$  (counted with multiplicities) are equal to  $\Lambda_1 \leq \dots \leq \Lambda_n$ . By (22) and (24), we have  $\mathfrak{e}(\mathcal{A})_{ij} \geq A_{ij}$ , for all  $i, j \in [n]$ . Then, applying Corollary 5.3 to the ring  $\mathcal{C}$ , and the matrices  $\mathcal{A}$  and  $A$ , we deduce that  $\Lambda \prec^w \Gamma$ .  $\square$

*Remark 7.2.* If the coefficients of  $\mathcal{A}_\epsilon$  have Puiseux series expansions in  $\epsilon$ , the coefficients  $\mathcal{Q}_j$  of the characteristic polynomial  $\mathcal{Q}(\epsilon, Y)$  of  $\mathcal{A}$  and thus the eigenvalues of  $\mathcal{A}$  belong to  $\mathcal{C}$  and have first order asymptotics, so that Theorem 7.1 applies. If we only assume that  $\mathcal{A} \in \mathcal{C}^{n \times n}$  satisfies (24), then the coefficients  $\mathcal{Q}_j$ , which are elements of  $\mathcal{C}$ , need not have first order asymptotics (even if  $a_{ij} \neq 0$  for all  $i, j$ ) due to cancellations. However, they satisfy the conditions  $\mathcal{Q}_n = 1$  and  $\mathcal{Q}_j(\epsilon) \simeq q_j \epsilon^{Q_j}$  for some exponents  $Q_j \in \mathbb{R}_{\min}$  computed using the exponents  $A_{ij}$  (see Section 4), so that Theorem 6.2 can be applied, leading to some sufficient conditions for the first order asymptotics of the eigenvalues of  $\mathcal{A}_\epsilon$  to exist. This will be used in Theorem 7.4.

We next show that the weak majorization inequality is an inequality in generic circumstances. Formally, we shall consider the following notion of genericity. We will say that a property  $\mathcal{P}(y)$  depending on the variable  $y = (y_1, \dots, y_n) \in \mathbb{C}^n$  holds for *generic values* of  $y$  if the set of elements  $y \in \mathbb{C}^n$  such that the property  $\mathcal{P}(y)$  is false is included in a proper algebraic set. This means that there exists  $Q \in \mathbb{C}[Y_1, \dots, Y_n] \setminus \{0\}$  such that  $\mathcal{P}(y)$  is true if  $Q(y) \neq 0$ . When the parameter  $y$  will be obvious, we shall simply say that  $\mathcal{P}$  is *generic* or holds *generically*. It is clear that if  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are both generic, then “ $\mathcal{P}_1$  and  $\mathcal{P}_2$ ” is also generic.

Since any polynomial  $q = \sum_{i_1, \dots, i_n \in \mathbb{N}} q_{i_1, \dots, i_n} Y_1^{i_1} \dots Y_n^{i_n} \in \mathbb{C}[Y_1, \dots, Y_n]$  in  $n$  indeterminates can be seen as an element of  $\mathcal{C}[Y_1, \dots, Y_n]$  whose coefficients are constant with respect to  $\epsilon$ , we have:

$$(25) \quad \mathfrak{e}(q) = \bigoplus_{\substack{i_1, \dots, i_n \in \mathbb{N} \\ q_{i_1, \dots, i_n} \neq 0}} Y_1^{i_1} \dots Y_n^{i_n} \in \mathbb{R}_{\min}[Y_1, \dots, Y_n].$$

We also define, for any  $Y \in \mathbb{R}_{\min}^n$ ,

$$(26) \quad q_Y^{\text{Sat}} := \sum_{\substack{i_1, \dots, i_n \in \mathbb{N} \\ \mathfrak{e}(q)(Y_1, \dots, Y_n) = Y_1^{i_1} \dots Y_n^{i_n} \neq 0}} q_{i_1, \dots, i_n} Y_1^{i_1} \dots Y_n^{i_n} \in \mathbb{C}[Y_1, \dots, Y_n].$$

The following result is clear from the above definitions of  $\mathbf{e}(q)$  and  $q_Y^{\text{Sat}}$ , since when  $y \neq 0$  or  $Y = 0$ ,  $\mathcal{Y} \simeq y\epsilon^Y \iff \mathcal{Y} \sim y\epsilon^Y$ . Note that there and in the sequel, we use the same notation for any formal polynomial over  $\mathbb{C}$  or  $\mathcal{C}$  and its associated polynomial function.

**Lemma 7.3.** *Let  $q \in \mathbb{C}[Y_1, \dots, Y_n]$  be non zero and let  $Q = \mathbf{e}(q)$  and  $q_Y^{\text{Sat}}$  be defined by (25) and (26), respectively. Let  $\mathcal{Y} \in \mathcal{C}^n$ ,  $y \in \mathbb{C}^n$  and  $Y \in \mathbb{R}_{\min}^n$  be such that  $\mathcal{Y}_i \simeq y_i \epsilon^{Y_i}$  for  $i = 1, \dots, n$ . Then,*

$$(27) \quad q(\mathcal{Y}_1, \dots, \mathcal{Y}_n) \simeq q_Y^{\text{Sat}}(y) \epsilon^{\widehat{Q}(Y)} ,$$

and for any fixed  $Y$  such that  $\widehat{Q}(Y) \neq 0$ , we have  $q_Y^{\text{Sat}}(y) \neq 0$  for generic values of  $y \in \mathbb{C}^n$ . Then, for generic values of  $y \in \mathbb{C}^n$ , the relation  $\simeq$  in (27) can be replaced by an the asymptotic equivalence relation  $\sim$ .

**Theorem 7.4.** *Let  $\mathcal{A} \in \mathcal{C}^{n \times n}$  satisfy (24) and let  $\Gamma = (\gamma_1 \leq \dots \leq \gamma_n)$  be the sequence of min-plus algebraic eigenvalues of  $A$ . For generic values of  $a = (a_{ij}) \in \mathbb{C}^{n \times n}$ , the eigenvalues  $\mathcal{L}_\epsilon^1, \dots, \mathcal{L}_\epsilon^n$  of  $\mathcal{A}_\epsilon$  (counted with multiplicities) have first order asymptotics,*

$$\mathcal{L}_\epsilon^i \sim \lambda_i \epsilon^{\Lambda_i} ,$$

and  $\Lambda = (\Lambda_1 \leq \dots \leq \Lambda_n)$  satisfies  $\Lambda = \Gamma$ .

*Proof.* Since  $\mathcal{A} = \mathcal{A}_\epsilon \in \mathcal{C}^{n \times n}$  and  $\mathcal{C}$  is a ring, the characteristic polynomial of  $\mathcal{A}$ ,  $\mathcal{Q}(\epsilon, Y) := \det(YI - \mathcal{A}_\epsilon)$  belongs to  $\mathcal{C}[Y]$ . Let  $Q = \mathbf{e}(\mathcal{Q}) \in \mathbb{R}_{\min}[Y]$ , and denote by  $P = \text{per}(YI \oplus A) \in \mathbb{R}_{\min}[Y]$  the min-plus characteristic polynomial of  $A$ . The coefficients of  $\mathcal{Q}$  are given by  $\mathcal{Q}_k(\epsilon) = (-1)^k \text{tr}_{n-k}(\mathcal{A}_\epsilon)$ , for  $k = 0, \dots, n-1$  and  $\mathcal{Q}_n = 1$ , where  $\text{tr}_k$  is the usual  $k$ -th trace (11). The coefficients of  $P$  are given by  $P_k = \text{tr}_{n-k}^{\min}(A)$ , for  $k = 0, \dots, n-1$  and  $P_n = 1$ , where here  $\text{tr}_k^{\min}$  is the min-plus  $k$ -th trace (7). By Lemma 7.3, we obtain that for any fixed matrix  $A \in \mathbb{R}_{\min}^{n \times n}$ , and any  $\mathcal{A} \in \mathcal{C}^{n \times n}$  satisfying (24) with  $a \in \mathbb{C}^{n \times n}$  and  $A$ ,  $\text{tr}_k(\mathcal{A}_\epsilon) \sim (\text{tr}_k)_A^{\text{Sat}}(a) \epsilon^{\text{tr}_k^{\min}(A)}$ , for generic values of  $a \in \mathbb{C}^{n \times n}$ . In particular, generically,  $\mathcal{Q}_k(\epsilon)$  has first order asymptotics and  $\mathbf{e}(\mathcal{Q}_k) = P_k$ , for all  $k = 0, \dots, n$ . This implies that  $Q = P$ , thus  $R(Q) = R(P) = \Gamma$ . Moreover,  $\mathcal{Q}$  satisfies Point (2) of Theorem 6.2. Hence, by Theorem 6.2, the eigenvalues  $\mathcal{L}_\epsilon^1, \dots, \mathcal{L}_\epsilon^n$  of  $\mathcal{A}_\epsilon$  have first order asymptotics and their exponents  $\Lambda_1 \leq \dots \leq \Lambda_n$  are equal to the roots of  $Q$ , hence  $\Lambda = \Gamma$ .  $\square$

*Remark 7.5.* Since min-plus eigenvalues can be computed in polynomial time, see Section 3, Theorem 7.4 shows that the sequence  $\Lambda$  of generic exponents of the eigenvalues can be computed in polynomial time.

## 8. EIGENVALUES OF MATRIX POLYNOMIALS

We consider now a matrix polynomial

$$(28) \quad \mathcal{A} = \mathcal{A}_0 + Y\mathcal{A}_1 + \dots + Y^d \mathcal{A}_d ,$$

with coefficients  $\mathcal{A}_k \in \mathcal{C}^{n \times n}$ ,  $k = 0, \dots, d$ . Making explicit the dependence in the parameter  $\epsilon$ , we shall write

$$\mathcal{A}_\epsilon = \mathcal{A}_{\epsilon,0} + Y\mathcal{A}_{\epsilon,1} + \dots + Y^d \mathcal{A}_{\epsilon,d} ,$$

where, for every  $k = 0, \dots, d$ ,  $\mathcal{A}_{\epsilon,k}$  is a  $n \times n$  matrix whose coefficients,  $(\mathcal{A}_{\epsilon,k})_{ij}$ , are complex valued continuous functions of the nonnegative parameter  $\epsilon$ . We shall

assume that for every  $0 \leq k \leq d$ , matrices  $a_k = ((a_k)_{ij}) \in \mathbb{C}^{n \times n}$  and  $A_k = ((A_k)_{ij}) \in \mathbb{R}_{\min}^{n \times n}$  are given, so that

$$(29) \quad (\mathcal{A}_{\epsilon,k})_{ij} \simeq (a_k)_{ij} \epsilon^{(A_k)_{ij}} \quad , \quad \text{for all } 1 \leq i, j \leq n \quad .$$

We shall also assume that the matrix polynomial  $\mathcal{A}_\epsilon$  is *regular*, which means that the characteristic polynomial  $\det(\mathcal{A}_\epsilon)$  (or  $\det(\mathcal{A})$ ) of the matrix polynomial  $\mathcal{A}_\epsilon$  is non identically zero. Then, the *eigenvalues* of  $\mathcal{A}_\epsilon$  (or  $\mathcal{A}$ ) are by definition the roots of this polynomial. If  $\deg \det(\mathcal{A}) < nd$ , then we also say that  $\infty$  is an eigenvalue of  $\mathcal{A}$  with multiplicity  $nd - \deg \det(\mathcal{A})$ . When  $\mathcal{A} = \mathcal{A}_0 - YI$ , these are the usual eigenvalues of  $\mathcal{A}_{0,\epsilon}$ . We shall study here the first order asymptotics of the eigenvalues of  $\mathcal{A}_\epsilon$  in the same spirit as in the previous section.

To the matrix polynomial  $\mathcal{A}$ , one can associate the *min-plus matrix polynomial*  $\mathfrak{e}(\mathcal{A}) := \mathfrak{e}(\mathcal{A}_0) \oplus Y\mathfrak{e}(\mathcal{A}_1) \oplus \cdots \oplus Y^d \mathfrak{e}(\mathcal{A}_d)$ . Here, we shall rather consider the min-plus matrix polynomial associated to the asymptotics (29):

$$(30) \quad A = A_0 \oplus Y A_1 \oplus \cdots \oplus Y^d A_d \in \mathbb{R}_{\min}^{n \times n}[Y] \quad .$$

The min-plus matrix polynomial  $A$  can be seen either as a (formal) polynomial with coefficients in  $\mathbb{R}_{\min}^{n \times n}$ , namely the  $A_k$ ,  $k = 0, \dots, d$ , or as a matrix with entries in  $\mathbb{R}_{\min}[Y]$ , denoted  $A_{ij}$ . We denote by  $\widehat{A}$  the function which associates to  $y \in \mathbb{R}_{\min}$ , the matrix  $\widehat{A}(y) := A_0 \oplus y A_1 \oplus \cdots \oplus y^d A_d \in \mathbb{R}_{\min}^{n \times n}$ . We call *min-plus characteristic polynomial* of the matrix polynomial  $A$ , the permanent  $P_A = \text{per } A$ . This is a formal polynomial, the associated polynomial function of which is equal to  $\widehat{P}_A(y) = \text{per } \widehat{A}(y)$ .

We shall say that the matrix polynomial  $A$  is *regular* if  $P_A$  is not identically zero. Then, the min-plus roots of  $P_A$  will be called the *algebraic eigenvalues* of the min-plus matrix polynomial  $A$ . Note that the valuation  $\text{val } P_A$  can be computed by introducing the matrix  $\text{val } A \in \mathbb{R}_{\min}^{n \times n}$ , such that  $(\text{val } A)_{ij} = \text{val } A_{ij}$ , where  $A_{ij}$  denotes the  $(i, j)$  entry of the min-plus polynomial  $A$ . Then,  $\text{val } P_A$  is equal to the min-plus permanent of the matrix  $\text{val } A$ . By symmetry, the degree  $\deg P_A$  is equal to the max-plus permanent of the matrix  $\deg A \in \mathbb{R}_{\max}^{n \times n}$ , such that  $(\deg A)_{ij} = \deg A_{ij}$ . When  $\text{val } P_A > 0$ ,  $0 = +\infty$  is an eigenvalue of  $A$  with multiplicity  $\text{val } P_A$ . When  $\deg P_A < nd$ ,  $P_A$  has only  $\deg P_A$  eigenvalues (belonging to  $\mathbb{R}_{\min}$ ). One can say that  $-\infty$  (the infinity of  $\mathbb{R}_{\min}$ ) is an eigenvalue of  $A$  with multiplicity  $nd - \deg P_A$ . Recall that, for any scalar  $y \in \mathbb{R}_{\min}$ , we have  $\widehat{P}_A(y) = \text{per } \widehat{A}(y)$  which is the value of the optimal assignment associated to the matrix  $\widehat{A}(y)$ . Hence, the algebraic eigenvalues of the matrix polynomial  $A$  (and so, the polynomial function  $\widehat{P}_A$ ) can be computed in  $O(n^4 d)$  time by adapting the method of Burkard and Butkovič [BB03]. Moreover, by adapting the parametric method of Gassner and Klinz [GK10], Hook [HT] showed that this can be reduced to a  $O(n^3 d^2)$  time.

The following result generalizes Theorem 7.1.

**Theorem 8.1.** *Let  $\mathcal{A}$  be as in (28) and (29) and denote by  $A$  the min-plus matrix polynomial (30) with coefficients  $A_k$  as in (29). Assume also that  $\mathcal{A}_d = I$ , the identity matrix in  $\mathbb{C}^{n \times n}$  so that  $A_d = I$  the identity matrix in  $\mathbb{R}_{\min}^{n \times n}$ . Then,  $A$  has  $nd$  min-plus algebraic eigenvalues (belonging to  $\mathbb{R}_{\min}$ ). Let  $\Gamma = (\gamma_1 \leq \cdots \leq \gamma_{nd})$  be the sequence of these eigenvalues. Assume that the eigenvalues  $\mathcal{L}_\epsilon^1, \dots, \mathcal{L}_\epsilon^{nd}$  of  $\mathcal{A}_\epsilon$  (counted with multiplicities) have first order asymptotics,*

$$\mathcal{L}_\epsilon^i \sim \lambda_i \epsilon^{\Lambda_i} \quad ,$$

and let  $\Lambda = (\Lambda_1 \leq \dots \leq \Lambda_n)$ . Then,  $\Lambda \prec^w \Gamma$ .

*Proof.* Let us denote by  $\mathcal{Q} = \det(\mathcal{A}) \in \mathcal{C}[Y]$  the characteristic polynomial of  $\mathcal{A}$  and by  $P = \text{per } A$  the min-plus characteristic polynomial of  $A$ . Since  $\mathcal{A}_d = I$ , we have  $\mathcal{Q}_{nd} = 1$  and  $P_{nd} = \mathbb{1}$ . The eigenvalues  $\mathcal{L}_\epsilon^1, \dots, \mathcal{L}_\epsilon^{nd}$  of  $\mathcal{A}_\epsilon$  are the roots of the polynomial  $\mathcal{Q}$ . Assume that they have first order asymptotics with exponents  $\Lambda_1 \leq \dots \leq \Lambda_{nd}$ . From (20), these are elements of  $\mathcal{C}^\dagger$  and their images by  $\mathbf{e}$  (counted with multiplicities) are equal to  $\Lambda_1 \leq \dots \leq \Lambda_{nd}$ . By Proposition 5.1 applied to the ring  $\mathcal{C}$ ,  $\Lambda = (\Lambda_1 \leq \dots \leq \Lambda_{nd})$  coincide with the roots of the min-plus polynomial  $Q = \mathbf{e}(\mathcal{Q})$ . Then, computing the coefficients of the formal polynomial  $\mathcal{Q}$ , we can deduce that  $Q \geq P$ . Since  $Q_{nd} = P_{nd} = \mathbb{1}$ , we get that  $\Lambda = R(Q) \prec^w R(P) = \Gamma$ .

Note that another way to prove this result is to apply a block companion transformation to the matrix polynomials  $\mathcal{A}$  and  $A$ , yielding linear pencils  $\mathcal{B}$  and  $B$  with leading terms  $\mathcal{B}_1$  and  $B_1$  equal to the identity matrix, then to apply Theorem 7.1 to  $\mathcal{B}_0$ .  $\square$

For any matrix  $B \in \mathbb{R}_{\min}^{n \times n}$  such that  $\text{per } B \neq \emptyset$ , we define the graph  $\text{Opt}(B)$  as the set of arcs belonging to optimal assignments: the set of nodes of  $\text{Opt}(B)$  is  $[n]$  and there is an arc from  $i$  to  $j$  if there is a permutation  $\sigma$  such that  $j = \sigma(i)$  and  $|\sigma|_B = \text{per } B$ . One can compute  $\text{Opt}(B)$  by using the following construction.

We shall say that two vectors  $U, V$  of dimension  $n$  with entries in  $\mathbb{R} = \mathbb{R}_{\min} \setminus \{0\}$  form a *Hungarian pair* with respect to  $B$  if, for all  $i, j$ , we have  $B_{ij} \geq U_i V_j$ , and  $U_1 \cdots U_n V_1 \cdots V_n = \text{per } B$ , the products being understood in the min-plus sense (since  $U, V$  are seen as elements of  $\mathbb{R}_{\min}^n$ ). Thus,  $(U, V)$  coincides with the optimal dual variable in the linear programming formulation of the optimal assignment problem. In particular, a Hungarian pair always exists if the optimal assignment problem is feasible, i.e., if  $\text{per } B \neq \emptyset$ , and it can be computed in  $O(n^3)$  time by the Hungarian algorithm (see for instance [Sch03, § 17]). For any Hungarian pair  $(U, V)$ , we now define the *saturation graph*,  $\text{Sat}(B, U, V)$ , which has set of nodes  $[n]$  and an arc from  $i$  to  $j$  if  $B_{ij} = U_i V_j$ . We shall see in the following section that  $\text{Opt}(B)$  can be computed easily from  $\text{Sat}(B, U, V)$ .

For any min-plus matrix polynomial  $A$  and any scalar  $\gamma \in \mathbb{R}_{\min}$ , we denote by  $\mathcal{G}_k(A, \gamma)$  the graph with set of nodes  $[n]$ , and an arc from  $i$  to  $j \in [n]$  if  $\gamma^k (A_k)_{ij} = \hat{A}_{ij}(\gamma) \neq 0$ . This is a subgraph of the graph of  $\hat{A}(\gamma)$ . For any graphs  $G$  and  $G'$ , the *intersection*  $G \cap G'$  is the graph whose set of nodes (resp. arcs) is the intersection of the sets of nodes (resp. arcs) of  $G$  and  $G'$ . Finally, if  $G$  is any graph with set of nodes  $[n]$ , and if  $b \in \mathbb{C}^{n \times n}$ , we define the matrix  $b^G$  by  $(b^G)_{ij} = b_{ij}$  if  $(i, j) \in G$ , and  $(b^G)_{ij} = 0$  otherwise. The following results generalize Theorem 6.1 and 7.4, respectively. Their proof will be given in the next section.

**Theorem 8.2.** *Let  $\mathcal{A}$  be a regular matrix polynomial over  $\mathcal{C}$  as in (28) satisfying (29) and denote by  $A$  the min-plus matrix polynomial (30) with coefficients  $A_k$  as in (29). Then  $A$  is a regular matrix polynomial and we have:*

- (1)  $\text{val } P_{A_\epsilon} \geq \text{val } P_A$ , which means that if  $0 = +\infty$  is an eigenvalue of  $A$  with multiplicity  $m_{0,A}$ , then  $0$  is an eigenvalue of  $A_\epsilon$  with a multiplicity at least  $m_{0,A}$ .
- (2)  $\text{deg } P_{A_\epsilon} \leq \text{deg } P_A$ , which means that if  $-\infty$  is an eigenvalue of  $A$  with multiplicity  $m_{-\infty,A}$ , then  $\infty$  is an eigenvalue of  $A_\epsilon$  with a multiplicity at least  $m_{-\infty,A}$ .

- (3) Let  $\gamma$  denote any finite ( $\neq \pm\infty$ ) algebraic eigenvalue of  $A$ , and denote by  $m_{\gamma,A}$  its multiplicity. Let  $G$  be equal either to  $\text{Opt}(\widehat{A}(\gamma))$  or  $\text{Sat}(\widehat{A}(\gamma), U, V)$ , for any choice of the Hungarian pair  $(U, V)$  with respect to  $\widehat{A}(\gamma)$ , and let  $G_k = \mathcal{G}_k(A, \gamma) \cap G$  for all  $0 \leq k \leq d$ . Assume that the matrix polynomial

$$(31) \quad a^{(\gamma)} := a_0^{G_0} + \Upsilon a_1^{G_1} + \dots + \Upsilon^d a_d^{G_d} .$$

is regular. Let  $m_\gamma \geq 0$  be the number of non-zero eigenvalues of the matrix polynomial  $a^{(\gamma)}$  and let  $\lambda_1, \dots, \lambda_{m_\gamma}$  be these eigenvalues. Then  $m_\gamma \leq m_{\gamma,A}$  and the matrix polynomial  $\mathcal{A}_\epsilon$  has  $m_\gamma$  eigenvalues  $\mathcal{L}_{\epsilon,1}, \dots, \mathcal{L}_{\epsilon,m_\gamma}$  with first order asymptotics of the form  $\mathcal{L}_{\epsilon,i} \sim \lambda_i \epsilon^\gamma$ .

Let us denote by  $m'_{\gamma,A}$  the sum of the multiplicities of all the algebraic eigenvalues of  $A$  greater than  $\gamma$  ( $+\infty$  comprised), putting  $m'_{\gamma,A} = 0$  if no such eigenvalues exist, and let  $m''_{\gamma,A} = nd - m_{\gamma,A} - m'_{\gamma,A}$ . Let  $m'_\gamma = \text{val det}(a^{(\gamma)})$  be the multiplicity of  $0$  as an eigenvalue of the matrix polynomial  $a^{(\gamma)}$ , with  $m'_\gamma = 0$  if  $0$  is not an eigenvalue. Let also  $m''_\gamma = nd - m_\gamma - m'_\gamma$  be the multiplicity of  $\infty$  as an eigenvalue of the matrix polynomial  $a^{(\gamma)}$ , as it were of degree  $d$ . Then  $m'_\gamma \geq m'_{\gamma,A}$  (resp.  $m''_\gamma \geq m''_{\gamma,A}$ ) and the matrix polynomial  $\mathcal{A}_\epsilon$  has precisely  $m'_\gamma$  (resp.  $m''_\gamma$ ) eigenvalues  $\mathcal{L}_\epsilon$  such that  $\mathcal{L}_\epsilon \simeq 0\epsilon^\gamma$  (resp.  $\mathcal{L}_\epsilon^{-1} \simeq 0\epsilon^{-\gamma}$ , that is the modulus of  $\epsilon^{-\gamma} \mathcal{L}_\epsilon$  converges to infinity).

**Theorem 8.3.** *Let  $A$  be a matrix polynomial over  $\mathcal{C}$  as in (28) satisfying (29) and denote by  $A$  the min-plus matrix polynomial (30) with coefficients  $A_k$  as in (29). Assume that  $A$  is a regular matrix polynomial. Then, for generic values of the parameters  $(a_k)_{i,j}$ , the matrix polynomial  $\mathcal{A}$  is regular, the inequalities in Theorem 8.2 are equalities, and all eigenvalues of the matrix polynomial  $\mathcal{A}_\epsilon$  have first order asymptotics. More precisely, we have:*

- (1)  $\text{val } P_{\mathcal{A}_\epsilon} = \text{val } P_A$ , which means that  $0 = +\infty$  is an eigenvalue of  $A$  if and only if  $0$  is an eigenvalue of  $\mathcal{A}_\epsilon$  and they have same multiplicity.
- (2)  $\text{deg } P_{\mathcal{A}_\epsilon} = \text{deg } P_A$ , which means that  $-\infty$  is an eigenvalue of  $A$  if and only if  $\infty$  is an eigenvalue of  $\mathcal{A}_\epsilon$  and they have same multiplicity.
- (3) for any finite ( $\neq \pm\infty$ ) algebraic eigenvalue  $\gamma$  of  $A$ , the matrix polynomial  $a^{(\gamma)}$  of (31) is regular, and has  $m_{\gamma,A}$  non-zero eigenvalues. Moreover,  $m'_{\gamma,A} = \text{val det}(a^{(\gamma)})$  is the multiplicity of  $0$  as an eigenvalue of the matrix polynomial  $a^{(\gamma)}$ , and  $m''_{\gamma,A} = nd - m_{\gamma,A} - m'_{\gamma,A}$  is the multiplicity of  $\infty$  as an eigenvalue of the matrix polynomial  $a^{(\gamma)}$ , as it were of degree  $d$ .

*Example 8.4.* Consider  $\mathcal{A} = \mathcal{A}_0 - \Upsilon I$ , with

$$\mathcal{A}_0 = \begin{bmatrix} b_{11}\epsilon & b_{12} & b_{13}\epsilon \\ b_{21}\epsilon & b_{22}\epsilon & b_{23} \\ b_{31}\epsilon & b_{32}\epsilon & b_{33}\epsilon \end{bmatrix}, \text{ where } b_{ij} \in \mathbb{C}.$$

When  $b_{12} = b_{23} = 1$ , the matrix  $\mathcal{A}_0$  corresponds to the perturbation of a Jordan block of size 3 and zero-eigenvalue. Višik, Ljusternik and Lidskiĭ theory predicts that the eigenvalues of  $\mathcal{A}_0$  (thus of  $\mathcal{A}$ ) have first order asymptotics of the form  $\mathcal{L}_{\epsilon,i} \sim \lambda_i \epsilon^{1/3}$ ,  $i = 1, \dots, 3$ , where  $\lambda_i$  are the roots of  $\lambda_i^3 = b_{12}b_{23}b_{31}$ . Assume now that  $b_{31} = 0$ , so that we are in a singular case of the Višik, Ljusternik and Lidskiĭ theory, and let us show that the results of the present section apply.

The associated min-plus matrix polynomial and characteristic polynomial function are

$$A = \begin{bmatrix} 1 \oplus Y & 0 & 1 \\ 1 & 1 \oplus Y & 0 \\ +\infty & 1 & 1 \oplus Y \end{bmatrix}, \quad \widehat{P}_A(x) = (x \oplus 0.5)^2(x \oplus 1),$$

so that the eigenvalues of  $A$  are  $\gamma_1 = \gamma_2 = 0.5$ , with multiplicity 2, and  $\gamma_3 = 1$ , with multiplicity 1. Hence, Theorem 8.3 predicts that two of the eigenvalues of  $\mathcal{A}_0$  have first asymptotics of the form  $\mathcal{L} \sim \lambda\epsilon^{1/2}$ , with  $\lambda \neq 0$ , and that one of them has first asymptotics of the form  $\mathcal{L} \sim \lambda\epsilon^1$ . Note that the eigenvalue  $\gamma = 0.5$  is the unique geometric min-plus eigenvalue of  $\mathcal{A}_0$  and that the associated critical graph covers all nodes  $\{1, 2, 3\}$ , so that the results of [ABG04a] can only predict the two first asymptotics of the form  $\mathcal{L} \sim \lambda\epsilon^{1/2}$ .

Let us now detail the results of Theorem 8.3. We first consider the eigenvalue  $\gamma = 0.5$ . Then  $U = (0, 0.5, 1)$  and  $V = (0.5, 0, -0.5)$  yields a Hungarian pair with respect to the matrix

$$\widehat{A}(0.5) = \begin{bmatrix} 0.5_1 & 0_0 & 1 \\ 1_0 & 0.5_1 & 0_0 \\ +\infty & 1_0 & 0.5_1 \end{bmatrix},$$

where we adopt the following convention to visualize the graphs  $G_k = \mathcal{G}_k(A, \gamma) \cap \text{Sat}(\widehat{A}(\gamma), U, V)$ : an arc  $(i, j)$  belongs to  $G_k$  if  $k$  is put as a subscript of the entry  $\widehat{A}_{ij}(\gamma)$ . For instance,  $\widehat{A}_{11}(0.5) = 0.5$ , and  $(1, 1)$  belongs  $G_1$ . Entries without subscripts, like  $\widehat{A}_{13}(0.5) = 1$ , correspond to arcs which do not belong to  $\text{Sat}(\widehat{A}(0.5), U, V)$ . The eigenvalues of the matrix polynomial  $a^{(0)}$  are the roots of

$$\det \begin{bmatrix} -\lambda & b_{12} & 0 \\ b_{21} & -\lambda & b_{23} \\ 0 & b_{32} & -\lambda \end{bmatrix} = \lambda(-\lambda^2 + b_{12}b_{21} + b_{32}b_{23}) = 0.$$

Theorem 8.3 predicts that this equation has, for generic values of the parameters  $b_{ij}$ , two non-zero roots,  $\lambda_1, \lambda_2$ , which yields two eigenvalues of  $\mathcal{A}_\epsilon$ ,  $\mathcal{L}_{\epsilon,i} \sim \lambda_i\epsilon^{1/2}$ , for  $i = 1, 2$ . Here  $\lambda_1$  and  $\lambda_2$  are the square roots of  $b_{12}b_{21} + b_{32}b_{23}$ .

Consider finally the eigenvalue  $\gamma = 1$ . We can take  $U = (0, 0, 1)$ ,  $V = (1, 0, 0)$ , and the previous computations become

$$\widehat{A}(1) = \begin{bmatrix} 1_{01} & 0_0 & 1 \\ 1_0 & 1 & 0_0 \\ +\infty & 1_0 & 1_{01} \end{bmatrix},$$

$$\det \begin{bmatrix} b_{11} - \lambda & b_{12} & 0 \\ b_{21} & 0 & b_{23} \\ 0 & b_{32} & b_{33} - \lambda \end{bmatrix} = \lambda(b_{12}b_{21} + b_{23}b_{32}) - (b_{12}b_{21}b_{33} + b_{11}b_{23}b_{32}) = 0.$$

Theorem 8.3 predicts that this equation has, for generic values of the parameters  $b_{ij}$ , a unique nonzero root,  $\lambda_3$ , and that there is a branch  $\mathcal{L}_{\epsilon,3} \sim \lambda_3\epsilon$ . Here  $\lambda_3 = (b_{12}b_{21}b_{33} + b_{11}b_{23}b_{32}) / (b_{12}b_{21} + b_{23}b_{32})$ .

## 9. PRELIMINARIES ON HUNGARIAN PAIRS

To prove Theorems 8.2 and 8.3, we need to establish some properties of the saturation graph associated to Hungarian pairs.

We shall adopt the following notation. For any  $U \in \mathbb{R}^n$ , we denote by  $\mathbf{d}(U)$  the diagonal  $n \times n$  matrix over  $\mathbb{R}_{\min}$  such that  $(\mathbf{d}(U))_{ii} = U_i$ , so that  $(\mathbf{d}(U))_{ij} = +\infty$  for  $i \neq j$ . For any permutation  $\sigma$  of  $[n]$ , we denote by  $P_{\min}^\sigma$  its associated min-plus permutation matrix:  $(P_{\min}^\sigma)_{ij} = \mathbb{1}$  if  $j = \sigma(i)$  and  $(P_{\min}^\sigma)_{ij} = \mathbb{0}$  otherwise. We reserve the simpler notation  $P^\sigma$  for the ordinary permutation matrix which is such that  $P_{ij}^\sigma = 1$  if  $j = \sigma(i)$  and  $P_{ij}^\sigma = 0$  otherwise.

Moreover, if  $U \in \mathbb{R}^n$ , we use the notation  $U_\sigma := (U_{\sigma(i)})_{i=1, \dots, n}$ . We shall say that a matrix  $M \in \mathbb{R}_{\min}^{n \times n}$  is *monomial* if it can be written as  $M = \mathbf{d}(U)P_{\min}^\sigma$  for some  $U \in \mathbb{R}^n$  and  $\sigma \in \mathfrak{S}_n$ . We have equivalently  $M = P_{\min}^\sigma \mathbf{d}(V)$ , by taking  $V = U_{\sigma^{-1}}$ . Recall that the monomial matrices are the only invertible matrices over  $\mathbb{R}_{\min}$ , and that  $M^{-1} = P_{\min}^{\sigma^{-1}} \mathbf{d}(-U)$ . For all matrices  $A$ ,  $A^T$  denotes the transpose of  $A$ . Now if  $G$  is a graph with set of nodes  $[n]$ , and  $\sigma, \tau \in \mathfrak{S}_n$ , we denote by  $G_{\sigma, \tau}$  the graph with the same set of nodes, and an arc  $(i, j)$  if and only if  $(\sigma(i), \tau^{-1}(j))$  is an arc of  $G$ .

The following elementary result will allow one to normalize matrices in a suitable way.

**Lemma 9.1.** *Let  $B \in \mathbb{R}_{\min}^{n \times n}$  such that  $\text{per } B \neq \mathbb{0}$ , and let  $M = \mathbf{d}(W)P_{\min}^\sigma$  and  $N = \mathbf{d}(X)P_{\min}^\tau$  be monomial matrices, with  $W, X \in \mathbb{R}^n$  and  $\sigma, \tau \in \mathfrak{S}_n$ . Then  $\text{per}(MBN) = W_1 \cdots W_n X_1 \cdots X_n \text{per } B = (\text{per } M)(\text{per } B)(\text{per } N)$ ,  $\nu$  is an optimal permutation for  $MBN$  if and only if  $\tau^{-1} \circ \nu \circ \sigma^{-1}$  is an optimal permutation for  $B$ , and  $\text{Opt}(MBN) = \text{Opt}(B)_{\sigma, \tau}$ .*

*Let  $(U, V)$  be a Hungarian pair with respect to  $B$ . Then,  $(MU, N^T V)$  is a Hungarian pair with respect to  $MBN$ , and we have  $\text{Sat}(MBN, MU, N^T V) = \text{Sat}(B, U, V)_{\sigma, \tau}$ .*

*Proof.* Let  $B, M$ , and  $N$  be as in the lemma. For all  $i, j \in [n]$ , we have  $(MBN)_{ij} = W_i B_{\sigma(i), \tau^{-1}(j)} X_{\tau^{-1}(j)}$ . Hence, for all  $\nu \in \mathfrak{S}_n$ , the weight of  $\nu$  with respect to  $MBN$  is equal to

$$|\nu|_{MBN} = W_1 \cdots W_n X_1 \cdots X_n |\tau^{-1} \circ \nu \circ \sigma^{-1}|_B .$$

Hence,  $\text{per}(MBN) = W_1 \cdots W_n X_1 \cdots X_n \text{per } B$ , and  $\nu$  is an optimal permutation for  $MBN$  if and only if  $\tau^{-1} \circ \nu \circ \sigma^{-1}$  is an optimal permutation for  $B$ . Moreover, since  $\text{per } M = W_1 \cdots W_n$  and  $\text{per } N = X_1 \cdots X_n$ , we deduce that  $\text{per}(MBN) = (\text{per } M)(\text{per } B)(\text{per } N)$ .

Let  $(i, j)$  be an arc of  $\text{Opt}(MBN)$ . Then, by definition, there exists an optimal permutation  $\nu$  for  $MBN$  such that  $j = \nu(i)$ , hence  $\nu' = \tau^{-1} \circ \nu \circ \sigma^{-1}$  is an optimal permutation for  $B$  such that  $\tau^{-1}(j) = \nu'(\sigma(i))$ , which implies that  $(\sigma(i), \tau^{-1}(j))$  is an arc of  $\text{Opt}(B)$ , so  $(i, j)$  is an arc of  $\text{Opt}(B)_{\sigma, \tau}$ . Since the reverse implication is also true (replace  $M$  and  $N$  by their inverse matrices), this shows the first assertion of the lemma.

Now let  $(U, V)$  be a Hungarian pair with respect to  $B$ . We have  $B_{ij} \geq U_i V_j$  for all  $i, j \in [n]$ . Hence,  $(MBN)_{ij} = W_i B_{\sigma(i), \tau^{-1}(j)} X_{\tau^{-1}(j)} \geq W_i U_{\sigma(i)} V_{\tau^{-1}(j)} X_{\tau^{-1}(j)} = (MU)_i (N^T V)_j$ , and

$$\begin{aligned} \text{per}(MBN) &= W_1 \cdots W_n X_1 \cdots X_n \text{per } B \\ &= W_1 \cdots W_n U_1 \cdots U_n X_1 \cdots X_n V_1 \cdots V_n \\ &= (MU)_1 \cdots (MU)_n (N^T V)_1 \cdots (N^T V)_n , \end{aligned}$$

so that  $(MU, N^T V)$  is a Hungarian pair with respect to  $MBN$ . Finally,  $(i, j)$  is an arc of  $\text{Sat}(MBN, MU, N^T V)$  if and only if  $(MBN)_{ij} = (MU)_i (N^T V)_j$ , which is

equivalent to  $B_{\sigma(i), \tau^{-1}(j)} = U_{\sigma(i)} V_{\tau^{-1}(j)}$ , then to the property that  $(\sigma(i), \tau^{-1}(j))$  is an arc of  $\text{Sat}(B, U, V)$ , hence to the one that  $(i, j)$  is an arc of  $\text{Sat}(B, U, V)_{\sigma, \tau}$ .  $\square$

We denote by  $\mathbb{1}$  the vector of  $\mathbb{R}^n$  with all its entries equal to  $\mathbb{1} = 0$ .

**Corollary 9.2.** *Let  $B \in \mathbb{R}_{\min}^{n \times n}$  be such that  $\text{per } B \neq 0$ , and let  $(U, V)$  be a Hungarian pair with respect to  $B$ . Then,  $(\mathbb{1}, \mathbb{1})$  is a Hungarian pair with respect to the matrix  $C := d(U)^{-1} B d(V)^{-1} \in \mathbb{R}_{\min}^{n \times n}$ ,  $\text{Sat}(B, U, V) = \text{Sat}(C, \mathbb{1}, \mathbb{1})$  and  $\text{Opt}(B) = \text{Opt}(C)$ . In particular,  $\text{per } C = \mathbb{1}$ ,  $C_{ij} \geq \mathbb{1} = 0$  for all  $i, j \in [n]$  and  $(i, j)$  is an arc of  $\text{Sat}(B, U, V)$  if and only if  $C_{ij} = \mathbb{1}$ .*

*Proof.* Let  $B, U, V, C$  be as in the corollary. Then, by Lemma 9.1 applied to  $M = d(U)^{-1}$  and  $N = d(V)^{-1}$ , we obtain the first assertion of the corollary, together with  $\text{per } C = \mathbb{1}$ . Then,  $C_{ij} \geq \mathbb{1} = 0$  for all  $i, j \in [n]$  and  $(i, j)$  is an arc of  $\text{Sat}(B, U, V)$  if and only if  $C_{ij} = \mathbb{1}$ .  $\square$

**Corollary 9.3.** *Let  $B \in \mathbb{R}_{\min}^{n \times n}$  be such that  $\text{per } B \neq 0$ , and let  $(U, V)$  be a Hungarian pair with respect to  $B$ . Then,  $\text{Opt}(B) \subset \text{Sat}(B, U, V)$ . Moreover, the following are equivalent for  $\sigma \in \mathfrak{S}_n$ : (i)  $\sigma$  is a permutation of  $\text{Sat}(B, U, V)$ ; (ii)  $\sigma$  is a permutation of  $\text{Opt}(B)$ ; (iii)  $\sigma$  is an optimal permutation of  $B$ .*

*Proof.* Let  $C$  be as in Corollary 9.2. Then, if  $(i, j)$  is an arc of  $\text{Opt}(C)$ , there exists  $\sigma \in \mathfrak{S}_n$  such that  $j = \sigma(i)$  and  $|\sigma|_C = \text{per } C = \mathbb{1}$ . Since all entries of  $C$  are greater or equal to  $\mathbb{1}$ , this implies that  $C_{ij} = \mathbb{1}$ , and so  $(i, j)$  is an arc of  $\text{Sat}(C, \mathbb{1}, \mathbb{1})$ . Since  $\text{Opt}(C) = \text{Opt}(B)$  and  $\text{Sat}(C, \mathbb{1}, \mathbb{1}) = \text{Sat}(B, U, V)$ , this shows that  $\text{Opt}(B) \subset \text{Sat}(B, U, V)$ .

Let us show that for  $\sigma \in \mathfrak{S}_n$ , (i)  $\Rightarrow$  (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i). Let  $\sigma$  be a permutation of  $\text{Sat}(B, U, V)$ , this means that  $(i, \sigma(i))$  is an arc of  $\text{Sat}(B, U, V)$ , for all  $i \in [n]$ . Then  $C_{i\sigma(i)} = \mathbb{1}$ , for all  $i \in [n]$ , which implies that  $|\sigma|_C = \mathbb{1} = \text{per } C$ , and so  $\sigma$  is an optimal permutation of  $C$ , hence an optimal permutation of  $B$ , by Lemma 9.1, which shows (i)  $\Rightarrow$  (iii). By definition of  $\text{Opt}(B)$ , if  $\sigma$  be an optimal permutation of  $B$ , then  $(i, \sigma(i))$  is an arc of  $\text{Opt}(B)$ , for all  $i \in [n]$ , so  $\sigma$  is a permutation of  $\text{Opt}(B)$ , which shows (iii)  $\Rightarrow$  (ii). If now  $\sigma$  be a permutation of  $\text{Opt}(B)$ , then, for all  $i \in [n]$ ,  $(i, \sigma(i))$  is an arc of  $\text{Opt}(B) \subset \text{Sat}(B, U, V)$ , hence,  $\sigma$  is a permutation of  $\text{Sat}(B, U, V)$ , which shows (ii)  $\Rightarrow$  (i).  $\square$

**Corollary 9.4.** *Let  $B \in \mathbb{R}_{\min}^{n \times n}$  such that  $\text{per } B \neq 0$ , let  $\sigma$  be an optimal permutation for  $B$ , and let  $(U, V)$  be a Hungarian pair with respect to  $B$ . Then, the identity map is an optimal permutation for  $P_{\min}^{\sigma^{-1}} B$  and  $(U_{\sigma^{-1}}, V)$  is a Hungarian pair with respect to  $P_{\min}^{\sigma^{-1}} B$ . Moreover, we have  $\text{per}(P_{\min}^{\sigma^{-1}} B) = \text{per}(B)$ ,  $\text{Opt}(P_{\min}^{\sigma^{-1}} B) = \text{Opt}(B)_{\sigma^{-1}, \text{id}}$ , and  $\text{Sat}(P_{\min}^{\sigma^{-1}} B, U_{\sigma^{-1}}, V) = \text{Sat}(B, U, V)_{\sigma^{-1}, \text{id}}$ .*

*Proof.* From Lemma 9.1,  $\nu$  is an optimal permutation for  $P_{\min}^{\sigma^{-1}} B$  if and only if  $\nu \circ \sigma$  is an optimal permutation for  $B$ . Hence if  $\sigma$  is an optimal permutation for  $B$ , the identity map is an optimal permutation for  $P_{\min}^{\sigma^{-1}} B$ . The other properties follow from Lemma 9.1.  $\square$

**Proposition 9.5.** *Let  $B \in \mathbb{R}_{\min}^{n \times n}$  such that  $\text{per } B = \mathbb{1}$ ,  $(\mathbb{1}, \mathbb{1})$  is a Hungarian pair with respect to  $B$  and the identity map is an optimal permutation of  $B$ . Then  $B\mathbb{1} = \mathbb{1}$ ,  $\rho_{\min}(B) = \mathbb{1}$ , and  $\text{Opt}(B)$  is the critical graph of  $B$ .*



*Proof.* Let  $B$  be as in the proposition. Since  $(\mathbb{1}, \mathbb{1})$  is a Hungarian pair with respect to  $B$ , Corollary 9.2 shows that all entries of  $B$  are greater or equal to  $\mathbb{1}$ , and that  $B_{ij} = \mathbb{1}$  for all arcs  $(i, j)$  of  $\text{Sat}(B, \mathbb{1}, \mathbb{1})$ . Then, by Corollary 9.3,  $B_{ij} = \mathbb{1}$  for all arcs  $(i, j)$  of an optimal permutation of  $B$ . Since the identity map is an optimal permutation of  $B$ , this implies that  $B_{ii} = \mathbb{1}$  for all  $i$ , which implies  $B\mathbb{1} = \mathbb{1}$ . Moreover, all circuits have a weight with respect to  $B$  greater or equal to  $\mathbb{1}$  and the circuits of an optimal permutation have a weight equal to  $\mathbb{1}$ . This implies that  $\rho_{\min}(B) = \mathbb{1}$ .

Let  $(i, j)$  be an arc of the critical graph of  $B$ . By definition (see Section 3), there exists a circuit passing through  $(i, j)$  with mean weight with respect to  $B$  equal to  $\mathbb{1}$ , so with weight equal to  $\mathbb{1}$ . Since the identity map is an optimal permutation of  $B$ , any circuit with weight  $\mathbb{1}$  can be completed into a permutation with weight  $\mathbb{1}$ , by taking the identity on the complementary of this circuit, hence,  $(i, j)$  is an arc of  $\text{Opt}(B)$ . Conversely, let  $(i, j)$  be an arc of  $\text{Opt}(B)$ , then there exists an optimal permutation  $\sigma$  for  $B$  such that  $j = \sigma(i)$ . This implies (as above) that  $B_{k\sigma(k)} = \mathbb{1}$ , for all  $k \in [n]$ . Then, all the circuits of  $\sigma$  have a weight equal to  $\mathbb{1}$ , and so are critical circuits. Since  $(i, j)$  belongs to one of them, this shows that  $(i, j)$  is an arc of the critical graph of  $B$ .  $\square$

**Corollary 9.6.** *Let  $B \in \mathbb{R}_{\min}^{n \times n}$  be as in Proposition 9.5. Then,  $\text{Opt}(B)$  is the disjoint union of the strongly connected components of  $\text{Sat}(B, \mathbb{1}, \mathbb{1})$ .*

*Proof.* By Proposition 9.5,  $\mathbb{1}$  is a fixed point of  $B$ , and  $\text{Opt}(B)$  is the critical graph of  $B$ . It is known that the critical graph of  $B$  is the disjoint union of the strongly connected components of the saturation graph of  $B$  on any eigenvector  $U$  of  $B$ , that is the set of arcs  $(i, j)$  such that  $B_{ij}U_j = U_i$  (see for instance [ABG04a]). When  $U = \mathbb{1}$ , this saturation graph coincides with  $\text{Sat}(B, \mathbb{1}, \mathbb{1})$ , which shows the corollary. Let us reproduce the proof in that case. Indeed, let  $(i, j)$  be an arc of a strongly connected component of the saturation graph of  $B$  on the eigenvector  $\mathbb{1}$ . Then there exists a path from  $j$  to  $i$  in this saturation graph. Concatenating this path with the arc  $(i, j)$ , we get a circuit, the weights with respect to  $B$  of which are all equal to  $\mathbb{1}$ , hence a critical circuit of  $B$ . This shows that  $(i, j)$  is an arc of the critical graph of  $B$ . Since the critical graph of  $B$  is the union of its strongly connected components, and is always included in the saturation graph, we get the desired property.  $\square$

**Corollary 9.7.** *Let  $B \in \mathbb{R}_{\min}^{n \times n}$  such that  $\text{per } B \neq 0$ ,  $(U, V)$  be a Hungarian pair with respect to  $B$ , and suppose that the identity map is an optimal permutation of  $B$ . Then,  $\text{Opt}(B)$  is the disjoint union of the strongly connected components of  $\text{Sat}(B, U, V)$ .*

*Proof.* Corollary 9.2 shows that the matrix  $C$  defined there satisfies  $\text{per } C = \mathbb{1}$ , that  $(\mathbb{1}, \mathbb{1})$  is a Hungarian pair with respect to  $C$ , together with  $\text{Opt}(B) = \text{Opt}(C)$  and  $\text{Sat}(B, U, V) = \text{Sat}(C, \mathbb{1}, \mathbb{1})$ . Moreover, by Lemma 9.1, the optimal permutations for  $B$  and  $C$  coincide, so that the identity map is an optimal permutation of  $C$ . Hence,  $C$  satisfies the assumptions of Proposition 9.5, so that, by Corollary 9.6,  $\text{Opt}(C)$  is the disjoint union of the strongly connected components of  $\text{Sat}(C, \mathbb{1}, \mathbb{1})$ . With the above equalities, this shows the corollary.  $\square$

The above results show that  $\text{Opt}(B)$  can be easily be constructed from any Hungarian pair with respect to  $B$ , as follows.

**Corollary 9.8.** *Let  $B \in \mathbb{R}_{\min}^{n \times n}$  such that  $\text{per } B \neq 0$ . Let  $(U, V)$  be a Hungarian pair with respect to  $B$ , and  $\sigma$  be a permutation of  $\text{Sat}(B, U, V)$ . Then,  $\text{Opt}(B)_{\sigma^{-1}, \text{id}}$  is the disjoint union of the strongly connected components of  $\text{Sat}(B, U, V)_{\sigma^{-1}, \text{id}}$ .*

## 10. PROOF OF THEOREMS 8.2 AND 8.3

We prove here the different assertions in Theorems 8.2 and 8.3. Since the intersection of generic sets is a generic set, it suffices to prove separately that each point of Theorem 8.3 holds for generic values of the parameters  $(a_k)_{ij}$ .

Let us introduce some additional notation. For any  $n \geq 1$  and  $d \geq 0$ , the characteristic polynomial  $\det(a)$  of the matrix polynomial  $a = a_0 + Y a_1 + \dots + Y^d a_d$  with coefficients in  $\mathbb{C}^{n \times n}$  can be thought of as a (formal) complex polynomial, denoted  $\text{ch}$ , in the variables  $Y$  and  $(a_k)_{ij}$  with  $k = 0, \dots, d$  and  $i, j = 1, \dots, n$ , the coefficients of which are integers. Hence, the coefficient  $\text{ch}_\ell$ ,  $0 \leq \ell \leq nd$ , of  $Y^\ell$  in  $\text{ch}$  can be thought of as a (formal) complex polynomial in the variables  $(a_k)_{ij}$  with  $k = 0, \dots, d$  and  $i, j = 1, \dots, n$ . In the sequel, we shall write  $p(a)$  when  $p$  is a polynomial in the variables  $(a_k)_{ij} \in \mathbb{C}$  with  $k = 0, \dots, d$  and  $i, j = 1, \dots, n$ , and  $a = a_0 + Y a_1 + \dots + Y^d a_d$ . We have in particular  $\text{ch}_\ell(a_0 + YI) = \text{tr}_{n-\ell}(a_0)$ . Similarly, the characteristic polynomial  $\text{per } a$  of the min-plus (formal) matrix polynomial  $a = a_0 \oplus Y a_1 \oplus \dots \oplus Y^d a_d$ , with coefficients in  $\mathbb{R}_{\min}^{n \times n}$  can be thought of as a (formal) min-plus polynomial, denoted  $\text{ch}^{\min}$ , in the variables  $Y$  and  $(a_k)_{ij}$  with  $k = 0, \dots, d$  and  $i, j = 1, \dots, n$ , the coefficients of which are all equal to  $\mathbb{1}$ , and the coefficient  $\text{ch}_\ell^{\min}$ ,  $0 \leq \ell \leq nd$ , of  $Y^\ell$  in  $\text{ch}^{\min}$  can be thought of as a (formal) min-plus polynomial in the variables  $(a_k)_{ij}$  with  $k = 0, \dots, d$  and  $i, j = 1, \dots, n$ . Again, we shall write  $p(a)$  instead of  $p$ , when  $p$  is a min-plus formal polynomial in the variables  $(a_k)_{ij} \in \mathbb{R}_{\min}$  with  $k = 0, \dots, d$  and  $i, j = 1, \dots, n$ , and  $a = a_0 \oplus Y a_1 \oplus \dots \oplus Y^d a_d$ . We shall also use the notation  $\widehat{p}(a)$  for the associated polynomial function. We have in particular  $\text{ch}_\ell^{\min}(a_0 \oplus YI) = \text{tr}_{n-\ell}^{\min}(a_0)$ , where  $\text{tr}_k^{\min}$  is the min-plus (formal)  $k$ -th trace given by the formula (7). In the sequel,  $\widehat{\text{ch}}_\ell^{\min}$  will denote the polynomial function associated to the formal polynomial  $\text{ch}_\ell^{\min}$ .

In all the proofs of the present section, we consider a matrix polynomial  $\mathcal{A}$  over  $\mathbb{C}$  as in (28) satisfying (29) and denote by  $A$  the min-plus matrix polynomial (30) with coefficients  $A_k$  as in (29).

**10.1. Proof of the first properties in Theorems 8.2 and 8.3.** Let us prove the following implications which correspond to the first properties stated in Theorems 8.2 and 8.3.

- (1) If  $\mathcal{A}$  is regular, then so does  $A$ .
- (2) If  $A$  is regular, then for generic values of the parameters  $(a_k)_{ij}$ , the matrix polynomial  $\mathcal{A}$  is regular.

From the above notations, it is easy to see that the min-plus polynomial  $\text{e}(\text{ch})$  (defined as in (25)) is equal to  $\text{ch}^{\min}$  and that  $\text{e}(\text{ch}_\ell) = \text{ch}_\ell^{\min}$ . Then, by Lemma 7.3, we obtain that for  $a$ ,  $A$  and  $\mathcal{A}$  as above, we have  $\text{ch}_\ell(\mathcal{A}) \simeq (\text{ch}_\ell)_A^{\text{Sat}}(a) \epsilon^{\widehat{\text{ch}}_\ell^{\min}(A)}$ . Moreover, by Lemma 7.3 again, for any fixed  $A$  such that  $\widehat{\text{ch}}_\ell^{\min}(A) \neq 0$ , we have  $(\text{ch}_\ell)_A^{\text{Sat}}(a) \neq 0$  for generic values of  $a$  (so of the  $(a_k)_{ij}$ ), hence the equivalence  $\text{ch}_\ell(\mathcal{A}) \sim (\text{ch}_\ell)_A^{\text{Sat}}(a) \epsilon^{\widehat{\text{ch}}_\ell^{\min}(A)}$ .

Using these properties, we see that if  $A$  is singular, then  $\widehat{\text{ch}}_\ell^{\min}(A) = 0$  for all  $\ell \geq 0$ , hence  $\text{ch}_\ell(\mathcal{A}) \simeq (\text{ch}_\ell)_A^{\text{Sat}}(a) \epsilon^{+\infty}$  and so  $\text{ch}_\ell(\mathcal{A}) = 0$ , which implies that

$\det(\mathcal{A}) = \text{ch}(\mathcal{A}) = 0$  and so  $\mathcal{A}$  is singular. Conversely, if  $A$  is regular, then there exists  $\ell \geq 0$  (for instance the degree of  $\text{per } A$ ) such that  $\alpha = \widehat{\text{ch}}_\ell^{\min}(A) \neq 0$ , and since for generic values of  $a$ ,  $(\text{ch}_\ell)_A^{\text{Sat}}(a) \neq 0$ , and  $\text{ch}_\ell(\mathcal{A}) \sim (\text{ch}_\ell)_A^{\text{Sat}}(a)\epsilon^\alpha$ , we get that  $\text{ch}_\ell(\mathcal{A}) \neq 0$ . Hence,  $\det(\mathcal{A}) = \text{ch}(\mathcal{A})$  is non identically zero, so the matrix polynomial  $\mathcal{A}$  is regular for generic values of  $a$ .

**10.2. Proof of Points (1) and (2) of Theorems 8.2 and 8.3.** Assume first that  $\mathcal{A}$  is regular as in Theorem 8.2. By the property 1 already proved in the previous section,  $A$  is also regular. Then,  $P_A = \text{per } A$  and  $P_{\mathcal{A}_\epsilon} = \det(\mathcal{A}_\epsilon)$  have finite valuations and degree, such that  $0 \leq \text{val } P_A \leq \deg P_A \leq nd$  and  $0 \leq \text{val } P_{\mathcal{A}_\epsilon} \leq \deg P_{\mathcal{A}_\epsilon} \leq nd$ . The inequalities  $\text{val } P_{\mathcal{A}_\epsilon} \geq \text{val } P_A$  and  $\deg P_{\mathcal{A}_\epsilon} \leq \deg P_A$  are trivial when  $\text{val } P_A = 0$  and  $\deg P_A = nd$ . When  $\text{val } P_A > 0$ ,  $0$  is an eigenvalue of  $A$  with multiplicity  $m_{0,A} = \text{val } P_A$ , and  $(P_A)_\ell = \widehat{\text{ch}}_\ell^{\min}(A) = 0$ , for  $\ell < \text{val } P_A$ . Then, by the same arguments as in the previous section, we have  $\text{ch}_\ell(\mathcal{A}) = 0$ , for  $\ell < \text{val } P_A$ , hence, for all  $\epsilon > 0$ ,  $\text{val } P_{\mathcal{A}_\epsilon} \geq \text{val } P_A$ , and  $0$  is an eigenvalue of  $\mathcal{A}_\epsilon$  with multiplicity  $\geq m_{0,A}$ . Similarly, when  $\deg P_A < nd$ ,  $-\infty$  is an eigenvalue of  $A$  with multiplicity  $m_{-\infty,A} = nd - d_A$ , and  $(P_A)_\ell = \widehat{\text{ch}}_\ell^{\min}(A) = 0$ , for  $\ell > \deg P_A$ . Then, by the same arguments as in the previous section, we have  $\text{ch}_\ell(\mathcal{A}) = 0$ , for  $\ell > \deg P_A$ , hence  $\deg P_{\mathcal{A}_\epsilon} \leq \deg P_A$ , and  $\infty$  is an eigenvalue of  $\mathcal{A}_\epsilon$  with multiplicity  $\geq m_{-\infty,A}$ . This proves Points (1) and (2) of Theorem 8.2.

Assume now that  $A$  is regular as in Theorem 8.3. By the property 2 already proved in the previous section,  $\mathcal{A}$  is regular, for generic values of the parameters  $(a_k)_{ij}$ . Then, by what is already proved above, for these generic values of the parameters  $(a_k)_{ij}$ , the inequalities  $\text{val } P_{\mathcal{A}_\epsilon} \geq \text{val } P_A$  and  $\deg P_{\mathcal{A}_\epsilon} \leq \deg P_A$  hold. By the definition of the valuation and degree, we have  $(P_A)_\ell = \widehat{\text{ch}}_\ell^{\min}(A) \neq 0$ , for  $\ell = \text{val } P_A$  and  $\ell = \deg P_A$ . Moreover, using Lemma 7.3, for both values of  $\ell$ , we have  $\text{ch}_\ell(\mathcal{A}) \simeq (\text{ch}_\ell)_A^{\text{Sat}}(a)\epsilon^{\widehat{\text{ch}}_\ell^{\min}(A)}$ , and for generic values of  $a$ , we have  $(\text{ch}_\ell)_A^{\text{Sat}}(a) \neq 0$ , and  $\text{ch}_\ell(\mathcal{A}) \sim (\text{ch}_\ell)_A^{\text{Sat}}(a)\epsilon^\alpha$ , so  $\text{ch}_\ell(\mathcal{A}) \neq 0$ . Hence, for these generic values of the parameters  $(a_k)_{ij}$ , we have  $\text{val } P_{\mathcal{A}_\epsilon} = \text{val } P_A$  and  $\deg P_{\mathcal{A}_\epsilon} = \deg P_A$ . Since the intersection of generic sets is a generic set, all the above properties hold for generic values of  $a$ :  $\mathcal{A}$  is regular,  $\text{val } P_{\mathcal{A}_\epsilon} = \text{val } P_A$  and  $\deg P_{\mathcal{A}_\epsilon} = \deg P_A$ .

**10.3. Proof of Point (3) of Theorems 8.2 and 8.3.** Let  $\gamma$ ,  $m_{\gamma,A}$ ,  $m'_{\gamma,A}$ ,  $m''_{\gamma,A}$  be as in Point (3) of Theorems 8.2. Let  $(U, V)$  be a Hungarian pair  $(U, V)$  with respect to  $\widehat{A}(\gamma)$ , and let us first consider the case where  $G = \text{Sat}(\widehat{A}(\gamma), U, V)$ , and  $G_k = \mathcal{G}_k(A, \gamma) \cap G$  for all  $0 \leq k \leq d$ .

Let us add some notations. For any  $U \in \mathbb{R}^n$ , we denote by  $\mathbf{d}_\epsilon(U)$  the diagonal  $n \times n$  matrix over  $\mathcal{C}$  such that  $(\mathbf{d}_\epsilon(U))_{ii} = \epsilon^{U_i}$  (and  $(\mathbf{d}_\epsilon(U))_{ij} = 0$  for  $i \neq j$ ). Then  $\mathbf{d}_\epsilon(U)$  is invertible (in  $\mathcal{C}^{n \times n}$ ) and  $\mathbf{d}_\epsilon(U)^{-1} = \mathbf{d}_\epsilon(-U)$ .

Consider the scaled matrix polynomial

$$\mathcal{B}_\epsilon = \mathbf{d}_\epsilon(-U)\mathcal{A}_\epsilon(\epsilon^\gamma Y)\mathbf{d}_\epsilon(-V) \text{ ,}$$

or, making explicit the coefficients of  $\mathcal{B}_\epsilon$ ,

$$\mathcal{B}_\epsilon = \mathcal{B}_{\epsilon,0} + Y\mathcal{B}_{\epsilon,1} + \cdots + Y^d\mathcal{B}_{\epsilon,d} \text{ .}$$

Then, using for instance Lemma 7.3, we get that for every  $k = 0, \dots, d$ ,

$$(32) \quad (\mathcal{B}_{\epsilon,k})_{ij} \simeq (a_k)_{ij}\epsilon^{(B_k)_{ij}} \text{ ,} \quad \text{for all } 1 \leq i, j \leq n \text{ ,}$$

where  $B_k = ((B_k)_{ij}) \in \mathbb{R}_{\min}^{n \times n}$  is such that  $B_k = \mathbf{d}(-U)\gamma^k A_k \mathbf{d}(-V) = \mathbf{d}(U)^{-1}\gamma^k A_k \mathbf{d}(V)^{-1}$ . Let us denote by  $B$  the min-plus matrix polynomial  $B = B_0 \oplus \cdots \oplus Y^d B_d$ . Then  $\mathbf{1}$  is an eigenvalue of  $B$  with multiplicity  $m_{\mathbf{1},B} = m_{\gamma,A}$ , and using the appropriate notations, we have  $m'_{\mathbf{1},B} = m'_{\gamma,A}$  and  $m''_{\mathbf{1},B} = m''_{\gamma,A}$ . Moreover,  $\widehat{B}(\mathbf{1}) = \mathbf{d}(U)^{-1}\widehat{A}(\gamma)\mathbf{d}(V)^{-1}$ , and since  $(U, V)$  is a Hungarian pair with respect to  $\widehat{A}(\gamma)$ , we get, by Corollary 9.2, that  $(\mathbf{1}, \mathbf{1})$  is a Hungarian pair with respect to  $\widehat{B}(\mathbf{1})$ , that  $\text{per } \widehat{B}(\mathbf{1}) = \mathbf{1}$ , and  $G = \text{Sat}(\widehat{A}(\gamma), U, V) = \text{Sat}(\widehat{B}(\mathbf{1}), \mathbf{1}, \mathbf{1})$ . We also have  $\mathcal{G}_k(A, \gamma) = \mathcal{G}_k(B, \mathbf{1})$ . So we are reduced to the case where  $\gamma = \mathbf{1}$ , and  $U = V = \mathbf{1}$ . Moreover,  $(B_k)_{ij} \geq \widehat{B}_{ij}(\mathbf{1}) \geq \mathbf{1}$  for all  $k = 0, \dots, d$  and  $i, j \in [n]$ , and we have  $(B_k)_{ij} = \mathbf{1}$  if and only if  $(i, j) \in G_k = \mathcal{G}_k(B, \mathbf{1}) \cap G$ . Using (32), this implies that

$$\lim_{\epsilon \rightarrow 0} \mathcal{B}_\epsilon = a^{(\gamma)},$$

where  $a^{(\gamma)}$  is given in (31). This implies that  $\lim_{\epsilon \rightarrow 0} \det(\mathcal{B}_\epsilon) = \det(a^{(\gamma)})$ .

Using the above notations, we have that the formal characteristic polynomials of  $\mathcal{B}$  and  $B$  are given by  $\det(\mathcal{B}_\epsilon) = \sum_{\ell=0}^{nd} \text{ch}_\ell(\mathcal{B}_\epsilon) Y^\ell$  and  $\text{per } B = \sum_{\ell=0}^{nd} \widehat{\text{ch}}_\ell^{\min}(B) Y^\ell$ . Moreover, by Lemma 7.3, we have  $\text{ch}_\ell(\mathcal{B}_\epsilon) \simeq (\text{ch}_\ell)_B^{\text{Sat}}(a) \epsilon^{\widehat{\text{ch}}_\ell^{\min}(B)}$ . Hence the polynomial  $\mathcal{P} = \det(\mathcal{B}) \in \mathcal{C}[Y]$  satisfies the assumptions of Theorem 6.1 with the degree  $nd$  instead of  $n$ ,  $P = \text{per } B$ , and  $p = \sum_{\ell=0}^{nd} (\text{ch}_\ell)_B^{\text{Sat}}(a)$ . Moreover the scalar  $c = \mathbf{1}$  is a finite root of  $P$  with multiplicity  $m = m_{\gamma,A}$ , and we have  $m' = m'_{\gamma,A}$  in Theorem 6.1. Since  $\widehat{P}(\mathbf{1}) = \text{per } \widehat{B}(\mathbf{1}) = \mathbf{1}$ , we get that  $p^{(\mathbf{1})} = \lim_{\epsilon \rightarrow 0} \det(\mathcal{B}_\epsilon)$  and so the above properties show that  $p^{(\mathbf{1})} = \det(a^{(\gamma)})$ .

Assume now that the matrix polynomial  $a^{(\gamma)}$  is regular. This means that  $p^{(\mathbf{1})}$  is non identically zero. Applying Theorem 6.1, we get that if it has  $m_\gamma \geq 1$  non-zero eigenvalues,  $\lambda_1, \dots, \lambda_{m_\gamma}$ , then, the matrix polynomial  $\mathcal{B}_\epsilon$  has  $m_\gamma$  eigenvalues  $\mathcal{M}_{\epsilon,1}, \dots, \mathcal{M}_{\epsilon,m_\gamma}$  with limits  $\lambda_1, \dots, \lambda_{m_\gamma}$ . This implies that  $\mathcal{A}_\epsilon$  has  $m_\gamma$  eigenvalues  $\mathcal{L}_{\epsilon,i} = \epsilon^\gamma \mathcal{M}_{\epsilon,i}$ ,  $i = 1, \dots, m_\gamma$ , with first order asymptotics of the form  $\mathcal{L}_{\epsilon,i} \sim \lambda_i \epsilon^\gamma$ . If 0 is an eigenvalue of the matrix polynomial  $a^{(\gamma)}$  with multiplicity  $m'_\gamma$ , then the valuation of  $p^{(\mathbf{1})}$  is equal to  $m'_\gamma$ , hence applying Theorem 6.1, we get that the matrix polynomial  $\mathcal{B}_\epsilon$  has precisely  $m'_\gamma$  eigenvalues converging to 0. Finally, the remaining  $m''_\gamma = nd - m_\gamma - m'_\gamma$  eigenvalues of  $\mathcal{B}_\epsilon$  have a modulus converging to infinity. Hence, the matrix polynomial  $\mathcal{A}_\epsilon$  has precisely  $m'_\gamma$  eigenvalues  $\mathcal{L}_\epsilon$  such that  $\mathcal{L}_\epsilon \simeq 0\epsilon^\gamma$ , and  $m''_\gamma = nd - m_\gamma - m'_\gamma$  eigenvalues  $\mathcal{L}_\epsilon$  such that the modulus of  $\epsilon^{-\gamma} \mathcal{L}_\epsilon$  converges to infinity. Finally the inequalities  $m_\gamma \leq m_{\gamma,A}$ ,  $m'_\gamma \geq m'_{\gamma,A}$  and  $m''_\gamma \geq m''_{\gamma,A}$  are deduced from Theorem 6.1. This finishes the proof of Point (3) of Theorem 8.2 in the case  $G = \text{Sat}(\widehat{A}(\gamma), U, V)$ .

Let us show Point (3) of Theorem 8.3. Let us fix the matrix polynomial  $A$  and so the matrix polynomial  $B$ . By Lemma 7.3, for all  $\ell$  such that  $P_\ell = \widehat{\text{ch}}_\ell^{\min}(B) \neq 0$ , we have that, for generic values of  $a$ ,  $p_\ell = (\text{ch}_\ell)_B^{\text{Sat}}(a) \neq 0$ . Since  $\mathbf{1}$  is a finite root of  $P$  with multiplicity  $m_{\gamma,A}$ , and  $m'_{\gamma,A}$  is the sum of the multiplicities of all the roots of  $P$  greater than  $\mathbf{1}$ , we obtain, from Corollary 2.7, that  $P_\ell = \widehat{P}(\mathbf{1}) = \mathbf{1} \neq 0$  for  $\ell = m'_{\gamma,A}$  and  $\ell = m'_{\gamma,A} + m_{\gamma,A}$ . Then, for generic values of  $a$ , we have  $p_\ell \neq 0$  for  $\ell = m'_{\gamma,A}$  and  $\ell = m'_{\gamma,A} + m_{\gamma,A}$ , which implies that the polynomial  $p^{(\mathbf{1})}$  of (23) is non zero and that its valuation is equal to  $m'_{\gamma,A}$  and its degree is equal to  $m'_{\gamma,A} + m_{\gamma,A}$ . Then, for generic values of  $a$ ,  $m_\gamma = m_{\gamma,A}$  and  $m'_\gamma = m'_{\gamma,A}$ , and so  $m''_\gamma = m''_{\gamma,A}$ , which finishes the proof of Point (3) of Theorem 8.3 in the case  $G = \text{Sat}(\widehat{A}(\gamma), U, V)$ .

To prove the same results when  $G = \text{Opt}(\widehat{A}(\gamma))$ , we shall use the following lemma.

**Lemma 10.1.** *Let  $b$  be a matrix polynomial with coefficients in  $\mathbb{C}^{n \times n}$  and degree  $d$ , let  $B \in \mathbb{R}_{\min}^{n \times n}$  be a min-plus matrix such that  $\text{per } B \neq 0$ , and let  $(U, V)$  be a Hungarian pair with respect to  $B$ . Then, the matrix polynomials  $b^G := b_0^G + \dots + \Upsilon^d b_d^G$  defined respectively with  $G = \text{Sat}(B, U, V)$  and with  $G = \text{Opt}(B)$  have the same eigenvalues.*

*Proof.* Let  $\sigma$  be an optimal permutation for  $B$ . Then, by Corollary 9.4, the identity map is an optimal permutation for  $C := P_{\min}^{\sigma^{-1}} B$ ,  $(U_{\sigma^{-1}}, V)$  is a Hungarian pair with respect to  $C$ , and we have  $\text{Opt}(C) = \text{Opt}(B)_{\sigma^{-1}, \text{id}}$ , and  $\text{Sat}(C, U_{\sigma^{-1}}, V) = \text{Sat}(B, U, V)_{\sigma^{-1}, \text{id}}$ . Then, by Corollary 9.7,  $\text{Opt}(C)$  is the disjoint union of the strongly connected components of  $\text{Sat}(C, U_{\sigma^{-1}}, V)$ . Multiplying the matrix polynomials  $b$  and  $b^G$  on the left by  $P^{\sigma^{-1}}$ , and using the above notation, we deduce that  $P^{\sigma^{-1}} b^G = c^{G'}$ , with  $c = P^{\sigma^{-1}} b$  and  $G' = G_{\sigma^{-1}, \text{id}}$ . Since  $P^{\sigma^{-1}}$  is invertible, the sequence of eigenvalues of  $b^G$  is the same as the one of  $P^{\sigma^{-1}} b^G$ , hence the same as the one of  $c^{G'}$ . If  $G'$  has several strongly connected components, then, up to a permutation, the matrix polynomial  $c^{G'}$  is bloc triangular, with diagonal blocs  $c^S$ , for each strongly connected component  $S$  of  $G'$ . Then, the sequence of eigenvalues of  $c^{G'}$  is the disjoint union of the sequences of eigenvalues of the matrix polynomials  $c^S$ . Since the sequence of eigenvalues of  $b^G$  is the same as the one of  $c^{G'}$ , and, by the above properties, the strongly connected components of  $G' = G_{\sigma^{-1}, \text{id}}$  are the same for  $G = \text{Opt}(B)$  and  $G = \text{Sat}(B, U, V)$ , we deduce that the sequence of eigenvalues of  $b^G$  are the same for  $G = \text{Opt}(B)$  and for  $G = \text{Sat}(B, U, V)$ .  $\square$

Applying this result to the matrix polynomial  $b = a_0^{\mathcal{G}_0(A, \gamma)} + \dots + \Upsilon^d a_d^{\mathcal{G}_d(A, \gamma)}$ , and the min-plus matrix  $B = \widehat{A}(\gamma)$ , we get that the matrix polynomials  $a^{(\gamma)}$  given in (31), with  $G_k = \mathcal{G}_k(A, \gamma) \cap G$  for all  $0 \leq k \leq d$ , and respectively  $G = \text{Sat}(\widehat{A}(\gamma), U, V)$  or  $G = \text{Opt}(\widehat{A}(\gamma))$  have same eigenvalues. So the assertion of Point (3) of Theorem 8.2 (resp. Theorem 8.3) for  $G = \text{Sat}(\widehat{A}(\gamma), U, V)$  is equivalent to the one for  $G = \text{Opt}(\widehat{A}(\gamma))$ , which finishes the proof of this point.

#### REFERENCES

- [ABG98] M. Akian, R. B. Bapat, and S. Gaubert. Asymptotics of the Perron eigenvalue and eigenvector using max algebra. *C. R. Acad. Sci. Paris, Sér. I* 327:927–932, 1998.
- [ABG04a] M. Akian, R. Bapat, and S. Gaubert. Min-plus methods in eigenvalue perturbation theory and generalised Lidskii-Vishik-Ljusternik theorem. arXiv:math.SP/0402090, 2004.
- [ABG04b] M. Akian, R. Bapat, and S. Gaubert. Perturbation of eigenvalues of matrix pencils and optimal assignment problem. *C. R. Acad. Sci. Paris, Série I*, 339:103–108, 2004. arXiv:math.SP/0402438.
- [ABG13] M. Akian, R. Bapat, and S. Gaubert. Max-plus algebras. In L. Hogben, editor, *Handbook of Linear Algebra (Discrete Mathematics and Its Applications)*, volume 39. Chapman & Hall/CRC, 2013. Chapter 25, Second Edition (First edition, 2006).
- [AGM14] Marianne Akian, Stéphane Gaubert, and Andrea Marchesini. Tropical bounds for eigenvalues of matrices. *Linear Algebra and its Applications*, 446:281–303, 2014.
- [AGS13] M. Akian, S. Gaubert, and M. Sharify. Log-majorization of the moduli of the eigenvalues of a matrix polynomial by tropical roots, 2013.
- [AGW05] M. Akian, S. Gaubert, and C. Walsh. Discrete max-plus spectral theory. In G. L. Litvinov and V. P. Maslov, editors, *Idempotent Mathematics and Mathematical Physics*,

- Contemporary Mathematics, pages 19–51. American Mathematical Society, 2005. Also ESI Preprint 1485.
- [Ale13] D. Alessandrini. Logarithmic limit sets of real semi-algebraic sets. *Adv. Geom.*, 13:155–190, 2013.
- [Bau85] H. Baumgärtel. *Analytic perturbation theory for matrices and operators*. Birkhäuser Verlag, Basel, 1985.
- [BB03] Rainer E. Burkard and Peter Butkovič. Finding all essential terms of a characteristic maxpolynomial. *Discrete Appl. Math.*, 130(3):367–380, 2003.
- [BCGG09] P. Butkovič, R. A. Cuninghame-Green, and S. Gaubert. Reducible spectral theory with applications to the robustness of matrices in max-algebra. *SIAM J. Matrix Anal. Appl.*, 31(3):1412–1431, 2009.
- [BCOQ92] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity — an Algebra for Discrete Event Systems*. Wiley, 1992.
- [BNS13] D.A. Bini, V. Noferini, and M. Sharify. Locating the Eigenvalues of Matrix Polynomials. *SIAM J. Matrix Anal. Appl.*, 34(4):1708–1727, 2013.
- [Bou89] N. Bourbaki. *Commutative Algebra*. Springer, 1989.
- [BR97] R. B. Bapat and T. E. S. Raghavan. *Nonnegative Matrices and Application*. Cambridge University Press, 1997.
- [BSvdD95] R. B. Bapat, D. Stanford, and P. van den Driessche. Pattern properties and spectral inequalities in max algebra. *SIAM Journal of Matrix Analysis and Applications*, 16(3):964–976, 1995.
- [But10] P. Butkovič. *Max-linear systems: theory and algorithms*. Springer Monographs in Mathematics. Springer-Verlag, London, 2010.
- [CDQV83] G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot. Analyse du comportement périodique des systèmes de production par la théorie des dioïdes. Rapport de recherche 191, INRIA, Le Chesnay, France, 1983.
- [CG79] R.A. Cuninghame-Green. *Minimax Algebra*. Number 166 in Lecture notes in Economics and Mathematical Systems. Springer, 1979.
- [CGM80] R.A. Cuninghame-Green and P.F.J. Meijer. An algebra for piecewise-linear minimax problems. *Discrete Appl. Math.*, 2:267–294, 1980.
- [Die68] J. Dieudonné. *Calcul infinitésimal*. Hermann, Paris, 1968.
- [DKM92] S. Yu. Dobrokhotov, V. N. Kolokol'tsov, and V. P. Maslov. Quantization of the Bellman equation, exponential asymptotics and tunneling. In *Idempotent analysis*, volume 13 of *Adv. Soviet Math.*, pages 1–46. Amer. Math. Soc., Providence, RI, 1992.
- [EKL06] M. Einsiedler, M. Kapranov, and D. Lind. Non-Archimedean amoebas and tropical varieties. *J. Reine Angew. Math.*, 601:139–157, 2006.
- [Fri86] S. Friedland. Limit eigenvalues of nonnegatives matrices. *Linear Alg. and Appl.*, 74:173–178, 1986.
- [GK10] E. Gassner and B. Klinz. A fast parametric assignment algorithm with applications in max-algebra. *Networks*, 55(2):61–77, 2010.
- [GM77] M. Gondran and M. Minoux. Valeurs propres et vecteurs propres dans les dioïdes et leur interprétation en théorie des graphes. *EDF, Bulletin de la Direction des Etudes et Recherches, Serie C, Mathématiques Informatique*, 2:25–41, 1977.
- [GS09] S. Gaubert and M. Sharify. Tropical scaling of polynomial matrices. In Rafael Bru and Sergio Romero-Vivó, editors, *Proceedings of the third Multidisciplinary International Symposium on Positive Systems: Theory and Applications (POSTA 09)*, volume 389 of *LNCIS*, pages 291–303, Valencia, Spain, 2009. Springer. arXiv:arXiv:0905.0121.
- [HMT13] S. Hammarling, C. J. Munro, and F. Tisseur. An algorithm for the complete solution of quadratic eigenvalue problems. *ACM Transactions on Mathematical Software (TOMS)*, 39(3), 2013.
- [HT] J. Hook and F. Tisseur. Eigensolvers for tropical standard and polynomial eigenvalue problems. In preparation.
- [IMS07] I. Itenberg, G. Mikhalkin, and E. Shustin. *Tropical algebraic geometry*. Oberwolfach seminars. Birkhäuser, 2007.
- [Kat95] Tosio Kato. *Perturbation theory for linear operators*. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
- [KM97] V. Kolokoltsov and V. Maslov. *Idempotent analysis and applications*. Kluwer Acad. Publisher, 1997.

- [Lid65] V.B. Lidskiĭ. Perturbation theory of non-conjugate operators. *U.S.S.R. Comput. Math. and Math. Phys.*, 1:73–85, 1965. (Ž. Vychisl. Mat. i Mat. Fiz. 6, no. 1, 52–60, 1965).
- [LMS01] G. L. Litvinov, V. P. Maslov, and G. B. Shpiz. Idempotent functional analysis: An algebraical approach. *Mat. Notes*, 69(5):696–729, 2001.
- [Mas73] V.P. Maslov. *Méthodes Operatorielles*. Mir, Moscou, 1973. French Transl. 1987.
- [MBO97] Julio Moro, James V. Burke, and Michael L. Overton. On the Lidskii-Vishik-Lyusternik perturbation theory for eigenvalues of matrices with arbitrary Jordan structure. *SIAM J. Matrix Anal. Appl.*, 18(4):793–817, 1997.
- [ME98] Yanyuan Ma and Alan Edelman. Nongeneric eigenvalue perturbations of Jordan blocks. *Linear Algebra Appl.*, 273:45–63, 1998.
- [MO79] A.W. Marshall and Olkin. *Inequalities: theory of majorization and its applications*. Number 143 in Serie Math. in Science and Engineering. Academic Press, 1979.
- [Mon34] Paul Montel. Sur quelques limites pour les modules des zéros des polynômes. *Commentarii Mathematici Helvetici*, 7(1):178–200, 1934.
- [MS92] V. Maslov and S. Samborskii, editors. *Idempotent analysis*, volume 13 of *Adv. in Sov. Math.* AMS, RI, 1992.
- [MS15] D. Maclagan and B. Sturmfels. *Introduction to tropical geometry*. 2015. To appear as volume 161 of AMS Graduate Studies in Mathematics.
- [Mur90] Kazuo Murota. Computing Puiseux-series solutions to determinantal equations via combinatorial relaxation. *SIAM J. Comput.*, 19(6):1132–1161, 1990.
- [Naj99] Branko Najman. The asymptotic behavior of the eigenvalues of a singularly perturbed linear pencil. *SIAM J. Matrix Anal. Appl.*, 20(2):420–427, 1999.
- [NST14] V. Noferini, M. Sharify, and F. Tisseur. Tropical roots as approximations to eigenvalues of matrix polynomials. [http://eprints.ma.man.ac.uk/2126/01/covered/MIMS\\_ep2014\\_16.pdf](http://eprints.ma.man.ac.uk/2126/01/covered/MIMS_ep2014_16.pdf), to appear in *SIAM J. Matrix Anal. Appl.*, 2014.
- [Roc70] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, New Jersey, 1970.
- [Rom67] I.V. Romanovskii. Optimization of stationary control of discrete deterministic process in dynamic programming. *Kibernetika*, 3(2):66–78, 1967.
- [Sch03] Alexander Schrijver. *Combinatorial Optimization*, volume A. Springer, 2003.
- [Sha11] M. Sharify. *Scaling Algorithms and Tropical Methods in Numerical Matrix Analysis*. Phd thesis, École Polytechnique (France), September 2011.
- [vdD99] Lou van den Dries. o-minimal structures and real analytic geometry. In *Current developments in mathematics, 1998 (Cambridge, MA)*, pages 105–152. Int. Press, Somerville, MA, 1999.
- [Vir01] Oleg Viro. Dequantization of real algebraic geometry on logarithmic paper. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 of *Progr. Math.*, pages 135–146. Birkhäuser, Basel, 2001.
- [VL60] M. I. Višik and L. A. Ljusternik. Solution of some perturbation problems in the case of matrices and self-adjoint or non-selfadjoint differential equations. I. *Russian Math. Surveys*, 15(3):1–73, 1960.
- [Vor67] N.N. Vorobyev. Extremal algebra of positive matrices. *Elektron. Informationsverarbeitung und Kybernetik*, 3, 1967. in Russian.

MARIANNE AKIAN, INRIA AND CMAP, ÉCOLE POLYTECHNIQUE. ADDRESS: CMAP, ÉCOLE POLYTECHNIQUE, ROUTE DE SACLAY, 91128 PALAISEAU CEDEX, FRANCE  
*E-mail address:* `Marianne.Akian@inria.fr`

RAVINDRA BAPAT, INDIAN STATISTICAL INSTITUTE, NEW DELHI, 110016, INDIA  
*E-mail address:* `rbb@isid1.isid.ac.in`

STÉPHANE GAUBERT, INRIA AND CMAP, ÉCOLE POLYTECHNIQUE. ADDRESS: CMAP, ÉCOLE POLYTECHNIQUE, ROUTE DE SACLAY, 91128 PALAISEAU CEDEX, FRANCE  
*E-mail address:* `Stephane.Gaubert@inria.fr`