# SYMMETRIC NORMS AND THE LEIBNIZ PROPERTY

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Abstract. We prove that symmetric norms on the space of bounded centered random variables, defined on uniform discrete spaces, have the strong Leibniz property. As an application, we shall obtain that the $p^{\text{th}}$ central seminorms on arbitrary probability spaces are strongly Leibniz.

## 1. Introduction

We say that a seminorm $L$ on a unital normed algebra $(\mathcal{A}, \|\cdot\|)$ is strongly Leibniz if (i) $L(1_{\mathcal{A}}) = 0$, (ii) the Leibniz property

$$L(ab) \leq \|a\|L(b) + \|b\|L(a)$$

holds for every $a, b \in \mathcal{A}$ and, furthermore, (iii) for every invertible $a$,

$$L(a^{-1}) \leq \|a^{-1}\|^2 L(a)$$

follows. The study of strong Leibniz seminorms regarded as non-commutative metrics on quantum metric spaces was initiated by M. Rieffel in his seminal papers [6] and [7]. Several examples show that property (ii) and (iii) are independent, see [7]. Recently, Rieffel has observed that the standard deviation is a strongly Leibniz seminorm, see [8]. For a probability space $(\Omega, \mathcal{F}, \mu)$ this means that for every $f$ and $g \in L^\infty(\Omega, \mu)$, we have the inequalities

$$\|fg - \mathbb{E}(fg)\|_2 \leq \|g\|_\infty \|f - \mathbb{E}f\|_2 + \|f\|_\infty \|g - \mathbb{E}g\|_2$$

and

$$\|f^{-1} - \mathbb{E}(f^{-1})\|_2 \leq \|f^{-1}\|_\infty^2 \|f - \mathbb{E}f\|_2 \quad \text{ if } f^{-1} \in L^\infty(\Omega, \mu).$$

In addition, with a proper notion of non-commutative (or quantum) deviation in unital $C^*$-algebras the non-commutative versions of the above inequalities can be proved as well (see [2] and [8]).

It seems to be a natural problem to investigate whether seminorms determined by higher order moments, or fractional moments, have the strong Leibniz property or not. A few particular answer related to this question has already been given in [2]. For instance, the seminorm $\|f - \mathbb{E}f\|_\infty$ possesses the previous properties in the real Banach space $L^\infty(\Omega, \mu)$. However, we were only able to prove the general case in discrete spaces containing at most 5 atoms.

In this paper we shall present a completely different approach. It turns out that the problem has nothing to do with the $L^p$ norms but symmetric norms on $\mathbb{R}^n$. We prove that every centered symmetric norm on the real $\ell_n^\infty$ is strongly Leibniz. Applying a simple approximation and uniformization method (see [2]), we can prove the similar result for the $p^{\text{th}}$ central seminorms in the real $L^\infty(\Omega, \mu)$ for any probability measure $\mu$.

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## 2. Preliminaries

We say that a norm $\|\cdot\|$ on $\mathbb{R}^n$ is symmetric if it is invariant under sign-changes and permutations of the components. Symmetric norms are monotone which means that

$$\|x\| \le \|y\| \quad \text{if} \quad |x|^{\downarrow} \le |y|^{\downarrow}$$

holds, where $|x|^{\downarrow}$ is the usual non-increasing rearrangement of the vector $|x|$. Furthermore, $\|\cdot\|$ is absolute:

$$\|x\| = \||x|\|$$

for every $x \in \mathbb{R}^n$ (see [1, Section 2]).

The vector $k$-norms (or Ky Fan $k$-norms) are special examples of symmetric norms. In fact, the vector $k$-norm of $x$ is defined by

$$\|x\|_{(k)} = \sum_{i=1}^{k} |x_i|^{\downarrow}.$$

In the case of $k = n$ and $k = 1$, we obtain the usual $\ell^1$ and $\ell^\infty$ norms on $\mathbb{R}^n$, denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively. We recall that the dual norm of any symmetric norm is symmetric as well.

A celebrated theorem of Ky Fan says that, for any $x, y \in \mathbb{R}^n_+$, the inequalities

$$\|x\|_{(k)} \le \|y\|_{(k)}$$

hold for every $1 \le k \le n$ if and only if

$$\|x\| \le \|y\|$$

for every symmetric norm $\|\cdot\|$ on $\mathbb{R}^n$ (see [1]). Hence one can look upon the vector $k$-norms as the cornerstones of symmetric norms.

Additionally, if one can assure a proper linear connection between the vectors $x$ and $y$, i.e. $Sy = x$ for some $S \in \mathbb{R}^{n\times n}$, interpolation methods are at our disposal to obtain the previous inequalities. Actually, the Calderón–Mityagin theorem (see [3], [4]) tells us that if $S$ is an $\ell^1$–$\ell^\infty$ contraction, that is,

$$\|Sy\|_1 \le \|y\|_1 \qquad \text{and} \qquad \|Sy\|_\infty \le \|y\|_\infty$$

hold for every $y \in \mathbb{R}^n$, then

$$\|Sy\| \le \|y\|$$

follows for every symmetric norm $\|\cdot\|$.

## 3. Leibniz inequality for symmetric norms

Let $x\colon \mathbb{Z}_n \to \mathbb{R}$. Define the Hermitian matrix $I_x \in \mathbb{R}^{n\times n}$ as follows

$$I_x = \begin{pmatrix} 1 - \frac{1}{2n}\sum\limits_{1\le i\le n}(x_1+x_i) & \frac{x_1+x_2}{2n} & \cdots & \frac{x_1+x_n}{2n} \\ \frac{x_1+x_2}{2n} & 1 - \frac{1}{2n}\sum\limits_{1\le i\ne 2\le n}(x_2+x_i) & & \frac{x_2+x_n}{2n} \\ \vdots & & \ddots & \vdots \\ \frac{x_1+x_n}{2n} & \frac{x_2+x_n}{2n} & \cdots & 1 - \frac{1}{2n}\sum\limits_{1\le i\le n}(x_n+x_i) \end{pmatrix},$$

that is, $(I_x)_{ij} = (x_i + x_j)/2n$ if $i \ne j$ and $(I_x)_{ii} = 1 - \sum_{j\ne i}(I_x)_{ij}$. Our first proposition connects the product of two functions $f, g\colon \mathbb{Z}_n \to \mathbb{R}$ with the matrices $I_{f+1}$ and $I_{g+1}$.

**Proposition 1.** *For any $f$, $g\colon \mathbb{Z}_n \to \mathbb{R}$,*

$$I_{f+1}(g - \mathbb{E}g) + I_{g+1}(f - \mathbb{E}f) = \mathbb{E}(fg) - fg.$$

*Proof.* Clearly, it is enough to show that

$$I_f(g - \mathbb{E}g) + I_g(f - \mathbb{E}f) = \mathbb{E}((f-1)(g-1)) - (f-1)(g-1)$$

holds. A straightforward calculation gives for every index $1 \le m \le n$ that

$$\begin{aligned}
n(I_f(g - \mathbb{E}g) &+ I_g(f - \mathbb{E}f))_m \\
&= \frac{1}{2n} \sum_{1\le i\ne m\le n} \sum_{1\le j\le n} (f_i + f_m)(g_i - g_j) \\
&\quad + \Big(1 - \frac{1}{2n} \sum_{1\le i\ne m\le n} (f_m + f_i)\Big) \sum_{1\le i\le n} (g_m - g_i) \\
&\quad + \frac{1}{2n} \sum_{1\le i\ne m\le n} \sum_{1\le j\le n} (g_i + g_m)(f_i - f_j) \\
&\quad + \Big(1 - \frac{1}{2n} \sum_{1\le i\ne m\le n} (g_m + g_i)\Big) \sum_{1\le i\le n} (f_m - f_i) \\
&= \frac{1}{2n} \sum_{1\le i\ne m\le n} (f_i + f_m)\left(\sum_{1\le j\le n} (g_i - g_j) - \sum_{1\le i\le n} (g_m - g_i)\right) \\
&\quad + \frac{1}{2n} \sum_{1\le i\ne m\le n} (g_i + g_m)\left(\sum_{1\le j\le n} (f_i - f_j) - \sum_{1\le i\le n} (f_m - f_i)\right) \\
&\quad + \sum_{1\le i\le n} (g_m - g_i) + \sum_{1\le i\le n} (f_m - f_i) \\
&= \frac{1}{2} \sum_{1\le i\le n} ((f_i + f_m)(g_i - g_m) + (g_i + g_m)(f_i - f_m)) \\
&\quad + \sum_{1\le i\le n} (g_m - g_i + f_m - f_i) \\
&= \sum_{1\le i\le n} (f_i g_i - f_m g_m + g_m - g_i + f_m - f_i) \\
&= \sum_{1\le i\le n} (f_i - 1)(g_i - 1) - n(f_m - 1)(g_m - 1) \\
&= n(\mathbb{E}((f-1)(g-1)) - (f-1)(g-1))_m,
\end{aligned}$$

which is what we intended to have. ☐

We recall that the dual norm of the vector $k$-norm is

$$\|x\|_{(k)^*} = \max\left(\|x\|_\infty, \frac{\|x\|_1}{k}\right) \qquad x \in \mathbb{R}^n$$

(e.g. [1, Exercise IV.1.18]).

Let $\mathfrak{B}_{(k)^*} = \{x \in \mathbb{R}^n \colon \|x\|_{(k)^*} \le 1\}$ denote the closed unit ball of the dual space $(\mathbb{R}^n, \|\cdot\|_{(k)})^*$. The set of extreme points of $\mathfrak{B}_{(k)^*}$ can be readily described. The result is well-known, however, for the sake of completeness, we sketch a proof.

**Lemma 1.**

$$\operatorname{ext} \mathfrak{B}_{(k)^*} = \left\{ \sum_{i\in S} \pm e_i \colon S \subseteq \{1, \dots, n\} \ \textit{and}\ |S| = k \right\},$$

*where $e_i$-s denote the standard basis elements of $\mathbb{R}^n$.*

*Proof.* Denote $\mathfrak{K}_0$ the points of the $n$-cube $[-1,1]^n$ which has at most $k$ non-zero coordinates. It is not difficult to see that

$$\text{conv } \mathfrak{K}_0 = \mathfrak{B}_{(k)^*}.$$

In fact, pick a point $v$ in $\mathfrak{B}_{(k)^*}$ which has at most $k+1$ non-zero coordinates. Denote $v_i$ a coordinate of $v$ which has the smallest non-zero modulus. Obviously, $|v_i| \leq 1$. Now choose a vector $c \in \{-1,0,1\}^n$ such that the support of $c$ has cardinality $k$, $i \in \text{supp } c$ and sign $c_j =$ sign $v_j$ for every $j \in$ supp $c$. Then it is simple to see that

$$\frac{v - |v_i|c}{1 - |v|_i} \in \mathfrak{B}_{(k)^*}.$$

Iterating the previous process, we arrive a point which has at most $k$ non-zero coordinates. This point is the convex combination of vertices of a proper $k$-cube in $[-1,1]^n$. ☐

Now we are ready to prove the following proposition.

**Proposition 2.** *For every $f \in [-1,1]^n$ and $1 \leq k \leq n$, the operator*

$$I^*_{f+1} \colon (\mathbb{R}^n, \|\cdot\|_{(k)^*}) \to (\mathbb{R}^n, \|\cdot\|_{(k)^*})/\mathbb{R}, \quad x \quad \mapsto I_{f+1}x + \lambda\mathbf{1}$$

*is a contraction.*

*Proof.* First, to get an upper bound on the norm of $I^*_{f+1}$, it is enough to calculate the norm of the class $I_{f+1}v$ for every extreme point $v$ of the unit ball $(\mathbb{R}^n, \|\cdot\|_{(k)^*})$. From Lemma 1, we can assume that

$$v = \sum_{i \in S_+} e_i - \sum_{i \in S_-} e_i$$

for some disjoint sets $S_+, S_- \subseteq \mathbb{Z}_n$ such that $|S_-| + |S_+| = k$. For any $x, y \in \mathbb{R}^n$ and $0 \leq s \leq 1$, we have $I^*_{sx+(1-s)y} = sI^*_x + (1-s)I^*_y$. Furthermore, since the quotient norm is convex, one has

$$\begin{aligned} \|I_{f+1}v\|_{(k)^*} &= \min_{\lambda \in \mathbb{R}} \|I_{f+1}v - \lambda\mathbf{1}\|_{(k)^*} \\ &\leq \max_{x \in [0,2]^n} \min_{\lambda \in \mathbb{R}} \|I_x v - \lambda\mathbf{1}\|_{(k)^*} \\ &= \max_{x \in \{0,2\}^n} \min_{\lambda \in \mathbb{R}} \|I_x v - \lambda\mathbf{1}\|_{(k)^*}. \end{aligned}$$

Next, pick an $x \in \{0,2\}^n$. Set

$$r_v = \frac{1}{n}\langle x, v\rangle.$$

In order to prove that $I_x v$ is in the unit ball of the quotient space, it is enough to show that

$$\|I_x v - r_v\mathbf{1}\|_{(k)^*} \leq 1.$$

In fact,

$$\begin{aligned} \|I_x v - r_v\mathbf{1}\|_\infty &= \max_{1 \leq i \leq n} \left|\left\langle I_x e_i - n^{-1}x, v\right\rangle\right| \\ &\leq \max_{1 \leq i \leq n} \left\|(I_x - n^{-1}x \otimes \mathbf{1})e_i\right\|_{(k)} \|v\|_{(k)^*} \\ &\leq \max_{1 \leq i \leq n} \left\|(I_x - n^{-1}x \otimes \mathbf{1})e_i\right\|_1. \end{aligned}$$

Let $s = \mathrm{card}\{i : x_i = 2\}$. For any $1 \le i \le n$, note that

$$\begin{aligned}\left\|(I_x - n^{-1}x \otimes \mathbf{1})e_i\right\|_1 &= \left|1 - \frac{1}{2n}\sum_{j=1}^{n}(x_i + x_j)\right| + \frac{1}{2n}\sum_{j=1}^{n}|x_i - x_j| \\ &= \begin{cases} \frac{s}{n} + \frac{n-s}{n} & \text{if } x_i = 2, \\ \left(1 - \frac{s}{n}\right) + \frac{s}{n} & \text{if } x_i = 0 \end{cases} \\ &= 1.\end{aligned}$$

Thus

$$\|I_x v - r_v \mathbf{1}\|_\infty \le 1.$$

Now, let $P_S$ denote the projection $\sum_{i=1}^{n} x_i e_i \mapsto \sum_{i \in S} x_i e_i$ on $\mathbb{R}^n$, where $S = S_- \cup S_+$ is the support of $v$. Then

$$\begin{aligned}\|I_x v - r_v \mathbf{1}\|_1 &= \sum_{i=1}^{n}\left|\left\langle P_S\left(I_x e_i - \frac{1}{n}x\right), v\right\rangle\right| \\ &\le \sum_{i=1}^{n}\left\|P_S\left(I_x e_i - \frac{1}{n}x\right)\right\|_{(k)} \|v\|_{(k)^*} \\ &\le \sum_{i=1}^{n}\left\|P_S\left(I_x e_i - \frac{1}{n}x\right)\right\|_1 \\ &= \sum_{i \in S}\left(\left|1 - \frac{1}{2n}\sum_{j=1}^{n}(x_i + x_j)\right| + \frac{1}{2n}\sum_{j \in S}|x_i - x_j|\right) \\ &\quad + \sum_{i \notin S}\frac{1}{2n}\sum_{j \in S}|x_i - x_j| \\ &= \sum_{i \in S}\left(\left|1 - \frac{1}{2n}\sum_{j=1}^{n}(x_i + x_j)\right| + \frac{1}{2n}\sum_{j=1}^{n}|x_i - x_j|\right),\end{aligned}$$

that is,

$$\begin{aligned}\|I_x v - r_v \mathbf{1}\|_1 &\le \sum_{i \in S}\left\|(I_x - n^{-1}x \otimes \mathbf{1})e_i\right\|_1 \\ &= |S|.\end{aligned}$$

Hence

$$\|I_x v - r_v \mathbf{1}\|_{(k)^*} \le 1,$$

and the proof is complete. □

Let $\mathfrak{X}_0$ denote the hyperplane $\{x \in \mathbb{R}^n : \mathbb{E}x = 0\} \subseteq \mathbb{R}^n$. Obviously, the dual of the Banach space $(\mathfrak{X}_0, \|\cdot\|_{(k)})$ is the quotient space $(\mathbb{R}^n, \|\cdot\|_{(k)^*})/\mathbb{R}$. In fact, $\mathfrak{X}_0$ is a one co-dimensional subspace of $\mathbb{R}$, whilst $\langle y, x - \mathbb{E}x\rangle = 0$ holds for every $y \in \mathbb{R}\mathbf{1}$.

Clearly, $I_{f+1}\mathbf{1} = \mathbf{1}$. Hence the adjoint of $I_{f+1} \colon (\mathfrak{X}_0, \|\cdot\|_{(k)}) \to (\mathbb{R}^n, \|\cdot\|_{(k)})$ is the operator

$$I_{f+1}^* : (\mathbb{R}^n, \|\cdot\|_{(k)^*}) \to (\mathbb{R}^n, \|\cdot\|_{(k)^*})/\mathbb{R}, \quad x \mapsto I_{f+1}x + \lambda\mathbf{1}$$

defined in Proposition 2. Since $\|I_{f+1}|\mathfrak{X}_0\| = \|(I_{f+1}|\mathfrak{X}_0)^*\|$ (see e.g. [5, Proposition 2.3.10]), a straightforward corollary of the previous result is

**Proposition 3.** *For every $f \in [-1,1]^n$, the operator $I_{f+1}$ is a contraction on the normed space $(\mathfrak{X}_0, \|\cdot\|_{(k)})$.*

Furthermore, we have the following

**Proposition 4.** *For every symmetric* $\|\cdot\|$ *on* $\mathbb{R}^n$ *and* $f \in [-1,1]^n$, $I_{f+1}$ *is a contraction on* $(\mathfrak{X}_0, \|\cdot\|)$.

*Proof.* For every $x \in \mathfrak{X}_0$ and $1 \le k \le n$, Proposition 3 tells us that

$$\sum_{i=1}^{k} |I_{f+1}x|_i^{\downarrow} \le \sum_{i=1}^{k} |x|_i^{\downarrow}$$

Thus the vector $|I_{f+1}x|$ is weakly majorized by $|x|$. Now Ky Fan's theorem for symmetric norms gives that

$$\|I_{f+1}x\| = \||I_{f+1}x|\| \le \||x|\| = \|x\|,$$

which is what we intended to have. □

Now one can readily prove the following Leibniz inequality for symmetric norms.

**Theorem 1.** *Let* $\|\cdot\|$ *be a symmetric norm on* $\mathbb{R}^n$. *For every* $f, g \colon \mathbb{Z}_n \to \mathbb{R}$, *we have*

$$\|fg - \mathbb{E}(fg)\| \le \|g\|_\infty \|f - \mathbb{E}f\| + \|f\|_\infty \|g - \mathbb{E}g\|.$$

*Proof.* Without loss of generality, we can assume that $\|f\|_\infty = \|g\|_\infty = 1$. Applying Proposition 1 and Proposition 4, it follows that

$$\begin{aligned}\|fg - \mathbb{E}(fg)\| &= \|I_{f+1}(g - \mathbb{E}g) + I_{g+1}(f - \mathbb{E}f)\| \\ &\le \|I_{f+1}|\mathfrak{X}_0\| \|g - \mathbb{E}g\| + \|I_{g+1}|\mathfrak{X}_0\| \|f - \mathbb{E}f\|_p \\ &= \|g - \mathbb{E}g\| + \|f - \mathbb{E}f\|,\end{aligned}$$

and the proof is complete. □

3.1. **Remark.** The operator $I_x$ leaves invariant the subspace $\mathfrak{X}_0$, since

$$\begin{aligned}\mathbb{E}(I_x(f - \mathbb{E}f)) &= \frac{1}{n}\langle I_x(f - \mathbb{E}f), \mathbf{1}\rangle \\ &= \frac{1}{n}\langle f - \mathbb{E}f, I_x\mathbf{1}\rangle \\ &= \frac{1}{n}\langle f - \mathbb{E}f, \mathbf{1}\rangle \\ &= 0.\end{aligned}$$

3.2. **Remark.** One can give a short proof of Proposition 4 via the Calderón–Mityagin interpolation result as we briefly indicate. For an $x \in [0,2]^n$, let us consider the matrix

$$L_x = I_x - \frac{1}{n} x \otimes \mathbf{1}.$$

We note that the off-diagonal part of $L_x$ is skew-symmetric: $(L_x)_{i,j} = -(L_x)_{j,i}$ for every $i \neq j$, hence $\|L_x^T\|_{1\to 1} = \|L_x^T\|_{\infty\to\infty}$. From the proof of Proposition 2, it follows that

$$\|L_x^T\|_{1\to 1} \le 1 \quad \text{and} \quad \|L_x^T\|_{\infty\to\infty} \le 1.$$

Moreover, for any symmetric norm $\|\cdot\|$, the adjoint of $I_x \colon (\mathfrak{X}_0, \|\cdot\|) \to (\mathbb{R}^n, \|\cdot\|)$, $v \mapsto I_x v$, is the operator

$$I_x^* \colon (\mathbb{R}^n, \|\cdot\|_*) \to (\mathbb{R}^n, \|\cdot\|_*)/\mathbb{R},$$

where

$$I_x^* v = I_x v + \lambda \mathbf{1}$$

and $\|\cdot\|_*$ denotes the dual norm. Again, for any $v \in \mathbb{R}^n$, let $r_v = \frac{1}{n}\langle x, v\rangle$. Then

$$\begin{aligned}\|I_x v - r_v \mathbf{1}\|_* &= \|I_x v - \frac{1}{n}\langle x, v\rangle\|_* \\ &= \|\langle (I_x - \frac{1}{n} x \otimes \mathbf{1}) e_i, v\rangle_i\|_* \\ &= \|L_x^T v\|_*.\end{aligned}$$

Since the dual norm $\|\cdot\|_*$ is symmetric, the Calderón–Mityagin theorem tells us that

$$\min_{\lambda\in\mathbb{R}} \|I_x v - \lambda \mathbf{1}\|_* \le \|L_x^T v\|_* \le \|v\|_*.$$

That is,

$$\|I_x^*\| \le 1,$$

and the operator $I_x$ is a contraction on $(\mathfrak{X}_0, \|\cdot\|)$ as well.

3.3. **Remark.** It is worth to note that if $x \in [0,1]^n$ then $I_x$ is doubly stochastic. Hence, the Birkhoff–von Neumann theorem gives that $\|I_x\|_{\|\cdot\|\to\|\cdot\|} \le 1$ for any permutation invariant norm $\|\cdot\|$ on $\mathbb{R}^n$. Now assume that $f, g$ are nonnegative and $\|f\|_\infty = \|g\|_\infty = 1$ Then

$$I_{-f+1}(\mathbb{E}g - g) + I_{-g+1}(\mathbb{E}f - f) = \mathbb{E}(fg) - fg,$$

and the matrices $I_{-f+1}$, $I_{-g+1}$ are doubly stochastic as well. A simple corollary is

**Theorem 2.** *Let $\|\cdot\|$ be a permutation invariant norm on $\mathbb{R}^n$. For any nonnegative functions $f$ and $g$ on $\mathbb{Z}_n$, we have*

$$\|fg - \mathbb{E}(fg)\| \le \|g\|_\infty \|f - \mathbb{E}f\| + \|f\|_\infty \|g - \mathbb{E}g\|.$$

## 4. The strong property

With a change of the matrix $I_x$, we shall prove the inequality

$$\|f^{-1} - \mathbb{E}(f^{-1})\| \le \|f^{-1}\|_\infty^2 \|f - \mathbb{E}f\|$$

for every symmetric norm on $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$ such that $x_i \ne 0$ for every $1 \le i \le n$. Let us consider the Hermitian

$$S_x = \begin{pmatrix} y_1 & \frac{1+x_1x_2}{nx_1x_2} & \cdots & \frac{1+x_1x_n}{nx_1x_n} \\ \frac{1+x_1x_2}{nx_1x_2} & y_2 & \cdots & \frac{1+x_2x_n}{nx_2x_n} \\ \vdots & & \ddots & \vdots \\ \frac{1+x_1x_n}{nx_1x_n} & \frac{1+x_2x_n}{nx_2x_n} & \cdots & y_n \end{pmatrix},$$

where

$$y_i = \frac{1}{n} - \frac{1}{n}\sum_{1<k\ne i\le n} \frac{1}{x_i x_k}.$$

Note that $S_x \mathbf{1} = \mathbf{1}$ and $S_x \mathfrak{X}_0 \subseteq \mathfrak{X}_0$ follows again. A simple calculation gives

**Lemma 2.** *For any $f\colon \mathbb{Z}_n \to \mathbb{R}$,*

$$S_f(f - \mathbb{E}f) = f^{-1} - \mathbb{E}(f^{-1}).$$

*Proof.* For every index $1 \le m \le n$,

$$
\begin{aligned}
(S_f(f-\mathbb{E}f))_m =& \frac{1}{n^2}\sum_{1\le i\ne m\le n}\sum_{j=1}^{n}\left(1+\frac{1}{f_if_m}\right)(f_i-f_j)\\
&+\frac{1}{n^2}\left(n-\sum_{1<i\ne m\le n}\left(1+\frac{1}{f_mf_i}\right)\right)\sum_{j=1}^{n}(f_m-f_j)\\
=&\frac{1}{n^2}\sum_{1\le i\ne m\le n}\left(1+\frac{1}{f_mf_i}\right)\left(\sum_{j=1}^{n}(f_i-f_j)-\sum_{j=1}^{n}(f_m-f_j)\right)\\
&+\frac{1}{n}\sum_{j=1}^{n}(f_m-f_j)\\
=&\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{f_m}-\frac{1}{f_i}\right),
\end{aligned}
$$

which completes the proof. □

**Lemma 3.** *For any real numbers $y_0, y_1, \ldots, y_{n-1}$,*

$$\Big|\sum_{i=0}^{n-1} y_i\Big| + \sum_{i=0}^{n-1}|y_0-y_i| \le n \max_{0\le i\le n-1}|y_i|.$$

*Proof.* Clearly, we can assume that $0 \le \sum_{i=0}^{n-1} y_i$. Let $\mathcal{I} \subseteq \{0,1,\ldots,n-1\}$ denote the index set such that $y_i \le y_0$ for all $i \in \mathcal{I}$. Hence,

$$
\begin{aligned}
\sum_{i=0}^{n-1} y_i + \sum_{i=1}^{n-1}|y_0-y_i| &= (2|\mathcal{I}|-n)y_0 + 2\sum_{i\notin\mathcal{I}} y_i\\
&\le n\max_{0\le i\le n-1}|y_i|.
\end{aligned}
$$

□

**Proposition 5.** *For any $f\colon \mathbb{Z}_n \to \mathbb{R}$ and symmetric norm $\|\cdot\|$ on $\mathbb{R}^n$,*

$$\|S_f x\| \le \|f^{-1}\|_\infty^2\|x\|, \quad \text{if } x \in \mathfrak{X}_0.$$

*Proof.* Fix a $1 \le k \le n$. The dual of $S_f\colon (\mathfrak{X}_0, \|\cdot\|_{(k)}) \to (\mathbb{R}^n, \|\cdot\|_{(k)})$ is

$$S_f^*\colon (\mathbb{R}^n, \|\cdot\|_{(k)^*}) \to (\mathbb{R}^n, \|\cdot\|_{(k)^*})/\mathbb{R}, \quad x \mapsto S_f x + \lambda\mathbf{1}.$$

For any $v$ with $\|v\|_{(k)^*} = 1$, set

$$r_v = \frac{1}{n}\langle 1 - f^{-2}, v\rangle.$$

Then

$$
\begin{aligned}
\|S_f v - r_v\mathbf{1}\|_\infty &= \max_{1\le i\le n}\left|\left\langle S_f e_i - n^{-1}(1-f^{-2}), v\right\rangle\right|\\
&\le \max_{1\le i\le n}\left\|S_f e_i - n^{-1}(1-f^{-2})\right\|_1\|v\|_{(k)^*}\\
&= \left\|S_f - n^{-1}(1-f^{-2})\otimes\mathbf{1}\right\|_{1\to1}.
\end{aligned}
$$

However, for every $1 \le i \le n$,

$$\left\|(S_f - n^{-1}(1-f^{-2})\otimes\mathbf{1})e_i\right\|_1 = \frac{1}{n|f_i|}\left|\sum_{k=1}^{n}\frac{1}{f_k}\right| + \sum_{k=1}^{n}\frac{1}{n|f_i|}\left|\frac{1}{f_i}-\frac{1}{f_k}\right|.$$

Moreover, Lemma 3 gives that

$$\left\|(S_f - n^{-1}(1-f^{-2})\otimes\mathbf{1})e_i\right\|_1 \le \|f^{-1}\|_\infty^2.$$

On the other hand, let us consider a vector $v = \sum_{i\in S} \pm e_i$, where $|S| = k$. Again, $P_S$ denote the projection $\sum_{i=1}^n x_i e_i \mapsto \sum_{i\in S} x_i e_i$ on $\mathbb{R}^n$. Then

$$\begin{aligned}\|S_f v - r_v \mathbf{1}\|_1 &= \sum_{1\le i\le n} \left|\left\langle P_S\left(S_f e_i - n^{-1}(1-f^{-2})\right), v\right\rangle\right| \\ &= \sum_{i\in S}\left(\frac{1}{n|f_i|}\left|\sum_{k=1}^n \frac{1}{f_k}\right| + \sum_{k\in S}\frac{1}{n|f_i|}\left|\frac{1}{f_i}-\frac{1}{f_k}\right|\right) \\ &\quad + \sum_{i\notin S}\sum_{k\in S}\frac{1}{n|f_i|}\left|\frac{1}{f_i}-\frac{1}{f_k}\right| \\ &\le \max_{1\le i\le n}\left(\frac{1}{|f_i|}\right)\sum_{i\in S}\left(\frac{1}{n}\left|\sum_{k=1}^n\frac{1}{f_k}\right| + \sum_{k=1}^n\frac{1}{n}\left|\frac{1}{f_i}-\frac{1}{f_k}\right|\right) \\ &\le \max_{1\le i\le n}\left(\frac{1}{|f_i|}\right)\max_{1\le i\le n}\left(\frac{1}{|f_i|}\right)|S|,\end{aligned}$$

where we used Lemma 3 in the last inequality. The previous arguments with Lemma 1 readily imply that

$$\min_{\lambda\in\mathbb{R}^n}\|S_f v - \lambda\mathbf{1}\|_{(k)^*} \le \|S_f v - r_v\mathbf{1}\|_{(k)^*} \le \|f^{-1}\|_\infty^2\|v\|_{(k)^*}.$$

Since $\|S_f|\mathfrak{X}_0\| = \|(S_f|\mathfrak{X}_0)^*\|$, the inequality $\|S_f|\mathfrak{X}_0\| \le \|f^{-1}\|_\infty^2$ follows as well on the Banach space $(\mathfrak{X}_0, \|\cdot\|_{(k)})$. A simple application of the Ky Fan dominance theorem tells us that

$$\|S_f|\mathfrak{X}_0\| \le \|f^{-1}\|_\infty^2$$

for every Banach space $(\mathfrak{X}_0, \|\cdot\|)$. □

A straightforward corollary of the previous proposition and Lemma 2 is the main result of the section.

**Theorem 3.** *For any $f\colon \mathbb{Z}_n \to \mathbb{R}$ such that $f^{-1}$ does exist and $\|\cdot\|$ symmetric norm on $\mathbb{R}^n$,*

$$\|f^{-1} - \mathbb{E}(f^{-1})\| \le \|f^{-1}\|_\infty^2\|f - \mathbb{E}f\|.$$

## 5. An application

Rieffel observed that the standard deviation is a strongly Leibniz seminorm in commutative and non-commutative probability spaces as well [8]. Now we can prove the strong Leibniz inequality for central seminorms of bounded real-valued random variables. One can prove analogues of the result for rearrangement invariant Banach function spaces as well, however, we do not pursue this direction here.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $1 \le p < \infty$. Then the $p^{\text{th}}$ central seminorm of $f \in L^\infty(\Omega,\mu)$ is

$$\sigma_p(f) = \|f - \mathbb{E}f\|_p = \left(\int_\Omega\left|f - \int_\Omega f\,d\mu\right|^p d\mu\right)^{1/p}.$$

Here is one of the main results of the paper.

**Theorem 4.** *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $1 \le p < \infty$. For any real $f$ and $g \in L^\infty(\Omega,\mu)$, we have*

$$\|fg - \mathbb{E}(fg)\|_p \le \|g\|_\infty\|f - \mathbb{E}f\|_p + \|f\|_\infty\|g - \mathbb{E}g\|_p$$

*and*

$$\|f^{-1} - \mathbb{E}(f^{-1})\|_p \le \|f^{-1}\|_\infty^2\|f - \mathbb{E}f\|_p, \qquad \text{if } f^{-1} \in L^\infty(\Omega,\mu).$$

*Proof.* To prove that $\sigma_p$ has the strong Leibniz property, we can derive the general case from the uniform case on $\mathbb{Z}_n$ as in [2, Proposition 2.1]. Indeed, let us consider the measurable simple functions $f_n = \sum_{k=1}^n a_k \chi_{S_k}$ and $g_n = \sum_{k=1}^n b_k \chi_{S_k}$, where $\chi_{S_k}$ denotes the characteristic function of the set $S_k$. Moreover, assume that the sets $S_k$ $(1 \le k \le n)$ are disjoint and $\bigcup_{k=1}^n S_k = \Omega$, so that $\mu(S_k)$-s define a probability measure $\mu_n$ on $\mathbb{Z}_n$. Then for any $\varepsilon > 0$ we can find a probability measure $\nu_n = (p_1, \ldots, p_n)$ such that $p_i \in \mathbb{Q}$ $(1 \le i \le n)$ and the inequalities

$$|\sigma_p(f_n; \mu_n) - \sigma_p(f_n; \nu_n)| \le \varepsilon$$
$$|\sigma_p(g_n; \mu_n) - \sigma_p(g_n; \nu_n)| \le \varepsilon$$
$$|\sigma_p(f_n g_n; \mu_n) - \sigma_p(f_n g_n; \nu_n)| \le \varepsilon$$

hold. Now let us choose the integers $m$ and $r_i$ such that $p_i = r_i/m$ for every $1 \le i \le n$. Then the map

$$\Phi\colon (c_1, \ldots, c_n) \mapsto (\underbrace{c_1, \ldots, c_1}_{r_1}, \ldots, \underbrace{c_n, \ldots, c_n}_{r_n})$$

defines an isometric algebra homomorphism from $\ell_n^\infty$ into $\ell_m^\infty$. Let $\lambda_m$ denote the uniform distribution on the set $\mathbb{Z}_m$. We clearly have, for instance, $\sigma_p(f_n; \nu_n) = \sigma_p(\Phi(f_n); \lambda_m)$. Hence

$$\sigma_p(f_n g_n; \nu_n) \le \|f_n\|_\infty \sigma_p(g_n; \nu_n) + \|g_n\|_\infty \sigma_p(f_n; \nu_n)$$

follows from Theorem 1. Since $\varepsilon$ can be arbitrary small, we obtain that $\sigma_p$ is a Leibniz seminorm on $\ell_n^\infty(\mu_n)$. Now if we choose sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ of measurable simple functions, where $f_n \to f$ and $g_n \to g$ in $L^p$ norm, such that $\|f_n\|_\infty = \|f\|_\infty$ and $\|g_n\|_\infty = \|g\|_\infty$ hold for every $n$, we infer that $\sigma_p$ has the Leibniz property.

A very similar argument with Theorem 3 at hand gives the strong part of the theorem. □

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