The largest Erdős-Ko-Rado sets in $2-(v,k,1)$ designs

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Abstract

An Erdős-Ko-Rado set in a block design is a set of pairwise intersecting blocks. In this article we study Erdős-Ko-Rado sets in $2-(v,k,1)$ designs, Steiner systems. The Steiner triple systems and other special classes are treated separately. For $k \geq 4$, we prove that the largest Erdős-Ko-Rado sets cannot be larger than a point-pencil if $r \geq k^2 - 3k + \frac{3}{4}\sqrt{k} + 2$ and that the largest Erdős-Ko-Rado sets are point-pencils if also $r \neq k^2 - k + 1$ and $(r,k) \neq (8,4)$. For unitals we also determine an upper bound on the size of the second-largest maximal Erdős-Ko-Rado sets.

Keywords: Erdős-Ko-Rado set, block design, Steiner system, unital

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1 Introduction

1.1 Block designs

Definition 1.1. A $t-(v,k,\lambda)$ block design, $v > k > 1$, $k \geq t \geq 1$, $\lambda > 0$, is an incidence geometry $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ with incidence relation $\mathcal{I}$, such that $|\mathcal{P}| = v$, such that any element of $\mathcal{B}$ (blocks) is incident with k elements of $\mathcal{P}$ (points) and such that any t points are contained in λ common blocks. A block can be identified with the k-subset of $\mathcal{P}$ which it determines.

Block designs have been widely studied for many years, see for example [1, 6, 7, 10, 12] for an overview.

The following counting results are widely known.

Theorem 1.2. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a $t-(v,k,\lambda)$ block design. Then,

- the number of blocks through an arbitrary set of i points equals $\lambda_i = \lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$;
- in particular, the number of blocks through a fixed point equals $r = \lambda_1 = \lambda\binom{v-1}{t-1}/\binom{k-1}{t-1}$;
- $b = |\mathcal{B}| = \frac{vr}{k}$.

The most studied class of block designs are the $2-(v,k,1)$ designs, which are called Steiner systems. Among them we mention especially the $2-(n^2+n+1, n+1, 1)$ designs (the projective planes of order n), $n \geq 2$, the $2-(n^2, n, 1)$ designs (the affine planes of order n), $n \geq 3$, and the $2-(n^3+1, n+1, 1)$ designs (the unitals of order n), $n \geq 2$.

By the above results, a $2-(v,k,1)$ design contains $b = \frac{v(v-1)}{k(k-1)}$ blocks, $r = \frac{v-1}{k-1}$ of them through a fixed point. Note that a $2-(v,k,1)$ design can only exist if $v \equiv 1 \pmod{k-1}$ and $k(k-1) \mid v(v-1)$.

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1.2 Erdős-Ko-Rado theorems

In 1961, the original Erdős-Ko-Rado theorem solved a problem in extremal combinatorics.

Theorem 1.3 ([11])**.** Let Ω be a set of size n and $\mathcal{S}$ a family of subsets of size k such that the elements of $\mathcal{S}$ are pairwise not disjoint. If $n \geq 2k$, then $|\mathcal{S}| \leq \binom{n-1}{k-1}$. If $n \geq N_0(k)$, then equality holds if and only if $\mathcal{S}$ is the set of all subsets of size k containing a fixed element of Ω.

In 1984, Wilson showed that the bound $n \geq 2k+1$ is both sufficient and necessary for the above classification: the families $\mathcal{S}$ meeting the upper bound are sets of all subsets of size k containing a fixed element of Ω ([20]).

Many generalisations of this problem have been investigated. The set Ω has often been replaced by a geometry, such as a vector space or a polar space, simultaneously replacing the subsets by subspaces. In general, an *Erdős-Ko-Rado set* is a set of subsets (subspaces of fixed dimension) which are pairwise non-disjoint. It is called *maximal* if it can not be extended to a larger Erdős-Ko-Rado set. Hence, an Erdős-Ko-Rado set on a design is a set of pairwise intersecting blocks. The Erdős-Ko-Rado problem asks for the classification of the (largest) Erdős-Ko-Rado sets.

In [4, Section 2] and [9], surveys of Erdős-Ko-Rado theorems in geometrical settings can be found. Recent results on Erdős-Ko-Rado sets in projective and polar spaces can be found in e.g. [2, 3, 8, 13, 14, 16, 19].

The most important type of Erdős-Ko-Rado sets are the sets of all subsets (blocks, subspaces, ...) through a fixed point. They are called *point-pencils*. In a block design a point-pencil is a maximal Erdős-Ko-Rado set if $r > k$.

For general block designs, the following Erdős-Ko-Rado result was obtained by Rands.

Theorem 1.4 ([18])**.** Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a $t-(v,k,\lambda)$ block design and let $\mathcal{S}$ be a subset of $\mathcal{B}$ such that the blocks of $\mathcal{S}$ have pairwise at least s points in common, $0 < s < t \leq k$.

- If $s < t-1$ and $v \geq s + \binom{k}{s}(k-s+1)(k-s)$, or
- if $s = t-1$ and $v \geq s + \binom{k}{s}^2(k-s)$,

then $|\mathcal{S}| \leq \lambda_s$ and equality is obtained if and only if $\mathcal{S}$ is the set of blocks through s fixed points.

For an Erdős-Ko-Rado set in a $2-(v,k,1)$ design, this implies the following corollary.

Corollary 1.5. Let $\mathcal{D}$ be a $2-(v,k,1)$ block design and let $\mathcal{S}$ be an Erdős-Ko-Rado set of $\mathcal{D}$, $k \geq 2$. If $v \geq 1 + k^2(k-1)$, then $|\mathcal{S}| \leq r$ and $|\mathcal{S}| = r$ if and only if $\mathcal{S}$ is a point-pencil.

In the same article ([18]), it is claimed that the bound $v \geq 1 + k^2(k-1)$ can be improved to $v > k^3 - 2k^2 + 2k$, but there is no proof of this statement. However, it is shown that the bound $v > k^3 - 2k^2 + 2k$ is sharp. If $v = k^3 - 2k^2 + 2k$ and $k-1$ is a prime power, the $2-(v,k,1)$ design consisting of the points and lines of $\mathrm{PG}(3, k-1)$ contains two different types of Erdős-Ko-Rado sets of size $r = k^2 - k + 1$: the set of all blocks through a fixed point and the set of blocks arising from the set of lines in a fixed plane.

In this article, we will prove the result about the bound $v > k^3 - 2k^2 + 2k$ (see Theorem 5.1). It follows from two easy observations. The main part of this paper is devoted to the investigation of $2-(v,k,1)$ designs with $v < k^3 - 2k^2 + 2k$ (see Theorem 5.5). It turns out that $v = k^3 - 2k^2 + 2k$ is an isolated case. For $2-(v,k,1)$ designs with v smaller than $k^3 - 2k^2 + 2k$ but not much, the largest Erdős-Ko-Rado sets are also point-pencils. The results are summarized in Theorem 4.4 and Corollary 5.6.

2 Some special Steiner systems

Remark 2.1. Let $\mathcal{D}$ be a $2-(v,k,1)$ design. For every point P in $\mathcal{D}$, there is a block not containing this point since $v > k$. Each of the points on this block determines a different block through P. Hence, $r \geq k$. If $r = k$, then $\mathcal{D}$ is a projective plane of order $k-1$; if $r = k+1$, then $\mathcal{D}$

is an affine plane of order k. So, the projective and affine planes are the two 'smallest' $2-(v,k,1)$ designs.

We look at the projective and affine planes is detail.

Remark 2.2. In a projective plane, every two blocks have a point in common. Hence, in a projective plane there is only one maximal Erdős-Ko-Rado set of blocks, namely the set of all blocks. Recall that we mentioned in the introduction that a point-pencil is only maximal if $r>k$.

Remark 2.3. In an affine plane of order n, the set of blocks can be partitioned in $n+1$ classes of n blocks, such that the blocks in the same class pairwise have no point in common. These are commonly called parallel classes. Two blocks of different classes always meet in a point. An Erdős-Ko-Rado set contains necessarily at most one block of each parallel class. A maximal Erdős-Ko-Rado set contains precisely one block of each parallel class. Consequently, every maximal Erdős-Ko-Rado set contains $n+1$ blocks.

It should be noted that not all these maximal Erdős-Ko-Rado sets are isomorphic. Also note that the point-pencil can be described in this way.

Now we turn our attention to $2-(v,k,1)$ designs with a special property.

Definition 2.4. The *O'Nan configuration* in a design $\mathcal{D}$ is a set of four blocks, pairwise non-disjoint, such that no three contain a common point.

We will show that we can find a complete classification of the maximal Erdős-Ko-Rado sets on designs not containing an O'Nan configuration. Note that all projective planes and all affine planes of order at least 3 do contain O'Nan configurations.

We already know the point-pencil, a maximal Erdős-Ko-Rado set of size r. We now give an example of a maximal Erdős-Ko-Rado set on a design without an O'Nan configuration.

Example 2.5. Let $\mathcal{D}$ be a $2-(v,k,1)$ design without an O'Nan configuration. Let P be a point and let B be a block of $\mathcal{D}$ such that $P\notin B$. Let $\mathcal{S}$ be the union of $\{B\}$ and the set of all blocks through P meeting B. It is obvious that all blocks of $\mathcal{S}$ meet each other, hence that $\mathcal{S}$ is an Erdős-Ko-Rado set. We call it the *triangle*. It contains $k+1$ blocks. We prove that it is maximal.

Let L be a block of $\mathcal{D}$ not in $\mathcal{S}$, meeting all blocks of $\mathcal{S}$. The block L cannot pass through P, hence meets all blocks of $\mathcal{S}$ through P in a different point. Since $L\neq B$, we know $k\geq 3$. Let P' and P'' be two points on $B\setminus\{L\cap B\}$ and let B' and B'' be the blocks of $\mathcal{S}$ through P, respectively meeting B in the points P' and P''. Then the blocks B, L, B' and B'' determine an O'Nan configuration, a contradiction.

Theorem 2.6. Let $\mathcal{D}$ be a $2-(v,k,1)$ design without an O'Nan configuration and let $\mathcal{S}$ be a maximal Erdős-Ko-Rado set on $\mathcal{D}$. Then, $\mathcal{S}$ is a point-pencil or a triangle.

Proof. Assume that $\mathcal{S}$ is not a point-pencil; then we can find three blocks in $\mathcal{S}$, say B_1, B_2 and B_3, not through a common point. Denote $P_1=B_2\cap B_3$, $P_2=B_3\cap B_1$ and $P_3=B_1\cap B_2$. Any block $B\in\mathcal{S}$ should have a non-empty intersection with as well B_1, B_2 as B_3. Since $\mathcal{D}$ does not contain an O'Nan configuration, B must pass through P_1, P_2 or P_3.

If the block $B_i'\in\mathcal{S}$ passes through P_i, $B_i'\notin\{B_1,B_2,B_3\}$, and the block $B_j'\in\mathcal{S}$ passes through P_j, $B_j'\notin\{B_1,B_2,B_3\}$, $1\leq i\neq j\leq 3$, then the blocks B_i, B_j, B_i' and B_j' determine an O'Nan configuration, a contradiction. Hence, all blocks of $\mathcal{S}\setminus\{B_1,B_2,B_3\}$ pass through the same point P_i, $1\leq i\leq 3$. Since $\mathcal{S}$ is maximal, it has to be a triangle based on the point P_i and the block B_i. □

Note that $r>k+1$ for all $2-(v,k,1)$ designs without an O'Nan configuration, but the affine plane of order 2. Hence, the point-pencil is the largest Erdős-Ko-Rado set in these designs. Of course, the above result only makes sense if $2-(v,k,1)$ designs without an O'Nan configuration exist. We give an example.

Example 2.7. Let $\mathcal{H}(2,q^2)$ be a non-singular Hermitian variety in $\mathrm{PG}(2,q^2)$, the Desarguesian projective plane of order q^2. Up to projective transformations it is defined by $X_0^{q+1}+X_1^{q+1}+X_2^{q+1}=0$. The set of points on $\mathcal{H}(2,q^2)$ and the secant lines to $\mathcal{H}(2,q^2)$ in $\mathrm{PG}(2,q^2)$, determine a unital. This unital is known as the *classical unital* or *Hermitian unital*.

Theorem 2.8 ([15])**.** A classical unital $\mathcal{U}$ does not contain an O'Nan configuration.

It is conjectured that the classical unitals are the only unitals not containing an O'Nan configuration, see [5, 17]. In [5] this conjecture is proven to be true for unitals of order 3. The unique unital of order 2 is also classical.

Corollary 2.9. On a classical unital there are only two types of maximal Erdős-Ko-Rado sets, the point-pencil and the triangle.

3 The counting arguments

In this section we will study maximal Erdős-Ko-Rado sets in $2-(v,k,1)$ designs that are not point-pencils.

Notation 3.1. Let $\mathcal{D}$ be a $2-(v,k,1)$ design and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$. Denote the set of points of $\mathcal{D}$ covered by the blocks of $\mathcal{S}$ by $\mathcal{P}'$.

We denote the number of points of $\mathcal{P}'$ that are contained in precisely i blocks of $\mathcal{S}$ by k_i. Furthermore we denote $k_{\mathcal{S}}=\max\{i \mid k_i>0\}$.

Lemma 3.2. Let $\mathcal{D}$ be a $2-(v,k,1)$ design and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$. Then $|\mathcal{S}|\leq k_{\mathcal{S}}k-k+1$. If $\mathcal{S}$ is maximal and different from the point-pencil, then $k_{\mathcal{S}}\leq k$.

Proof. Fix a block $C\in\mathcal{S}$. All blocks of $\mathcal{S}$ have a nontrivial intersection with C, so

$$|\mathcal{S}|\leq 1+k(k_{\mathcal{S}}-1)=k_{\mathcal{S}}k-k+1\ .$$

Now we prove the second part of the lemma. For every point $P\in\mathcal{P}'$, we can find a block $B\in\mathcal{S}$ not passing through P, since $\mathcal{S}$ is maximal but not a point-pencil. Any block of $\mathcal{S}$ through P should meet B and there is at most one block in $\mathcal{S}$ through P and a given point of B. Hence, there are at most k blocks in $\mathcal{S}$ passing through P. Consequently, $k_{\mathcal{S}}\leq k$. ☐

Lemma 3.3. Choose $l\in\mathbb{N}\setminus\{0,1\}$, and $a,b\in\mathbb{Z}$ with

$$a\geq\max\left\{-l(r-l-1)+1-\frac{br}{l+1},-\frac{b(b-1)}{(l+1)l}-2(b-1)\right\}\ ,$$
$$a\leq\frac{rl-l^2+l-1}{l-1}-\frac{b(2l^2+2l-r+b-1)}{l^2-1}\ .$$

Let $n_1,\dots,n_l\in\mathbb{N}$ be such that $\sum_{i=1}^{l}in_i=(a-1)(l+1)+br+l(l+1)(r-l-1)$ and $\sum_{i=2}^{l}i(i-1)n_i=b(b-1)+l(l+1)(a+2b-2)$. Then $\sum_{i=2}^{l}(i-1)n_i\leq\binom{b}{2}+(a+2b-2)\binom{l+1}{2}$.

Proof. Note that the inequalities $-l(r-l-1)+1-\frac{br}{l+1}\leq a$ and $-\frac{b(b-1)}{(l+1)l}-2(b-1)\leq a$ are present to ensure that both $a(l+1)+br+l(l+1)(r-l-1)-l-1$ and $b(b-1)+l(l+1)(a+2b-2)$ are nonnegative.

Using the first equality, we can express n_1 as a function of l, a, b and $n_2, \ldots, n_l$. Note that

$$\begin{aligned} n_1 &= (a-1)(l+1) + br + l(l+1)(r-l-1) - \sum_{i=2}^{l} i n_i \\ &\geq (a-1)(l+1) + br + l(l+1)(r-l-1) - \sum_{i=2}^{l} i(i-1) n_i \\ &= (a-1)(l+1) + br + l(l+1)(r-l-1) - b(b-1) - l(l+1)(a+2b-2) \\ &= -a(l^2-1) - b(b-1) - b(2l^2+2l-r) + (l+1)(rl-l^2+l-1) \\ &\geq 0 \;, \end{aligned}$$

by the assumption. Hence, for every choice of $n_2, \ldots, n_l$, we can find a value $n_1 \in \mathbb{N}$ such that the first equality holds. Now, we focus on the second equality. Assume that $n_j > 0$ for a value $j \geq 3$. Then define $n'_j = n_j - 1$, $n'_2 = n_2 + \frac{j(j-1)}{2}$ and $n'_k = n_k$ for $k \notin \{2, j\}$. It follows that

$$\sum_{i=2}^{l} i(i-1) n'_i = \sum_{i=2}^{l} i(i-1) n_i = b(b-1) + l(l+1)(a+2b-2) \;.$$

However,

$$\sum_{i=2}^{l} (i-1) n'_i = \left(\sum_{i=2}^{l} (i-1) n_i \right) - (j-1) + \frac{j(j-1)}{2} > \sum_{i=2}^{l} (i-1) n_i \;,$$

since $j \geq 3$. So, repeatedly applying the above construction, we find that $\sum_{i=2}^{l} (i-1) n_i$ is maximal if $n_i = 0$ for all $i \geq 3$ and $n_2 = \binom{b}{2} + (a+2b-2)\binom{l+1}{2}$. The lemma follows. □

Lemma 3.4. Let $\mathcal{D}$ be a $2-(v,k,1)$ design with replication number $r = \frac{v-1}{k-1}$, $k \geq 3$, and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$ such that $|\mathcal{P}'| = k(k-1) + b$. Then

$$\begin{aligned} |\mathcal{S}| \leq \max \Bigg\{ & k^2 - k + 1 - 2\frac{(r-k)(k^2-k+1-r)}{k(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} + 2\frac{(b-1)(k^2-k-r)}{(k-1)(k-2)}, \\ & k^2 - r - \frac{r-1}{k-2} + \frac{b(b-1-r+2k(k-1))}{k(k-2)} \Bigg\} \;. \end{aligned}$$

Proof. Recall that $\mathcal{B}$ is the set of blocks of $\mathcal{D}$. We denote the subset of $\mathcal{B}$ containing precisely i points of $\mathcal{P}'$ by $\mathcal{B}_i$ and we also denote $m_i = |\mathcal{B}_i|$. Note that $\mathcal{S} \subseteq \mathcal{B}_k$. We define $a := k^2 - k + 1 - |\mathcal{B}_k|$. Counting the tuples (P, B) with $P \in \mathcal{P}'$, $B \in \mathcal{B}$ and P on B, we find

$$\sum_{i=1}^{k} i m_i = (k(k-1) + b) r \;.$$

Now applying $m_k = k^2 - k + 1 - a$, we find

$$m_1 = (k(k-1)+b)r - \sum_{i=2}^{k-1} i m_i - k(k^2-k+1-a) = k(k-1)(r-k) + (a-1)k + br - \sum_{i=2}^{k-1} i m_i \;.$$

Counting the tuples (P, P', B) with $P, P' \in \mathcal{P}'$, $B \in \mathcal{B}$, $P \neq P'$ and both P and P' on B, we find

$$\sum_{i=1}^{k} i(i-1) m_i = (k(k-1)+b)(k(k-1)+b-1) \;.$$

Hence,

$$\sum_{i=2}^{k-1} i(i-1) m_i = (k(k-1)+b)(k(k-1)+b-1) - k(k-1)(k^2-k+1-a) = b(b-1) + (a+2b-2)k(k-1) \;.$$

Now we consider the set T of triples (P, P', B) with $P, P' \in \mathcal{P} \setminus \mathcal{P}'$, $B \in \mathcal{B}_1$, $P, P' \in B$ and $P \neq P'$. On the one hand we know

$$\begin{aligned} &|T| = m_1(k-1)(k-2) \\ &= k(k-1)^2(k-2)(r-k) + (a-1)k(k-1)(k-2) + br(k-1)(k-2) - (k-1)(k-2)\sum_{i=2}^{k-1} im_i \,. \end{aligned}$$

On the other hand, using $|\mathcal{P} \setminus \mathcal{P}'| = v - k(k-1) - b = (r-k)(k-1) - (b-1)$, we can also find that

$$|T| \leq ((r-k)(k-1) - (b-1))\,((r-k)(k-1) - b) - \sum_{i=2}^{k-1}(k-i)(k-i-1)m_i \,.$$

Comparing this equality and inequality for $|T|$, we find

$$\begin{aligned} \sum_{i=2}^{k-1}(k(k-1)(i-1) - i(i-1))m_i &\geq k(k-1)^2(k-2)(r-k) + (a-1)k(k-1)(k-2) \\ &+ br(k-1)(k-2) - b(b-1) - (r-k)^2(k-1)^2 + (2b-1)(r-k)(k-1) \,. \end{aligned}$$

Using the formula for $\sum_{i=2}^{k-1} i(i-1)m_i$, and dividing both sides by $k-1$, it follows that

$$k\sum_{i=2}^{k-1}(i-1)m_i \geq ak(k-1) + bkr - k^2 + (r-k)(k^3 - 2k^2 - (r-1)(k-1)) \,.$$

We distinguish between two cases. If $a > r - k + 1 + \frac{r-1}{k-2} - \frac{b(b-1-r+2k(k-1))}{k(k-2)}$, then $|\mathcal{S}| \leq |\mathcal{B}_k| \leq k^2 - r - \frac{r-1}{k-2} + \frac{b(b-1-r+2k(k-1))}{k(k-2)}$. If $a \leq r - k + 1 + \frac{r-1}{k-2} - \frac{b(b-1-r+2k(k-1))}{k(k-2)}$, we can apply Lemma 3.3 with $l = k-1$. Note that the conditions $-l(r-l-1) + 1 - \frac{br}{l+1} \leq a$ and $-\frac{b(b-1)}{(l+1)l} - 2(b-1) \leq a$ are fulfilled since $\sum_{i=2}^{k-1} i(i-1)m_i$ and $\sum_{i=1}^{k-1} im_i$ are nonnegative. We find

$$k\binom{b}{2} + k(a+2b-2)\binom{k}{2} \geq ak(k-1) + bkr - k^2 + (r-k)(k^3 - 2k^2 - (r-1)(k-1)) \,,$$

hence

$$a \geq \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} - \frac{2(b-1)(k^2-k-r)}{(k-1)(k-2)} - \frac{b(b-1)}{(k-1)(k-2)} \,.$$

We find thus that

$$|\mathcal{S}| \leq |\mathcal{B}_k| \leq k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} + \frac{2(b-1)(k^2-k-r)}{(k-1)(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} \,,$$

which finishes the proof. □

Using the substitution $R = (k-1)^2 - r$, we can rewrite this lemma.

Corollary 3.5. Let $\mathcal{D}$ be a $2-(v,k,1)$ design, $k \geq 3$, and denote $(k-1)^2 - r = (k-1)^2 - \frac{v-1}{k-1}$ by R. Let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$ such that $|\mathcal{P}'| = k(k-1) + b$. Then

$$\begin{aligned} |\mathcal{S}| \leq \max\Bigg\{&k^2 - k + 1 - 2\frac{(k^2-3k+1-R)(k+R)}{k(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} + 2\frac{(b-1)(k-1+R)}{(k-1)(k-2)}, \\ &k - 1 + R + \frac{R}{k-2} + \frac{b(b+k^2+R-2)}{k(k-2)}\Bigg\} \,. \end{aligned}$$

Lemma 3.6. Let $\mathcal{D}$ be a $2-(v,k,1)$ design and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$ with $k_{\mathcal{S}} = k$. Then $|\mathcal{P}'| = k^2 - k + 1$.

Proof. Since $k_{\mathcal{S}} = k$, we can find a point $P \in \mathcal{P}'$ lying on k blocks of $\mathcal{S}$. Denote these blocks by $B_1, \ldots, B_k$ and denote the set of points covered by these blocks by $\mathcal{P}''$. Any block of $\mathcal{S}$ not through P contains a point on each of the blocks B_i, $i = 1, \ldots, k$. Since a block contains precisely k points, all points on such a block are contained in $\mathcal{P}''$. Hence, $\mathcal{P}'' = \mathcal{P}'$ and

$$|\mathcal{P}''| = |\bigcup_{i=1}^{k} B_i| = 1 + k(k-1) = k^2 - k + 1 \,. \qquad \square$$

Lemma 3.7. Let $\mathcal{D}$ be a $2-(v,k,1)$ design and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$ with $k_{\mathcal{S}} = k-1$. Write $a' = (k-1)^2 - |\mathcal{S}|$. If $a' < k-1$, then $k(k-1) \leq |\mathcal{P}'| \leq k(k-1) + \frac{a'^2 - a'}{k-1-a'}$.

Proof. First we will prove that there is a block in $\mathcal{S}$ containing at least two points that are on $k-1$ blocks of $\mathcal{S}$. Assume there is no such block and choose a block C. At most one point on C belongs to $k-1$ blocks of $\mathcal{S}$. However, all blocks of $\mathcal{S}$ have a nontrivial intersection with C, so

$$|\mathcal{S}| \leq 1 + (k-2) + (k-1)(k-3) = (k-1)(k-2) \,,$$

hence $a' \geq k-1$, which contradicts the assumption $a' < k-1$.

Let B_1 be a block of $\mathcal{S}$ through the points Q_1 and Q_2, both on $k-1$ blocks of $\mathcal{S}$, and let $B_1, B_2, \ldots, B_{k-1}$ and $B_1 = C_1, C_2, \ldots, C_{k-1}$ be the blocks of $\mathcal{S}$, respectively through Q_1 and Q_2. There are $(k-2)^2$ points which lie on a block B_j and also on a block $C_{j'}$, $2 \leq j, j' \leq k-1$; there are $k-2$ points which lie on a block B_i, but not on a block $C_{i'}$, and there are also $k-2$ points which lie on a block C_i, but not on a block $B_{i'}$; the block $B_1 = C_1$ contains k points. Hence, $|\mathcal{P}'| \geq (k-2)^2 + 2(k-2) + k = k(k-1)$.

Now, recall the notation k_i. By standard counting arguments we know that

$$\sum_{i=1}^{k-1} i k_i = ((k-1)^2 - a')k \quad \text{and} \quad \sum_{i=1}^{k-1} i(i-1)k_i = ((k-1)^2 - a')(k(k-2) - a') \,.$$

Let $j \in \mathbb{N} \setminus \{0\}$ be the smallest value such that $k_j \neq 0$ and let R be a point of $\mathcal{P}'$ on j blocks of $\mathcal{S}$. Let $B \in \mathcal{S}$ be a block through R. All blocks of $\mathcal{S}$ meet B, hence

$$|\mathcal{S}| = (k-1)^2 - a' \leq 1 + (k-1)(k-2) + (j-1) \,.$$

It follows that $j \geq k-1-a'$. Therefore, the following inequality holds:

$$\sum_{i=1}^{k-1} (i - (k-a'-1))(k-1-i)k_i \geq 0 \,.$$

So,

$$\begin{aligned}
0 &\leq -\sum_{i=1}^{k-1} i(i-1)k_i + (2k-a'-3)\sum_{i=1}^{k-1} ik_i - (k-a'-1)(k-1)\sum_{i=1}^{k-1} k_i \\
&= -((k-1)^2 - a')(k(k-2) - a') + (2k-a'-3)((k-1)^2 - a')k - (k-a'-1)(k-1)\sum_{i=1}^{k-1} k_i \\
&= ((k-1)^2 - a')(k-1)(k-a') - (k-a'-1)(k-1)\sum_{i=1}^{k-1} k_i \,.
\end{aligned}$$

Consequently,

$$|\mathcal{P}'| = \sum_{i=1}^{k-1} k_i \leq \frac{((k-1)^2 - a')(k-a')}{k-a'-1} = k(k-1) + \frac{a'^2 - a'}{k-a'-1}$$

and the lemma follows. $\square$

4 Classification results for $k = 3$

For $k = 2$, a $2 - (v, k, 1)$ design is a complete graph K_v on v vertices, the edges being the blocks. It can immediately be seen that there are precisely two different types of maximal Erdős-Ko-Rado sets on K_v, namely the point-pencil, which contains $v - 1$ blocks, and the triangle, a set $\{\{p_1, p_2\}, \{p_1, p_3\}, \{p_2, p_3\}\}$ for three points $p_1, p_2, p_3 \in \mathcal{P}$, which contains 3 blocks.

So, the first nontrivial case is $k = 3$. A $2 - (v, 3, 1)$ design is called a Steiner triple system of size v. Steiner triple systems exist if and only if $v \equiv 1, 3 \pmod 6$ and $v \geq 7$. Up to isomorphism, there is only one Steiner triple system for $v = 7$, namely the Fano plane, the projective plane of order 2; there is only one Steiner triple system for $v = 9$, namely the affine plane of order 3; and there are two Steiner triple systems for $v = 13$. For more details, we refer the interested reader to [7, Section II.1, Section II.2].

Theorem 4.1. Let $\mathcal{D}$ be a $2 - (v, 3, 1)$ design and let $\mathcal{S}$ be a maximal Erdős-Ko-Rado set of $\mathcal{D}$. Then $\mathcal{S}$ belongs to one of five types. The maximal Erdős-Ko-Rado sets contain $\frac{v-1}{2}$, 4, 5, 6 or 7 blocks. Each type corresponds to a size and vice versa.

Proof. If all blocks of $\mathcal{S}$ pass through a common point, then $\mathcal{S}$ is a point-pencil and it contains $\frac{v-1}{2}$ blocks. So, from now on we assume that there is no point on all blocks of $\mathcal{S}$. Let $B_1, B_2, B_3 \in \mathcal{S}$ be three blocks such that $B_1 \cap B_2 = \{P_3\}$, $B_1 \cap B_3 = \{P_2\}$ and $B_2 \cap B_3 = \{P_1\}$, with P_1, P_2, P_3 three different points. Let Q_i be the third point on the block B_i, $i = 1, 2, 3$. There is precisely one block through the points P_i and Q_i. We denote it by B'_i and we denote the third point on this block by R_i, $i = 1, 2, 3$.

If the three points Q_1, Q_2 and Q_3 are contained in a common block B', then this block has to be contained in $\mathcal{S}$ by the maximality condition. The only other blocks that could be contained in $\mathcal{S}$ are B'_1, B'_2 and B'_3. If all three points R_1, R_2 and R_3 are different, then only one of these blocks belongs to $\mathcal{S}$. We find an Erdős-Ko-Rado set of size 4 or 5, depending on whether the block B' exists. If two of the points R_1, R_2 and R_3 coincide, then we find an Erdős-Ko-Rado set of size 5 or 6. If $R_1 = R_2 = R_3$, then we find an Erdős-Ko-Rado set of size 6 or 7.

Note that the two constructions of Erdős-Ko-Rado sets of size 5 give rise to isomorphic sets, so there is only one type of Erdős-Ko-Rado sets of size 5. Analogously, there is also only one type of Erdős-Ko-Rado sets of size 6. □

Remark 4.2. The five types of maximal Erdős-Ko-Rado sets in $2 - (v, 3, 1)$ designs are explicitly described in the above theorem. Apart from the point-pencil, these block sets can be embedded in a Fano plane. However, they cannot be extended to a Fano plane by blocks of the design, due to the maximality condition. Note that the Erdős-Ko-Rado set of size 7 is a Fano plane that is embedded in the design.

Since the four types of maximal Erdős-Ko-Rado sets different from the point-pencil are determined by their size, we can denote them by EKR_i, $i = 4, \ldots, 7$, the index referring to their size. Note that each of the maximal Erdős-Ko-Rado sets different from the point-pencil, cover precisely 7 points of the design.

Remark 4.3. In a given $2 - (v, 3, 1)$ design $\mathcal{D}$, not necessarily all five types occur. For example, if $\mathcal{D}$ is the Fano plane ($v = 7$), then there is only one maximal Erdős-Ko-Rado set, namely EKR_7, which is the set of all blocks in this case. If $\mathcal{D}$ is not a projective plane, at least two types occur, one of which is the point-pencil.

We list the results for Erdős-Ko-Rado sets on Steiner triple systems of size v. For small values of v, the results are more detailed.

Theorem 4.4. Let $\mathcal{D}$ be a $2 - (v, 3, 1)$ design.

- If $v = 7$, there is only one maximal Erdős-Ko-Rado set in $\mathcal{D}$.
- If $v = 9$, there are two types of maximal Erdős-Ko-Rado sets in $\mathcal{D}$, the point-pencil and EKR_4. Both contain 4 blocks.

- If $v = 13$, there are three types of maximal Erdős-Ko-Rado sets in $\mathcal{D}$, the point-pencil, EKR_4 and EKR_5. The largest Erdős-Ko-Rado sets are the point-pencils.
- If $v = 15$, the largest Erdős-Ko-Rado sets contain 7 blocks. There are 23 nonisomorphic $2-(15,3,1)$ designs containing an EKR_7, and 57 nonisomorphic $2-(15,3,1)$ designs not containing an EKR_7. The former have two types of maximal Erdős-Ko-Rado sets of size 7; for the latter all Erdős-Ko-Rado sets of size 7 are point-pencils.
- If $v \geq 19$, the largest Erdős-Ko-Rado sets are point-pencils.

Proof. The case $v = 7$ has been treated in Remark 4.3. If $v = 9$, then $\mathcal{D}$ is an affine plane of order 3. One can see immediately that only two of the above types of maximal Erdős-Ko-Rado sets occur, the point-pencil and the smallest one of the others, the EKR_4. Both contain four blocks. Compare this result with Remark 2.3

If $v = 13$, there are two nonisomorphic $2-(v,3,1)$ designs. Their point sets can be denoted by $\{0,1,\dots,9,a,b,c\}$. Using [7, Table II.1.27], we can write the block sets as in Table 1.

0	0	0	0	0	0	1	1	1	1	1	2	2	2	2	2	3	3	3	4	4	4	5	5	5	6
1	3	5	7	9	b	3	4	6	9	a	3	4	6	7	8	6	7	8	6	8	a	7	8	9	7
2	4	6	8	a	c	5	7	8	b	c	9	5	a	c	b	b	a	c	c	9	b	b	a	c	9

0	0	0	0	0	0	1	1	1	1	1	2	2	2	2	2	3	3	3	4	4	4	5	5	5	6
1	3	5	7	9	b	3	4	6	9	a	3	4	6	7	8	6	7	8	6	8	a	7	8	9	7
2	4	6	8	a	c	5	7	8	b	c	9	5	a	b	c	b	c	a	c	9	b	a	b	c	9

Table 1: Block sets

We know that the point-pencil contains 6 blocks. By Theorem 3.4, applied for $k = 3$, $b = 1$ and $r = 6$, we know that any other maximal Erdős-Ko-Rado set contains at most 5 blocks. So, on both $2-(13,3,1)$ designs, at most three types of maximal Erdős-Ko-Rado sets occur. Using the above notation, the two sets $\{\{0,1,2\},\{0,3,4\},\{1,3,5\},\{2,3,9\},\{2,4,5\}\}$ and $\{\{0,1,2\},\{0,3,4\},\{0,9,a\},\{2,3,9\}\}$ are maximal Erdős-Ko-Rado sets for both $2-(13,3,1)$ designs. Hence, there are precisely three types of maximal Erdős-Ko-Rado sets on $2-(13,3,1)$ designs.

There are 80 nonisomorphic $2-(15,3,1)$ designs, see [7, Table II.1.28] for an overview. The point-pencil contains 7 blocks in these designs. In [7, Table II.1.29] it is mentioned which of these 80 designs contains a Fano plane as subdesign; 23 of them do, and 57 do not. The statement follows.

If $v \geq 19$, then $r \geq 9$, hence the point-pencil contains more blocks than the Erdős-Ko-Rado sets of type EKR_i, $i = 4,\dots,7$. $\square$

Note that one of the 23 different $2-(15,3,1)$ designs having a Fano plane as subdesign, is the design consisting of the points and lines of PG(3, 2). Also note that the last part of Theorem 4.4 is a special case of Corollary 1.5.

5 Classification results for $k \geq 4$

In this section we present the main classification theorems for Erdős-Ko-Rado sets in $2-(v,k,1)$ designs. In Theorem 5.1 we will provide a proof for the result claimed in [18] about $2-(v,k,1)$ designs with large v. Theorem 5.5 contains a classification theorem for $2-(v,k,1)$ designs with v a little smaller. A survey result can be found in Corollary 5.6.

In this section we will use the parameter $k_{\mathcal{S}}$, introduced in Notation 3.1.

Theorem 5.1. Let $\mathcal{D}$ be a $2-(v,k,1)$ design and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$. If $r \geq k^2-k+1$, then $|\mathcal{S}| \leq r$. If $r = \frac{v-1}{k-1} > k^2-k+1$ and $|\mathcal{S}| = r$, then $\mathcal{S}$ is a point-pencil.

Proof. Without loss of generality, we can assume that $\mathcal{S}$ is a maximal Erdős-Ko-Rado set. If $\mathcal{S}$ is a point-pencil, then $|\mathcal{S}| = r$. So, from now on, we can assume that $\mathcal{S}$ is not a point-pencil. By Lemma 3.2 we know that $k_{\mathcal{S}} \leq k$. However, by the same lemma we also know that $|\mathcal{S}| \leq k^2 - k + 1$, if $k_{\mathcal{S}} \leq k$.

Both statements in the theorem immediately follow. □

As mentioned at the end of Section 1, there are $2-(v,k,1)$ designs with $r = k^2 - k + 1$, having a second type of Erdős-Ko-Rado sets of size r.

Now, we look at Erdős-Ko-Rado sets in $2-(v,k,1)$ designs with $r \leq k^2 - k$. A classification result will be proven in Theorem 5.5. Before we prove some preparatory lemmas. In these lemmas we distinguish between the case $4 \leq k \leq 13$ and the case $k \geq 14$.

First, we have a look at the small cases, $4 \leq k \leq 13$.

k	4	5	6	7	8	9	10	11	12	13
R_k	1	2	3	4	4	5	6	7	8	9

Table 2: The values R_k.

Lemma 5.2. Let $\mathcal{D}$ be a $2-(v,k,1)$ design, $4 \leq k \leq 13$, and denote $(k-1)^2 - r = (k-1)^2 - \frac{v-1}{k-1}$ by R. Let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$ with $k_{\mathcal{S}} = k-1$. If $0 \leq R \leq R_k$, then $|\mathcal{S}| < (k-1)^2 - R$.

Proof. We denote $(k-1)^2 - |\mathcal{S}|$ by a', as in Lemma 3.7. By Lemma 3.2 we know that $a' \geq 0$. If $R < a'$, then $|\mathcal{S}| < (k-1)^2 - R$. So, now we assume that $a' \leq R$. Since $R_k < k-1$, also $a' < k-1$ and we know by Lemma 3.7 that $k(k-1) \leq |\mathcal{P}'| \leq k(k-1) + \frac{R(R-1)}{k-1-R}$. Denoting $|\mathcal{P}'| - k(k-1)$ by b, it follows that $0 \leq b \leq \frac{R(R-1)}{k-1-R}$. By Lemma 3.5 we know that

$$|\mathcal{S}| \leq \max\left\{k^2 - k + 1 - 2\frac{(k^2-3k+1-R)(k+R)}{k(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} + 2\frac{(b-1)(k-1+R)}{(k-1)(k-2)},\right.$$
$$\left. k - 1 + R + \frac{R}{k-2} + \frac{b(b+k^2+R-2)}{k(k-2)}\right\} .$$

By hand or by using a computer algebra package, it can be checked that the above maximum is smaller than $(k-1)^2 - R = r$ for all choices of k, R, b fulfilling $4 \leq k \leq 13$, $0 \leq R \leq R_k$ and $0 \leq b \leq \frac{R(R-1)}{k-1-R}$. □

Extending the calculations in the above proof, we can see that the values R_k are optimal; enlarging one of these values leads to a contradiction.

Now, we look at the more general case $k \geq 14$. We start with some inequalities which we will need in the proof of Lemma 5.4

Lemma 5.3. Choose $b, c, k \in \mathbb{N}$, with $k \geq 14$, $1 \leq c \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k}$ and $0 \leq b \leq c$. Then

$$\frac{k^3 - 7k^2 + 10k - 2bk - 2 - \sqrt{D(b,k)}}{4(k-1)} < \frac{1 - c + \sqrt{(c-1)^2 + 4c(k-1)}}{2} ,$$

with $D(b,k) = (k^3 - 3k^2 - 2bk + 6k - 2)^2 - 8k(k-1)(b-1)(b-2)$. Furthermore, for $k \in \mathbb{N}$ with $k \geq 14$,

$$\frac{k^3 - 7k^2 + 10k - 2 - \sqrt{D(0,k)}}{4(k-1)} < 0 .$$

Proof. First, note that $D(b,k) \geq 0$ for all $0 \leq b \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k} =: C_k$, hence the above functions exist.

The second part of the lemma is immediate, so we focus on the first part. Note that

$$\frac{k^3-7k^2+10k-2(b+1)k-2-\sqrt{D(b+1,k)}}{4(k-1)}-\frac{k^3-7k^2+10k-2bk-2-\sqrt{D(b,k)}}{4(k-1)}$$
$$=\frac{\sqrt{D(b,k)}-\sqrt{D(b+1,k)}-2k}{4(k-1)}\ .$$

Now,

$$\begin{aligned}
& \frac{\sqrt{D(b,k)}-\sqrt{D(b+1,k)}-2k}{4(k-1)}\geq 0 \\
\Leftrightarrow\quad & \sqrt{D(b,k)}-\sqrt{D(b+1,k)}\geq 2k \\
\Leftrightarrow\quad & D(b,k)-D(b+1,k)\geq 2k\left(\sqrt{D(b,k)}+\sqrt{D(b+1,k)}\right) \\
\Leftrightarrow\quad & 2k^3-6k^2+4bk+2k-8b+4\geq \sqrt{D(b,k)}+\sqrt{D(b+1,k)}\ .
\end{aligned}$$

This final inequality is valid since $\sqrt{D(b,k)}+\sqrt{D(b+1,k)}\leq 2k^3-6k^2-4bk+10k-4$. These calculations show that

$$\frac{k^3-7k^2+10k-2(b+1)k-2-\sqrt{D(b+1,k)}}{4(k-1)}\geq\frac{k^3-7k^2+10k-2bk-2-\sqrt{D(b,k)}}{4(k-1)}\ .$$

Hence, it is sufficient to prove that

$$\frac{k^3-7k^2+10k-2ck-2-\sqrt{D(c,k)}}{4(k-1)}<\frac{1-c+\sqrt{(c-1)^2+4c(k-1)}}{2}\ .$$

Since $c\leq\frac{4}{3}k\sqrt{k}-2k-2\sqrt{k}<\frac{k^3-7k^2+8k}{2}$ for $k\geq 14$, this is equivalent to

$$\begin{aligned}
& \left(2(k-1)\sqrt{(c-1)^2+4c(k-1)}+\sqrt{D(c,k)}\right)^2>(k^3-7k^2+8k-2c)^2 \\
\Leftrightarrow\quad & \sqrt{(c-1)^2+4c(k-1)}\sqrt{D(c,k)}>-2k^4+(9+c)k^3-(7c+9)k^2+(14c-2)k+2-6c\ .
\end{aligned} \tag{1}$$

Considering the left-hand side of the inequality (1) as a function of c, for a fixed value of k, we can compute its second derivative. We find that this second derivative is negative on $[0,C_k]$, hence the function on the left-hand side is concave on $[0,C_k]$. Therefore, it dominates the function

$$c\mapsto\sqrt{D(0,k)}+c\frac{\sqrt{(C_k-1)^2+4C_k(k-1)}\sqrt{D(C_k,k)}-\sqrt{D(0,k)}}{C_k}\ .$$

The slope of this line is smaller than $k^3-7k^2+14k-6$. So, we only need to check the inequality for the largest considered value for c, namely C_k. It turns out that this inequality is valid if $k\geq 14$. □

In the final step of the argument we needed that $k\geq 14$. This is why the cases $4\leq k\leq 13$ had to be treated separately. We now discuss $2-(v,k,1)$ designs with $k_{\mathcal{S}}=k-1$. These are the hardest case in the proof of Theorem 5.5.

Lemma 5.4. Let $\mathcal{D}$ be a $2-(v,k,1)$ design, $k\geq 14$, and denote $(k-1)^2-r=(k-1)^2-\frac{v-1}{k-1}$ by R. Let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$ with $k_{\mathcal{S}}=k-1$. If $0\leq R<\sqrt{k-1}$ or $\frac{1-c+\sqrt{(c-1)^2+4c(k-1)}}{2}\leq R<\frac{-c+\sqrt{c^2+4(c+1)(k-1)}}{2}$ for a value $c\in\mathbb{N}$, with $1\leq c\leq\frac{4}{3}k\sqrt{k}-2k-2\sqrt{k}$, then $|\mathcal{S}|<(k-1)^2-R$.

Proof. Denote the interval $\left[\frac{1-c+\sqrt{(c-1)^2+4c(k-1)}}{2}, \frac{-c+\sqrt{c^2+4(c+1)(k-1)}}{2}\right[$ by I_c, $c \in \mathbb{N}$ and $1 \leq c \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k} := C_k$, and the interval $\left[0, \sqrt{k-1}\right[$ by I_0. Recall the notation $\mathcal{P}'$. We assume that $R \in I_c$. From Lemma 3.7, it follows that $|\mathcal{P}'| \leq k(k-1) + c$. Hence, by Corollary 3.5,

$$|\mathcal{S}| \leq \max\left\{k^2 - k + 1 - 2\frac{(k^2-3k+1-R)(k+R)}{k(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} + 2\frac{(b-1)(k-1+R)}{(k-1)(k-2)}, \right.$$
$$\left. k - 1 + R + \frac{R}{k-2} + \frac{b(b+k^2+R-2)}{k(k-2)}\right\},$$

with $b = k(k-1) - |\mathcal{P}'|$, hence $0 \leq b \leq c$. Since $c \leq C_k$ and $R < k-2$, the inequality

$$k - 1 + R + \frac{R}{k-2} + \frac{b(b+k^2+R-2)}{k(k-2)} < (k-1)^2 - R$$

clearly holds in all cases. Now, we consider the inequality

$$(k-1)^2 - R > k^2 - k + 1 - 2\frac{(k^2-3k+1-R)(k+R)}{k(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} + 2\frac{(b-1)(k-1+R)}{(k-1)(k-2)}$$
$$\Leftrightarrow \qquad 0 > k + R - 2\frac{(k^2-3k+1-R)(k+R)}{k(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} + 2\frac{(b-1)(k-1+R)}{(k-1)(k-2)}.$$

This inequality is valid if and only if

$$\frac{k^3 - 7k^2 + 10k - 2bk - 2 - \sqrt{D(b,k)}}{4(k-1)} < R < \frac{k^3 - 7k^2 + 10k - 2bk - 2 + \sqrt{D(b,k)}}{4(k-1)}, \qquad (2)$$

with $D(b,k) = (k^3 - 3k^2 - 2bk + 6k - 2)^2 - 8k(k-1)(b-1)(b-2)$. The double inequality in (2) should hold for all b, with $0 \leq b \leq c$. Now,

$$R < \frac{-c + \sqrt{c^2 + 4(c+1)(k-1)}}{2} \qquad \text{and}$$
$$\frac{k^3 - 7k^2 + 10k - 2ck - 2}{4(k-1)} \leq \frac{k^3 - 7k^2 + 10k - 2bk - 2 + \sqrt{D(b,k)}}{4(k-1)},$$

but the inequality $\frac{-c+\sqrt{c^2+4(c+1)(k-1)}}{2} < \frac{k^3-7k^2+10k-2ck-2}{4(k-1)}$ holds for all $0 \leq c \leq C_k$ since $k \geq 14$. Hence, the right inequality in (2) always holds. Using

$$R \geq \frac{1 - c + \sqrt{(c-1)^2 + 4c(k-1)}}{2}$$

and Lemma 5.3, also the left inequality in (2) follows. This finishes the proof. □

Theorem 5.5. Let $\mathcal{D}$ be a $2-(v,k,1)$ design, $k \geq 4$, and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$. If $k^2 - k \geq r = \frac{v-1}{k-1} \geq k^2 - 3k + \frac{3}{4}\sqrt{k} + 2$, then $|\mathcal{S}| \leq r$. If $(r,k) \neq (8,4)$, equality is obtained if and only if $\mathcal{S}$ is a point-pencil.

Proof. Without loss of generality, we can assume that $\mathcal{S}$ is a maximal Erdős-Ko-Rado set. Recall the notation $k_{\mathcal{S}}$. If $\mathcal{S}$ is a point-pencil, then $|\mathcal{S}| = r$. So, from now on, we can assume that $\mathcal{S}$ is not a point-pencil. By Lemma 3.2 we know that $k_{\mathcal{S}} \leq k$. We distinguish between three cases.

- If $k_{\mathcal{S}} = k-1$, then $|\mathcal{S}| \leq k^2 - 2k + 1$ by Lemma 3.2. In this case, if $k^2 - 2k + 1 < r \leq k^2 - k$, the theorem clearly holds, so we assume $r \leq k^2 - 2k + 1$. As before, we denote $R = (k-1)^2 - r$. First, assume that $k \geq 14$. In this case, $0 \leq R \leq k - \frac{3}{4}\sqrt{k} - 1$. So, $0 \leq R < \sqrt{k-1}$ or there is a value $c \in \mathbb{N}$, with $1 \le c \le \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k}$, such that $\frac{1-c+\sqrt{(c-1)^2+4c(k-1)}}{2} \le R < \frac{-c+\sqrt{c^2+4(c+1)(k-1)}}{2}$. Applying Lemma 5.4 we find that $|\mathcal{S}| < (k-1)^2 - R = r$.

Now assume that $4 \le k \le 13$. In this case, $0 \le R \le R_k = \left\lfloor k - \frac{3}{4}\sqrt{k} - 1 \right\rfloor$. Applying Lemma 5.2, we find that $|\mathcal{S}| < (k-1)^2 - R = r$.

- If $k_{\mathcal{S}} = k$, then $|\mathcal{P}'| = k^2 - k + 1$ by Lemma 3.6. So, we can apply Lemma 3.4 with $b = 1$. We find that

$$|\mathcal{S}| \le \max\left\{k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)}, k^2 - r - \frac{r-1}{k-2} + \frac{2k(k-1)-r}{k(k-2)}\right\}.$$

The inequality $k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} < r$ holds if and only if $\frac{k^2}{2} < r < k^2 - k + 1$. If $k \ge 5$, this condition is fulfilled since $k^2 - k + 1 > k^2 - k$ and $\frac{k^2}{2} < k^2 - 3k + \frac{3}{4}\sqrt{k} + 2$. If $k = 4$ and $R = 0$, hence $r = 9$, then $k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} = 8 < r$; if $k = 4$ and $R = 1$, hence $r = 8$, then $k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} = 8 = r$.

Since $k^2 - 3k + \frac{3}{4}\sqrt{k} + 2 > \frac{k^2}{2} - \frac{k}{4} + \frac{3}{8}$ for all $k \ge 4$, the inequality $k^2 - r - \frac{r-1}{k-2} + \frac{2k(k-1)-r}{k(k-2)} < r$ is fulfilled in all cases.

- If $k_{\mathcal{S}} \le k-2$, then $|\mathcal{S}| \le k^2 - 3k + 1$ by Lemma 3.2. Clearly, $k^2 - 3k + 1 < k^2 - 3k + \frac{3}{4}\sqrt{k} + 2 \le r$.

Hence, for $k \ge 5$, in all three cases $|\mathcal{S}| < r$; for $k = 4$, in all three cases $|\mathcal{S}| \le r$ and moreover $|\mathcal{S}| < r$ if $r \ne 8$. The theorem follows. □

We now summarize the results of this section.

Corollary 5.6. Let $\mathcal{D}$ be a $2-(v,k,1)$ design, $k \ge 4$, with $r = \frac{v-1}{k-1} \ge k^2 - 3k + \frac{3}{4}\sqrt{k} + 2$, and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$. Then $|\mathcal{S}| \le r$. If $r \ne k^2 - k + 1$ and $(r,k) \ne (8,4)$, then $|\mathcal{S}| = r$ if and only if $\mathcal{S}$ is a point-pencil.

Proof. This follows immediately from Theorem 5.1 and Theorem 5.5. □

6 Maximal Erdős-Ko-Rado sets in unitals

The results from Lemma 3.2, Lemma 3.4, Lemma 3.6 and Lemma 3.7 can also be used in a different way. For a fixed class of designs, with v (or equivalently r) a function of k, an upper bound on the size of the largest maximal Erdős-Ko-Rado set different from a point-pencil can be computed. We show this for the unitals. Recall that a $2-(q^3+1, q+1, 1)$ design is a unital of order q. First we state Lemma 3.4 for a unital of order q.

Lemma 6.1. Let $\mathcal{U}$ be a unital of order q and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{U}$ such that $|\mathcal{P}'| = q(q+1) + b$, whereby $\mathcal{P}'$ is the set of points covered by the elements of $\mathcal{S}$. Then

$$|\mathcal{S}| \le \max\left\{q^2 - q + 1 + \frac{b(b-1)}{q(q-1)} + \frac{2b}{q-1}, q + \frac{bq(q+2)}{q^2-1} + \frac{b(b-1)}{q^2-1}\right\}.$$

Lemma 6.2. Let $\mathcal{U}$ be a unital of order q and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{U}$ with $k_{\mathcal{S}} = q+1$. If $q \ge 4$, then $|\mathcal{S}| \le q^2 - q + 1$. If $q = 3$, then $|\mathcal{S}| \le 8$.

Proof. By Lemma 3.6 we know that $|\mathcal{P}'| = q^2 + q + 1$. We apply Lemma 6.1 and we find that $|\mathcal{S}| \le \max\left\{q^2 - q + 1 + \frac{2}{q-1}, q + \frac{q(q+2)}{q^2-1}\right\}$. The lemma immediately follows. □

Lemma 6.3. Let $\mathcal{U}$ be a unital of order q and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{U}$ with $k_{\mathcal{S}} = q$. If $q \ge 5$, then $|\mathcal{S}| \le q^2 - q + \sqrt[3]{q^2} - \frac{2}{3}\sqrt[3]{q} + 1$. If $q = 3$, then $|\mathcal{S}| \le 7$; if $q = 4$, then $|\mathcal{S}| \le 13$.

Proof. Denote $q^2 - |\mathcal{S}|$ by a'. We can assume $a' < q$ since otherwise the lemma clearly holds. By Lemma 3.2, we know that $a' \geq 0$, and by Lemma 3.7 we know that $|\mathcal{P}'| = q^2 + q + b$, with $0 \leq b \leq \frac{a'^2 - a'}{q - a'}$. We apply Lemma 6.1 and we find that

$$|\mathcal{S}| \leq q^2 - q + 1 + 2\frac{a'(a'-1)}{(q-a')(q-1)} + \frac{a'(a'-1)(a'^2-q)}{q(q-1)(q-a')^2}$$

$$\text{or} \quad |\mathcal{S}| \leq q + \frac{qa'(q+2)(a'-1)}{(q^2-1)(q-a')} + \frac{a'(a'-1)(a'^2-q)}{(q-a')^2(q^2-1)} \; .$$

Using $|\mathcal{S}| = q^2 - a'$, the first inequality can be rewritten as

$$q(q - a' - 1)(q - a')^2(q - 1) \leq a'(a' - 1)(2q^2 - 2qa' + a'^2 - q) \; .$$

For $q = 3$, this implies $a' \geq 2$ and for $q = 4$, this implies $a' \geq 3$. For general q, it implies $a' \geq q - \sqrt[3]{q^2} + \frac{2}{3}\sqrt[3]{q} - 1$.

Now we look at the second inequality. Using $|\mathcal{S}| = q^2 - a'$, it can be rewritten as

$$(q^2 - q - a')(q - a')^2(q^2 - 1) \leq a'(a' - 1)(q^3 - (a' - 2)q^2 - (2a' + 1)q + a'^2) \; .$$

Using that $0 \leq a' < q$, it follows that $a' = q - 1$.

Only one of the inequalities needs to hold, but $q - \sqrt[3]{q^2} + \frac{2}{3}\sqrt[3]{q} - 1 \leq q - 1$. The lemma follows. □

Theorem 6.4. Let $\mathcal{U}$ be a unital of order q and let $\mathcal{S}$ be a maximal Erdős-Ko-Rado set on $\mathcal{U}$. If $q \geq 5$, then either $|\mathcal{S}| = q^2$ and $\mathcal{S}$ is a point-pencil, or else $|\mathcal{S}| \leq q^2 - q + \sqrt[3]{q^2} - \frac{2}{3}\sqrt[3]{q} + 1$. If $q = 4$, then either $|\mathcal{S}| = 16 = q^2$ and $\mathcal{S}$ is a point-pencil, or else $|\mathcal{S}| \leq 13 = q^2 - q + 1$. If $q = 3$, then either $|\mathcal{S}| = 9 = q^2$ and $\mathcal{S}$ is a point-pencil, or else $|\mathcal{S}| \leq 8$.

Proof. If $\mathcal{S}$ is a point-pencil, then it contains q^2 elements. From now on, we assume that $\mathcal{S}$ is not a point-pencil. Recall the definition of $k_{\mathcal{S}}$. By Lemma 3.2, $k_{\mathcal{S}} \leq q + 1$. Moreover, if $k_{\mathcal{S}} \leq q - 1$, then $|\mathcal{S}| \leq q^2 - q - 1$.

First, we assume $q \geq 5$. If $k_{\mathcal{S}} = q$, then $|\mathcal{S}| \leq q^2 - q + \sqrt[3]{q^2} - \frac{2}{3}\sqrt[3]{q} + 1$ by Lemma 6.3. If $k_{\mathcal{S}} = q + 1$, then $|\mathcal{S}| \leq q^2 - q + 1$ by Lemma 6.2.

The results for $q = 3, 4$ are obtained in the same way, using the results from Lemma 6.2 and Lemma 6.3. □

Remark 6.5. Note that these results correspond with the result for classical unitals in Corollary 2.9 since the triangle contains only $q + 2$ blocks.

Note that the unitals are not covered by Corollary 1.5. However, they are covered by Theorem 5.6. So we already knew that the point-pencils are the largest Erdős-Ko-Rado sets. The above theorem thus gives a bound on the size of the second-largest maximal Erdős-Ko-Rado set.

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