Analytical results for non-linear Compton scattering in short intense laser pulses

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We study in detail the strong-field QED process of non-linear Compton scattering in short intense laser pulses. Our main focus is placed on how the spectrum of the back-scattered laser light depends on the shape and duration of the initial short intensive pulse. Although this pulse shape dependence is very complicated and highly non-linear, and has never been addressed explicitly, our analysis reveals that all the dependence on the laser pulse shape is contained in a three-parameter master integral. Here we present completely analytical expressions the non-linear Compton spectrum in terms of a master integral. Moreover, we analyse the universal behaviour of the shape of the spectrum for very high harmonic lines.

1. Introduction

The non-linear Compton scattering of high-intensity laser pulses off counterpropagating high-energetic electrons is one of the fundamental processes in strong-field QED. Its theoretical description goes back to the 1960's where many strong-field QED processes had been studied in a series of seminal papers (Nikishov & Ritus 1964*b*,*a*, 1965; Goldman 1964; Brown & Kibble 1964). For instance, these authors predicted the emission of high harmonics and a non-linear intensity-dependent red-shift of the emitted radiation, which is proportional to a_0^2, where a_0 is the dimensionless normalized laser amplitude that is related to the laser intensity (I) and wavelength (λ) via $a_0^2 = 0.73 \times I[10^{18}\ \mathrm{W/cm^2}]\lambda^2[\mathrm{\mu m}]$. In a classical picture the generation of high harmonics and the non-linear red-shift of the emitted radiation can be understood as the influence of the laser's magnetic field due to the $\boldsymbol{v} \times \boldsymbol{B}$ term in the classical Lorentz force equation.

However, while most contemporary high-intensity laser facilities generate laser pulses with femtosecond duration (Mourou *et al.* 2006; Korzhimanov *et al.* 2011; Di Piazza *et al.* 2012), most of the early papers on non-linear Compton scattering did not consider the effect of the finite duration of the intensive laser pulse. For a realistic laser pulse with a finite duration the laser intensity gradually increases from zero to its maximum value. Consequently, the non-linear red-shift is not constant during the course of the laser pulse and the harmonic lines of the emitted radiation are considerably broadened, with a large number of spectral lines for each harmonic (Narozhnyi & Fofanov 1996; Boca & Florescu 2009; Seipt & Kämpfer 2011; Hartemann & Kerman 1996; Hartemann *et al.* 1996, 2010; Mackenroth & Di Piazza 2011). In the classical picture this broadening is caused by a gradual slow-down of the longitudinal electron motion as the laser intensity ramps up (Seipt *et al.* 2015*a*). The occurrence of the additional line structure can be interpreted as interference of the radiation that is emitted during different times (Seipt & Kämpfer

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2013*b*). The broadening of the spectral lines is especially important with regard to the application of non-linear Compton scattering as an x- and gamma-ray radiation source (Jochmann *et al.* 2013; Sarri *et al.* 2014; Rykovanov *et al.* 2014; Seipt *et al.* 2015*a*; Khrennikov *et al.* 2015).

For ultra-high laser intensities $a_0 \gg 1$ the formation time of the emitted photon is much shorter than the laser period and the interference of radiation that is emitted at different times during the course of the pulse is suppressed (Dinu *et al.* 2015). In this regime, where the Compton emission becomes vital for the formation of QED cascades, the spectrum can be effectively simulated using the photon emission probabilities in a constant crossed field (Ritus 1985; Fedotov *et al.* 2010; King *et al.* 2013; Harvey *et al.* 2015; Narozhny & Fedotov 2015). We therefore focus in this paper on the intermediate intensity region $a_0 \sim 1$, where the interference matters and a general relation between the shape and duration of the laser pulse and the shape of the spectrum of the backscattered light is very complicated and highly non-linear. The shape of the harmonic lines is determined by an interplay between the laser pulse duration, (ii) spectral composition of the pulse and (iii) the non-linear ponderomotive broadening which depends on the laser intensity ramps.

In this paper we analytically analyse for the first time the non-linear Compton scattering process in a short intensive laser pulse. In particular, we investigate how the duration and the shape of the short intensive laser pulse affect the spectrum of the emitted radiation. We derive a scale invariant master integral that contains all dependence on the shape of the laser pulse, and we give explicit analytical expressions for several specific laser pulse shapes. Our paper is organized as follows: In Section 2 we briefly outline the calculation of the transition amplitude and the energy and angular differential emission probability for non-linear Compton scattering using Volkov states in a pulsed laser field. The transition amplitude is analysed further in Section 3 where we extract the dependence on the duration and shape of the laser pulse in the form of a master integral. Explicit analytic expressions for the master integral for different pulse shapes are given in Section 3.4. Throughout the paper we use units with $\hbar = c = 1$. Scalar products between four-vectors are denoted by $a \cdot b \equiv a^\mu b_\mu = a^0 b^0 - \boldsymbol{ab}$, and the Feynman slash notation is used for scalar products between four-vectors and the Dirac matrices: $\not{a} \equiv \gamma \cdot a$.

2. Theoretical Background

The non-linear Compton scattering process, i.e. the emission of a photon by an electron under the action of an intense laser field, is conveniently described theoretically in the Furry picture. The interaction of the electrons with the laser pulse is treated non-perturbatively by using Volkov electron states Ψ as solutions of the Dirac equation $(i\not{\partial} - e\not{A} - m)\Psi = 0$ in the plane-wave background laser field A. Here m and $e = -|e|$ denote the mass and charge of the electron, respectively. By employing these Volkov states and the strong-field S matrix in the Furry picture can be represented by the Feynman diagram in Fig. 1. It is given by the expression

$$S = -ie \int \mathrm{d}^4 x \, \bar{\Psi}_{p'}(x) \gamma_\mu \mathcal{A}^\mu_{k'}(x) \Psi_p(x) \,, \tag{2.1}$$

where $\mathcal{A}^\mu_{k'}(x) = (\varepsilon'^*)^\mu e^{ik' \cdot x}$ is the amplitude for the emission of a (non-laser) photon with four momentum k' and polarization ε', while p and p' are the asymptotic four-momenta of the electron before and after the photon emission.

In the following, we shall restrict our discussion to the case of a circularly polarized

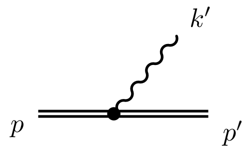

FIGURE 1. Feynman diagram for the emission of a photon with four-momentum k' (wiggly line) by a laser-dressed Volkov electron with asymptotic four-momentum p (double-line). After the photon emission the electron has the asymptotic four-momentum p'. The double-lines indicate the non-perturbative interaction of the electron with the intense short laser pulse.

laser pulse with the four-vector potential in the axial gauge ($k \cdot A = 0$)

$$A^\mu(\phi) = A_0\, g(\phi)\, \mathrm{Re}\, \epsilon_+^\mu e^{-i\phi} . \tag{2.2}$$

It depends only on the phase variable $\phi = k \cdot x$ with the laser photon four-momentum $k = (\omega, 0, 0, -\omega)$, and with the normalized polarization four-vector $\epsilon_\pm^\mu = (0, 1, \pm i, 0)/\sqrt{2}$, with $\epsilon_+ \cdot \epsilon_- = -1$. The dimensionless normalized laser amplitude is given by $a_0 = |e|A_0/m$. The shape of the laser pulse is described by an envelope function $g(\phi)$, that depends on ϕ only via the ratio $\phi/\Delta\phi$ with the pulse duration $\Delta\phi$. Moreover, we use symmetric pulse envelopes with $g(\phi) = g(-\phi)$, $g(0) = 1$ and $g(\pm\infty) = 0$. For the following we shall assume that the laser pulse consists of several optical cycles such that $\Delta\phi \gg 1$ and we may employ the slowly varying envelope approximation.

One can perform the spatial integrations in the S matrix most conveniently in light-front coordinates, defined as $x^\pm = x^0 \pm x^3$ and $\boldsymbol{x}^\perp = (x^1, x^2)$, such that the laser phase is proportional to $\phi = \omega x^+$, and $\mathrm{d}^4x = (2\omega)^{-1}\mathrm{d}\phi \mathrm{d}x^- \mathrm{d}^2\boldsymbol{x}^\perp$. After integrating over three light-front coordinates, the S matrix can be represented in a form (Seipt & Kämpfer 2013*a*)

$$S = -ie(2\pi)^3 \delta_{\mathrm{l.f.}}(p - p' - k')\, \mathscr{M}(s) \tag{2.3}$$

with the light-front delta function $\delta_{\mathrm{l.f.}}(p-p'-k') = \frac{1}{\omega}\delta^2(\boldsymbol{p}_\perp - \boldsymbol{p}'_\perp - \boldsymbol{k}'_\perp)\delta(p^+ - p'^+ - k'^+)$, enforcing the conservation of three momentum components, and the transition amplitude

$$\mathscr{M} = \mathscr{T}_0\mathscr{C}_0 + \mathscr{T}_+\mathscr{C}_+ + \mathscr{T}_-\mathscr{C}_- + \mathscr{T}_2\mathscr{C}_2 . \tag{2.4}$$

Here, the quantities $\mathscr{T}_j$ denote the transition operators

$$\begin{aligned} \mathscr{T}_0 &= \bar{u}_{p'} \not{\varepsilon}'^* u_p , \\ \mathscr{T}_\pm &= \frac{m a_0}{4}\, \bar{u}_{p'} \left(\frac{\not{\epsilon}_\pm \not{k} \not{\varepsilon}'^*}{(k \cdot p')} + \frac{\not{\varepsilon}'^* \not{k} \not{\epsilon}_\pm}{(k \cdot p)} \right) u_p , \\ \mathscr{T}_2 &= \frac{m^2 a_0^2\, (\varepsilon'^* \cdot k)}{4(k \cdot p)(k \cdot p')}\, \bar{u}_{p'} \not{k} u_p , \end{aligned} \tag{2.5}$$

which are sensitive to the spin of the incident and final electrons (via the spinors u_p and $\bar{u}_{p'}$) and the polarization of the emitted photon. However, they are only weakly dependent on the energy and momentum of the emitted photon. The dependence on the dynamics of the scattering process is mainly contained in the so-called dynamic integrals over the laser phase

$$\mathscr{C}_{\pm 1} = \int_{-\infty}^{\infty} \mathrm{d}\phi\, g(\phi) e^{\mp i\phi} \exp\left\{ i\ell \left[\phi + \alpha g(\phi) \sin(\phi + \varphi) + \beta \int_0^{\phi} \mathrm{d}\phi'\, g^2(\phi') \right] \right\} ,$$

$$\mathscr{C}_{2} = \int_{-\infty}^{\infty} \mathrm{d}\phi\, g^2(\phi) \exp\left\{ i\ell \left[\phi + \alpha g(\phi) \sin(\phi + \varphi) + \beta \int_0^{\phi} \mathrm{d}\phi'\, g^2(\phi') \right] \right\} . \tag{2.6}$$

Here we employed the slowly varying envelope approximation for the laser pulse and we use the definition $\varphi = \arctan k'_y/k'_x$. The fourth dynamic integral $\mathscr{C}_0$ is represented as a combination of the other three integrals, defined in Eqs. (2.6), as

$$\mathscr{C}_0 = -\frac{\alpha}{2} \left(e^{-i\varphi} \mathscr{C}_+ + e^{i\varphi} \mathscr{C}_- \right) - \beta \mathscr{C}_2 , \tag{2.7}$$

by the requirement of the gauge invariance of the S matrix (Ilderton 2011; Seipt 2012).

Here we have defined ℓ as the amount of four-momentum that is absorbed from the laser field

$$\ell \equiv \frac{k' \cdot p}{k \cdot p'} = \frac{p'^- + k'^- - p^-}{k^-} , \tag{2.8}$$

and provides a Lorentz-invariant way to parametrise the frequency of the emitted photon

$$\omega'(\ell) = \frac{\ell \omega}{1 + \ell \frac{\omega}{m} (1 + \cos\vartheta)} . \tag{2.9}$$

For convenience, we moved to the rest-frame of the incident electron, where $p = (m, 0, 0, 0)$. Moreover, we defined the coefficients

$$\alpha = \frac{a_0}{\sqrt{2}} \sin\vartheta , \tag{2.10}$$

$$\beta = \frac{a_0^2}{4} (1 + \cos\vartheta) . \tag{2.11}$$

Using the the transition amplitude (2.4), the angular- and energy-differential photon emission probability is given by (Seipt & Kämpfer 2011; Seipt 2012)

$$\frac{\mathrm{d}W}{\mathrm{d}\omega' \mathrm{d}\Omega} = \frac{e^2 \omega' |\mathscr{M}|^2}{64\pi^3 (k \cdot p)(k \cdot p')} . \tag{2.12}$$

In order to study how the differential emission probability depends on the laser pulse parameters—the laser strength a_0, pulse duration $\Delta\phi$, or the shape of the pulse envelope g—it is required to evaluate the dynamic integrals $\mathscr{C}_j$.

For many purposes it is sufficient to perform the integrations over the laser phase numerically, as was done for instance in Refs. (Seipt & Kämpfer 2011; Mackenroth & Di Piazza 2011; Krajewska & Kamiński 2012; Twardy *et al.* 2014). Another approach to calculate the spectrum (2.12) relies on a saddle point analysis of the highly oscillating phase integrals in (2.6) (Narozhnyi & Fofanov 1996; Seipt & Kämpfer 2013*b*; Mackenroth *et al.* 2010; Seipt *et al.* 2015*a*,*b*). A third possibility to evaluate the dynamic integrals would be an attempt to find analytic solutions. Such a completely analytical evaluation has been done for instance in (Hartemann *et al.* 1996) for the on-axis radiation. This approach will be pursued in this paper for arbitrary emission angles and general pulse shapes. In the following we will present a completely analytical evaluation of the dynamic integrals to gain more insight on the dependence of the spectrum of the backscattered light on the laser pulse duration and envelope shape.

3. Analytic Evaluation of the Dynamic Integrals

In this section we further analyse the properties of the spectrum of the emitted radiation. We apply a detailed mathematical analysis to the transition amplitude $\mathscr{M}$, and in particular the dynamic integrals $\mathscr{C}_j$, from the previous section in order to extract how they depend on the laser pulse duration and pulse shape. Here, we aim to provide explicit analytic expressions for the dynamic integrals (2.6).

It is known from previous studies (Narozhnyi & Fofanov 1996; Seipt & Kämpfer 2013*b*; Seipt *et al.* 2015*a*) that the oscillating term ($\propto \alpha$) in the exponent of the dynamic integrals, Eq. (2.6), is responsible for the emission of high harmonics. The term containing the integral over the squared pulse envelope ($\propto \beta$) changes only slowly as a function of ϕ. This so-called ponderomotive term is responsible for the broadening of the harmonic lines and the spectral structures seen within each harmonic. In the following, we first disentangle these two effects (Section 3.1) and later on analyse the ponderomotive broadening for each harmonic line (Sections 3.2 et seq.)

3.1. *Expansion into Harmonics*

Let us first expand the dynamic integrals into a sum of partial terms which can be interpreted as the emission of higher harmonics in analogy to the case of infinite plane waves. Following Narozhnyi & Fofanov (1996) and Seipt & Kämpfer (2013*b*), we define a generalized floating-window Fourier series for a non-periodic function $f(\phi)$

$$f(\phi) = \sum_{n=-\infty}^{\infty} c_n(\phi) e^{-in\phi} , \qquad c_n(\phi) = \frac{1}{2\pi} \int_{\phi-\pi}^{\phi+\pi} \mathrm{d}\phi' \, f(\phi') \, e^{in\phi'} \tag{3.1}$$

with Fourier coefficients $c_n(\phi)$ that depend on the location of the window centre. Applying this floating window Fourier series to the integrand of the dynamic integrals, and using the slowly varying envelope approximation (Narozhnyi & Fofanov 1996), yields a generalized Jacobi-Anger type expansion

$$e^{i\ell\alpha g(\phi) \sin{(\phi+\varphi)}} = \sum_n (-1)^n \, J_n(\ell\alpha g(\phi)) \, e^{-in(\phi+\varphi)} , \tag{3.2}$$

with the Bessel function of the first kind $J_n(z)$ (Watson 1922). This expansion strongly resembles the expansion into harmonics known from the well-studied case of infinite plane waves (Berestetzki *et al.* 1980), where $g = 1$. Note, however, that here the argument of the Bessel functions depends on the laser phase ϕ via laser pulse envelope $g(\phi)$.

Employing the above expansion we can cast the dynamic integrals into a form

$$\begin{aligned} \mathscr{C}_2(\ell) &= \sum_n (-1)^n \, e^{-in\varphi} \, C_2^{(n)}(\ell) , \\ \mathscr{C}_\pm(\ell) &= \sum_n (-1)^n \, e^{-i(n\mp1)\varphi} \, C_\pm^{(n)}(\ell) , \end{aligned} \tag{3.3}$$

with

$$\begin{aligned} C_2^{(n)}(\ell) &= \int_{-\infty}^{\infty} \mathrm{d}\phi \, g^2(\phi) J_n(\ell\alpha g) e^{i(\ell-n)\phi + i\ell\beta \int \mathrm{d}\phi \, g^2} , \\ C_\pm^{(n)}(\ell) &= \int_{-\infty}^{\infty} \mathrm{d}\phi \, g(\phi) J_{n\mp1}(\ell\alpha g) e^{i(\ell-n)\phi + i\ell\beta \int \mathrm{d}\phi \, g^2} . \end{aligned} \tag{3.4}$$

Note that for symmetric laser pulse envelopes, as we use in this paper, all the coefficients $C_j^{(n)}(\ell)$ are purely real-valued. Making use of the expansions (3.3), the transition amplitude $\mathscr{M}$ can be written as a sum of partial amplitudes $\mathscr{M} = \sum_{n=1}^{\infty} \mathscr{M}^{(n)}$ representing the emission of the n-th harmonic. The shape of each of the harmonic lines is determined by the integrals $C_j^{(n)}(\ell)$, which might be called the partial dynamic integrals for the n-th harmonic. Unfortunately, in these integrals the pulse envelope g appears as the argument of the Bessel function, preventing their immediate analytic evaluation.

The parameter α that appears in the argument of the Bessel functions goes to zero for on-axis radiation, $\vartheta = 0$. Since the Bessel functions behave as $J_n(z) \approx \frac{z^n}{2^n n!}$ for small argument z this means that only the first harmonic $n = 1$ is emitted on-axis for a circularly polarized laser pulse, with the only contribution coming from $C_+^{(1)}$. The result that no higher harmonics occur on-axis is known from the case of infinitely long plane waves as “blind spot” or “dead cone” in the literature (see e.g. Harvey *et al.* (2009) and references therein).

3.2. *Reduction to a Master Integral*

The next important step is to extract the pulse shape function g from the argument of the Bessel functions in the definition of the integrals $C_j^{(n)}$. This will eventually allow to define a master integral that contains all the dependence on the laser pulse envelope. Such an extraction is achieved by applying the multiple argument expansion for the Bessel functions (Watson 1922)

$$J_n\left(\ell\alpha g(\phi)\right) = g^n(\phi) \sum_{k=0}^{\infty} \left[1 - g^2(\phi)\right]^k \frac{J_{n+k}(\ell\alpha)}{k!} \left(\frac{\ell\alpha}{2}\right)^k . \tag{3.5}$$

Thus, instead of having to deal with the pulse envelope g as an argument of the n-th Bessel function we now get a power series in $(1 - g^2)$, with the coefficients containing higher-order Bessel functions. We should note that the overlap of the functions g^n and some power of $(1 - g^2)^k$ rapidly gets small for increasing n and k. The powers of g^n are localized at the origin $\phi = 0$ more strongly for larger values of n, while the powers of $(1 - g^2)$ vanish at the origin. Their product in the expansion (3.5) samples the edges of the laser pulse.

Employing the above expansions we obtain for the partial dynamic integrals for the n-th harmonic the series

$$C_2^{(n)}(\ell) = \sum_{k=0}^{\infty} \frac{J_{n+k}(\ell\alpha)}{k!} \left(\frac{\ell\alpha}{2}\right)^k B_{n+2}^k(\ell - n, \ell\beta) , \tag{3.6}$$

$$C_\pm^{(n)}(\ell) = \sum_{k=0}^{\infty} \frac{J_{n+k\mp1}(\ell\alpha)}{k!} \left(\frac{\ell\alpha}{2}\right)^k B_{n+1\mp1}^k(\ell - n, \ell\beta) , \tag{3.7}$$

where we have defined the ponderomotive integrals

$$B_r^k(\ell - n, \ell\beta) = \int_{-\infty}^{\infty} \mathrm{d}\phi\, g^r(\phi) \left[1 - g^2(\phi)\right]^k e^{i(\ell-n)\phi + i\ell\beta \int \mathrm{d}\phi\, g^2} . \tag{3.8}$$

They contain all dependence on the laser pulse shape and pulse duration and its influence on the longitudinal electron motion and spectral broadening.

Before evaluating these ponderomotive integrals further, let us first discuss the limit of infinite plane waves, $g \to 1$, where the laser intensity is switched on adiabatically at past infinity and then stays constant. As a consequence there is no ponderomotive broadening for the infinite plane waves. Because of $1 - g^2 = 0$ we find

$$B_r^k(\ell - n, \ell\beta) \stackrel{g=1}{\longrightarrow} \delta_{k0} \int \mathrm{d}\phi \, e^{i(\ell-n)\phi + i\ell\beta\phi} = 2\pi \, \delta_{k0} \, \delta(\ell - n + \ell\beta) \, . \tag{3.9}$$

The delta function here restricts the generally continuous variable ℓ to discrete values $\ell_n = n/(1+\beta)$. Thus, the frequency of the emitted photon, Eq. (2.9), becomes discrete as well:

$$\omega'_n \equiv \omega'(\ell_n) = \frac{n\omega}{1 + \beta + \frac{n\omega}{m}(1 + \cos\vartheta)} = \frac{n\omega}{1 + \left(\frac{n\omega}{m} + \frac{a_0^2}{4}\right)(1 + \cos\vartheta)} \tag{3.10}$$

with the well-known non-linear intensity dependent red-shift (Berestetzki *et al.* 1980), but no spectral broadening. Eq. (3.10) is usually interpreted as the absorption of n laser photons and the emission of high harmonics. Moreover, in the expansion (3.5) all terms with $k > 0$ vanish and we re-obtain the well known result that the partial matrix element of the n-th harmonic contains the Bessel functions J_n and $J_{n\pm1}$ (Berestetzki *et al.* 1980; Ritus 1985).

It is possible to find a recurrence relation for the ponderomotive integrals:

$$B_r^k = B_r^{k-1} - B_{r+2}^{k-1} \, . \tag{3.11}$$

Subsequent application of this relation helps to reduce the order of the upper index to zero:

$$B_r^k = \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} B_{r+2\nu}^0 \, . \tag{3.12}$$

Thus, we have to calculate only those ponderomotive integrals with the upper index $k = 0$. Let us now re-scale the integration variable in (3.8) as $\phi \to t = \phi/\Delta\phi$, in order to define the three-parameter master integral as

$$\mathcal{B}_r(\xi, \eta) \; \equiv \; \frac{B_r^0(\ell - n, \ell\beta)}{\Delta\phi} \, , \tag{3.13}$$

as a function of the rescaled variables $\xi = (\ell - n)\Delta\phi$ and $\eta = \ell\beta\Delta\phi$ and for positive integer values of r. The master integral explicitly reads

$$\mathcal{B}_r(\xi, \eta) \equiv \int_{-\infty}^{\infty} \mathrm{d}t \, g^r(t) \, e^{i\xi t + i\eta \int \mathrm{d}t \, g^2(t)} \, . \tag{3.14}$$

It only depends on the shape of the laser pulse and is completely independent of the pulse duration. Note that the master integrals are real-valued functions for all symmetric laser pulse shapes.

From stationary phase arguments one can deduce that the master integrals are essentially different from zero only in the regions bounded by the η–axis and the line $\eta = -\xi$. In this region the master integral is an oscillating function for all values of r. This region is visualized as a grey shaded area in Fig. 2. Outside of this region it rapidly approaches zero.

Before we continue our discussion of the properties of the master integral and its pulse shape dependence, let us first represent the partial transition amplitudes $\mathscr{M} = \sum_n \mathscr{M}^{(n)}$

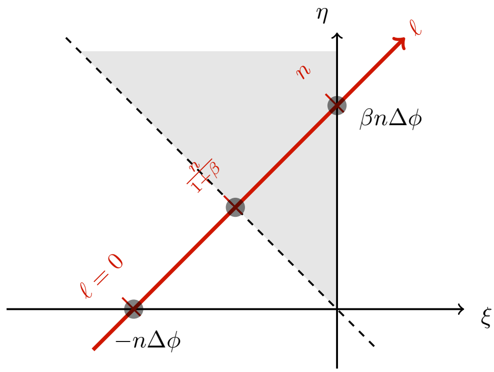

FIGURE 2. Illustration how to cut the ξ–η plane in order to obtain the shape of the spectral lines as a function of ℓ.

in terms of the $\mathcal{B}_r(\xi,\eta)$ by putting together all the expansions:

$$\begin{aligned}\mathscr{M}^{(n)} = \Delta\phi(-1)^n e^{-in\varphi} \sum_{k=0}^{\infty}\sum_{\nu=0}^{k} \frac{(-1)^\nu}{k!}\binom{k}{\nu}\left(\frac{\ell\alpha}{2}\right)^k \\ \times \Big\{ e^{i\varphi} J_{n+k-1}(\ell\alpha)\left[\mathscr{T}_+ - \frac{\alpha e^{-i\phi}}{2}\mathscr{T}_0\right]\mathcal{B}_{n+2\nu}(\xi,\eta) \\ + e^{i\varphi} J_{n+k+1}(\ell\alpha)\left[\mathscr{T}_- - \frac{\alpha e^{+i\phi}}{2}\mathscr{T}_0\right]\mathcal{B}_{n+2+2\nu}(\xi,\eta) \\ + J_{n+k}(\ell\alpha)\Big[\mathscr{T}_2 - \beta\mathscr{T}_0\Big]\mathcal{B}_{n+2+2\nu}(\xi,\eta)\Big\}. \end{aligned} \tag{3.15}$$

To obtain the spectrum of non-linear Compton scattering we have to plug this transition amplitude into Eq. (2.12).

For the sake of completeness let us now briefly discuss the partial transition amplitudes in the limit of infinite plane wave laser fields, $g = 1$. In this case the master integral turns into $\mathcal{B}_r(\xi,\eta) \to 2\pi\delta(\xi+\eta)$, i.e. they are localized along the diagonal $\eta = -\xi$. Moreover, the summation over ν yields just a Kronecker delta δ_{k0} such that only the $k = 0$ term survives in the sum over k. Therefore we obtain for the transition amplitude for infinite plane waves

$$\begin{aligned}\mathscr{M}^{(n)} = 2\pi\delta(\ell - n + \ell\beta)\Delta\phi(-1)^n e^{-in\varphi}\Big\{ e^{i\varphi} J_{n-1}(\ell_n\alpha)\left[\mathscr{T}_+ - \frac{\alpha e^{-i\phi}}{2}\mathscr{T}_0\right] \\ + e^{i\varphi} J_{n+1}(\ell_n\alpha)\left[\mathscr{T}_- - \frac{\alpha e^{+i\phi}}{2}\mathscr{T}_0\right] + J_n(\ell_n\alpha)\Big[\mathscr{T}_2 - \beta\mathscr{T}_0\Big]\Big\}, \end{aligned} \tag{3.16}$$

with the argument of the Bessel functions now being $\ell_n\alpha = n\alpha/(1+\beta)$, which reproduces the well-known textbook result (Berestetzki *et al.* 1980). The direct comparison between (3.15) and (3.16) impressively demonstrates how much more complex and intricate the case of the pulsed laser fields is, as compared to infinite plane waves. The master integrals $\mathcal{B}_r(\xi,\eta)$, which become trivial in the case of infinite plane waves, cause the increased complexity of the transition amplitude for pulsed plane wave laser fields. They can be considered as a fingerprint of the laser pulse shape.

Let us now return to our discussion of the properties of the master integral Eq. (3.8) by first discussing how the rescaled arguments ξ and η relate to the frequency ω' and

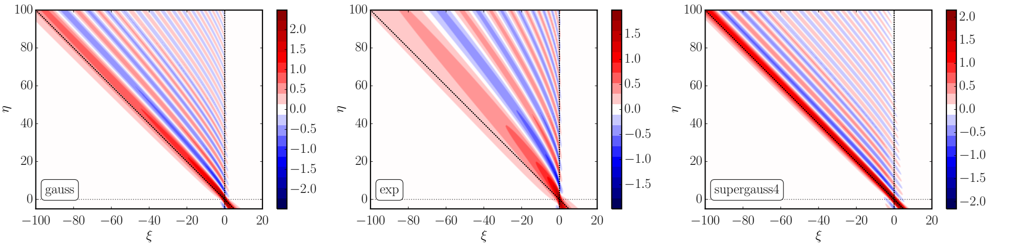

FIGURE 3. Numeric evaluation of the master integral $\mathcal{B}_1(\xi,\eta)$ for for different pulse shapes: a Gaussian $g(t)=e^{-t^2/2}$ (left), an exponential $g=e^{-|t|}$ (centre), and a Supergaussian $g=e^{-t^4/2}$ (right).

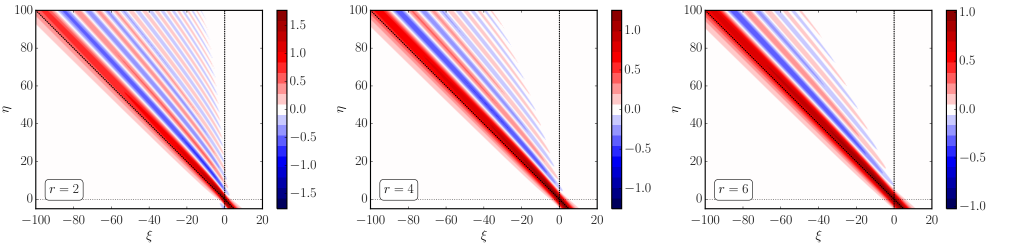

FIGURE 4. Numeric evaluation of the master integral (3.14) for a Gaussian pulse envelope $g=e^{-t^2/2}$ in the ξ–η plane for increasing values of r from left to right.

scattering angle ϑ of the emitted photon. In order to obtain the shape of the frequency spectrum as a function of ℓ (or the photon frequency ω' by means of Eq. (2.9)) one has to cut the functions $\mathcal{B}_r(\xi,\eta)$ along the straight line $\eta=\beta\xi+n\beta\Delta\phi$ in the ξ–η plane, depicted as a red diagonal line in Fig. 2. This line intersects the ξ–axis at $\xi=-n\Delta\phi$, the slope is just β (i.e. it depends on the laser intensity), and the intersection with the η–axis is at $\eta=\beta n\Delta\phi$. Note that the dependence on the scattering angle ϑ is entirely contained in the parameter β, see Eq. (2.11).

It is important to note which values of ℓ lie inside the grey shaded area where the master integral is non-zero: They are exactly those values between the red-shifted and unshifted n-th harmonic lines in the infinite plane wave: $n/(1+\beta)\leqslant\ell\leqslant n$, see Fig. 2. Moreover, we see that the diagonal $\eta=-\xi$ represents the red-shifted harmonics in the infinite monochromatic plane wave. That means, the region close to the diagonal line $\eta=-\xi$ is formed close to the centre of the laser pulse where the intensity is largest. In the region close to the η–axis the master integral is formed at the very edges of the laser pulse where the intensity is very low.

Numeric evaluations of the master integral $\mathcal{B}_1(\xi,\eta)$ are depicted in Fig. 3 for three different pulse envelopes. We see that each pulse shape generates a distinct pattern of oscillations in the triangular region bounded by the η–axis and the diagonal $\eta=-\xi$. For a Gaussian pulse envelope $g(t)=e^{-t^2/2}$ numeric evaluations of the master integrals are depicted in Fig. 4 for different values of r. One can see that the larger the value of r the stronger the function is localized close to the line $\eta=-\xi$ (i.e. the non-linear Compton edge in the limit of infinite plane waves). By recalling how we need to cut the ξ–η plane to obtain the frequency spectrum we easily deduce that for longer pulses or higher harmonics the spectral lines contain more oscillations. This observation is in line with results using the saddle point method (Seipt & Kämpfer 2013b).

3.3. *Universal Behaviour of High Order Harmonics*

Let us now draw some conclusions about the shape of very high order harmonics. One can notice that for large values of r, the powers of $g^r(t)$ become strongly localized around $t \approx 0$. Thus, the main contribution to the master integral, Eq. (3.14), is provided by the region of $g(t)$ around the maximum at $t = 0$. For sufficiently smooth pulse envelopes† $g(t)$ a Taylor expansion around the point $t = 0$ reads

$$g^r(t) \approx \left(1 - |g''(0)|\frac{t^2}{2}\right)^r . \tag{3.17}$$

Defining $\tau^2 = 1/|g''(0)|$ and employing the known limit $(1 + x/r)^r \overset{r\to\infty}{\to} e^x$, we find

$$g^r(t) \approx e^{-\frac{r\,t^2}{2\tau^2}} . \tag{3.18}$$

which is of Gaussian shape and does not depend on any details of the primary pulse shape, except for the curvature at the maximum, i.e. the second derivative $g''(0)$ at $t = 0$. Note that if one is concerned about laser pulses with a flat top envelope (in the sense that $g''(0) = 0$), the Taylor series in (3.17) can be extended further, and will eventually lead to a Supergaussian shape of $g^r(t)$.

The conclusion is that for sufficiently high harmonic order the master integral is approximately given by

$$\mathcal{B}_r(\xi, \eta) \approx \int_{-\infty}^{\infty} \mathrm{d}t\, e^{-\frac{r}{2}\left(\frac{t}{\tau}\right)^2 + i\xi t + i\eta \int \mathrm{d}t\, g^2(t)} . \tag{3.19}$$

Because for large values of r the main contributions to this integral come from a narrow region around $t = 0$, we can approximate $\int dt\, g^2(t)$ polynomially around $t = 0$ up to the third order. This allows us to give an analytic expression for the master integral $\mathcal{B}_r$ for large values of $r \to \infty$ as

$$\mathcal{B}_r(\xi, \eta) \approx \frac{2\pi\tau^{2/3}}{\eta^{1/3}} \exp\left(-\frac{r}{2\eta}(\xi + \eta) + \frac{r^3}{12(\tau\eta)^2}\right) \mathrm{Ai}\left(\frac{r^2}{4(\tau\eta)^{4/3}} - \frac{\tau^{2/3}}{\eta^{1/3}}(\xi + \eta)\right) \tag{3.20}$$

in terms of the Airy function Ai (Erdélyi *et al.* 1953). The central conclusion here is that for very large values of r the shape of the harmonic lines approximate the shape of the harmonics for a Gaussian laser pulse with temporal duration $\tau/\sqrt{r} = 1/\sqrt{r|g''(0)|}$. The only specific input from the original pulse shape g that affects the shape of the high-harmonic spectral lines is the curvature of the pulse envelope at the maximum.

3.4. *Explicit Closed Form Analytic Results for the Master Integral*

We finally provide explicit closed form analytic expressions for the master integral $\mathcal{B}_r(\xi, \eta)$ for several different laser pulse shapes g.

3.4.1. *Hyperbolic Secant Pulse Shape*

For a hyperbolic secant pulse, $g(t) = 1/\cosh(t)$, we have $\int g^2 dt = \tanh t$, and the master integral

$$\mathcal{B}_r(\xi, \eta) = \int_{-\infty}^{\infty} \mathrm{d}t\, \frac{e^{i\xi t + i\eta \tanh t}}{\cosh^r t} \tag{3.21}$$

† This means the first derivative of the pulse envelope at $t = 0$ has to exist. Due to the symmetry of the pulse envelope it is then equal to zero: $g'(0) = 0$.

can be evaluated by transforming the integration variable according to $z = 1/(2e^t \cosh t)$, yielding

$$\mathcal{B}_r(\xi,\eta) = 2^{r-1} e^{-i\eta} \int_0^1 \mathrm{d}z\; z^{\frac{r+i\xi}{2}-1} (1-z)^{\frac{r-i\xi}{2}-1}\, e^{2i\eta z} \,. \tag{3.22}$$

This integral can be evaluated as

$$\mathcal{B}_r(\xi,\eta) = \frac{2^{r-1} e^{-i\eta}}{\Gamma(r)} \Gamma\left(\frac{r+i\xi}{2}\right) \Gamma\left(\frac{r-i\xi}{2}\right) {}_1F_1\left(\frac{r+i\xi}{2}, r, 2i\eta\right) , \tag{3.23}$$

with the Gamma function $\Gamma(z)$ and the confluent hyperbolic function ${}_1F_1(a,b;z)$ (Erdélyi *et al.* 1953). An equivalent representation of (3.23) can be given in terms of the generalized Laguerre functions $\mathcal{L}_\nu^\lambda(z)$ (Erdélyi *et al.* 1953) as

$$\mathcal{B}_r(\xi,\eta) = \frac{2^{r-1} e^{-i\eta}\pi}{\sin\left(\frac{\pi}{2}(r+i\xi)\right)} \mathcal{L}^{r-1}_{-\frac{r+i\xi}{2}}(2i\eta) \,. \tag{3.24}$$

3.4.2. *Exponential Pulse Shape*

For a pulse shape of the form $g(t) = e^{-|t|}$ the master integral takes the form

$$\mathcal{B}_r(\xi,\eta) = \int_0^\infty \mathrm{d}t\, e^{-(r-i\xi)t}\, e^{i\eta e^{-t}\sinh t} \;+\; \text{c.c.} \,. \tag{3.25}$$

After a substitution $z = \frac{i\eta}{2} e^{-2t}$ we get

$$\mathcal{B}_r(\xi,\eta) = \frac{e^{\frac{i\eta}{2}}}{2} \left(\frac{2}{i\eta}\right)^{\frac{r-i\xi}{2}} \int_0^{\frac{i\eta}{2}} \mathrm{d}z\, z^{\frac{r-i\xi}{2}-1}\, e^{-z} \;+\; \text{c.c.} \,, \tag{3.26}$$

which can be expressed as

$$\mathcal{B}_r(\xi,\eta) = \frac{e^{i\eta/2}}{2} \left(\frac{2}{i\eta}\right)^{\frac{r-i\xi}{2}} \gamma\left(\frac{r-i\xi}{2}, \frac{i\eta}{2}\right) \;+\; \text{c.c.} \,, \tag{3.27}$$

with the lower incomplete gamma function $\gamma(a,z)$ (Erdélyi *et al.* 1953),

3.4.3. *Staircase Pulse Shapes*

Let us assume the pulse envelope is staircase, defined as $g(t) = \sum_{k=1}^N \nu_k \chi_{I_k}(t)$ for $t > 0$, with ν_k being the height of the k-th step (as measured from the ground level) and the characteristic function $\chi_{I_k}(t) = 1$ if $t \in I_k = [(k-1)/N, k/N)$, and zero otherwise. (For $t < 0$ the envelope is fully defined by the symmetry $g(-t) = g(t)$.) For the moment we assume that the step height increases uniformly, $\nu_k = (N-k+1)/N$, but a generalization to arbitrary steps is obvious. A compelling feature of the staircase pulse is the possibility to approximate many different smooth pulse shapes in the limit of infinite steps $N \to \infty$, just by adjusting the step heights. For instance, the uniform staircase discussed here would converge to a smooth triangle pulse.

By splitting the t-integration range into the intervals I_k where g is constant we evaluate the master integral as

$$\mathcal{B}_r(\xi,\eta) = \sum_{k=1}^N 2(\nu_k)^r \,\frac{\sin\frac{\xi+\eta\,\nu_k^2}{2}}{\frac{\xi+\eta\,\nu_k^2}{2}}\; \cos\left(\eta\,\Phi_k + \frac{2k-1}{N}\frac{\xi+\eta\,\nu_k^2}{2}\right) \tag{3.28}$$

with

$$\Phi_k = \frac{1}{N}\sum_{\kappa=1}^{k}\nu_\kappa^2 - \frac{k}{N}\nu_k^2 \,. \tag{3.29}$$

For $N = 1$ we recover the well known result of of a sinc profile for the box pulse which is aligned along the $\eta = -\xi$ diagonal, i.e. it corresponds to the usual infinite plane wave red-shift. While the bandwidth of the laser pulse translates to the Compton scattered light, we see no indication of the ponderomotive broadening due to a gradual ramp up of the laser intensity. How this ponderomotive broadening effect develops can be seen quite instructively when going to a pulse with more than one step. For N steps we observe a total of N strips in the η–ξ plane that are centred along the lines $\eta = -\xi/\nu_k^2$. On each of the steps the radiation is emitted with with their respective red-shift, determined by the square of the k-th step height as $\ell = n/(1 + \beta\nu_k^2)$. With increasing N these strips eventually are overlapping, reproducing the typical picture from the smooth pulses discussed before. Thus, the staircase pulse model discussed here helps to investigate the transition from the case of a constant amplitude laser pulse to the case of smooth pulses where the ponderomotive broadening sets in and strongly influences the non-linear Compton spectrum.

4. Conclusions

In summary, we provided in this paper a comprehensive and completely analytical evaluation of the non-linear Compton transition amplitude. It was found that the dependence on the *shape* of the strong laser pulse can be traced back to a three-parameter master integral. In addition, all the dependence on the *pulse duration* can be conveniently scaled out from the master integral. For certain shapes of the laser pulse envelope we provided explicit analytical expressions for the master integral. In addition, for very high harmonics we find a universal behaviour of the shape of the harmonic lines.

In this paper we studied only the case of circularly polarized laser light. The laser polarization affects the form of the Jacobi-Anger type expansion (3.2) and the subsequent extraction of the laser pulse envelope from the argument of the Bessel functions via Eq. (3.5). In the case of an elliptic or linear laser polarization we would encounter generalized two-argument Bessel functions (Seipt 2012; Korsch *et al.* 2006). But eventually the laser pulse shape dependence is described by exactly the same master integrals (3.8) as for circular laser polarization discussed in this paper.

We would like to stress that the analytical structure of the strong-field S matrix is similar also for other first-order strong-field QED processes like Breit-Wheeler pair production, or pair annihilation. Thus, our analytic results could be easily translated to these processes too.

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