A POLYNOMIAL DEFINED BY THE $SL(2;\mathbb{C})$-REIDEMEISTER TORSION FOR A HOMOLOGY 3-SPHERE OBTAINED BY DEHN-SURGERY ALONG A TORUS KNOT

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Abstract. Let M_n be a homology 3-sphere obtained by $\frac{1}{n}$-Dehn surgery along a (p,q)-torus knot. We consider a polynomial $\sigma_{(p,q,n)}(t)$ whose zeros are the inverses of the Reideimeister torsion of M_n for $SL(2;\mathbb{C})$-irreducible representations. We give an explicit formula of this polynomial by using Tchebychev polynomials of the first kind. Further we also give a 3-term relations of these polynomials.

1. Introduction

Let $T(p,q)$ be a (p,q)-torus knot in S^3. Here p,q are coprime and positive integers. Let M_n be a homology 3-sphere obtained by $\frac{1}{n}$-Dehn surgery along $T(p,q)$. It is well known that M_n is a Brieskorn homology 3-sphere $\Sigma(p,q,N)$ where we write N for $|pqn+1|$. Here $\Sigma(p,q,N)$ is defined as

$$\{(z_1,z_2,z_3)\in\mathbb{C}^3 \mid z_1^p+z_2^q+z_3^N=0,\ |z_1|^2+|z_2|^2+|z_3|^2=1\}.$$

In this paper we consider the Reidemeister torsion $\tau_\rho(M_n)$ of M_n for an irreducible representation $\rho:\pi_1(M_n)\to SL(2;\mathbb{C})$.

In the 1980's Johnson [1] gave an explicit formula for any non-trivial value of $\tau_\rho(M_n)$. Furthermore, he proposed to consider the polynomial whose zero set coincides with the set of all non-trivial values $\{\frac{1}{\tau_\rho(M_n)}\}$, which is denoted by $\sigma_{(2,3,n)}(t)$. Under some normalization of $\sigma_{(2,3,n)}(t)$, he gave a 3-term relation among $\sigma_{(2,3,n+1)}(t)$, $\sigma_{(2,3,n)}(t)$ and $\sigma_{(2,3,n-1)}(t)$ by using Tchebychev polynomials of the first kind.

Recently in [5] we gave one generalization of the Johnson's formula for a $(2p',q)$-torus knot. Here p',q are coprime odd integers. In this paper, we show the formula for any torus knot $T(p,q)$.

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2. Definition of Reidemeister torsion

First let us describe definitions and properties of the Reidemeister torsion for $SL(2;\mathbb{C})$-representations. See Johnson [1], Kitano [2, 3] and Porti [7] for details.

Let $\mathbf{b} = (b_1, \cdots, b_d)$ and $\mathbf{c} = (c_1, \cdots, c_d)$ be two bases for a d-dimensional vector space W over $\mathbb{C}$. Setting $b_i = \sum_{j=1}^{d} p_{ji} c_j$, we obtain a nonsingular matrix $P = (p_{ij}) \in GL(d;\mathbb{C})$. Let $[\mathbf{b}/\ \mathbf{c}]$ denote the determinant of P.

Suppose

$$C_* : 0 \to C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

is an acyclic chain complex of finite dimensional vector spaces over $\mathbb{C}$. We assume that a preferred basis $\mathbf{c}_i$ for C_i is given for each i. That is, C_* is a based acyclic chain complex over $\mathbb{C}$.

Choose any basis $\mathbf{b}_i$ for $B_i = \mathrm{Im}(\partial_{i+1})$ and take a lift of it in C_{i+1}, which is denoted by $\tilde{\mathbf{b}}_i$. Since $B_i = Z_i = \mathrm{Ker}\partial_i$, the basis $\mathbf{b}_i$ can serve as a basis for Z_i. Furthermore since the sequence

$$0 \to Z_i \to C_i \xrightarrow{\partial_i} B_{i-1} \to 0$$

is exact, the vectors $(\mathbf{b}_i, \tilde{\mathbf{b}}_{i-1})$ form a basis for C_i. Here $\tilde{\mathbf{b}}_{i-1}$ is a lift of $\mathbf{b}_{i-1}$ in C_i. It is easily shown that $[\mathbf{b}_i, \tilde{\mathbf{b}}_{i-1}/\mathbf{c}_i]$ does not depend on a choice of a lift $\tilde{\mathbf{b}}_{i-1}$. Hence we can simply denote it by $[\mathbf{b}_i, \mathbf{b}_{i-1}/\mathbf{c}_i]$.

Definition 2.1. *The torsion $\tau(C_*)$ of a based chain complex C_* with $\{\boldsymbol{c}_*\}$ is given by the alternating product*

$$\tau(C_*) = \prod_{i=0}^{k} [\boldsymbol{b}_i, \boldsymbol{b}_{i-1}/\boldsymbol{c}_i]^{(-1)^{i+1}}.$$

Remark 2.2. *It is easy to see that $\tau(C_*)$ does not depend on choices of the bases $\{\boldsymbol{b}_0, \cdots, \boldsymbol{b}_k\}$.*

Now we apply this torsion invariant of chain complexes to geometric situations as follows. Let X be a finite CW-complex and $\tilde{X}$ a universal covering of X with the lifted CW-complex structure. The fundamental group $\pi_1 X$ acts on $\tilde{X}$ from the right-hand side as deck transformations. We may assume that this action is free and cellular by taking a subdivision if we need. Then the chain complex $C_*(\tilde{X};\mathbb{Z})$ has the structure of a chain complex of free $\mathbb{Z}[\pi_1 X]$-modules.

Let $\rho : \pi_1 X \to SL(2;\mathbb{C})$ be a representation. We denote the 2-dimensional vector space $\mathbb{C}^2$ by V. Using the representation ρ, V admits the structure of a $\mathbb{Z}[\pi_1 X]$-module and then we denote it by V_ρ.

Define the chain complex $C_*(X; V_\rho)$ by $C_*(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1 X]} V_\rho$ and choose a preferred basis

$$(\tilde{u}_1 \otimes \mathbf{e}_1, \tilde{u}_1 \otimes \mathbf{e}_2, \cdots, \tilde{u}_d \otimes \mathbf{e}_1, \tilde{u}_d \otimes \mathbf{e}_2)$$

of $C_i(X; V_\rho)$ where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a canonical basis of $V = \mathbb{C}^2$, $\{u_1, \cdots, u_d\}$ are the i-cells giving a basis of $C_i(X; \mathbb{Z})$ and $\{\tilde{u}_1, \cdots, \tilde{u}_d\}$ are lifts of them on $\tilde{X}$. Now we suppose that $C_*(X; V_\rho)$ is acyclic, namely all homology groups $H_*(X; V_\rho)$ are vanishing. In this case ρ is called an acyclic representation.

Definition 2.3. *Let $\rho : \pi_1(X) \to SL(2; \mathbb{C})$ be an acyclic representation. Then the Reidemeister torsion $\tau_\rho(X) \in \mathbb{C} \setminus \{0\}$ is defined by the torsion $\tau(C_*(X; V_\rho))$ of $C_*(X; V_\rho)$.*

Remark 2.4.

(1) *We define $\tau_\rho(X) = 0$ for a non-acyclic representation ρ.*
(2) *The definition of $\tau_\rho(X)$ depends on several choices. However it is well known that it is a piecewise linear invariant in the case of $SL(2; \mathbb{C})$-representations.*

3. Johnson's theory

Let $T(p, q) \subset S^3$ be a (p, q)-torus knot with coprime integers p, q. Now we write M_n to a closed orientable 3-manifold obtained by a $\frac{1}{n}$-Dehn surgery along $T(p, q)$. Here the fundamental group of $S^3 \setminus T(p, q)$ has the presentation as follows;

$$\pi_1(S^3 \setminus T(p, q)) = \langle x, y \mid x^p = y^q \rangle.$$

Furthermore $\pi_1(M_n)$ admits the presentation as follows;

$$\pi_1(M_n) = \langle x, y \mid x^p = y^q, ml^n = 1 \rangle$$

where $m = x^{-r} y^s$ $(r, s \in \mathbb{Z},\ ps - qr = 1)$ is a meridian of $T(p, q)$ and similarly $l = x^{-p} m^{pq} = y^{-q} m^{pq}$ is a longitude.
It is seen [1, 5] that the set of the conjugacy classes of the irreducible representations of $\pi_1(M_n)$ in $SL(2; \mathbb{C})$ is finite. Any conjugacy class can be represented by $\rho_{(a,b,k)} : \pi_1(M_n) \to SL(2; \mathbb{C})$ for some triple (a, b, k) such that

(1) $0 < a < p, 0 < b < q, a \equiv b \bmod 2$,
(2) $0 < k < N = |pqn + 1|, k \equiv na \bmod 2$,
(3) $\mathrm{tr}(\rho_{(a,b,k)}(x)) = 2 \cos \frac{a\pi}{p}$,
(4) $\mathrm{tr}(\rho_{(a,b,k)}(y)) = 2 \cos \frac{b\pi}{q}$,
(5) $\mathrm{tr}(\rho_{(a,b,k)}(m)) = 2 \cos \frac{k\pi}{N}$.

Furthermore Johnson computed $\tau_{\rho_{(a,b,k)}}(M_n)$ as follows.

Theorem 3.1 (Johnson)**.**

(1) *A representation $\rho_{(a,b,k)}$ is acylic if and only if $a \equiv b \equiv 1$.*

(2) *For any acyclic representation $\rho_{(a,b,k)}$ with $a \equiv b \equiv 1$, then one has*

$$\tau_{\rho_{(a,b,k)}}(M_n) = \frac{1}{2\left(1-\cos\frac{a\pi}{p}\right)\left(1-\cos\frac{b\pi}{q}\right)\left(1+\cos\frac{pqk\pi}{N}\right)}.$$

4. Main theorem

In this section we give a formula of the torsion polynomial $\sigma_{(p,q,n)}(t)$ for $M_n = \Sigma(p,q,N)$ obtained by a $\frac{1}{n}$-Dehn surgery along $T(p,q)$. Now we define torsion polynomials as follows.

Definition 4.1. *A one variable polynomial $\sigma_{(p,q,n)}(t)$ is called the torsion polynomial of M_n if the zero set coincides with the set of all non trivial values $\left\{\frac{1}{\tau_\rho(M_n)} \mid \tau_\rho(M_n) \neq 0\right\}$ and it satisfies the following normalization condition as*

$$\sigma_{(p,q,n)}(0) = \begin{cases} (-1)^{\frac{(N-1)p(q-1)}{8}} & p \text{ is even}, q \text{ is odd}, \\ (-1)^{\frac{(N-1)(p-1)q}{8}} & q \text{ is even}, q \text{ is odd}, \\ (-1)^{\frac{(N-1)(p-1)(q-1)}{8}} & p, q \text{ are odd}, n \text{ is even}, \\ (-1)^{\frac{N(p-1)(q-1)}{8}} & p, q \text{ are odd}, n \text{ is odd} \end{cases}$$

where $N = |pqn+1|$.

Remark 4.2.

(1) *For $M_0 = S^3$, the torsion polynomial $\sigma_{(p,q,0)}(t)$ is defined by $\sigma_{(p,q,0)}(t) = 1$.*
(2) *In the case that $p = 2p'$ is even and p' is odd, then this normalization condition coincides with the one in* [5].

From here assume $n \neq 0$. Recall Johnson's formula

$$\frac{1}{\tau_{\rho_{(a,b,k)}}(M_n)} = 2\left(1-\cos\frac{a\pi}{p}\right)\left(1-\cos\frac{b\pi}{q}\right)\left(1+\cos\frac{pqk\pi}{N}\right)$$

where $0 < a < p, 0 < b < q, a \equiv b \equiv 1 \bmod 2, k \equiv n \bmod 2$. Here by putting

$$C_{(p,q,a,b)} = \left(1-\cos\frac{a\pi}{p}\right)\left(1-\cos\frac{b\pi}{q}\right),$$

one has

$$\frac{1}{\tau_{\rho_{(a,b,k)}}(M_n)} = 4C_{(p,q,a,b)} \cdot \frac{1}{2}\left(1+\cos\frac{pqk\pi}{N}\right).$$

Main result is the following.

Theorem 4.3. *The torsion polynomial of M_n is given by*

$$\sigma_{(p,q,n)}(t) = \prod_{(a,b)} Y_{(n,a,b)}(t)$$

where

$$Y_{(n,a,b)}(t) = \begin{cases} \frac{T_{N+1}(s)-T_{N-1}(s)}{2(s^2-1)^2} & (p \text{ or } q \text{ is even}, n > 0), \\ -\frac{T_{N+1}(s)-T_{N-1}(s)}{2(s^2-1)^2} & (p \text{ or } q \text{ is even}, n < 0), \\ \frac{T_{N+1}(s)-T_{N-1}(s)}{2(s^2-1)^2} & (p, q \text{ are odd}, n \text{ is even}, n > 0), \\ -\frac{T_{N+1}(s)-T_{N-1}(s)}{2(s^2-1)^2} & (p, q \text{ are odd}, n \text{ is even}, n < 0). \\ T_N(s) & (p, q, n \text{ are odd}). \end{cases}$$

Here

- $T_l(x)$ *is the l-th Tchebychev polynomial of the first kind.*
- $s = \dfrac{\sqrt{t}}{2\sqrt{C_{(p,q,a,b)}}}$.
- $C_{(p,q,a,b)} = \left(1 - \cos\frac{a\pi}{p}\right)\left(1 - \cos\frac{b\pi}{q}\right)$.
- *a pair of integers* (a, b) *is satisfying the following conditions;*
  - $0 < a < p, 0 < b < q$,
  - $a \equiv b \equiv 1 \mod 2$.

Remark 4.4. *Recall that the l-th Tchebychev polynomial* $T_l(x)$ *is defined by* $T_l(\cos\theta) = \cos(l\theta)$.

Proof. We consider the following;

$$X_n(x) = \begin{cases} \frac{T_{N+1}(x)-T_{N-1}(x)}{2(x^2-1)} & (n > 0) \\ -\frac{T_{N+1}(x)-T_{N-1}(x)}{2(x^2-1)} & (n < 0). \end{cases}$$
$$X'_n(x) = T_N(x).$$

First we assume $p = 2p'$ is even. For the case that p' is odd, then it is proved in [5]. Then we suppose that p' is even. Here $N = |2p'qn + 1|$ is always odd.

Case 1:$p = 2p'$, p' is even and $n > 0$

We modify one factor $(1 + \cos\frac{2p'qk\pi}{N})$ of $\dfrac{1}{\tau_\rho(M_n)}$ as follows. See [5] for the proof.

Lemma 4.5. *The set* $\{\cos\frac{2p'qk\pi}{N} \mid 0 < k < N, k \equiv n \mod 2\}$ *is equal to the set* $\{\cos\frac{2p'k\pi}{N} \mid 0 < k < \frac{N-1}{2}\}$.

Now we can modify

$$\frac{1}{2}\left(1 + \cos\frac{2p'k\pi}{N}\right) = \frac{1}{2} \cdot 2\cos^2\frac{2p'k\pi}{2N}$$
$$= \cos^2\frac{p'k\pi}{N}.$$

We put

$$z_k = \cos\frac{p'k\pi}{N}\ (1 \le k \le N-1).$$

By the definition, it is seen

$$\begin{aligned} z_{N-k} &= \cos\frac{p'(N-k)\pi}{N} \\ &= \cos(p'\pi - \frac{p'k\pi}{N}) \\ &= z_k \end{aligned}$$

because p' is even.
Therefore it is enough to consider only z_k $(1 \le k \le \frac{N-1}{2})$.
Now we substitute $x = z_k$ to $T_{N+1}(x)$. Then one has

$$\begin{aligned} T_{N+1}(z_k) &= \cos\left((N+1)\frac{p'k\pi}{N}\right) \\ &= \cos\frac{p'k\pi}{N} \\ &= z_k \end{aligned}$$

and

$$\begin{aligned} T_{N-1}(z_k) &= \cos\left((N-1)\frac{p'k\pi}{N}\right) \\ &= \cos\frac{p'k\pi}{N} \\ &= z_k. \end{aligned}$$

Hence it holds

$$T_{N+1}(z_k) - T_{N-1}(z_k) = 0.$$

By properties of Tchebychev polynomials, it is seen that

- $T_{N+1}(1) - T_{N-1}(1) = 0$,
- $T_{N+1}(-1) - T_{N-1}(-1) = 0$.

We remark that the degree of $X_n(x) = \frac{T_{N+1}(x)-T_{N-1}(x)}{2(x^2-1)}$ is $N-1$ and $z_1, \cdots, z_{\frac{N-1}{2}}$ are zeros. Because both of $T_{N+1}(x)$ and $T_{N-1}(x)$ are even functions, then $-z_1, \cdots, -z_{\frac{N-1}{2}}$ are also zeros of $X_n(x)$. Hence $X_n(x)$ is a functions of x^2. Here by replacing x by $\frac{\sqrt{t}}{2\sqrt{C_{(p,q,a,b)}}}$, the degree of $Y_{(n,a,b)}(t)$ is $\frac{N-1}{2}$, and the roots of $Y_{(n,a,b)}(t)$ are

$$4C_{(p,q,a,b)}z_k^2 = 4C_{(p,q,a,b)}\cos^2\frac{\pi k}{N} \quad \left(0 < k < \frac{N-1}{2}\right),$$

which are all non trivial values of $\frac{1}{\tau_{\rho_{(a,b,k)}}(M_n)}$.

Here we check the normalization condition. By the definition of $Y_{(n,a,b)}(t)$ and properties of $T_{N+1}(x), T_{N-1}(x)$, one has

$$\begin{aligned} Y_{(n,a,b)}(0) &= \frac{T_{N+1}(0) - T_{N-1}(0)}{2(0-1)} \\ &= -\frac{(-1)^{\frac{N+1}{2}} - (-1)^{\frac{N-1}{2}}}{2} \\ &= (-1)^{\frac{N-1}{2}}. \end{aligned}$$

Hence it can be seen

$$\begin{aligned} \sigma_{(p,q,n)}(0) &= \prod_{(a,b)} (-1)^{\frac{N-1}{2}} \\ &= \prod_{(a,b)} \left((-1)^{\frac{N-1}{2}}\right)^{\frac{p(q-1)}{4}} \\ &= (-1)^{\frac{(N-1)p(q-1)}{8}}. \end{aligned}$$

Therefore we obtain the formula.

Case 2:$p = 2p'$ and $n < 0$
In this case we modify $N = |2p'qn + 1| = 2p'q|n| - 1$. By the same arguments, it is easy to see the claim of the theorem is proved.
Next assume both of p, q are odd integers.

Case 3:p, q are odd and n is even
If n is even, then $N = |pqn + 1|$ is odd. Then the similar arguments in [5] work well. Then it can be proved.

Case 4:p, q are odd and n is odd
Suppose n is positive. First note that $N = |pqn + 1|$ is even. We can modify one factor $(1 + \cos\frac{pqk\pi}{N})$ of $\dfrac{1}{\tau_\rho(M_n)}$ as follows. It is clear because $(q, N) = 1$.

Lemma 4.6. *The set* $\{\cos\frac{pqk\pi}{N} \mid 0 < k < N, k \equiv n \textit{ mod } 2\}$ *is equal to the set* $\{\cos\frac{pk\pi}{N} \mid 0 < k < N, k \equiv 1 \textit{ mod } 2\}$.

Now we can modify

$$\begin{aligned} \frac{1}{2}\left(1 + \cos\frac{pk\pi}{N}\right) &= \frac{1}{2} \cdot 2\cos^2\frac{pk\pi}{2N} \\ &= \cos^2\frac{pk\pi}{2N}. \end{aligned}$$

We put

$$z'_k = \cos\frac{pk\pi}{2N} \ (1 \le k \le N-1,\ k \equiv 1 \bmod 2).$$

Here we subsitute $x = z'_k$ $(1 \leq k \leq \frac{N-1}{2},\ k \equiv 1 \bmod 2)$ to $T_N(x)$. Then one has

$$\begin{aligned}T_N(z'_k) &= \cos\left(\frac{N(pk\pi)}{2N}\right)\\ &= \cos\left(\frac{pk\pi}{2}\right)\\ &= 0\end{aligned}$$

because pk is odd.
Similarly it can be also seen that

$$T_N(-z'_k) = 0.$$

We mention that the degree of $X'_n(x) = T_N(x)$ is N and $\pm z'_1, \cdots, \pm z'_{N-1}$ are the zeros. Because $X'_n(x)$ is a functions of x^2. Here by replacing x by $\frac{\sqrt{t}}{2\sqrt{C_{(p,q,a,b)}}}$, Here it holds that its degree of $Y_{(n,a,b)}(t)$ is $\frac{N-1}{2}$, and the roots of $Y_{(n,a,b)}(t)$ are

$$4C_{(p,q,a,b)}{z'_k}^2 = 4C_{(p,q,a,b)}\cos^2\frac{\pi k}{N} \quad \left(0 < k < \frac{N-1}{2}\right),$$

which are all non trivial values of $\frac{1}{\tau_{\rho_{(a,b,k)}}(M_n)}$.
Finally we can check the normalization condition as follows. By the definition of $Y_{(n,a,b)}(t)$, one has

$$\begin{aligned}Y_{(n,a,b)}(0) &= T_N(0)\\ &= (-1)^{\frac{N}{2}}\end{aligned}$$

and

$$\begin{aligned}\sigma_{(p,q,n)}(0) &= \prod_{(a,b)}(-1)^{\frac{N}{2}}\\ &= \left((-1)^{\frac{N}{2}}\right)^{\frac{(p-1)(q-1)}{4}}\\ &= (-1)^{\frac{N(p-1)(q-1)}{8}}.\end{aligned}$$

Therefore we obtain the formula.
In the case that n is negative, then it can be proved by similar arguments. Therefore this completes the proof. □

Remark 4.7. *By defining as $X_0(t) = 1$, it implies $Y_{(0,a,b)}(t) = 1$. Then the above statement is true for $n = 0$.*

By direct computation, one obtains the following corollary.

Corollary 4.8. *The degree $deg(\sigma_{(p,q,n)}(t))$ is given by*

$$deg(\sigma_{(p,q,n)}(t)) = \begin{cases} \frac{(N-1)p(q-1)}{8} & (p \text{ even}, q \text{ odd}), \\ \frac{(N-1)(p-1)q}{8} & (p \text{ odd}, q \text{ even}), \\ \frac{(N-1)(p-1)(q-1)}{8} & (p, q \text{ odd}, n \text{ even}), \\ \frac{N(p-1)(q-1)}{8} & (p, q \text{ odd}, n \text{ odd}). \end{cases}$$

We mention the 3-term relations. For each factor of $Y_{(n,a,b)}(t)$ of $\sigma_{(p,q,n)}(t)$, there exists the following relation.

Proposition 4.9.

(1) *Assume one of p and q is even. For any n, it holds that*

$$Y_{(n+1,a,b)}(t) = D(t)Y_{(n,a,b)}(t) - Y_{(n-1,a,b)}(t)$$

where $D(t) = 2T_{pq}\left(\frac{\sqrt{t}}{2\sqrt{C_{p,q,a,b}}}\right)$.

(2) *Assume both of p, q are odd. For any n, it holds that*

$$Y_{(n+2,a,b)}(t) = D(t)Y_{(n,a,b)}(t) - Y_{(n-2,a,b)}(t)$$

where $D(t) = 2T_{2pq}\left(\frac{\sqrt{t}}{2\sqrt{C_{2p,q,a,b}}}\right)$.

Proof. Here we need to consider $N = |pqn + 1|$ is a function of $n \in \mathbb{Z}$ for fixed p, q. Then we write $N(n)$ for N in this proof.
The proof for the first case is essentially the same one for the 3-term relations [5]. We give the proof only for the second case.
Recall the following property of Tchebychev polynomials

$$2T_m(x)T_n(x) = T_{m+n}(x) + T_{m-n}(x)$$

for any $m, n \in \mathbb{Z}$.

Case 1: n is even
If $n > 0$ one has

$$\begin{aligned} 2T_{2pq}(x)X_n(x) &= 2T_{2pq}(x)\frac{T_{N(n)+1}(x) - T_{N(n)-1}(x)}{2(x^2-1)} \\ &= \frac{T_{(pqn+1)+1+2pq}(x) + T_{(pqn+1)+1-2pq}(x) - (T_{(pqn+1)-1+2pq}(x) + T_{(pqn+1)-1-2pq}(x))}{2(x^2-1)} \\ &= \frac{T_{pq(n+2)+1+1}(x) - T_{pq(n+2)+1-1}(x) + T_{pq(n-2)+1+1}(x) - T_{pq(n-2)+1-1}(x)}{2(x^2-1)} \\ &= \frac{T_{N(n+2)+1}(x) - T_{N(n+2)-1}(x) + T_{N(n-2)+1}(x) - T_{N(n-2)-1}(x)}{2(x^2-1)} \\ &= X_{n+2}(x) + X_{n-2}(x). \end{aligned}$$

Therefore it can be seen that

$$X_{n+2}(x) = 2T_{2pq}(x)X_n(x) - X_{n-2}(x)$$

and

$$Y_{(n+2,a,b)}(t) = 2T_{2pq}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right)Y_{(n,a,b)}(t) - Y_{(n-2,a,b)}(t).$$

If $n < 0$, it can be also proved by the above argument.

Case 2: n is odd
If $n > 0$, one has

$$\begin{aligned}
2T_{2pq}(x)X'_n(x) &= 2T_{2pq}(x)T_{N(n)}(x)\\
&= T_{pqn+1+2pq}(x) + T_{pqn+1-2pq}(x)\\
&= T_{pq(n+2)+1}(x) + T_{pq(n-2)+1}(x)\\
&= T_{N(n+2)}(x) + T_{N(n-2)}(x)\\
&= X'_{n+2}(x) + X'_{n-2}(x).
\end{aligned}$$

Therefore it can be seen that

$$X'_{n+2}(x) = 2T_{2pq}(x)X'_n(x) - X'_{n-2}(x)$$

and

$$Y_{(n+2,a,b)}(t) = 2T_{2pq}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right)Y_{(n,a,b)}(t) - Y_{(n-2,a,b)}(t).$$

If $n < 0$, it can be also proved.
This completes the proof of this proposition. □

5. EXAMPLES

Finally we give some examples.

Example 5.1. *Put* $p = 4, q = 3$. *Now* $N = |12n + 1|$. *In this case* $(a, b) = (1, 1), (3, 1)$. *By applying the main theorem, one has*

$$\begin{aligned}
\sigma_{(4,3,-1)}(t) &= 34359738368t^{10} - 77309411328t^9 + 66840428544t^8\\
&\quad - 28655484928t^7 + 6677331968t^6 - 882900992t^5 + 66371584t^4\\
&\quad - 2723840t^3 + 55680t^2 - 480t + 1.\\
\sigma_{(4,3,0)}(t) &= 1.\\
\sigma_{(4,3,1)}(t) &= 4398046511104t^{12} - 12094627905536t^{11} + 13434657701888t^{10}\\
&\quad - 7859790151680t^9 + 2670664351744t^8 - 552909930496t^7\\
&\quad + 71319945216t^6 - 5727322112t^5 + 278757376t^4\\
&\quad - 7741440t^3 + 110208t^2 - 672t + 1.
\end{aligned}$$

Example 5.2. *Put* $p = 3, q = 5$. *Now* $N = |15n + 1|$. *In this case* $(a, b) = (1, 1), (1, 3)$. *For any odd number n, one has*

$$\begin{aligned}\sigma_{(3,5,n)}(t) &= Y_{(n,1,1)}(t)Y_{(n,1,3)}(t)\\ &= T_N\left(\frac{\sqrt{t}}{2\sqrt{C_{(3,5,1,1)}}}\right)Y_N\left(\frac{\sqrt{t}}{2\sqrt{C_{(3,5,1,3)}}}\right).\end{aligned}$$

By applying the main theorem, we obtain

$$\begin{aligned}\sigma_{(3,5,-1)}(t) &= 18014398509481984t^{14} - 47287796087390208t^{13} + 51721026970583040t^{12}\\ &\quad - 30847898228883456t^{11} + 11085001353330688t^{10} - 2520389888507904t^{9}\\ &\quad + 372923420377088t^{8} - 36436086620160t^{7} + 2352597696512t^{6}\\ &\quad - 98837200896t^{5} + 2605023232t^{4} - 40341504t^{3} + 329280t^{2} - 1176t + 11.\\ \sigma_{(3,5,0)}(t) &= 1.\\ \sigma_{(3,5,1)}(t) &= 4611686018427387904t^{16} - 13835058055282163712t^{15}\\ &\quad + 17726168133330272256t^{14} - 12754194144713244672t^{13}\\ &\quad + 5718164151876976640t^{12} - 1682516673287946240t^{11}\\ &\quad + 334779300425236480t^{10} - 45872724622442496t^{9}\\ &\quad + 4367893693202432t^{8} + -288911712583680t^{7}\\ &\quad + 13126896451584t^{6} - 399582953472t^{5}\\ &\quad + 7798652928t^{4} - 90832896t^{3} + 563200t^{2} - 1536t + 1.\end{aligned}$$

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