Critical scaling in the large-N $O(N)$ model in higher dimensions and its possible connection to quantum gravity

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We analyze the critical scaling of the large-N $O(N)$ model in higher dimensions using the exact renormalization group equations, motivated by the recently found non-trivial fixed point in $4 < d < 6$ dimensions with metastable critical potential. Particular attention is paid to the case of $d = 5$ where the scaling exponent ν has the value 1/3, which coincides with the scaling exponent of quantum gravity in one fewer dimensions. Convincing results show that this relation could be generalized to arbitrary number of dimensions above five. Some aspects of the AdS/CFT correspondence are also discussed.

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The non-trivial critical behavior in $O(N)$ theories are well-known for dimensions $d < 4$ [1]. Thus, a statement on the existence of interacting critical theories beyond four space-time dimensions is rather unusual since one would expect the triviality of the $O(N)$ vector model in general [2]. In the recent works [3, 4], however, exhaustive one and three loop analyses of the $O(N)$ theory with cubic interactions and $N + 1$ scalars show that the large-N $O(N)$ theory could follow the asymptotically safe scenario under the renormalization group in the UV. More precisely, it was argued that the IR fixed point found in the aforementioned $O(N)$ theory with the cubic interaction is equivalent to a perturbatively unitary UV fixed point in the large-N $O(N)$ model for dimensions $4 < d < 6$. The presence of such UV fixed point could be particularly interesting due to the conjectured $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ duality between a higher-spin $d+1$-dimensional massless gauge theory in AdS space with an appropriate boundary condition and the large-N critical $O(N)$ model in d dimensions [5]. The former is called the Vasiliev theory, which is a minimal interacting theory with gravity and higher-spin fields in its spectrum. However, a negative cosmological constant needs to be introduced via an AdS vacuum in order to make this model consistent. The Vasiliev theory can be obtained as the tensionless limit of string theory, where the infinite tower of higher-spin string modes are massless, and since there is no energy scale it can be considered as a toy model describing physics beyond the Planck scale [6]. The existence of the UV fixed point in the large-N $O(N)$ model has been subject to studies using conformal bootstrap analysis [7, 8] and exact (or functional) renormalization group (ERG or FRG) methods [9, 10], too.

First we give a brief review of the analytical results from [9]. Let us consider the effective average action of the O(N) symmetric theory in d dimensions within the Local Potential Approximation (LPA):

$$\Gamma_k = \int d^dx \left[\frac{1}{2}(\partial\bar\phi)^2 + U_k(\bar\phi^2)\right]. \tag{1}$$

U_k is the dimensionful potential depending on $\bar\phi^2$, where $\bar\phi$ is the dimensionful vacuum expectation value (VEV) of the field. The subscript k stands for the RG scale (i.e. the Wilsonian cutoff), on which the effective theory is defined. In the large-N limit the anomalous dimension of the Goldstone modes vanish, therefore, setting the wave function renormalization constant to unity in (1) gives a well-justified approximation. In fact, in the large-N limit of the $O(N)$ model the LPA is considered to be exact [11–13]. The flow of the effective action is given by the exact functional differential equation [14]

$$\partial_t\Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2)} + R_k\right)^{-1}\partial_t R_k. \tag{2}$$

Here, we have introduced the logarithmic flow parameter $t = \ln(k/\Lambda)$ (with the initial UV scale Λ), and a momentum dependent regulating function $R_k(q^2)$, ensuring that only the fluctuations above the Wilsonian cutoff scale are integrated out. Further, we have $\Gamma_k^{(2)}[\bar\phi]$ as a shorthand notation for the second derivative with respect to the field, and the trace denotes the integration over all momenta as well as the summation over internal indices. The integral can be evaluated by choosing $R_k(q^2)$ in a way that it satisfies some basic requirements: particularly Γ_k approaches the bare action in the limit $k \to \Lambda$ and the full quantum effective action when $k \to 0$ [14]. For a detailed study of an extensive class of regulator functions see e.g. [15]. In the present case we will pick the so called optimized regulator $R_k(q^2) = (k^2-q^2)\,\theta(k^2-q^2)$ which provides an analytic result for the momentum integral [16]. It is convenient to introduce $\bar\rho \equiv \frac{1}{2}\bar\phi^2$, which we will use throughout this paper. By inserting (1) into (2), and taking the limit $N \to \infty$, yields the flow for the effective potential in the large-N [13]:

$$\partial_t u = -du + (d-2)\rho u' + \frac{1}{1+u'}. \tag{3}$$

Here we switched to dimensionless quantities $u = Uk^{-d}$, $\rho = \bar\rho k^{-d+2}$ and $u' = \partial_\rho u$. There exists an exact solution for the ρ derivative of (3) which can be obtained by using the method of characteristics [13, 17]. In fact, the fixed point solutions associated to (3) can be given as an implicit function $\rho = \rho(u'_*)$.

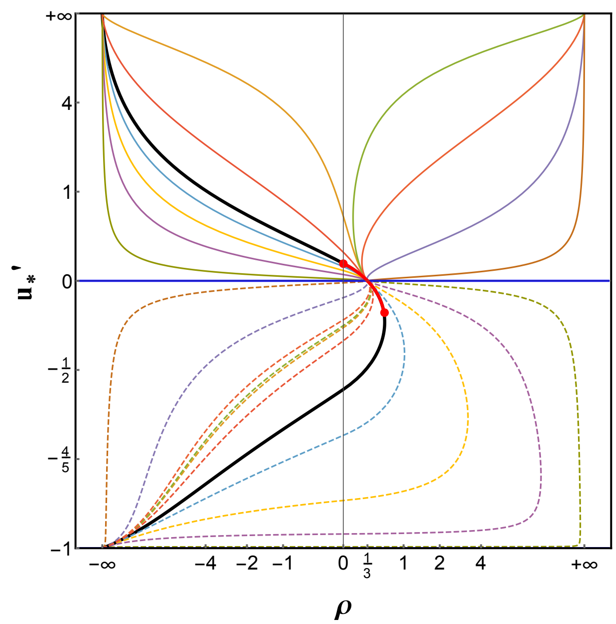

FIG. 1. Each curve corresponds to a critical potential derivative with a particular choice of c in $d = 5$. The thick black line is for the $c = 0$ solution. We will consider the red line segment as the physical branch. The axes are rescaled for better display.

Here we introduced u_* which denotes the dimensionless effective potential at the fixed point. Their most compact form for $d = 2n+1$ and $d = 2n$ ($n \in \mathbb{Z}$) are respectively

$$\rho = c {u'_*}^{\frac{d}{2}-1} + \frac{1}{(d-2)}\, {}_2F_1\left(2, 1-\frac{d}{2}; 2-\frac{d}{2}; -u'_*\right) \quad (4)$$

and

$$\rho = \bar{c} {u'_*}^{\frac{d}{2}-1} + \frac{1}{(d+2)(1+u'_*)^2}\, {}_2F_1\left(1, 2; 2+\frac{d}{2}; \frac{1}{1+u'_*}\right), \quad (5)$$

where c is an arbitrary constant obtained from the integration, $\bar{c} = c - \frac{d\pi}{4}\sin(d\pi/2)$ and ${}_2F_1$ is the hypergeometric function. The solutions for the $d = 5$ case are shown in Fig. 1, where each curve corresponds to a solution with a particular value of the parameter c. For $u'_* \geq 0$ (4) holds for every $c \in \mathbb{R}$, however, to obtain a continuation of the solutions to $u'_* \leq 0$ one needs to give imaginary values for the constant c, except for one solution, namely, which corresponds to $c = 0$. This latter is depicted in Fig. 1 as the thick black curve, that smoothly goes through $u_* = 0$, and intersects the horizontal and vertical axes at $\rho = 1/3$ and $u'_*(0) \approx 0.1392$ on the upper plane, respectively. It is tempting to consider this fixed point potential as the physical one, as it is analytic at its extremum. On the other hand, this curve still has the problem that all the others have (with their continuation): u'_* can be considered only as a multivalued function of ρ as it was pointed out in [9]. Nevertheless, we can still define two branches of this solution on a restricted interval of the field, namely, for $\rho \in [0, 0.6214]$. We only need to decide which one will we choose.

Now, we turn to the results of [10]. Here another technique was used to solve the flow equation of the effective potential, which is based on its polynomial expansion, hence it is assumed to be analytic around zero:

$$u(\rho) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{u^{(i)}(0)}{i!}\rho^i. \quad (6)$$

The derivatives of the potential can be considered as the couplings of the theory: $u'(0) = g_1 = m^2$ (squared mass), $u''(0) = g_2 = \lambda$ (quartic coupling), etc... In [10] an efficient algorithm was worked out for finding the true fixed points of the theory up to expansion order 50 if necessary, based on the observation that all the couplings can be expressed through the squared mass of the system at the fixed point ($g_i^* = g_i^*(m_*^2)$).

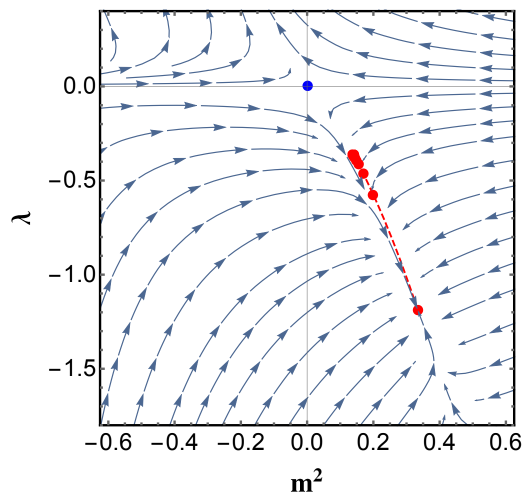

FIG. 2. The RG flow in the $\{m^2, \lambda\}$ hyperplane of the theory space. The drifting of the non-trivial fixed point is shown towards $\{m_*^2, \lambda_*\} \approx \{0.1392, -0.3613\}$ as the expansion order is increased. The blue dot is the GFP.

In the case of the five-dimensional large-N $O(N)$ model the fixed point structure shows besides the non-interacting Gaussian (GFP) a non-trivial fixed point, too. The position of the fixed point drifts as we increase the order of the Taylor-expansion, and it converges to the value $m_*^2 \approx 0.1392$, see Fig. 2. As $m^2 = u'(0)$ we can safely state that the same fixed point solution was recovered than the one found by using the analytic solution of the flow when $c = 0$ in (4). In fact, it seems that this technique naturally singles out a fixed point solution from all the others that are present in the analytical case. In Fig. 1 this corresponds to the red line segment on the thick black curve and we will consider it as the physical solution.

Since both the analytic and the polynomial fixed point solutions are now available, we can compute the corresponding effective potential. The exact critical potential computed from (4) is shown in Fig. 3, where it can be compared to the one that is obtained by the polynomial expansion, $u_*(\rho) = \sum_{i=1}^n \frac{g_i^*(m_*^2)}{i!}\rho^i$. The matching between the two curves is excellent, however, the non-analytic nature of the exact potential is obvious as it is restricted to the finite interval of the field, contrary to the polynomial case which is of course analytic as it was assumed.

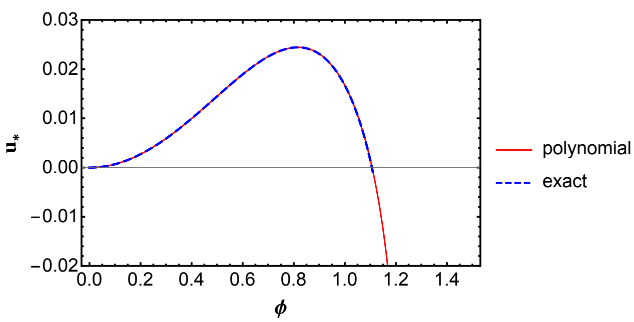

FIG. 3. The metastable critical potential in $d = 5$. Here we used the variable $\phi = \sqrt{2\rho}$ and hence the exact potential is valid between $\phi \in [0, 1.1148]$.

Perhaps the most important piece of information is that this potential has a metastable ground state, however, it does not have a true vacuum. At first sight, this fact might discourage us from investigating it further, but metastable and unstable vacua are not unknown in the world of physics. First of all there is the question on the electroweak vacuum stability. As of today it is still a question if the Higgs potential exhibits a ground state or we sit in a false vacuum (with a very long lifetime) that implies an unstable universe [18]. On the other hand the theory could be saved from the AdS side, too. Since, as it was mentioned above, the critical large-N $O(N)$ theory in $d = 5$ is possibly dual to a Vasiliev higher-spin theory in AdS_6 space, they must have the same energy spectrum. In AdS space, on the other hand, the so called Breitenlohner-Freedman (BF) bound gives a negative, dimension dependent lower bound for the squared mass of the field above which the theory can be considered as stable [19]. The BF bound can be generalized for massless higher-spin fields, too, which also depends on the value of the spin [20]. In turn, the same argument could hold for the other branch of $u_*'(\rho) < 0$ for $\rho \in [0, 0.6214]$ with $u_*'(0) = m_*^2 \approx -0.5776$, see in Fig. 1. In this case the potential does not have any metastable minimum, it is completely unstable in the restricted interval as $u'(\rho) < 0$ for these field values. However, this fixed point potential can be found only from the analytic solution, whereas the polynomial approach ignores it entirely.

Despite the fact that the critical potential is restricted to a finite interval of the field, it is still possible to extract the critical exponent ν from it, which is the scaling exponent of the correlation length (or inverse mass) and characterizes the system at criticality. For the exact determination of the exponents in d dimensions we will use the method of eigenperturbation, which is based on the linearized flow around the fixed point, i.e. $u(\rho,t) = u_*(\rho) + \delta u(\rho,t)$ [17, 21]. Using (3) we can derive the fluctuation equation for the derivative of the potential

$$\partial_t \delta u' = 2\frac{u_*'}{u_*''}\left(\partial_\rho - \frac{(u_*' u_*'')'}{u_*' u_*''} - \frac{d-4}{2}\frac{u_*''}{u_*'}\right)\delta u'. \quad (7)$$

Apparently, it can be thought of as an eigenvalue problem: $\partial_t \delta u' = \theta \delta u'$, where the smallest eigenvalue θ equals the negative inverse of the scaling exponent ν. Solving this PDE via the method of separation of variables yields

$$\delta u' \propto e^{t\theta} u_*'^{\frac{1}{2}(\theta+d-2)} u_*''. \quad (8)$$

We demand the regularity of the perturbation at the node ($u_*'(\rho_0) = 0$), that is we will have a restriction on the values of θ in order to keep $\delta u'$ analytic. It is not hard to show that for both formula in (4) and (5) the extremum is at $\rho_0 = 1/(d-2)$ and the behavior for the $c = 0$ and $\bar{c} = 0$ solutions in the vicinity of the node is linear in ρ, $u_*' \propto \left(\rho - \frac{1}{d-2}\right)$, which makes u_*'' a constant. Substituting back this expression into (8) gives

$$\delta u' \propto e^{t\theta}\left(\rho - \frac{1}{d-2}\right)^{\frac{1}{2}(\theta+d-2)}. \quad (9)$$

The allowed values are then $\theta = 2(l + 1 - d/2)$, where l is a non-negative integer, and the scaling exponent is obtained by the lowest value of θ, i.e. for $l = 0$. Thus, the scaling exponent for arbitrary dimensions in the large-N $O(N)$ model is

$$\nu = (d-2)^{-1}. \quad (10)$$

By using the polynomial expansion one can compute the critical exponent ν as the negative inverse of the lowest eigenvalue of the stability matrix at the fixed point $B_{ij} = \partial\beta_i/\partial g_j|_{g=g_*}$ [14]. Here, the beta functions are defined as the RG scale derivative of the couplings: $\beta_i = \partial g_i(t)/\partial t$. Since the LPA became exact in the large-N limit, the right value for the critical exponent can be obtained at every order of the expansion, i.e. we get back (10) for arbitrary dimensions. This relation, on the other hand, is well-known for the large-N $O(N)$ theories in $d \leq 4$ [11, 12, 22, 23]. However, it was not extended to higher dimensions as the upper critical dimension was considered to be $d = 4$. Yet, with an accurate analysis of the fixed point structure, it seems that we can find, with both the analytical method and the polynomial expansion, a non-trivial fixed point, where, instead of the mean-field scaling, the relation (10) still holds for $d > 4$. However, we need to be careful with this statement since the effective potentials defined at criticality are non-analytic and/or metastable for these values of d. In particular, for $d = 5$ we have $\nu = 1/3$ and the ground state seems to be metastable. In fact, in the papers [3, 4] also an unbounded critical potential is expected, and in that respect the results presented here are consistent with those. Although in the large-N the dimensionality is restricted to $4 < d < 6$ due to the unitarity bound [3, 4, 24], we can study further the higher dimensional cases. Fig. 4 displays the solutions (4) and (5) for $d > 5$ with $c = 0$ and $\bar{c} = 0$, respectively. We can make the following observations. In $d = 6, 8$ dimensions u'_* is singular at $\rho = 0$ and multivalued for $\rho < \rho_0$, in addition, the function (5) also gets complex for $u'_* \in [0, -1]$.

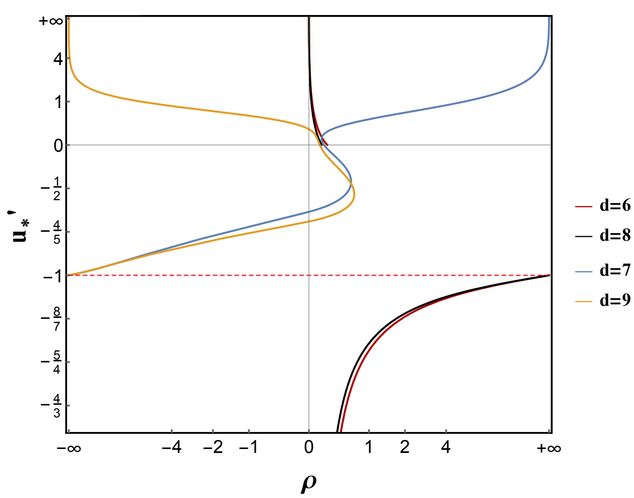

FIG. 4. The fixed point solutions given by (4) and (5) in $d \geq 6$ for $c = 0$ and $\bar{c} = 0$, respectively. The axes are rescaled for better display.

In $d = 7$ the potential seems to be stable, however, because of the turning points it becomes multivalued, although at $\rho = 0$ it is unique. And in $d = 9$ we pretty much have the same situation as in Fig. 1. These three categories seem to be preserved to all even, $d = 4n + 3$ and $d = 4n + 1$ $(n \geq 1)$ dimensions, respectively. To the even-dimensional case it is very hard to give a physical interpretation due to its singular structure and complex nature. For the $d = 4n + 3$ cases we can define three branches due to the "S" shape of the curve around $u'_* = 0$, which makes it challenging to understand its physical content, if there is any. Perhaps one could cut out certain parts of the "S" shape in the spirit of Maxwell's construction [25], which will allow us to define a bounded potential with a cusp at the position of the cut. In the last case, when $d = 4n+1$, one can simply use the same arguments as in $d = 5$ to define a metastable potential in a finite interval. It is also worth to mention that a similar convergence as in Fig. 2 can be observed clearly only for $4 < d < 6$ $(d \in \mathbb{R})$ by using the polynomial approximation, and from the analytical side, using (4), these solutions have the same structure as in Fig. 1. Considering these facts one can conclude that physically sensible fixed points exist in $4 < d < 6$, provided accepting the metastable potential. However, although the relation for the scaling exponent ν holds naively for all d, one would need to further investigate the $d \geq 6$ cases, both for integer and fractal dimensions.

In the following, we will present an interesting observation, which might link the large-N $O(N)$ model to quantum Einstein gravity (QEG). As of today many evidences suggest that QEG admits a continuous phase transition between physically two distinct phases described by a strong and weak Newton's coupling [26]. This phenomenon is naturally associated to a UV fixed point which is characterized by a non-trivial scaling of the correlation length: $\xi \propto |G_b - G_*|^{-\nu}$, where the dimensionless quantities G_b and G_* are the bare and the fixed point Newton's coupling, respectively. Within the framework of FRG in [27], using the optimized regulator and a special reparametrization of the metric fluctuation, that ensures the gauge independence, $\nu^{-1} = -6 + 4/d + 2d$ is obtained. Plugging in $d = 4$ yields $\nu = 1/3$. The scaling $\nu \simeq 1/3$ was found by using the Regge lattice action in the extensive numerical studies by Hamber [28–30]. In fact, in [29] a simple geometrical argument is given in support of the exact value of 1/3. It is based on the observation that the quantum correction to the static gravitational potential (due to the vacuum-polarization induced scale dependence of Newton's coupling) can be interpreted as a uniform mass distribution surrounding the original source only if $\nu^{-1} = d-1$ for $d \geq 4$. In particular, for $d = 4$ this gives $\nu = 1/3$. This conjecture can be compared to the results coming from $\nu^{-1} = -6+4/d+2d$ in [27] by inserting different values for d greater than four: $\nu(d = 5) \approx 0.2083$, $\nu(d = 6) = 0.15$. Moreover, these values for ν might improve by taking into account higher order curvature invariants in the approximation used in [27]. Having these results, it is apparent that an interesting correspondence can be revealed between the critical exponents of the large-N $O(N)$ model (ν_O) and QEG (ν_G) as a function of dimension:

$$\nu_O(d) \simeq \nu_G(d-1), \qquad \text{for } d \geq 5. \tag{11}$$

A similar dimensional reduction relates a classical field theory in a the presence of a random source to the corresponding quantum field theory in two fewer dimensions [31] as a consequence of a hidden supersymmetry which was pointed out by Parisi and Sourlas in [32], and proved rigorously by Klein *et al.* in [33]. The concept of the Parisi-Sourlas dimensional reduction was successfully applied to show that the dilute branched polymers and the Lee-Yang edge singularity of the Ising model in two fewer dimensions belong to the same universality class [34, 35]. However, in our case we would need to find a correspondence between the two theories with only one dimensional difference, but we also need to keep in mind that the classical Vasiliev theory, which is supposed to be dual to the large-N $O(N)$ model in $d = 5$, lives in a six dimensional AdS space. Despite the relation found in (11) the two theory does not necessarily fall into the same universality class. For that, one would need possibly to relate all the critical exponents in some way. For instance, from FRG studies in QEG the most conventional value for the anomalous dimension of the graviton propagator in four dimensions is $\eta_G = -2$, whereas in the large-N $O(N)$ model $\eta_O = 0$, i.e. they differ. However, if one assumes the usual scaling laws [22] to be valid in QEG, $\eta_G = -2$ in $d = 4$ [from $2-\eta_G = d(\delta_G-1)/(\delta_G+1)$] gives $\delta_G \to \infty$, which is physically rather questionable for a critical exponent. Perhaps modified scaling laws are required and/or $\eta_G = -2$ is not the right value for the anomalous dimension. It would be interesting to find out if (11) is a mere coincidence or there exists a deeper explanation that implies a correspondence between QEG_{d-1} and the large-N $O(N)$ theory in d dimensions which is in turn dual to the higher-spin Vasiliev theory in AdS_{d+1} space (where $d \geq 5$).

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