

A microwave realization of the Gaussian symplectic ensemble

A. Rehemanijiang,¹ M. Allgaier,^{1,2} C. H. Joyner,³ S. Müller,⁴ M. Sieber,⁴ U. Kuhl,^{1,5} and H.-J. Stöckmann¹

¹*Fachbereich Physik der Philipps-Universität Marburg, D-35032 Marburg, Germany*

²*Integrated Quantum Optics, Applied Physics, University of Paderborn, 33098 Paderborn, Germany*

³*School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, UK*

⁴*School of Mathematics, University of Bristol, Bristol BS8 1TW, UK*

⁵*Université de Nice - Sophia Antipolis, Laboratoire de la Physique de la Matière Condensée, CNRS, Parc Valrose, 06108 Nice, France*

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Following an idea by Joyner et al. [EPL, 107 (2014) 50004] a microwave graph with an antiunitary symmetry T obeying $T^2 = -1$ has been realized. The Kramers doublets expected for such systems have been clearly identified and could be lifted by a perturbation which breaks the antiunitary symmetry. The observed spectral level spacings distribution of the Kramers doublets is in agreement with the predictions from the Gaussian symplectic ensemble, expected for chaotic systems with such a symmetry.

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Random matrix theory has proven to be an extremely powerful tool to describe the spectra of chaotic systems [1–4]. For systems with time-reversal symmetry (TRS) and no half-integer spin in particular there is an abundant number of studies, theoretical, numerical and experimental, showing that the universal spectral properties are perfectly well reproduced by the corresponding properties of the Gaussian orthogonal random matrix ensemble (GOE) (see Ref. 5 for a review). This is the essence of the famous conjecture by Bohigas, Giannoni and Schmit [3], which has received strong theoretical support, see e.g. Refs. 6–8. For systems with TRS and half-integer spin and systems with no TRS the Gaussian symplectic ensemble (GSE) and the Gaussian unitary ensemble (GUE), respectively, hold instead. For the latter two of the classical ensembles the experimental situation is still very limited. There are three studies of the spectra of systems with broken TRS showing GUE statistics [9–11], all of them applying microwave techniques. For the GSE there is no experimental realization at all up to now. Only by using that a GSE spectrum is obtained by taking only every second level of a GOE spectrum [1], GSE statistics has been experimentally observed in a microwave hyperbola billiard [12].

In fact GUE statistics may be observed even in systems without broken TRS if there is a suitable geometrical symmetry. One example is the billiard with three-fold rotational symmetry [13], with microwave realizations [14, 15]. Another example is the constant width billiard [16], again with an experimental realization [17].

On the other hand, GOE statistics may be obtained in billiards with a magnetic field if there is an additional reflection symmetry [18]. This is because, even though time-reversal symmetry is broken, there exists an antiunitary symmetry that combines time-reversal with reflection. To be able to observe GSE statistics in a system without spin requires a similar non-conventional symme-

try. What is needed according to Dyson’s threefold way [19] is an antiunitary symmetry T with the property that $T^2 = -1$. This is sufficient to guarantee GSE statistics if the system is chaotic. In addition it leads to Kramer’s degeneracy, i. e. application of T to an energy eigenfunction yields an orthogonal eigenfunction with equal energy. A system with such a symmetry was recently found in the form of a quantum graph [20].

Quantum graphs were introduced by Kottos and Smilansky [21] as an ideal tool to study various aspects of quantum chaos. The wave function on a quantum graph satisfies a one-dimensional Schrödinger equation on each of the bonds with suitable matching conditions (implying current conservation) at the vertices. Just, as for quantum billiards, there is a one-to-one mapping onto the corresponding microwave graph. This analogy has been used in a number of experiments including one on graphs with and without broken TRS [11, 22, 23].

To realize graphs with GSE symmetry the graph shown in Fig. 1(top) was proposed in [20]. It contains two geometrically identical subgraphs, but with phase shifts by $+\pi/2$ and $-\pi/2$, respectively, along two corresponding bonds. The two subgraphs are connected by a pair of bonds yielding a graph with a geometric inversion center. In addition there is another phase shift of π along one of the two bonds, but not the other one! This is the crucial point: Due to this trick the total graph is symmetric with respect to an antiunitary operator T , squaring to minus one, $T^2 = -1$, where

$$\begin{aligned} T\psi(x_1) &= +\psi^*(x_2), \\ T\psi(x_2) &= -\psi^*(x_1), \end{aligned} \quad (1)$$

i. e. if ψ satisfies the Schrödinger equation as well as the vertex conditions then the same applies to $T\psi$. Here x_1 is a coordinate in subgraph 1, and x_2 the corresponding coordinate (related by inversion) in subgraph 2. Applying (1) twice shows $T^2 = -1$.

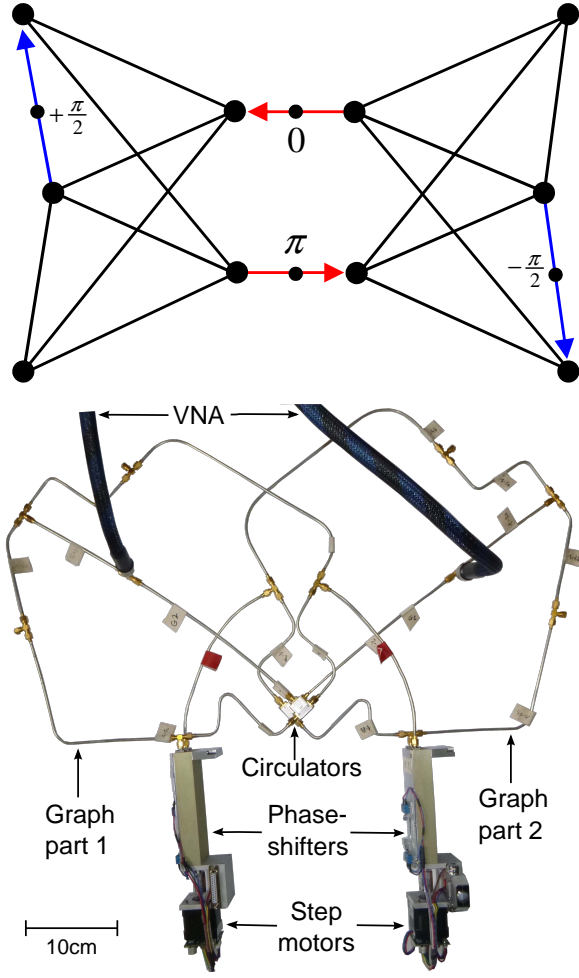


FIG. 1. (Color online) (Top) Sketch of the graph proposed in Ref. 20 to study GSE statistics without spin. The four arrows denote bonds along which additional phases are acquired. (Bottom) Photograph of the graph used in the experiment.

Since this fact is essential for the following, a complementary approach shall be given establishing a direct correspondence between the experiment and a spin 1/2 system. The wave function in a quantum graph is subject to two constraints, continuity at the vertices and current conservation. In a microwave graph these constraints correspond to the well-known Kirchhoff rules governing electric circuits. They lead to a secular equation system having a solution only if the determinant of the corresponding matrix vanishes,

$$\det[h(k)] = 0 \quad (2)$$

where the matrix elements of $h(k)$ are given by

$$h_{ij}(k) = \begin{cases} -\sum_{n \neq i} C_{in} \cot(kl_{in}) & i = j \\ C_{ij} e^{-i\varphi_{ij}} [\sin(kl_{ij})]^{-1} & i \neq j \end{cases} \quad (3)$$

where $C_{ij} = 1$, if nodes i and j are connected, and $C_{ij} = 0$ otherwise. l_{ij} is the length of the bond connecting

nodes i and j . φ_{ij} is a phase resulting, e.g., from a vector potential and breaks TRS if present. The equation holds for Neumann boundary conditions at all nodes, the situation found in the experiment. Details can be found in [24]. The solutions of the determinant condition (2) generate the spectrum of the graph.

Applied to the graph of Fig. 1 the secular matrix $h(k)$ may be written as

$$h = h_{\text{dis}} + v \quad (4)$$

where h_{dis} is the secular matrix for the disconnected sub-graphs, and v describes the connecting bonds. It is convenient to introduce an order of rows and columns according to $\{1, 2, \dots, n; \bar{1}, \bar{2}, \dots, \bar{n}\}$, where the numbers without bar refer to the vertices of subgraph 1, and the numbers with bar to those of subgraph 2. h_{dis} may then be written as

$$h_{\text{dis}} = \begin{pmatrix} h_0 & \cdot \\ \cdot & h_0^* \end{pmatrix} \quad (5)$$

where h_0 and h_0^* are the secular matrices for each of the two subgraphs. Since the only difference between the subgraphs is the sign of the $\pi/2$ phase shift in one of the bonds, their secular matrices are just complex conjugates of each other, see Eq. (3). Assuming for the sake of simplicity that there is just one pair of bonds connecting node 1 with node $\bar{2}$, and node $\bar{1}$ with node 2, the matrix elements of v are given by

$$v_{11} = v_{22} = v_{\bar{1}\bar{1}} = v_{\bar{2}\bar{2}} = -\cot(kl) \quad (6)$$

$$v_{1\bar{2}} = v_{\bar{2}1} = -v_{2\bar{1}} = -v_{\bar{1}2} = [\sin(kl)]^{-1} \quad (7)$$

$$v_{ij} = v_{i\bar{j}} = v_{\bar{i}j} = v_{\bar{i}\bar{j}} = 0 \quad \text{otherwise} \quad (8)$$

where l is the length of the bond connecting 1 with $\bar{2}$ and $\bar{1}$ with 2. The generalization to a larger number of bond pairs is straightforward.

Changing now the sequence of rows and columns to $\{1, \bar{1}; 2, \bar{2}; \dots; n, \bar{n}\}$, the resulting $2n \times 2n$ matrix $\tilde{h}(k)$ may be written in terms of a $n \times n$ matrix with quaternion matrix elements,

$$[\tilde{h}(k)]_{nm} = [\text{Re}(h_0)_{nm} + v_{nm}] \mathbf{1} - \text{Im}(h_0)_{nm} \tau_z - v_{n\bar{m}} \tau_y \quad (9)$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} -i & \cdot \\ \cdot & i \end{pmatrix}, \quad \tau_y = \begin{pmatrix} \cdot & -1 \\ 1 & \cdot \end{pmatrix} \quad (10)$$

The determinant is not changed by this rearrangement of rows and columns, $\det[h(k)] = \det[\tilde{h}(k)]$. The matrix elements $[\tilde{h}(k)]_{nm}$ commute with $C\tau_y$, where C denotes the complex conjugate, and hence the whole matrix commutes with $T = \text{diag}(C\tau_y, \dots, C\tau_y)$, where T squares to minus one, $T^2 = -1$. This is exactly the situation found for spin 1/2 systems, and just as in such systems a two-fold Kramers degenerate spectrum is expected showing

the signatures of the GSE provided the system is chaotic, see e. g. Chapter 2 of Ref. 4.

The requirements defined by Joyner et al. [20] to realize graphs with GSE symmetry pose some challenges. Since we did not know of a simple way to achieve phase shifts of $\pm\pi/2$ along the bonds, we proceeded in a somewhat different way and instead built two geometrically identical subgraphs, but with two circulators of opposite sense of rotation within the two subgraphs. A circulator is a microwave device introducing directionality: Microwaves pass from port 1 to port 2, from port 2 to port 3, and from port 3 to port 1. The result is the same as with the $\pm\pi/2$ shifts: the circulators break TRS, resulting in identical GUE spectra for the two disconnected subgraphs, but with an opposite sense of propagation within the respective subgraphs. Again the two subgraphs may thus be described in terms of a secular matrix h_0 and its complex conjugate h_0^* , respectively.

The phase difference between the two connecting bonds is adjustable by means of mechanical phase shifters which in reality, however, do not change the phase but the length. This approach has the obvious shortcoming that for a given length change Δl the phase shift $\Delta\varphi$ depends on frequency ν :

$$\Delta\varphi = k\Delta l = \frac{2\pi\nu}{c}\Delta l \quad (11)$$

where k is the wave number, and c is the vacuum velocity of light. $l = nl_0$ is the *optical* length where l_0 is the *geometrical* length and $n = 1.43$ the index of refraction of the dielectric within the coaxial cables.

Figure 1(b) shows a photograph of the graph used in the experiment. The bonds of the graphs were formed by Huber & Suhner EZ-141 coaxial semi-rigid cables with SMA connectors, coupled by T junctions at the vertices. The phase shifters (ATM, P1507) had been equipped with motors to allow for an automatic stepping. Reflection and transmission measurements were performed with an Agilent 8720ES vector network analyzer (VNA) with two ports at equivalent positions of the two subgraphs. The corresponding reflection and transmission amplitudes will be denoted in the following by S_{ij} , $i, j = 1, 2$. The operating range of the circulators (Aerotex I70-1FFF) extended from 6 GHz to 12 GHz. Therefore the evaluation of the spectra was restricted to this window.

We started by taking a series of spectra for constant Δl . Figure 2(top) shows the transmission for altogether 396 Δl values stacked onto each other between $\Delta l_{\min} \approx 0$ and $\Delta l_{\max} = 4.4$ cm in a gray scale. The lines for $\Delta\varphi = \pi$ and $\Delta\varphi = 3\pi$ are marked in red and green, respectively. Next, a variable transformation from Δl to $\Delta\varphi$ was performed, using Eq. (11), to obtain reflection spectra S_{11} for constant $\Delta\varphi$. The result is shown in Fig. 2(bottom). For a given frequency ν the maximum $\Delta\varphi$ accessible is, according to Eq. (11), given by $\Delta\varphi_{\max} = (2\pi\Delta l_{\max}/c)\nu$.

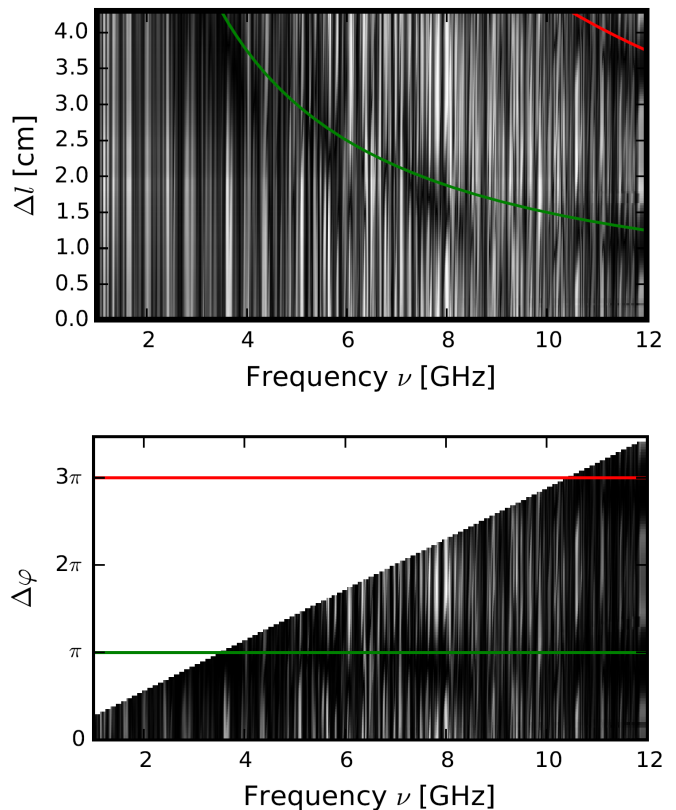


FIG. 2. (Color online) (Top) Transmission $|S_{12}|^2$ in dependence of frequency for constant Δl in a gray scale. The spectra for different Δl are stacked onto each other. (Bottom) The same data, but rearranged to yield spectra for constant $\Delta\varphi$.

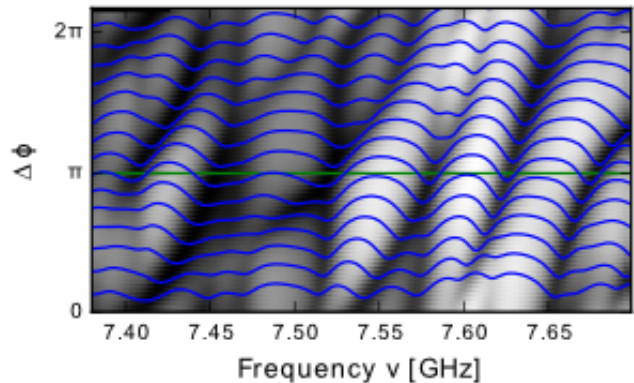


FIG. 3. Reflection $|S_{11}|^2$ in dependence of frequency in a shade plot. The spectra for different $\Delta\varphi$ are stacked onto each other.

The inaccessible regime above this limit is left white in Fig. 2(bottom). As expected the pattern is periodic in $\Delta\varphi$ with period 2π . Moreover, dark regions concentrate at the π and 3π lines, corresponding to low transmission, whereas in between for $\Delta\varphi = 0$ and $\Delta\varphi = 2\pi$ light regions are found. This is an interference effect: Trans-

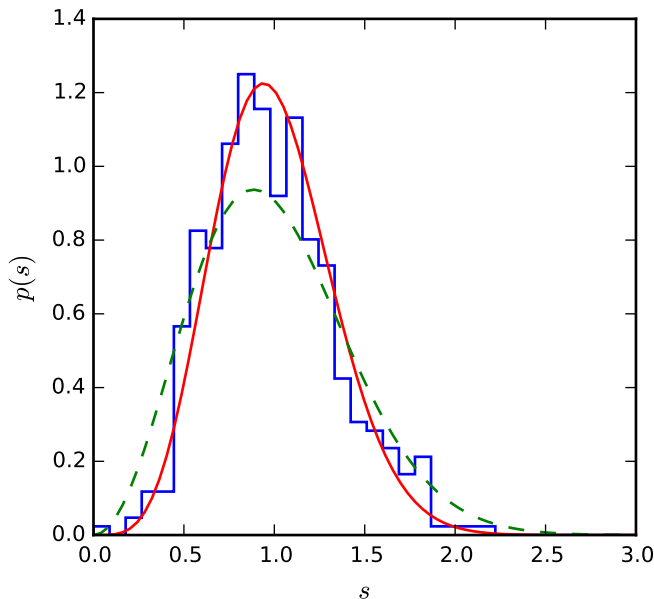


FIG. 4. (Color online) Spectral nearest neighbor distance distribution obtained by superimposing the results from four different spectra. The solid red and dotted green lines correspond to GSE and GUE Wigner distributions, respectively, see Eqs. (12) and (13).

mission paths always occur in pairs, both of them of the same length, but one of them passing through the bond with the phase shift, the other one through the other bond.

Figure 3 shows the reflection spectra for different $\Delta\varphi$, again stacked onto each other in a shade plot. One clearly observes the formation of Kramers doublets at the π line, and their splitting into singlets when departing from this line. There is a complete equivalence to the Zeeman splitting of spin doublets, where $\Delta\varphi$ takes the role of the magnetic field. This is a clear confirmation that we had been successful in constructing a graph with anti-unitary symmetry T with $T^2 = -1$. Furthermore there is a clear tendency of the levels towards an equal level spacings at the π line, one of the fingerprints for a GSE spectrum.

To obtain the complete spectrum, we did not use the reflection spectrum but its phase derivative. The phase of the reflection shows steps for each resonance leading to sharp peaks in the phase derivative. This allowed for an automatic determination of about 90% of the eigenvalues. With the additional information from the spectral level dynamics, see Fig. 3, the missing ones could be easily identified. About 10% of the Kramers doublets split due to experimental imperfections. Whenever this was evident from the level dynamics, the resulting two resonances were replaced by a single one in the middle.

The integrated density of states, with ν as the variable, may be written as $n(\nu) = n_{\text{Weyl}}(\nu) + n_{\text{fluc}}(\nu)$ where the average part is given by Weyl's law, $n_{\text{Weyl}}(\nu) = (\pi/L)(2\pi\nu/c)$ with L denoting the sum of all bond

lengths. The fluctuating part $n_{\text{fluc}}(\nu)$ reflects the influence of the periodic orbits [24]. We determined $n_{\text{fluc}}(\nu)$ by fitting a straight line to the experimental integrated density of states and subtracting the linear part. A small number of missing or misidentified resonances showing up in step-wise changes of $n_{\text{fluc}}(\nu)$ enabled the final correction of the spectrum. From the fit a length of $L = 2.93$ m was obtained. The fluctuating part $n_{\text{fluc}}(\nu)$ showed only the fluctuations typical for periodic orbits, thus illustrating the correctness of the procedure.

Figure 4 shows the resulting distribution of spacings between neighboring levels in units of the mean level spacing. To improve the statistics, the results from 4 different graphs were superimposed, leading to 477 Kramers doublets. The red solid and the green dotted line correspond to the Wigner prediction for the GSE,

$$p_{\text{GSE}}(s) = \frac{2^{18}}{3^6\pi^3} s^4 \exp\left(-\frac{64}{9\pi} s^2\right) \quad (12)$$

and the GUE,

$$p_{\text{GUE}}(s) = \frac{32}{\pi^2} s^2 \exp\left(-\frac{4}{\pi} s^2\right) \quad (13)$$

respectively. The experimental result fits well to the GSE distribution and, though the statistical evidence as yet is only moderate, it is clearly at odds with a GUE distribution.

It needed half a century after the establishment of random matrix theory by Wigner, Dyson, Mehta and others to arrive at the present experimental realization of the third of the three classical random matrix ensembles. It might be considered surprising that two bonds between the two subgraphs are already sufficient to turn the two GUE spectra of the disconnected subgraphs into a GSE spectrum for the total graph. On the other hand the present statistical evidence is not yet sufficient to determine whether more connecting bonds are needed in order to reduce the minor differences in the level spacing statistics. Further studies are thus required. The dependence of the level dynamics on $\Delta\varphi$ offers a promising research direction, due to the interesting feature that all three classical ensembles are present, namely the GSE for $\Delta\varphi = \pi$, the GOE for $\Delta\varphi = 0$, and the GUE in between. However, the most promising future aspect is undoubtedly that the whole spin 1/2 physics [25] is now accessible to microwave analogue studies.

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