POSITIVE SEMICLASSICAL STATES FOR A
FRACTIONAL SCHRÖDINGER-POISSON SYSTEM

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ABSTRACT. We consider a fractional Schrödinger-Poisson system in the whole space $\mathbb{R}^N$ in presence of a positive potential and depending on a small positive parameter ε. We show that, for suitably small ε (i.e. in the "semiclassical limit") the number of positive solutions is estimated below by the Ljusternick-Schnirelmann category of the set of minima of the potential.

1. INTRODUCTION

In the last decades a great attention has been given to the following Schrödinger-Poisson type system

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-2}u \\ -\Delta \phi = u^2, \end{cases}$$

which arises in non relativistic Quantum Mechanics. Such a system is obtained by looking for standing waves solutions in the purely electrostatic case to the Schrödinger-Maxwell system. For a deduction of this system, see e.g. [12]. Here the unknowns are u, the modulus of the wave function, and ϕ which represents the electrostatic potential. V is a given external potential and $p \geq 2$ a suitable given number.

The system has been studied by many authors, both in bounded and unbounded domains, with different assumptions on the data involved: boundary conditions, potentials, nonlinearities; many different type of solutions have been encountered (minimal energy, sign changing, radial, nonradial...), the behaviour of the solutions (e.g. concentration phenomena) has been studied as well as multiplicity results have been obtained. It is really difficult to give a complete list of references: the reader may see [13] and the references therein.

However it seems that results relating the number of positive solutions with topological invariants of the "objects" appearing in the problem are few in the literature. We cite the paper [34] where the system is studied in a (smooth and) bounded domain $\Omega \subset \mathbb{R}^3$ with $u = \phi = 0$ on $\partial\Omega$ and V constant. It is shown, by using variational methods that, whenever p is sufficiently near the critical Sobolev exponent 6, the number of positive solutions is estimated below by the *Ljusternick-Schnirelamnn category* of the domain Ω.

On the other hand it is known that a particular interest has the *semiclassical limit* of the Schrödinger-Poisson system (that is when the Plank constant $\hbar$ appearing in the system, see e.g. [12], tends to zero) especially due to the fact that this limit describes the transition from Quantum to Classical Mechanics. Such a situation is studied e.g. in [33], among many other papers. We cite also Fang and Zhao [23] which consider the following doubly perturbed system

in the whole space $\mathbb{R}^3$:

$$\begin{cases} -\varepsilon^2 \Delta w + V(x)w + \psi w = |w|^{p-2}w \\ -\varepsilon \Delta \psi = w^2. \end{cases}$$

Here V is a suitable potential, $4 < p < 6$, and ε is a positive parameter proportional to $\hbar$. In this case the authors estimate, whenever ε tends to zero, the number of positive solutions by the Ljusternick-Schnirelamnn category of the set of minima of the potential V, obtaining a result in the same spirit of [34].

Recently, especially after the formulation of the Fractional Quantum Mechanics, the derivation of the Fractional Schrödinger equation given by N. Laskin in [27–29], and the notion of fractional harmonic extension of a function studied in the pioneering paper [15], equations involving fractional operators are receiving a great attention. Indeed pseudodifferential operators appear in many problems in Physics and Chemistry, see e.g. [30, 31]; but also in obstacle problems [32, 35], optimization and finance [20], conformal geometry and minimal surfaces [14, 16, 17], etc.

Motivated by the previous discussion, we investigate in this paper the existence of positive solutions for the following doubly singularly perturbed fractional Schrödinger-Poisson system in $\mathbb{R}^N$:

$$(P_\varepsilon) \qquad \begin{cases} \varepsilon^{2s} (-\Delta)^s w + V(x)w + \psi w = f(w) \\ \varepsilon^\theta (-\Delta)^{\alpha/2} \psi = \gamma_\alpha w^2, \end{cases}$$

where $\gamma_\alpha := \frac{\pi^{N/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma(N/2-\alpha/2)}$ is a constant (Γ is the Euler function). By a positive solution of (P_ε) we mean a pair (w, ψ) where w is positive. To the best of our knowledge, there are only few recent papers dealing with a system like (P_ε): in [37] the author deals with $\varepsilon = 1$ proving under suitable assumptions on f the existence of infinitely many (but possibly sign changing) solutions by means of the Fountain Theorem. A similar system is studied in [36] and the existence of infinitely many (again, possibly sign changing) solutions is obtained by means of the Symmetric Mountain Pass Theorem.

In this paper we assume that

(H1) $s \in (0,1)$, $\alpha \in (0, N)$, $\theta \in (0, \alpha)$, $N \in (2s, 2s + \alpha)$,

moreover the potential V and the nonlinearity f satisfy the assumptions listed below:

(V1) $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function and

$$0 < \min_{\mathbb{R}^N} V := V_0 < V_\infty := \liminf_{|x| \to +\infty} V \in (V_0, +\infty];$$

(f1) $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 and $f(t) = 0$ for $t \le 0$;

(f2) $\lim_{t \to 0} f(t)/t = 0$;

(f3) there is $q_0 \in (2, 2_s^* - 1)$ such that $\lim_{t \to \infty} f(t)/t^{q_0} = 0$, where $2_s^* := 2N/(N - 2s)$;

(f4) there is $K > 4$ such that $0 < KF(t) := K \int_0^t f(\tau)d\tau \le tf(t)$ for all $t > 0$;

(f5) the function $t \mapsto f(t)/t^3$ is strictly increasing in $(0, +\infty)$.

The assumptions on the nonlinearity f are quite standard in order to work with variational methods, use the Nehari manifold and the Palais-Smale condition. The assumption (V1) will be fundamental in order to estimate the number of positive solutions and also to recover some compactness.

We recall, once for all, that a C^1 functional $\mathcal{J}$, defined on a smooth manifold $\mathfrak{M}$, is said to satisfy the Palais-Smale condition at level $c \in \mathbb{R}$ ($(PS)_c$ for brevity) if every sequence $\{u_n\} \subset \mathfrak{M}$ such that

$$(1.1) \qquad \mathcal{J}(u_n) \to c \quad \text{and} \quad \mathcal{J}'(u_n) \to 0$$

has a convergent subsequence. A sequence $\{u_n\}$ satisfying (1.1) is also named a $(PS)_c$ sequence.

To stay our result let us introduce

$$M := \left\{ x \in \mathbb{R}^N : V(x) = V_0 \right\}$$

the set of minima of V. Our result is the following

Theorem 1.1. *Under the above assumptions* (H1), (V1), (f1)-(f5), *there exists an $\varepsilon^* > 0$ such that for every $\varepsilon \in (0, \varepsilon^*]$ problem (P_ε) possesses at least $\operatorname{cat} M$ positive solutions.*

Moreover if $\operatorname{cat} M > 1$ and M is bounded, then (for suitably small ε) there exist at least $\operatorname{cat} M + 1$ positive solutions.

Hereafter, given a topological pair (X, Y), $\operatorname{cat}_X(Y)$ is the Ljusternick-Schnirelmann category of Y in X, and, if $X = Y$ this is just denoted with $\operatorname{cat} X$.

The proof of Theorem 1.1 is carried out by adapting some ideas of Benci, Cerami and Passaseo [10, 11] and using the Ljusternick-Schnirelmann Theory. We mention that these ideas and techniques have been extensively used to attack also other type of problems, and indeed similar results are obtained for other equations and operators, like the Schrödinger operator [18,19], the $p-$laplacian [3,4], the biharmonic operator [7], $p\&q-$laplacian, fractional laplacian [24,25], magnetic laplacian [5,6] or quasilinear operators [2,8,9].

The plan of the paper is the following. In Section 2 we recall some basic facts, we present some preliminaries and the variational setting for the problem. Section 3 is devoted to prove some compactness properties; as a byproduct we prove the existence of a ground state solution for our problem, that is a solution having minimal energy. In Section 4 we introduce the barycenter map, we show some of its properties and prove, by means of the Ljusternick-Schnirelamnn Theory, Theorem 1.1.

Notations. In the paper we will denote with $|\cdot|_p$ the usual L^p norm in $\mathbb{R}^N$; we denote with $B_r(x)$ the closed ball in $\mathbb{R}^N$ centered in x with radius $r > 0$, with $B_r^c(x)$ its complementary; if $x = 0$ we simply write B_r; moreover the letters $C, C_1, C_2, \dots$ will denote generic positive constants (whose value may change from line to line). Other notations will be introduced whenever we need.

2. Preliminaries

2.1. **Some well known facts.** Before to introduce the variational setting of our problem, we recall some basic facts concerning the fractional Sobolev spaces and their embeddings.

Given $\beta \in (0, 1)$, the fractional Laplacian $(-\Delta)^\beta$ is the pseudodifferential operator which can be defined via the Fourier transform

$$\mathcal{F}((-\Delta)^\beta u) = |\cdot|^{2\beta} \mathcal{F} u,$$

or, if u has sufficient regularity, by

$$(-\Delta)^\beta u(z) = -\frac{C_{N,\beta}}{2} \int_{\mathbb{R}^N} \frac{u(z+y) - u(z-y) - 2u(z)}{|y|^{N+2\beta}} dy, \quad z \in \mathbb{R}^N,$$

where $C_{N,\beta}$ is a suitable normalization constant.

For $s \in (0,1)$ let

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : (-\Delta)^{s/2} u \in L^2(\mathbb{R}^N) \right\}$$

be the Hilbert space with scalar product and (squared) norm given by

$$(u,v) = \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v + \int_{\mathbb{R}^N} uv, \qquad \|u\|^2 = |(-\Delta)^{s/2} u|_2^2 + |u|_2^2.$$

It is known that $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), p \in [2, 2_s^*]$ with $2_s^* := 2N/(N - 2s)$. Moreover the embedding of $H^s(\Omega)$ is compact if $\Omega \subset \mathbb{R}^N$ is bounded and $p \neq 2_s^*$.

We will consider also the homogeneous Sobolev spaces $\dot{H}^{\alpha/2}(\mathbb{R}^N)$ defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $|(-\Delta)^{\alpha/4} u|_2$. This is a Hilbert space with scalar product and (squared) norm

$$(u,v)_{\dot{H}^{\alpha/2}} = \int_{\mathbb{R}^N} (-\Delta)^{\alpha/4} u (-\Delta)^{\alpha/4} v, \qquad \|u\|_{\dot{H}^{\alpha/2}}^2 = |(-\Delta)^{\alpha/4} u|_2^2.$$

It is well known that $\dot{H}^{\alpha/2}(\mathbb{R}^N) \hookrightarrow L^{2_\alpha^*/2}(\mathbb{R}^N), 2_\alpha^* = 2N/(N - \alpha)$. For more general facts about the fractional Laplacian we refer the reader to the beautiful paper [22].

We recall here another fact that will be frequently used:

$$(2.1) \qquad \forall \varepsilon > 0 \; \exists M_\xi > 0 \; : \quad \int_{\mathbb{R}^N} f(u)u \leq \xi \int_{\mathbb{R}^N} u^2 + M_\xi \int_{\mathbb{R}^N} |u|^{q_0+1}, \quad \forall u \in H^s(\mathbb{R}^N).$$

This simply follows by (f2) and (f3).

2.2. **The variational setting.** It is easily seen that, just performing the change of variables $w(x) := u(x/\varepsilon), \psi(x) := \phi(x/\varepsilon)$, problem (P_ε) can be rewritten as

$$(P_\varepsilon^*) \qquad \begin{cases} (-\Delta)^s u + V(\varepsilon x)u + \phi(x)u = f(u) \\ (-\Delta)^{\alpha/2}\phi = \varepsilon^{\alpha-\theta}\gamma_\alpha u^2, \end{cases}$$

to which we will refer from now on.

A usual "reduction" argument can be used to deal with a single equation involving just u. Indeed for every $u \in H^s(\mathbb{R}^N)$ the second equation in (P_ε^*) is uniquely solved. Actually, for future reference, we will prove a slightly more general fact.

Let us fix two functions $u, w \in H^s(\mathbb{R}^N)$ and consider the problem

$$(Q_\varepsilon) \qquad \begin{cases} (-\Delta)^{\alpha/2}\phi = \varepsilon^{\alpha-\theta}\gamma_\alpha uw, \\ \phi \in \dot{H}^{\alpha/2}(\mathbb{R}^N) \end{cases}$$

whose weak solution is a function $\tilde{\phi} \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$ such that

$$\forall v \in \dot{H}^{\alpha/2}(\mathbb{R}^N) : \quad \int_{\mathbb{R}^N} (-\Delta)^{\alpha/4}\tilde{\phi}(-\Delta)^{\alpha/4} v = \varepsilon^{\alpha-\theta}\gamma_\alpha \int_{\mathbb{R}^N} uwv.$$

For every $v \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$, by the Hölder inequality and the continuous embeddings, we have

$$\left| \int_{\mathbb{R}^N} uwv \right| \leq |u|_{\frac{4N}{N+\alpha}} |w|_{\frac{4N}{N+\alpha}} |v|_{2_{\alpha/2}^*} \leq C\|u\|\|w\|\|v\|_{\dot{H}^{\alpha/2}}$$

deducing that the map

$$T_{u,w} : v \in \dot{H}^{\alpha/2}(\mathbb{R}^N) \longmapsto \int_{\mathbb{R}^N} uwv \in \mathbb{R}$$

is linear and continuous: then there exists a unique solution $\phi_{\varepsilon,u,w} \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$ to (Q_ε). Moreover this solution has the representation by means of the Riesz kernel $\mathcal{K}_\alpha(x) = \gamma_\alpha^{-1}|x|^{\alpha-N}$, hence

$$\phi_{\varepsilon,u,w} = \varepsilon^{\alpha-\theta} \frac{1}{|\cdot|^{N-\alpha}} \star (uw).$$

Furthermore

$$(2.2) \qquad \|\phi_{\varepsilon,u,w}\|_{\dot{H}^{\alpha/2}} = \varepsilon^{\alpha-\theta}\|T_{u,w}\|_{\mathcal{L}(\dot{H}^{\alpha/2};\mathbb{R})} \le \varepsilon^{\alpha-\theta}C\|u\|\|w\|$$

and then, for $\zeta, \eta \in H^s(\mathbb{R}^N)$

$$(2.3) \qquad \int_{\mathbb{R}^N} \phi_{\varepsilon,u,w}\zeta\eta \le |\phi_{\varepsilon,u,w}|_{2^*_{\alpha/2}}|\zeta|_{\frac{4N}{N+\alpha}}|\eta|_{\frac{4N}{N+\alpha}} \le \varepsilon^{\alpha-\theta}C_{\mathrm{e}}\|u\|\|w\|\|\zeta\|\|\eta\|$$

where C_{e} is a suitable embedding constant. Altough its value is not important, we will refer to this constant later on.

A particular case of the previous situation is when $u = w$. In this case we simplify the notation and write

- $T_u(v) := T_{u,u}(v) = \int_{\mathbb{R}^N} u^2 v$, and

- $\phi_{\varepsilon,u}$ for the unique solution of the second equation in (P_ε^*) for fixed $u \in H^s(\mathbb{R}^N)$. Then

$$\|\phi_{\varepsilon,u}\|_{\dot{H}^{\alpha/2}} \le \varepsilon^{\alpha-\theta}C\|u\|^2$$

and the map

$$u \in H^s(\mathbb{R}^N) \longmapsto \phi_{\varepsilon,u} \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$$

is bounded.

Observe also that

$$(2.4) \quad u_n^2 \to u^2 \text{ in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \Longrightarrow T_{u_n} \longrightarrow T_u \text{ as operators } \Longrightarrow \phi_{\varepsilon,u_n} \longrightarrow \phi_{\varepsilon,u} \text{ in } \dot{H}^{\alpha/2}(\mathbb{R}^N).$$

For convenience let us define the map (well defined by (2.3))

$$A : u \in H^s(\mathbb{R}^N) \longmapsto \int_{\mathbb{R}^N} \phi_{\varepsilon,u}u^2 \in \mathbb{R}.$$

Then

$$(2.5) \qquad |A(u)| \le \varepsilon^{\alpha-\theta}C_{\mathrm{e}}\|u\|^4$$

(where C_{e} is the same constant in (2.3)). Some relevant properties of $\phi_{\varepsilon,u}$ and A are listed below. Although these properties are known to be true, we are not able to find them explicitly in the literature; so we prefer to give a proof here.

Lemma 2.1. *The following propositions hold.*

(i) *For every $u \in H^s(\mathbb{R}^N) : \phi_{\varepsilon,u} \ge 0$;*

(ii) *for every $u \in H^s(\mathbb{R}^N), t \in \mathbb{R} : \phi_{\varepsilon,tu} = t^2\phi_{\varepsilon,u}$;*

(iii) *if $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ then $\phi_{\varepsilon,u_n} \rightharpoonup \phi_{\varepsilon,u}$ in $\dot{H}^{\alpha/2}(\mathbb{R}^N)$;*

(iv) *A is of class C^2 and for every $u, v, w \in H^s(\mathbb{R}^N)$*

$$A'(u)[v] = 4\int_{\mathbb{R}^N} \phi_{\varepsilon,u}uv, \quad A''(u)[v,w] = 4\int_{\mathbb{R}^N} \phi_{\varepsilon,u}vw + 8\int_{\mathbb{R}^N} \phi_{\varepsilon,u,w}uv,$$

(v) *if $u_n \to u$ in $L^r(\mathbb{R}^N)$, with $2 \le r < 2^*_s$, then $A(u_n) \to A(u)$;*

(vi) *if $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ then $A(u_n - u) = A(u_n) - A(u) + o_n(1)$.*

Proof. Items (i) and (ii) follow directly by the definition of $\phi_{\varepsilon,u}$.

To prove (iii), let $v \in C_c^\infty(\mathbb{R}^N)$; we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} (-\Delta)^{\alpha/4}(\phi_{\varepsilon,u_n} - \phi_{\varepsilon,u})(-\Delta)^{\alpha/4}v \;&=\; \int_{\mathbb{R}^N} (u_n^2 - u^2)v \\
&\leq\; |v|_\infty \Big(\int_{\operatorname{supp} v} (u_n - u)^2 \Big)^{1/2} \Big(\int_{\operatorname{supp} v} (u_n + u)^2 \Big)^{1/2} \\
&\to\; 0.
\end{aligned}
$$

The conclusion then follows by density.

The proof of (iv) is straightforward: we refer the reader to [23].

To show (v), recall that $2 < \frac{4N}{N+\alpha} < 2_s^*$. Since by assumption $|u_n^2|_{\frac{2N}{N+\alpha}} \to |u^2|_{\frac{2N}{N+\alpha}}$ and $u_n^2 \to u^2$ a.e. in $\mathbb{R}^N$, using the Brezis-Lieb Lemma, $u_n^2 \to u^2$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. But then using (2.4) we get $\phi_{\varepsilon,u_n} \to \phi_{\varepsilon,u}$ in $L^{2_{\alpha/2}^*}(\mathbb{R}^N)$. Consequently

$$
\begin{aligned}
|A(u_n) - A(u)| &\leq \int_{\mathbb{R}^N} |\phi_{\varepsilon,u_n} u_n^2 - \phi_{\varepsilon,u} u^2| \leq \int_{\mathbb{R}^N} |(\phi_{\varepsilon,u_n} - \phi_{\varepsilon,u}) u_n^2| + \int_{\mathbb{R}^N} |\phi_{\varepsilon,u}(u_n^2 - u^2)| \\
&\leq |\phi_{\varepsilon,u_n} - \phi_{\varepsilon,u}|_{2_{\alpha/2}^*} |u_n^2|_{\frac{2N}{N+\alpha}} + |\phi_{\varepsilon,u}|_{2_{\alpha/2}^*} |u_n^2 - u^2|_{\frac{2N}{N+\alpha}}
\end{aligned}
$$

from which we conclude.

To prove (vi), for the sake of simplicity we drop the factor $\varepsilon^{\alpha-\theta}$ in the expression of $\phi_{\varepsilon,u,v}$. Defining

$$
\sigma := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(y)u^2(x)}{|x-y|^{N-\alpha}} dy\, dx \,,
$$

$$
\sigma_n^1 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u_n^2(y)u^2(x)}{|x-y|^{N-\alpha}} dy\, dx\,, \qquad \sigma_n^2 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u_n(y)u(y)u_n(x)u(x)}{|x-y|^{N-\alpha}} dy\, dx
$$

$$
\sigma_n^3 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u_n^2(y)u_n(x)u(x)}{|x-y|^{N-\alpha}} dy\, dx\,, \qquad \sigma_n^4 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u_n(y)u(y)u^2(x)}{|x-y|^{N-\alpha}} dy\, dx,
$$

it is easy to check that

$$
A(u_n - u) - A(u_n) + A(u) = 2\sigma + 2\sigma_n^1 + 4\sigma_n^2 - 4\sigma_n^3 - 4\sigma_n^4.
$$

Now we claim that, whenever $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$,

$$
\lim_{n\to\infty} \sigma_n^i = \sigma, \quad i = 1,2,3,4
$$

which readily gives the conclusion.

We prove here only the cases $i = 1, 2$ since the proof of the other cases is very similar. Recall that

$$
\phi_{\varepsilon,u}(x) = \int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^{N-\alpha}} dy\,, \qquad \phi_{\varepsilon,u_n}(x) = \int_{\mathbb{R}^N} \frac{u_n^2(y)}{|x-y|^{N-\alpha}} dy.
$$

Since $u^2 \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) = L^{(2_{\alpha/2}^*)'}(\mathbb{R}^N)$ and by item (iii) it holds $\phi_{\varepsilon,u_n} \to \phi_{\varepsilon,u}$ in $L^{2_{\alpha/2}^*}(\mathbb{R}^N)$, we conclude that

$$
\sigma_n^1 = \int_{\mathbb{R}^N} \phi_{\varepsilon,u_n} u^2 \to \int_{\mathbb{R}^N} \phi_{\varepsilon,u} u^2 = \sigma
$$

and the claim is true for $i = 1$.

For $i = 2$ recall that

$$
\phi_{\varepsilon,u_n,u}(x) = \int_{\mathbb{R}^N} \frac{u_n(y)u(y)}{|x-y|^{N-\alpha}} dy.
$$

First we show that $\phi_{\varepsilon,u_n,u} \to \phi_{\varepsilon,u}$ a.e. in $\mathbb{R}^N$. Given $\xi > 0$ and choosing $R > 1/\xi$, $\frac{N}{2s} < p < \frac{N}{N-\alpha}$ and $\frac{N}{N-\alpha} < q$ (so that $2p', 2q' \in (2, 2_s^*)$), we have, for large n:

$$
\begin{aligned}
|\phi_{\varepsilon,u_n,u}(x) - \phi_{\varepsilon,u}(x)| &\leq |u_n - u|_{L^{2p'}(B_R(x))}|u|_{L^{2p'}(B_R(x))}\left(\int_{|y-x|<R}\frac{dy}{|x-y|^{p(N-\alpha)}}\right)^{1/p} \\
&+ |u_n - u|_{L^{2q'}(B_R^c(x))}|u|_{L^{2q'}(B_R^c(x))}\left(\int_{|y-x|\geq R}\frac{dy}{|x-y|^{q(N-\alpha)}}\right)^{1/q} \\
&\leq C_1\xi + C_2\xi^{N-\alpha},
\end{aligned}
$$

concluding the pointwise convergence. Moreover by the Sobolev embedding and using (2.2),

$$|\phi_{\varepsilon,u_n,u}u_n|_2 \leq |\phi_{\varepsilon,u_n,u}|_{2_{\alpha/2}^*}|u_n|_{2N/\alpha} \leq C_1\|u_n\|^2\|u\| \leq C_2$$

and therefore, up to subsequence, $\phi_{\varepsilon,u_n,u}u_n \rightharpoonup \phi_{\varepsilon,u}u$ in $L^2(\mathbb{R}^N)$, by [26, Lemma 4.8]. Since $u \in L^2(\mathbb{R}^N)$

$$\sigma_n^2 = \int_{\mathbb{R}^N}\phi_{\varepsilon,u_n,u}u_nu \to \int_{\mathbb{R}^N}\phi_{\varepsilon,u}u^2 = \sigma$$

and the claim is proved for $i = 2$. $\qquad\square$

We introduce now the variational setting for our problem. Let us define the Hilbert space

$$W_\varepsilon := \left\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N}V(\varepsilon x)u^2 < \infty\right\}$$

endowed with scalar product and (squared) norm given by

$$(u,v)_\varepsilon := \int_{\mathbb{R}^N}(-\Delta)^{s/2}u(-\Delta)^{s/2}v + \int_{\mathbb{R}^N}V(\varepsilon x)uv$$

and

$$\|u\|_\varepsilon^2 := \int_{\mathbb{R}^N}|(-\Delta)^{s/2}u|^2 + \int_{\mathbb{R}^N}V(\varepsilon x)u^2.$$

Then it is standard to see that the critical points of the C^2 functional (see Lemma 2.1 (iv))

$$I_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u|^2 + \frac{1}{2}\int_{\mathbb{R}^N}V(\varepsilon x)u^2 + \frac{1}{4}\int_{\mathbb{R}^N}\phi_{\varepsilon,u}u^2 - \int_{\mathbb{R}^N}F(u),$$

on W_ε are weak solutions of problem (P_ε^*).

By defining

$$\mathcal{N}_\varepsilon := \left\{u \in W_\varepsilon \setminus \{0\} : J_\varepsilon(u) = 0\right\},$$

where

$$J_\varepsilon(u) := I_\varepsilon'(u)[u] = \|u\|_\varepsilon^2 + \int_{\mathbb{R}^N}\phi_{\varepsilon,u}u^2 - \int_{\mathbb{R}^N}f(u)u,$$

we have, by standard arguments:

Lemma 2.2. *For every $u \in \mathcal{N}_\varepsilon$, $J_\varepsilon'(u)[u] < 0$ and there are positive constants $h_\varepsilon, k_\varepsilon$ such that $\|u\|_\varepsilon \geq h_\varepsilon$, $I_\varepsilon(u) \geq k_\varepsilon$. Furthermore, $\mathcal{N}_\varepsilon$ is diffeomorphic to the set*

$$\mathcal{S}_\varepsilon := \{u \in W_\varepsilon : \|u\|_\varepsilon = 1, \ u > 0 \ a.e.\}.$$

$\mathcal{N}_\varepsilon$ is the *Nehari manifold* associated to I_ε. By the assumptions on f, the functional I_ε has the Mountain Pass geometry. This is standard but we give the easy proof for completeness.

(MP1) $I_\varepsilon(0) = 0$;

(MP2) since, for every $\xi > 0$ there exists $M_\xi > 0$ such that $F(u) \leq \xi u^2 + M_\xi |u|^{q_0+1}$, we have

$$
\begin{aligned}
I_\varepsilon(u) &\geq \frac{1}{2}\|u\|_\varepsilon^2 - \int_{\mathbb{R}^N} F(u) \\
&\geq \frac{1}{2}\|u\|_\varepsilon^2 - \xi C_1\|u\|_\varepsilon^2 - M_\xi C_2 \|u\|_\varepsilon^{q_0+1}
\end{aligned}
$$

and we conclude I_ε has a strict local minimum at $u = 0$;

(MP3) finally, since (f4) implies $F(t) \geq Ct^K$ for $t > 0$, with $K > 4$ (and less then $q_0 + 1$), fixed $v \in C_c^\infty(\mathbb{R}^N), v > 0$ we have

$$
\begin{aligned}
I_\varepsilon(tv) &= \frac{t^2}{2}\|v\|_\varepsilon^2 + \frac{t^4}{4}\int_{\mathbb{R}^N} \phi_{\varepsilon,v} v^2 - \int_{\mathbb{R}^N} F(tv) \\
&\leq \frac{t^2}{2}\|v\|_\varepsilon^2 + \frac{t^4}{4}\int_{\mathbb{R}^N} \phi_{\varepsilon,v} v^2 - Ct^K \int_{\mathbb{R}^N} v^K
\end{aligned}
$$

concluding that the functional is negative for suitable large t.

Then denoting with

$$
c_\varepsilon := \inf_{\gamma \in \mathcal{H}_\varepsilon} \sup_{t \in [0,1]} I_\varepsilon(\gamma(t)), \qquad \mathcal{H}_\varepsilon = \left\{ \gamma \in C([0,1], W_\varepsilon) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0 \right\}
$$

the Mountain Pass level, and with

$$
m_\varepsilon := \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u)
$$

the *ground state level*, it holds, in a standard way, that

(2.6)
$$
c_\varepsilon = m_\varepsilon = \inf_{u \in W_\varepsilon \setminus \{0\}} \sup_{t \geq 0} I_\varepsilon(tu).
$$

It is known that for "perturbed" problems a major role is played by the problem at infinity that we now introduce.

2.3. **The problem at "infinity".** Let us consider the "limit" problem (the autonomous problem) associated to (P_ε^*), that is

(A_μ)
$$
\begin{cases}
(-\Delta)^s u + \mu u = f(u) \\
u \in H^s(\mathbb{R}^N)
\end{cases}
$$

where $\mu > 0$ is a constant. The solutions are critical points of the functional

$$
E_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + \frac{\mu}{2}\int_{\mathbb{R}^N} u^2 - \int_{\mathbb{R}^N} F(u).
$$

in $H^s(\mathbb{R}^N)$. Denoting with $H_\mu^s(\mathbb{R}^N)$ simply the space $H^s(\mathbb{R}^N)$ endowed with the (equivalent squared) norm

$$
\|u\|_{H_\mu^s}^2 := |(-\Delta)^{s/2} u|_2^2 + \mu |u|_2^2,
$$

by the assumptions of the nonlinearity f, it is easy to see that the functional E_μ has the Mountain Pass geometry with Mountain Pass level

$$
c_\mu^\infty := \inf_{\gamma \in \mathcal{H}_\mu} \sup_{t \in [0,1]} E_\mu(\gamma(t)), \qquad \mathcal{H}_\mu := \left\{ \gamma \in C([0,1], H_\mu^s(\mathbb{R}^N)) : \gamma(0) = 0, E_\mu(\gamma(1)) < 0 \right\}.
$$

Introducing the set

$$
\mathcal{M}_\mu := \left\{ u \in H^s(\mathbb{R}^N) \setminus \{0\} : \|u\|_{H_\mu^s}^2 = \int_{\mathbb{R}^N} f(u)u \right\}
$$

it is standard to see that

- $\mathcal{M}_\mu$ has a structure of differentiable manifold (said the *Nehari manifold* associated to E_μ),
- $\mathcal{M}_\mu$ is bounded away from zero and radially homeomorfic to the unit sphere,
- the mountain pass value c_μ^∞ coincide with the *ground state level*

$$m_\mu^\infty := \inf_{u \in \mathcal{M}_\mu} E_\mu(u) > 0.$$

The symbol "∞" in the notations is just to recall we are dealing with the limit problem. In the sequel we will mainly deal with $\mu = V_0$ and $\mu = V_\infty$ (whenever this last one is finite). Of course the inequality

$$m_\varepsilon \geq m_{V_0}^\infty$$

holds.

3. COMPACTNESS PROPERTIES FOR I_ε, E_μ: EXISTENCE OF A GROUND STATE SOLUTION

We begin by showing the boundedness of the Palais-Smale sequences for E_μ in $H_\mu^s(\mathbb{R}^N)$ and I_ε in W_ε. Let $\{u_n\} \subset H_\mu^s(\mathbb{R}^N)$ be a Palais-Smale sequence for E_μ, that is, $|E_\mu(u_n)| \leq C$ and $E_\mu'(u_n) \to 0$. Then, for large n,

$$C + \|u_n\|_{H_\mu^s} > E_\mu(u_n) - \frac{1}{K} E_\mu'(u_n)[u_n] = \left(\frac{1}{2} - \frac{1}{K}\right) \|u_n\|_{H_\mu^s}^2 + \frac{1}{K} \int_{\mathbb{R}^N} (f(u_n)u_n - KF(u_n))$$

$$\geq \left(\frac{1}{2} - \frac{1}{K}\right) \|u_n\|_{H_\mu^s}^2,$$

and thus $\{u_n\}$ is bounded. Similarly we conclude for I_ε, using that

$$I_\varepsilon(u_n) - \frac{1}{K} I_\varepsilon'(u_n)[u_n] = \left(\frac{1}{2} - \frac{1}{K}\right) \|u_n\|_\varepsilon^2 + \left(\frac{1}{4} - \frac{1}{K}\right) \int_{\mathbb{R}^N} \phi_{\varepsilon, u_n} u_n^2 + \frac{1}{K} \int_{\mathbb{R}^N} (f(u_n)u_n - KF(u_n))$$

$$\geq \left(\frac{1}{2} - \frac{1}{K}\right) \|u_n\|_\varepsilon^2.$$

In order to prove compactness, some preliminary work is needed. Let us recall the following Lions type lemma, whose proof can be found in [21, Lemma 2.3].

Lemma 3.1. *If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and for some $R > 0$ and $2 \leq r < 2_s^*$ we have*

$$\sup_{x \in \mathbb{R}^N} \int_{B_R(x)} |u_n|^r \to 0 \quad as \quad n \to \infty,$$

then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2_s^*$.

Then we can prove the following

Lemma 3.2. *Let $\{u_n\} \subset W_\varepsilon$ be bounded and such that $I_\varepsilon'(u_n) \to 0$. Then we have either*

a) *$u_n \to 0$ in W_ε, or*

b) *there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R, c > 0$ such that*

$$\liminf_{n \to +\infty} \int_{B_R(y_n)} u_n^2 \geq c > 0.$$

Proof. Suppose that b) does not occur. Using Lemma 3.1 it follows

$$u_n \to 0 \ \text{ in } \ L^p(\mathbb{R}^N) \ \text{ for } \ p \in (2, 2_s^*).$$

Using (2.1), the boundedness of $\{u_n\}$ in $L^2(\mathbb{R}^N)$ and the fact that $u_n \to 0$ in $L^{q_0+1}(\mathbb{R}^N)$, we conclude that

$$\int_{\mathbb{R}^N} f(u_n) u_n \to 0.$$

Finally, since

$$\|u_n\|_\varepsilon^2 - \int_{\mathbb{R}^N} f(u_n) u_n \le \|u_n\|_\varepsilon^2 + \int_{\mathbb{R}^N} \phi_{\varepsilon, u_n} u_n^2 - \int_{\mathbb{R}^N} f(u_n) u_n = I_\varepsilon'(u_n)[u_n] = o_n(1),$$

it follows that $u_n \to 0$ in W_ε. $\qquad\square$

In the rest of the paper we assume, without loss of generality, that $0 \in M$, that is, $V(0) = V_0$.

Lemma 3.3. *Assume that $V_\infty < \infty$ and let $\{v_n\} \subset W_\varepsilon$ be a $(PS)_d$ sequence for I_ε such that $v_n \rightharpoonup 0$ in W_ε. Then*

$$v_n \not\to 0 \ \text{ in } \ W_\varepsilon \implies d \ge m_{V_\infty}^\infty.$$

Proof. Observe, preliminarily, that by condition (V1) it follows that

$$(3.1) \qquad \forall \xi > 0 \ \exists \widetilde{R} = \widetilde{R}_\xi > 0 : \quad V(\varepsilon x) > V_\infty - \xi, \qquad \forall x \notin B_{\widetilde{R}}.$$

Let $\{t_n\} \subset (0, +\infty)$ be such that $\{t_n v_n\} \subset \mathcal{M}_{V_\infty}$. We start by showing the following

Claim: *The sequence $\{t_n\}$ satisfies $\limsup_{n\to\infty} t_n \le 1$.*

Supposing by contradiction that the claim does not hold, there exists $\delta > 0$ and a subsequence still denoted by $\{t_n\}$, such that

$$(3.2) \qquad t_n \ge 1 + \delta \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $\{v_n\}$ is a bounded $(PS)_d$ sequence for I_ε, $I_\varepsilon'(v_n)[v_n] = o_n(1)$, that is,

$$\|v_n\|_\varepsilon^2 + \int_{\mathbb{R}^n} \phi_{\varepsilon, v_n} v_n^2 = \int_{\mathbb{R}^n} f(v_n) v_n + o_n(1).$$

Moreover, since $\{t_n v_n\} \subset \mathcal{M}_{V_\infty}$, we get

$$\|t_n v_n\|_{H_{V_\infty}^s}^2 = \int_{\mathbb{R}^N} f(t_n v_n) t_n v_n.$$

These equalities imply that

$$\int_{\mathbb{R}^N} \left(\frac{f(t_n v_n)}{t_n} - f(v_n) \right) v_n = \int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x)) v_n^2 - \int_{\mathbb{R}^N} \phi_{\varepsilon, v_n} v_n^2 + o_n(1),$$

and thus

$$(3.3) \qquad \int_{\mathbb{R}^N} \left(\frac{f(t_n v_n)}{t_n} - f(v_n) \right) v_n \le \int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x)) v_n^2 + o_n(1).$$

Using (3.1), the fact that $v_n \to 0$ in $L^2(B_{\widetilde{R}})$ and that $\{v_n\}$ is bounded in W_ε, let us say by some constant $C > 0$, we deduce by (3.3)

$$(3.4) \qquad \forall \xi > 0 : \int_{\mathbb{R}^N} \left(\frac{f(t_n v_n)}{t_n} - f(v_n) \right) v_n \le \xi C + o_n(1).$$

Since $v_n \not\to 0$ in W_ε, we may invoke Lemma 3.2 to obtain $\{y_n\} \subset \mathbb{R}^N$ and $R, c > 0$ such that

$$(3.5) \qquad \int_{B_R(y_n)} v_n^2 \ge c.$$

Defining $\check{v}_n := v_n(\cdot + y_n)$, we may suppose that, up to a subsequence,

$$\check{v}_n \rightharpoonup \check{v} \quad \text{in} \quad H^s(\mathbb{R}^N)$$

and, in view of (3.5), there exists a subset $\Omega \subset \mathbb{R}^N$ with positive measure such that $\check{v} > 0$ in Ω. By (f5) and (3.2), (3.4) becomes

$$0 < \int_\Omega \left(\frac{f((1+\delta)\check{v}_n)}{(1+\delta)\check{v}_n} - \frac{f(\check{v}_n)}{\check{v}_n} \right) \check{v}_n^2 \leq \xi C + o_n(1).$$

Now passing to the limit and applying Fatou's Lemma, it follows that, for every $\xi > 0$

$$0 < \int_\Omega \left[\frac{f((1+\delta)\check{v})}{(1+\delta)\check{v}} - \frac{f(\check{v})}{\check{v}} \right] \check{v}^2 \leq \xi C,$$

which is absurd and proves the claim.

Now we distinguish two cases.

Case 1: $\limsup_{n\to\infty} t_n = 1$.

Up to subsequence we can assume that $t_n \to 1$. We have,

(3.6) $$d + o_n(1) = I_\varepsilon(v_n) \geq m_{V_\infty}^\infty + I_\varepsilon(v_n) - E_{V_\infty}(t_n v_n).$$

Moreover,

$$I_\varepsilon(v_n) - E_{V_\infty}(t_n v_n) = \frac{(1-t_n^2)}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v_n|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (V(\varepsilon x) - t_n^2 V_\infty) v_n^2$$

$$+ \frac{1}{4} \int_{\mathbb{R}^n} \phi_{\varepsilon,v_n} v_n^2 + \int_{\mathbb{R}^N} (F(t_n v_n) - F(v_n)),$$

and due to the boundedness of $\{v_n\}$ we get, for every $\xi > 0$,

$$I_\varepsilon(v_n) - E_{V_\infty}(t_n v_n) \geq o_n(1) - C\xi + \int_{\mathbb{R}^N} (F(t_n v_n) - F(v_n)),$$

where we have used again (3.1). By the Mean Value Theorem, $\int_{\mathbb{R}^N} (F(t_n v_n) - F(v_n)) = o_n(1)$, therefore (3.6) becomes

$$d + o_n(1) \geq m_{V_\infty}^\infty - C\xi + o_n(1),$$

and taking the limit in n, by the arbitrariness of ξ, we deduce $d \geq m_{V_\infty}^\infty$.

Case 2: $\limsup_{n\to\infty} t_n = t_0 < 1$.

We can assume $t_n \to t_0$ and $t_n < 1$. Since $t \mapsto \frac{1}{4} f(t) t - F(t)$ is increasing in $(0, \infty)$,

$$m_{V_\infty}^\infty \leq E_{V_\infty}(t_n v_n) = \int_{\mathbb{R}^N} \left(\frac{1}{2} f(t_n v_n) t_n v_n - F(t_n v_n) \right)$$

$$= \int_{\mathbb{R}^N} \frac{1}{4} f(t_n v_n) t_n v_n + \int_{\mathbb{R}^N} \left(\frac{1}{4} f(t_n v_n) t_n v_n - F(t_n v_n) \right)$$

$$= \frac{1}{4} \|t_n v_n\|_{H_{V_\infty}^s}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{4} f(t_n v_n) t_n v_n - F(t_n v_n) \right)$$

(3.7) $$\leq \frac{1}{4} \|t_n v_n\|_{H_{V_\infty}^s}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{4} f(v_n) v_n - F(v_n) \right).$$

But

(3.8) $$\|t_n v_n\|_{V_\infty}^2 \leq \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v_n|^2 + \int_{\mathbb{R}^N} t_n^2 V_\infty v_n^2.$$

Again by (3.1), given $\xi > 0$,

$$t_n^2 V_\infty - \xi < V_\infty - \xi < V(\varepsilon x) \quad \text{for } x \notin B_{\widetilde{R}}$$

and hence

$$\int_{\mathbb{R}^N} t_n^2 V_\infty v_n^2 \leq \int_{B_{\widetilde{R}}} V_\infty v_n^2 + \int_{|x| \geq \widetilde{R}} V(\varepsilon x) v_n^2 + \int_{|x| \geq \widetilde{R}} \xi v_n^2$$

$$\leq o_n(1) + \int_{\mathbb{R}^N} V(\varepsilon x) v_n^2 + C\xi.$$

From this and (3.8) we have

$$\|t_n v_n\|^2_{H^s_{V_\infty}} \leq \|v_n\|^2_\varepsilon + C\xi + o_n(1).$$

Therefore, using (3.7)

$$m_{V_\infty}^\infty \leq \frac{1}{4}\|v_n\|^2_\varepsilon + \int_{\mathbb{R}^N} \left(\frac{1}{4}f(v_n)v_n - F(v_n) \right) + C\xi + o_n(1)$$

$$= I_\varepsilon(v_n) - \frac{1}{4}I'_\varepsilon(v_n)[v_n] + C\xi + o_n(1)$$

$$= d + C\xi + o_n(1).$$

concluding the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 3.4. *The functional I_ε in W_ε satisfies the $(PS)_c$ condition*

    1. at any level $c < m_{V_\infty}^\infty$, if $V_\infty < \infty$,

    2. at any level $c \in \mathbb{R}$, if $V_\infty = \infty$.

Proof. Let $\{u_n\} \subset W_\varepsilon$ be such that $I_\varepsilon(u_n) \to c$ and $I'_\varepsilon(u_n) \to 0$. We have already seen that $\{u_n\}$ is bounded in W_ε. Thus there exists $u \in W_\varepsilon$ such that, up to a subsequence, $u_n \rightharpoonup u$ in W_ε. Note that $I'_\varepsilon(u) = 0$, since by Lemma 2.1 (iv), we have for every $w \in W_\varepsilon$

$$(u_n, w)_\varepsilon \to (u, w)_\varepsilon, \quad A'(u_n)[w] \to A'(u)[w] \quad \text{and} \quad \int_{\mathbb{R}^N} f(u_n)w \to \int_{\mathbb{R}^N} f(u)w.$$

Defining $v_n := u_n - u$, we have that $\int_{\mathbb{R}^N} F(v_n) = \int_{\mathbb{R}^N} F(u_n) - \int_{\mathbb{R}^N} F(u) + o_n(1)$ (see [1]) and by Lemma 2.1 (vi), we have $A(v_n) = A(u_n) - A(u) + o_n(1)$; hence arguing as in [4], we obtain also

(3.9) $$I'_\varepsilon(v_n) \to 0.$$

Moreover

(3.10) $$I_\varepsilon(v_n) = I_\varepsilon(u_n) - I_\varepsilon(u) + o_n(1) = c - I_\varepsilon(u) + o_n(1) =: d + o_n(1)$$

and (3.9) and (3.10) show that $\{v_n\}$ is a $(PS)_d$ sequence. By (f4),

$$I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{4}I'_\varepsilon(u)[u] = \frac{1}{4}\|u\|^2_\varepsilon + \int_{\mathbb{R}^N} \left(\frac{1}{4}f(u)u - F(u) \right)$$

$$\geq \frac{1}{4}\int_{\mathbb{R}^N} \left(f(u)u - 4F(u) \right) \geq 0$$

and then coming back in (3.10) we have

(3.11) $$d \leq c.$$

Then,

1. if $V_\infty < \infty$, and $c < m_{V_\infty}^\infty$, by (3.11) we obtain

$$d \le c < m_{V_\infty}^\infty.$$

It follows from Lemma 3.3 that $v_n \to 0$, that is $u_n \to u$ in W_ε.

2. If $V_\infty = \infty$, by the compact imbedding $W_\varepsilon \hookrightarrow\hookrightarrow L^r(\mathbb{R}^N), 2 \le r < 2_s^*$, up to a subsequence, $v_n \to 0$ in $L^r(\mathbb{R}^N)$ and since $I_\varepsilon'(v_n) \to 0$, we have

$$(3.12) \qquad I_\varepsilon'(v_n)[v_n] = \|v_n\|_\varepsilon^2 + \int_{\mathbb{R}^N} \phi_{\varepsilon,v_n} v_n^2 - \int_{\mathbb{R}^N} f(v_n)v_n = o_n(1).$$

By Lemma 2.1 (v), $A(v_n) = \int_{\mathbb{R}^N} \phi_{\varepsilon,v_n} v_n^2 = o_n(1)$, and since by (2.1) it holds again $\int_{\mathbb{R}^N} f(v_n)v_n = o_n(1)$, we have by (3.12) $\|v_n\|_\varepsilon^2 = o_n(1)$, that is $u_n \to u$ in W_ε.

The proof is thereby complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As a consequence it is standard to prove that

Proposition 3.5. *The functional I_ε restricted to $\mathcal{N}_\varepsilon$ satisfies the $(PS)_c$ condition*

  1. at any level $c < m_{V_\infty}^\infty$, if $V_\infty < \infty$,

  2. at any level $c \in \mathbb{R}$, if $V_\infty = \infty$.

Moreover, the constrained critical points of the functional I_ε on $\mathcal{N}_\varepsilon$ are critical points of I_ε in W_ε, hence solution of (P_ε^*).

Let us recall the following result (see [24, Lemma 6]) concerning problem (A_μ).

Lemma 3.6 (Ground state for the autonomous problem)**.** *Let $\{u_n\} \subset \mathcal{M}_\mu$ be a sequence satisfying $E_\mu(u_n) \to m_\mu^\infty$. Then, up to subsequences the following alternative holds:*

  a) $\{u_n\}$ strongly converges in $H^s(\mathbb{R}^N)$;

  b) there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + \tilde{y}_n)$ strongly converges in $H^s(\mathbb{R}^N)$.

In particular, there exists a minimizer $\mathfrak{w}_\mu \ge 0$ for m_μ^∞.

Now we can prove the existence of a ground state for our problem. Assumption (H1) is tacitly assumed.

Theorem 3.7. *Suppose that f verifies (f1)-(f5) and V verifies (V1). Then there exists a ground state solution $\mathfrak{u}_\varepsilon \in W_\varepsilon$ of (P_ε^*),*

  1. for every $\varepsilon \in (0, \bar{\varepsilon}]$, for some $\bar{\varepsilon} > 0$, if $V_\infty < \infty$;

  2. for every $\varepsilon > 0$, if $V_\infty = \infty$.

Proof. Since the functional I_ε has the geometry of the Mountain Pass Theorem in W_ε there exists $\{u_n\} \subset W_\varepsilon$ satisfying

$$I_\varepsilon(u_n) \to c_\varepsilon \text{ and } I_\varepsilon'(u_n) \to 0.$$

1. If $V_\infty < \infty$, in virtue of Proposition 3.4, we have only to show that $c_\varepsilon < m_{V_\infty}^\infty$ for every positive ε smaller than a certain $\bar{\varepsilon}$.

Let $\mu \in (V_0, V_\infty)$, so that

$$(3.13) \qquad m_{V_0}^\infty < m_\mu^\infty < m_{V_\infty}^\infty.$$

For $r > 0$ let η_r be a smooth cut-off function in $\mathbb{R}^N$ which equals 1 on B_r and with support in B_{2r}. Let $w_r := \eta_r \mathfrak{w}_\mu$ and $s_r > 0$ such that $s_r w_r \in \mathcal{M}_\mu$. If it were, for every

$r > 0 : E_\mu(s_r w_r) \geq m_{V_\infty}^\infty$, since $w_r \to \mathfrak{w}_\mu$ in $H^s(\mathbb{R}^N)$ for $r \to +\infty$, we would have $s_r \to 1$ and then

$$m_{V_\infty}^\infty \leq \liminf_{r \to +\infty} E_\mu(s_r w_r) = E_\mu(\mathfrak{w}_\mu) = m_\mu^\infty$$

which contradicts (3.13). This means that there exists $\overline{r} > 0$ such that $\omega := s_{\overline{r}} w_{\overline{r}} \in \mathcal{M}_\mu$ satisfies

$$(3.14) \qquad\qquad\qquad\qquad E_\mu(\omega) < m_{V_\infty}^\infty.$$

Given $\varepsilon > 0$, let $t_\varepsilon > 0$ the number such that $t_\varepsilon \omega \in \mathcal{N}_\varepsilon$. Therefore

$$t_\varepsilon^2 \|\omega\|_\varepsilon^2 + t_\varepsilon^4 \int_{\mathbb{R}^N} \phi_{\varepsilon,\omega}\, \omega^2 = t_\varepsilon \int_{\mathbb{R}^N} f(t_\varepsilon \omega)\omega$$

implying that

$$(3.15) \qquad\qquad \frac{\|\omega\|_\varepsilon^2}{t_\varepsilon^2} + \int_{\mathbb{R}^N} \phi_{\varepsilon,\omega}\, \omega^2 \geq \int_{B_{\overline{r}}} \frac{f(t_\varepsilon \omega)}{(t_\varepsilon \omega)^3}\omega^4.$$

Now we claim that there exists $T > 0$ such that $\limsup_{\varepsilon \to 0^+} t_\varepsilon \leq T$. If by contradiction there exists $\varepsilon_n \to 0^+$ with $t_{\varepsilon_n} \to \infty$, then by (3.15) and (f5) we have

$$(3.16) \qquad\qquad \frac{\|\omega\|_{\varepsilon_n}^2}{t_{\varepsilon_n}^2} + \int_{\mathbb{R}^N} \phi_{\varepsilon_n,\omega}\, \omega^2 \geq \frac{f(t_{\varepsilon_n} \omega(\overline{x}))}{(t_{\varepsilon_n} \omega(\overline{x}))^3} \int_{B_{\overline{r}}} \omega^4,$$

where $\omega(\overline{x}) := \min_{B_{\overline{r}}} \omega(x)$. The absurd is achieved by passing to the limit in n, since by (f4) the right hand side of (3.16) tends to ∞, while the left hand side tends to 0.

Then there exists $\varepsilon_1 > 0$ such that

$$(3.17) \qquad\qquad\qquad \forall \varepsilon \in (0, \varepsilon_1]: \quad t_\varepsilon \in (0, T].$$

Condition (V1) implies also that there exists some $\varepsilon_2 > 0$ such that

$$(3.18) \qquad\qquad \forall \varepsilon \in (0, \varepsilon_2]: \quad V(\varepsilon x) \leq \frac{V_0 + \mu}{2}, \quad \text{for all} \quad x \in \operatorname{supp}\omega.$$

Finally let

$$\varepsilon_3 := \left(\frac{(\mu - V_0)|\omega|_2^2}{C_e\, T^2 \|\omega\|^4} \right)^{1/(\alpha-\theta)},$$

where C_e is the same constant appearing in (2.5), hence in particular

$$(3.19) \quad \forall \varepsilon \in (0, \varepsilon_3]: \quad \int_{\mathbb{R}^N} \phi_{\varepsilon,\omega}\omega^2 \leq \varepsilon^{\alpha-\theta} C_e \|\omega\|^4 \quad \text{and} \quad T^2 \varepsilon^{\alpha-\theta} C_e \|\omega\|^4 \leq (\mu - V_0) \int_{\mathbb{R}^N} \omega^2.$$

Let $\bar\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. By using (3.17)-(3.19) we have, for every $\varepsilon \in (0, \bar\varepsilon]$:

$$\int_{\mathbb{R}^N} V(\varepsilon x)\omega^2 + \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} \phi_{\varepsilon,\omega}\, \omega^2 \leq \frac{V_0 + \mu}{2}|\omega|_2^2 + \frac{1}{2}T^2 \varepsilon^{\alpha-\theta} C_e \|\omega\|^4 \leq \mu \int_{\mathbb{R}^N} \omega^2,$$

from which we infer $I_\varepsilon(t_\varepsilon \omega) \leq E_\mu(t_\varepsilon \omega)$. Then by (2.6) and (3.14),

$$c_\varepsilon \leq I_\varepsilon(t_\varepsilon \omega) \leq E_\mu(t_\varepsilon \omega) \leq E_\mu(\omega) < m_{V_\infty}^\infty.$$

which concludes the proof in this case.

2. If $V_\infty = \infty$, by Proposition 3.4, $\{u_n\}$ strongly converges to some $\mathfrak{u}_\varepsilon$ in $H^s(\mathbb{R}^N)$, which satisfies

$$I_\varepsilon(\mathfrak{u}_\varepsilon) = c_\varepsilon \text{ and } I_\varepsilon'(\mathfrak{u}_\varepsilon) = 0.$$

and $\mathfrak{u}_\varepsilon$ is the ground state we were looking for. $\square$

4. Proof of Theorem 1.1

In this Section we introduce the barycenter map in order to study the "topological complexity" of suitable sublevels of the functional I_ε in the Nehari manifold. Let us start with the following

Proposition 4.1. *Let $\varepsilon_n \to 0^+$ and $u_n \in \mathcal{N}_{\varepsilon_n}$ be such that $I_{\varepsilon_n}(u_n) \to m_{V_0}^\infty$. Then there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + \tilde{y}_n)$ has a convergent subsequence in $H^s(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n := \varepsilon_n \tilde{y}_n \to y \in M$.*

Recall that M is the set where V achieves the minimum V_0.

Proof. We begin by showing that $\{u_n\}$ is bounded in $H_{V_0}^s(\mathbb{R}^N)$. By assumptions, $I'_{\varepsilon_n}(u_n)[u_n] = 0$ and $I_{\varepsilon_n}(u_n) \to m_{V_0}^\infty$ write as

$$(4.1) \qquad \|u_n\|_{\varepsilon_n}^2 + \int_{\mathbb{R}^N} \phi_{\varepsilon_n, u_n} u_n^2 = \int_{\mathbb{R}^N} f(u_n) u_n$$

and

$$\frac{1}{2}\|u_n\|_{\varepsilon_n}^2 + \frac{1}{4}\int_{\mathbb{R}^N} \phi_{\varepsilon_n, u_n} u_n^2 - \int_{\mathbb{R}^N} F(u_n) = m_{V_0}^\infty + o_n(1)$$

which combined together give

$$\frac{1}{4}\int_{\mathbb{R}^N} f(u_n)u_n - \int_{\mathbb{R}^N} F(u_n) = \frac{1}{4}\left(\|u_n\|_{\varepsilon_n}^2 + \int_{\mathbb{R}^N} \phi_{\varepsilon_n, u_n} u_n^2\right) - \int_{\mathbb{R}^N} F(u_n) \leq m_{V_0}^\infty + o_n(1).$$

Using (f4) we get

$$0 \leq \left(\frac{1}{4} - \frac{1}{K}\right)\int_{\mathbb{R}^N} f(u_n)u_n \leq m_{V_0}^\infty + o_n(1),$$

and therefore, coming back to (4.1), for some positive constant C (independent on n)

$$(4.2) \qquad \|u_n\|_{H_{V_0}^s} \leq \|u_n\|_{\varepsilon_n} \leq C.$$

We prove the following

Claim: there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ and $R, c > 0$ such that $\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} u_n^2 \geq c > 0$.

Indeed, if it were not the case then

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} u_n^2 = 0, \quad \text{for every } R > 0.$$

By Lemma 3.2, $u_n \to 0$ in $L^p(\mathbb{R}^N)$, for $2 < p < 2_s^*$ and then

$$\int_{\mathbb{R}^N} f(u_n)u_n \to 0.$$

Therefore $\|u_n\|_{\varepsilon_n}^2 + \int_{\mathbb{R}^N} \phi_{\varepsilon_n, u_n} u_n^2 = o_n(1)$, and also from

$$0 \leq \int_{\mathbb{R}^N} F(u_n) \leq \frac{1}{K}\int_{\mathbb{R}^N} f(u_n)u_n$$

we have $\int_{\mathbb{R}^N} F(u_n) = o_n(1)$. But then $\lim_{n\to\infty} I_{\varepsilon_n}(u_n) = m_{V_0}^\infty = 0$ which is a contradiction and proves our claim.

Then the sequence $v_n := u_n(\cdot + \tilde{y}_n)$ is also bounded in $H^s(\mathbb{R}^N)$ and

$$(4.3) \qquad v_n \rightharpoonup v \not\equiv 0 \quad \text{in} \quad H^s(\mathbb{R}^N)$$

since
$$\int_{B_R} v^2 = \liminf_{n\to\infty} \int_{B_R} v_n^2 = \liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} u_n^2 \geq c > 0,$$

by the claim.

Let now $t_n > 0$ be such that $\tilde{v}_n := t_n v_n \in \mathcal{M}_{V_0}$; the next step is to prove that

(4.4) $$E_{V_0}(\tilde{v}_n) \to m_{V_0}^{\infty}.$$

For this, note that

$$
\begin{aligned}
m_{V_0}^{\infty} \leq E_{V_0}(\tilde{v}_n) &= \frac{1}{2}\|\tilde{v}_n\|_{V_0}^2 - \int_{\mathbb{R}^N} F(u_n) \\
&= \frac{t_n^2}{2}\int_{\mathbb{R}^N}\left[|(-\Delta)^{s/2}u_n(x+\tilde{y}_n)|^2 + V_0 u_n^2(x+\tilde{y}_n)\right]dx - \int_{\mathbb{R}^N} F(t_n u_n(x+\tilde{y}_n))dx \\
&= \frac{t_n^2}{2}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_n(z)|^2 dz + \frac{t_n^2}{2}\int_{\mathbb{R}^N} V_0 u_n^2(z)dz - \int_{\mathbb{R}^N} F(t_n u_n(z))dz \\
&\leq \frac{t_n^2}{2}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_n|^2 + \frac{t_n^2}{2}\int_{\mathbb{R}^N} V(\varepsilon_n z)u_n^2 + \frac{t_n^4}{4}\int_{\mathbb{R}^N} \phi_{\varepsilon_n, u_n}u_n^2 - \int_{\mathbb{R}^N} F(t_n u_n) \\
&= I_{\varepsilon_n}(t_n u_n)
\end{aligned}
$$

and then
$$m_{V_0}^{\infty} \leq E_{V_0}(\tilde{v}_n) \leq I_{\varepsilon_n}(t_n u_n) \leq I_{\varepsilon_n}(u_n) = m_{V_0}^{\infty} + o_n(1)$$

which proves (4.4).

We can prove now that $v_n \to v$ in $H^s(\mathbb{R}^N)$. As in the first part of the proof (where we proved the boundedness of $\{u_n\}$ in $H_{V_0}^s(\mathbb{R}^N)$), it is easy to see that

$$\{\tilde{v}_n\} \subset \mathcal{M}_{V_0} \quad \text{and} \quad E_{V_0}(\tilde{v}_n) \to m_{V_0}^{\infty} \implies \|\tilde{v}_n\|_{H_{V_0}^s} \leq C$$

and an analogous claim as before holds for the sequence $\{\tilde{v}_n\}$. Then $\tilde{v}_n \rightharpoonup \bar{v}$ in $H_{V_0}^s(\mathbb{R}^N)$ and (as before) there exists $\delta > 0$ such that

(4.5) $$0 < \delta \leq \|\tilde{v}_n\|_{H_{V_0}^s}.$$

This implies
$$0 < t_n\delta \leq \|t_n v_n\|_{H_{V_0}^s} = \|\tilde{v}_n\|_{H_{V_0}^s} \leq C,$$

showing that, up to subsequence, $t_n \to t_0 \geq 0$. If now $t_0 = 0$ using (4.2) we derive

$$0 \leq \|\tilde{v}_n\|_{H_{V_0}^s} = t_n\|v_n\|_{H_{V_0}^s} \leq t_n C \to 0,$$

so that $\tilde{v}_n \to 0$ in $H_{V_0}^s(\mathbb{R}^N)$. From this and (4.4) it follows $m_{V_0}^{\infty} = 0$ which is absurd. So $t_0 > 0$. Then $t_n v_n \rightharpoonup t_0\bar{v} =: \tilde{v}$ in $H^s(\mathbb{R}^N)$ and by (4.5) $\tilde{v} \not\equiv 0$. By Lemma 3.6 applied to $\{\tilde{v}_n\}$ we get $\tilde{v}_n \to \tilde{v}$ in $H^s(\mathbb{R}^N)$ and then $v_n \to \bar{v}$. By (4.3) we deduce $v_n \to v$ and the first part of the proposition is proved.

We proceed to prove the second part. We first state that $\{y_n\}$ is bounded in $\mathbb{R}^N$ (here $y_n = \varepsilon_n\tilde{y}_n$ with $\tilde{y}_n$ given in the above claim). Assume the contrary; then

1. if $V_\infty < \infty$, since $\tilde{v}_n \to \tilde{v}$ in $H^s(\mathbb{R}^N)$ and $V_0 < V_\infty$, we have

$$m_{V_0}^\infty = \frac{1}{2}\|\tilde{v}\|_{H_{V_0}^s}^2 - \int_{\mathbb{R}^N} F(\tilde{v}) < \frac{1}{2}\|\tilde{v}\|_{H_{V_\infty}^s}^2 - \int_{\mathbb{R}^N} F(\tilde{v})$$

$$\leq \liminf_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\tilde{v}_n|^2 + \lim_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)\tilde{v}_n^2(x)dx - \int_{\mathbb{R}^N} F(\tilde{v}_n)\right)$$

$$= \liminf_{n\to\infty}\left(\frac{t_n^2}{2}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_n|^2 + \frac{t_n^2}{2}\int_{\mathbb{R}^N} V(\varepsilon_n z)u_n^2 - \int_{\mathbb{R}^N} F(t_n u_n)\right)$$

$$\leq \liminf_{n\to\infty}\left(\frac{1}{2}\|t_n u_n\|_{\varepsilon_n}^2 - \int_{\mathbb{R}^N} F(t_n u_n) + \frac{t_n^4}{4}\int_{\mathbb{R}^N}\phi_{\varepsilon_n,u_n}u_n^2\right)$$

from which

$$m_{V_0}^\infty < \liminf_{n\to\infty} I_{\varepsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = m_{V_0}^\infty$$

which is a contradiction.

2. If $V_\infty = \infty$, we have

$$\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)v_n^2(x)dx \leq \int_{\mathbb{R}^N}|(-\Delta)^{s/2}v_n(x)|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)v_n^2(x)dx$$

$$+ \int_{\mathbb{R}^N}\phi_{\varepsilon_n,v_n}(x)v_n^2(x)dx$$

$$= \int_{\mathbb{R}^N} f(v_n(x))v_n(x)dx,$$

and by the Fatou's Lemma we obtain the absurd

$$\infty = \liminf_{n\to\infty}\int_{\mathbb{R}^N} f(v_n)v_n = \int_{\mathbb{R}^N} f(v)v.$$

Then $\{y_n\}$ has to be bounded and we can assume $y_n \to y \in \mathbb{R}^N$. If $y \notin M$ then $V_0 < V(y)$, and similarly to the computation made in case 1. above (simply replace V_∞ with $V(y)$) we have a contradiction. Hence $y \in M$ and the proof is thereby complete. $\qquad\square$

For $\delta > 0$ (later it will be fixed conveniently) let η be a smooth nonincreasing cut-off function defined in $[0,\infty)$ such that

$$\eta(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \delta/2 \\ 0 & \text{if } s \geq \delta. \end{cases}$$

Let $\mathfrak{w}_{V_0}$ be a ground state solution given in Lemma 3.6 of problem (A_μ) with $\mu = V_0$ and for any $y \in M$, let us define

$$\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)\mathfrak{w}_{V_0}\left(\frac{\varepsilon x - y}{\varepsilon}\right).$$

Let $t_\varepsilon > 0$ verifying $\max_{t\geq 0} I_\varepsilon(t\Psi_{\varepsilon,y}) = I_\varepsilon(t_\varepsilon\Psi_{\varepsilon,y})$, so that $t_\varepsilon\Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$, and let

$$\Phi_\varepsilon : y \in M \mapsto t_\varepsilon\Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon.$$

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$ and it is easy to see that Φ_ε is a continuous map.

The next result will help us to define a map from M to a suitable sublevel in the Nehari manifold.

Lemma 4.2. *The function Φ_ε satisfies*

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = m_{V_0}^\infty, \quad \text{uniformly in } y \in M.$$

Proof. Suppose by contradiction that the lemma is false. Then there exist $\delta_0 > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0^+$ such that

$$(4.6) \qquad\qquad |I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - m_{V_0}^\infty| \geq \delta_0.$$

Using Lebesgue's Theorem, we have

$$(4.7) \qquad\qquad \lim_{n\to\infty} \|\Psi_{\varepsilon_n, y_n}\|_{\varepsilon_n}^2 = \|\mathfrak{w}_{V_0}\|_{H_{V_0}^s}^2,$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F\left(\Psi_{\varepsilon_n, y_n}\right) = \int_{\mathbb{R}^N} F\left(\mathfrak{w}_{V_0}\right),$$

$$\lim_{n\to\infty} \|\Psi_{\varepsilon_n, y_n}\|_{H_{V_0}^s}^2 = \|\mathfrak{w}_{V_0}\|_{H_{V_0}^s}^2.$$

This last convergence implies that $\{\|\Psi_{\varepsilon_n, y_n}\|\}$ is bounded. From (2.3)

$$\int_{\mathbb{R}^N} \phi_{\varepsilon_n, \Psi_{\varepsilon_n, y_n}} \Psi_{\varepsilon_n, y_n}^2 \leq \varepsilon_n^{\alpha-\theta} C_{\mathrm{e}} \|\Psi_{\varepsilon_n, y_n}\|^4,$$

and then

$$(4.8) \qquad\qquad \lim_{n\to\infty} \int_{\mathbb{R}^N} \phi_{\varepsilon_n, \Psi_{\varepsilon_n, y_n}} \Psi_{\varepsilon_n, y_n}^2 = 0.$$

Remembering that $t_{\varepsilon_n} \Psi_{\varepsilon_n, y} \in \mathcal{N}_{\varepsilon_n}$ (see few lines before the Lemma), the condition $I'_{\varepsilon_n}(t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n})[t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n}] = 0$ means

$$(4.9) \qquad \|\Psi_{\varepsilon_n, y_n}\|_{\varepsilon_n}^2 + t_{\varepsilon_n}^2 \int_{\mathbb{R}^N} \phi_{\varepsilon_n, \Psi_{\varepsilon_n, y_n}} \Psi_{\varepsilon_n, y_n}^2 = \int_{\mathbb{R}^N} \frac{f\left(t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n}\right)}{t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n}} \Psi_{\varepsilon_n, y_n}^2.$$

We now prove the following

Claim: $\lim_{n\to+\infty} t_{\varepsilon_n} = 1$.

We begin by showing the boundedness of $\{t_{\varepsilon_n}\}$. Since $\varepsilon_n \to 0^+$, we can assume $\delta/2 < \delta/(2\varepsilon_n)$ and then from (4.9), using (f5) and making the change of variable $z := (\varepsilon_n x - y_n)/\varepsilon_n$, we get

$$(4.10) \qquad \frac{\|\Psi_{\varepsilon_n, y_n}\|_{\varepsilon_n}^2}{t_{\varepsilon_n}^2} + \int_{\mathbb{R}^N} \phi_{\varepsilon_n, \Psi_{\varepsilon_n, y_n}} \Psi_{\varepsilon_n, y_n}^2 \geq \frac{f\left(t_{\varepsilon_n} \mathfrak{w}_{V_0}\left(\overline{z}\right)\right)}{\left(t_{\varepsilon_n} \mathfrak{w}_{V_0}\left(\overline{z}\right)\right)^3} \int_{B_{\delta/2}} \mathfrak{w}_{V_0}^4\left(z\right),$$

where $\mathfrak{w}_{V_0}(\overline{z}) := \min_{B_{\delta/2}} \mathfrak{w}_{V_0}(z)$. If $\{t_{\varepsilon_n}\}$ were unbounded, passing to the limit in n in (4.10), the left hand side would tend to 0 (due to (4.7) and (4.8)), the right hand side to $+\infty$ (due to (f4)). So we can assume that $t_{\varepsilon_n} \to t_0 \geq 0$.

For given $\xi > 0$, by (2.1), there exists $M_\xi > 0$ such that

$$(4.11) \qquad \int_{\mathbb{R}^N} \frac{f\left(t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n}\right)}{t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n}} \Psi_{\varepsilon_n, y_n}^2 \leq \xi \int_{\mathbb{R}^N} \Psi_{\varepsilon_n, y_n}^2 + M_\xi t_{\varepsilon_n}^{q-1} \int_{\mathbb{R}^N} \Psi_{\varepsilon_n, y_n}^{q+1}.$$

Since $\{\Psi_{\varepsilon_n, y_n}\}$ is bounded in $H^s(\mathbb{R}^N)$, if $t_0 = 0$, from (4.11) we deduce

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f\left(t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n}\right)}{t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n}} \Psi_{\varepsilon_n, y_n}^2 = 0,$$

which joint with (4.8) and (4.9) led to $\lim_{n\to\infty} \|\Psi_{\varepsilon_n, y_n}\|_{\varepsilon_n}^2 = 0$ contradicting (4.7). Then $t_{\varepsilon_n} \to t_0 > 0$. Now taking the limit in n in (4.9) we arrive at

$$\|\mathfrak{w}_{V_0}\|_{H_{V_0}^s}^2 = \int_{\mathbb{R}^N} \frac{f(t_0 \mathfrak{w}_{V_0})}{t_0} \mathfrak{w}_{V_0},$$

and since $\mathfrak{w}_{V_0} \in \mathcal{M}_{V_0}$, it has to be $t_0 = 1$, which proves the claim.

Finally, note that

$$
\begin{aligned}
I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\Psi_{\varepsilon_n,y_n}|^2 + \frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^N}V(\varepsilon_n x)\Psi_{\varepsilon_n,y_n}^2 \\
&+ \frac{t_{\varepsilon_n}^4}{4}\int_{\mathbb{R}^N}\phi_{\varepsilon_n,\Psi_{\varepsilon_n,y_n}}\Psi_{\varepsilon_n,y_n}^2 - \int_{\mathbb{R}^N}F\left(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}\right).
\end{aligned}
$$

and then (by using the claim) $\lim_{n\to\infty}I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = E_{V_0}(\mathfrak{w}_{V_0}) = m_{V_0}^\infty$, which contradicts (4.6). Thus the Lemma holds.   □

The remaining part of the paper mainly follows the arguments of [24].

By Lemma 4.2, $h(\varepsilon) := |I_\varepsilon(\Phi_\varepsilon(y)) - m_{V_0}^\infty| = o(1)$ for $\varepsilon \to 0^+$ uniformly in y, and then $I_\varepsilon(\Phi_\varepsilon(y)) - m_{V_0}^\infty \le h(\varepsilon)$. In particular the sublevel set in the Nehari manifold

$$
\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)} := \left\{ u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \le m_{V_0}^\infty + h(\varepsilon) \right\}
$$

is not empty, since for sufficiently small ε,

(4.12) $\forall y \in M : \Phi_\varepsilon(y) \in \mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}.$

From now on we fix a $\delta > 0$ in such a way that M and

$$
M_{2\delta} := \left\{ x \in \mathbb{R}^N : d(x,M) \le 2\delta \right\}
$$

are homotopically equivalent (d denotes the euclidean distance). Take a $\rho = \rho(\delta) > 0$ such that $M_{2\delta} \subset B_\rho$ and $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be defined as follows

$$
\chi(x) = \begin{cases} x & \text{if } |x| \le \rho \\ \rho\dfrac{x}{|x|} & \text{if } |x| \ge \rho. \end{cases}
$$

Define the *barycenter map* β_ε

$$
\beta_\varepsilon(u) := \frac{\displaystyle\int_{\mathbb{R}^N}\chi(\varepsilon x)u^2(x)}{\displaystyle\int_{\mathbb{R}^N}u^2(x)} \in \mathbb{R}^N
$$

for all $u \in W_\varepsilon$ with compact support.

We will take advantage of the following results (see [24, Lemma 8 and 9]).

Lemma 4.3. *The function β_ε satisfies*

$$
\lim_{\varepsilon\to 0^+}\beta_\varepsilon(\Phi_\varepsilon(y)) = y, \quad \text{uniformly in } y \in M.
$$

Lemma 4.4. *We have*

$$
\lim_{\varepsilon\to 0^+}\sup_{u\in\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}}\inf_{y\in M_\delta}\left|\beta_\varepsilon(u) - y\right| = 0.
$$

In virtue of Lemma 4.4, there exists $\varepsilon^* > 0$ such that

$$
\forall \varepsilon \in (0, \varepsilon^*] : \sup_{u\in\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}}d(\beta_\varepsilon(u), M_\delta) < \delta/2.
$$

Define now

$$
M^+ := \left\{ x \in \mathbb{R}^N : d(x,M) \le 3\delta/2 \right\}
$$

so that M and M^+ are homotopically equivalent.

Now, reducing $\varepsilon^* > 0$ if necessary, we can assume that Lemma 4.3, Lemma 4.4 and (4.12) hold. Then by standard arguments the composed map

$$M \xrightarrow{\Phi_\varepsilon} \mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)} \xrightarrow{\beta_\varepsilon} M^+ \quad \text{is homotopic to the inclusion map.}$$

In case $V_\infty < \infty$, we eventually reduce ε^* in such a way that also the Palais-Smale condition is satisfied in the interval $(m_{V_0}^\infty, m_{V_0}^\infty + h(\varepsilon))$, see Proposition 3.5. By well known properties of the category, it is

$$\text{cat}(\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}) \geq \text{cat}_{M^+}(M)$$

and the Ljusternik-Schnirelman theory ensures the existence of at least $\text{cat}_{M^+}(M) = \text{cat}(M)$ constraint critical points of I_ε on $\mathcal{N}_\varepsilon$. The proof of the main Theorem 1.1 then follows by Proposition 3.5.

If M is bounded and not contractible in itself, then the existence of another critical point of I_ε on $\mathcal{N}_\varepsilon$ follows from some ideas in [11]. We recall here the main steps for completeness.

The goal is to exhibit a subset $\mathcal{A} \subset \mathcal{N}_\varepsilon$ such that

   i) $\mathcal{A}$ is not contractible in $\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}$,

   ii) $\mathcal{A}$ is contractible in $\mathcal{N}_\varepsilon^{\bar{c}} = \{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \leq \bar{c}\}$, for some $\bar{c} > m_{V_0}^\infty + h(\varepsilon)$.

This would imply, since the Palais-Smale holds, that there is a critical level between $m_{V_0}^\infty + h(\varepsilon)$ and $\bar{c}$.

First note that when M is not contractible and bounded the compact set $\mathcal{A} := \Phi_\varepsilon(M)$ can not be contractible in $\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}$, proving i).

Let us denote, for $u \in W_\varepsilon \setminus \{0\}$, with $t_\varepsilon(u) > 0$ the unique positive number such that $t_\varepsilon(u)u \in \mathcal{N}_\varepsilon$. Choose a function $u^* \in W_\varepsilon$ be such that $u^* \geq 0$, $I_\varepsilon(t_\varepsilon(u^*)u^*) > m_{V_0}^\infty + h(\varepsilon)$ and consider the compact and contractible cone

$$\mathfrak{C} := \Big\{tu^* + (1-t)u : t \in [0,1], u \in \mathcal{A}\Big\}.$$

Observe that, since the functions in $\mathfrak{C}$ have to be positive on a set of nonzero measure, it is $0 \notin \mathfrak{C}$. Now we project this cone on $\mathcal{N}_\varepsilon$: let

$$t_\varepsilon(\mathfrak{C}) := \Big\{t_\varepsilon(w)w : w \in \mathfrak{C}\Big\} \subset \mathcal{N}_\varepsilon$$

and set

$$\bar{c} := \max_{t_\varepsilon(\mathfrak{C})} I_\varepsilon > m_{V_0}^\infty + h(\varepsilon)$$

(indeed the maximum is achieved being $t_\varepsilon(\mathfrak{C})$ compact). Of course $\mathcal{A} \subset t_\varepsilon(\mathfrak{C}) \subset \mathcal{N}_\varepsilon$ and $t_\varepsilon(\mathfrak{C})$ is contractible in $\mathcal{N}_\varepsilon^{\bar{c}}$: we deduce ii).

Then there is a critical level for I_ε greater than $m_{V_0}^\infty + h(\varepsilon)$, hence different from the previous ones we have found. The proof of Theorem 1.1 is complete.

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