

The property of maximal transcendentality: calculation of Feynman integrals

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Abstract

We review some results of calculations, having the property of maximal transcendentality.

1 Introduction

It is well known that the popular property of maximal transcendentality, which was introduced in [1] for the Balitsky-Fadin-Kuraev-Lipatov (BFKL) kernel [2, 3] in the $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) model [4], is also applicable for the anomalous dimension (AD) matrices of the twist-2 and twist-3 Wilson operators and for the coefficient functions of the “deep-inelastic scattering” (DIS) in this model. The property gives a possibility to recover the results for the ADs [1, 5, 6] and the coefficient functions [7] without any direct calculations by using the QCD corresponding values [8].

The very similar property appears also in the results of calculation of the large class of Feynman integrals (FIs), mostly for so-called master integrals [9]. The results for most of them can be reconstructed also without any direct calculations using a knowledge of several terms in their inverse-mass expansion [10]. Note that the properties of the results are related with the ones of the amplitudes, form-factors and correlation functions (see [11, 12, 13] and references therein) studied recently in the framework of the $\mathcal{N} = 4$ SYM.

In this brief review, we demonstrate the existence of the property of maximal transcendentality (or maximal complexity) in the results of two-loop two- and three-point FIs (see also [14]). Moreover, we show its manifestation for the eigenvalues of AD matrices of the twist-2 Wilson operators.

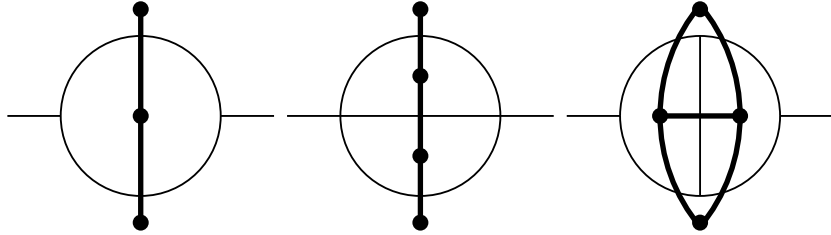


Figure 1: The examples of usual and dual FIs.

2 Calculation of Feynman integrals

The arguments based on the property of maximal transcendentality give a possibility to calculate a large class of FIs in a simplest way. Let us consider the results in some details.

1. At the beginning, we note that hereafter we will consider our FIs in the momentum space but it is rather convenient also to work in the *dual* coordinate space (see, for example, [15, 16, 17]), where all momentums of diagrams are replaced by the corresponding coordinates. Of course, the results of the integration of the diagrams do not changed during the procedure. However the graphic representations of the FIs are different. Shortly speaking, all loops (triangles, n -leg one-loop internal graphs) should be replaced by the corresponding chains (three-leg vertices, n -leg vertices). For some simplest cases, the replacement is shown on Fig. 1, where the thin lines correspond to the standard FIs and the thick ones show the corresponding dual graphs. More complicated cases were considered, for example, in Ref. [18]. The integration in dual graphs are doing on the internal points. The rules of the integration, including the integration-by-part (IBP) procedure [19], were considered in [15, 16].

With the usage of the dual technique, the evaluation of the α_s -corrections to the longitudinal DIS structure function has been done in [15, 20]. All the calculations were done for the massless diagrams. The extension of such calculations to the massive case were done in [21]. Some recent evaluations of the massive dual FIs can be found also in [22].

2. Now we will return to the momentum space. Application of the IBP procedure [19] to loop internal momenta leads to relations between differ-

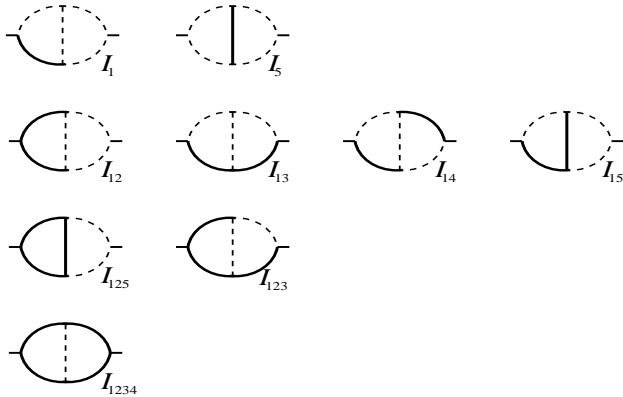


Fig. 2

ent FIs and, thus, to necessity to calculate only some of them, which in a sense, are independent. These independent diagrams (which were chosen quite arbitrary, of course) are called the master-integrals [9].

The application of the IBP procedure [19] to the master-integrals themselves leads to the differential equations (DEs) [21, 23] for them with the inhomogeneous terms (ITs) containing less complicated diagrams.¹ The application of the IBP procedure to the diagrams in ITs leads to the new DEs for them with the new ITs containing even farther less complicated diagrams (\equiv less² complicated ones). Repeating the procedure several times, at a last step one can obtain the ITs containing mostly tadpoles which can be calculated in-turn very easily (see also the discussions in the part **3** below).

Solving the DEs at this last step, one can reproduce the diagrams for ITs of the DEs at the previous step. Repeating the procedure several times one can obtain the results for the initial FIs.

This scheme has been used successfully for calculation of two-loop two-point [21, 23, 25] and three-point diagrams [26, 10] with one nonzero mass. This procedure is very powerful but quite complicated. There are, however, some simplifications, which are based on the series representations of FIs.

Indeed, the inverse-mass expansion of two-loop two-point (see Fig. 2) and three-point diagrams (see Fig. 3)² with one nonzero mass (massless and

¹The “less complicated diagrams” contain usually less number of propagators and sometimes they can be represented as diagrams with less number of loops and with some “effective masses” (see, for example, [10, 24] and references therein).

²The diagrams shown in Figs. 2 and 3, are complicated two-loop FIs, which have no three-massive-particle cuts. So, their results should be expressed as combinations of

massive propagators are shown as dashed and solid lines, respectively), can be considered as

$$\text{FI} = \frac{\hat{N}}{q^{2\alpha}} \sum_{n=1} C_n (\eta x)^n \left\{ F_0(n) + \left[\ln(-x) F_{1,1}(n) + \frac{1}{\varepsilon} F_{1,2}(n) \right] \right. \\ \left. + \left[\ln^2(-x) F_{2,1}(n) + \frac{1}{\varepsilon} \ln(-x) F_{2,2}(n) + \frac{1}{\varepsilon^2} F_{2,3}(n) + \zeta(2) F_{2,4}(n) \right] + \dots \right\}, \quad (1)$$

where $x = q^2/m^2$, $\eta = 1$ or -1 and $\alpha = 1$ and 2 for two-point and three-point cases, respectively.

Here the normalization $\hat{N} = (\bar{\mu}^2/m^2)^{2\varepsilon}$, where $\bar{\mu} = 4\pi e^{-\gamma_E} \mu$ is in the standard \overline{MS} -scheme and γ_E is the Euler constant. Moreover, the space-time dimension is $D = 4 - 2\varepsilon$ and

$$C_n = \frac{(n!)^2}{(2n)!} \equiv \hat{C}_n \quad (2)$$

for diagrams with two-massive-particle-cuts ($2m$ -cuts). For the diagrams with one-massive-particle-cuts (m -cuts) $C_n = 1$.

For m -cut case, the coefficients $F_{N,k}(n)$ should have the form

$$F_{N,k}(n) \sim \frac{S_{\pm a, \dots}}{n^b}, \frac{\zeta(\pm a)}{n^b}, \quad (3)$$

where $S_{\pm a, \dots} \equiv S_{\pm a, \dots}(j-1)$ are harmonic sums

$$S_{\pm a}(j) = \sum_{m=1}^j \frac{(\pm 1)^m}{m^a}, \quad S_{\pm a, \pm b, \pm c, \dots}(j) = \sum_{m=1}^j \frac{(\pm 1)^m}{m^a} S_{\pm b, \pm c, \dots}(m), \quad (4)$$

and $\zeta(\pm a)$ are the Euler-Zagier constants

$$\zeta(\pm a) = \sum_{m=1}^{\infty} \frac{(\pm 1)^m}{m^a}, \quad \zeta(\pm a, \pm b, \pm c, \dots) = \sum_{m=1}^{\infty} \frac{(\pm 1)^m}{m^a} S_{\pm b, \pm c, \dots}(m-1), \quad (5)$$

For $2m$ -cut case, the coefficients $F_{N,k}(n)$ can be more complicated

$$F_{N,k}(n) \sim \frac{S_{\pm a, \dots}}{n^b}, \frac{V_{a, \dots}}{n^b}, \frac{W_{a, \dots}}{n^b}, \quad (6)$$

polylogarithms. Note that we consider only three-point diagrams with independent upward momenta q_1 and q_2 , which obey the conditions $q_1^2 = q_2^2 = 0$ and $(q_1 + q_2)^2 \equiv q^2 \neq 0$, where q is downward momentum.

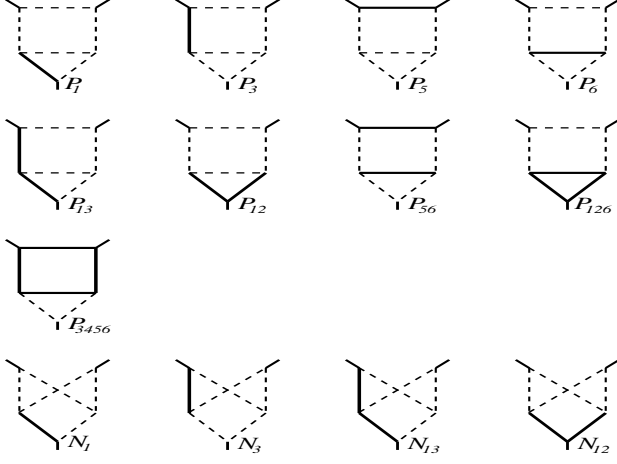


Fig. 3

where $V_{\pm a, \dots} \equiv V_{\pm a, \dots}(j-1)$ and $W_{\pm a, \dots} \equiv W_{\pm a, \dots}(j-1)$ with

$$V_a(j) = \sum_{m=1}^j \frac{\hat{C}_m}{m^a}, \quad V_{a,b,c,\dots}(j) = \sum_{m=1}^j \frac{\hat{C}_m}{m^a} S_{b,c,\dots}(m), \quad (7)$$

$$W_a(j) = \sum_{m=1}^j \frac{\hat{C}_m^{-1}}{m^a}, \quad W_{a,b,c,\dots}(j) = \sum_{m=1}^j \frac{\hat{C}_m^{-1}}{m^a} S_{b,c,\dots}(m), \quad (8)$$

The terms $\sim V_{a,\dots}$ and $\sim W_{a,\dots}$ can come only in the $2m$ -cut case. The origin of the appearance of these terms is the product of series (1) with the different coefficients $C_n = 1$ and $C_n = \hat{C}_n$.

As an example, consider two-loop two-point diagrams I_1 and I_{12} shown in Fig. 2 and studied in [10]

$$I_1 = \frac{\hat{N}}{q^2} \sum_{n=1} x^n \left\{ \frac{1}{2} \ln^2(-x) - \frac{2}{n} \ln(-x) + \zeta(2) + 2S_2 - 2\frac{S_1}{n} + \frac{3}{n^2} \right\}, \quad (9)$$

$$I_{12} = \frac{\hat{N}}{q^2} \sum_{n=1} \frac{x^n}{n^2} \left\{ \frac{1}{n} + \frac{(n!)^2}{(2n)!} \left(-2 \ln(-x) - 3W_1 + \frac{2}{n} \right) \right\}. \quad (10)$$

From (9) one can see that the corresponding functions $F_{N,k}(n)$ have the form

$$F_{N,k}(n) \sim \frac{1}{n^{3-N}}, \quad (N \geq 2), \quad (11)$$

if we introduce the following complexity of the sums ($\Phi = (S, V, W)$)

$$\Phi_{\pm a} \sim \Phi_{\pm a_1, \pm a_2} \sim \Phi_{\pm a_1, \pm a_2, \dots, \pm a_m} \sim \zeta_a \sim \frac{1}{n^a}, \quad \left(\sum_{i=1}^m a_i = a \right). \quad (12)$$

The number $3 - N$ defines the level of transcendentality (or complexity, or weight) of the coefficients $F_{N,k}(n)$. The property reduces strongly the number of the possible elements in $F_{N,k}(n)$. The level of transcendentality decreases if we consider the singular parts of diagrams and/or coefficients in front of ζ -functions and of logarithm powers. Thus, finding the parts we are able to predict the rest, using the ansatz based on the results already obtained but containing elements with a higher level of transcendentality.

Other I -type integrals in [10] have similar form. They have been calculated exactly by DE method [21, 23].

Now we consider two-loop three-point diagrams, P_5 and P_{12} shown in Fig. 3 and calculated in [10]:

$$\begin{aligned} P_5 &= \frac{\hat{N}}{(q^2)^2} \sum_{n=1} \frac{(-x)^n}{n} \left\{ -6\zeta_3 + 2(S_1\zeta_2 + 6S_3 - 2S_1S_2 + 4\frac{S_2}{n} - \frac{S_1^2}{n} + 2\frac{S_1}{n^2} \right. \\ &\quad \left. + \left(-4S_2 + S_1^2 - 2\frac{S_1}{n} \right) \ln(-x) + S_1 \ln^2(-x) \right\}, \quad (13) \\ P_{12} &= \frac{\hat{N}}{q^2} \sum_{n=1} \frac{x^n}{n^2} \frac{(n!)^2}{(2n)!} \left\{ \frac{2}{\varepsilon^2} + \frac{2}{\varepsilon} \left(S_1 - 3W_1 + \frac{1}{n} - \ln(-x) \right) + 12W_2 - 18W_{1,1} \right. \\ &\quad \left. - 13S_2 + S_1^2 - 6S_1W_1 + 2\frac{S_1}{n} + \frac{2}{n^2} - 2 \left(S_1 + \frac{1}{n} \right) \ln(-x) + \ln^2(-x) \right\}, \end{aligned}$$

Now the coefficients $F_{N,k}(n)$ have the form

$$F_{N,k}(n) \sim \frac{1}{n^{4-N}}, \quad (N \geq 3), \quad (14)$$

The diagram P_5 (and also P_1 , P_3 and P_6 in [10]) have been calculated exactly by DE method [21, 23]. To find the results for P_{12} (and also all others in [10]) we have used the knowledge of the several n terms in the inverse-mass expansion (1) (usually less than $n = 100$) and the following arguments:

- If a two-loop two-point diagram with the “similar topology” (for example, I_{12} for P_{12} and so on) has been already calculated, we should

consider a similar set of basic elements for corresponding $F_{N,k}(n)$ of two-loop three-point diagrams but with the higher level of complexity.

- Let the considered diagram contain singularities and/or powers of logarithms. Because in front of the leading singularity, or the largest power of logarithm, or the largest ζ -function the coefficients are very simple, they can be often predicted directly from the first several terms of expansion.

Moreover, often we can calculate the singular part using another technique (see [10] for extraction of $\sim W_1(n)$ part). Then we should expand the singular parts, find the basic elements and try to use them (with the corresponding increase of the level of complexity) to predict the regular part of the diagram. If we have to find the ε -suppressed terms, we should increase the level of complexity for the corresponding basic elements.

Later, using the ansatz for $F_{N,k}(n)$ and several terms (usually, less than 100) in the above expression, which can be calculated exactly, we obtain the system of algebraic equations for the parameters of the ansatz. Solving the system, we can obtain the analytical results for FI without exact calculations. To check the results, it is needed only to calculate a few more terms in the above inverse-mass expansion (1) and compare them with the predictions of our ansatz with the above fixed coefficients.

So, the considered arguments give a possibility to find the results for many complicated two-loop three-point diagrams without direct calculations. Some variations of the procedure have been successfully used for calculating the Feynman diagrams for many processes (see [26, 10, 24, 27]).

Note that the properties similar to (11) and (14) have been observed recently [13] in the so-called double operator-product-expansion limit of some four-point diagrams.

3. The coefficients have the structure (11) and (14) with the rule (12). We note that these conditions reduces strongly the number of possible harmonic sums. In turn, the restriction relates with the specific form of the DEs for the considered FIs. The DEs formally can be represented like [14, 28]

$$\left((x+a) \frac{d}{dx} - \bar{k}\varepsilon \right) \text{FI} = \text{less complicated diagrams} (\equiv \text{FI}_1), \quad (15)$$

with some number a and some function $\bar{k}(x)$. Such form is generated by IBP procedure for diagrams including an internal triangle with the index ³ equal to $D - 1 + O(\varepsilon)$ in D -dimensional space.

Taking the set of the less complicated Feynman integrals FI_1 as diagrams having internal triangles with the similar indices, we will have their result stucture similar to above one (14) but with the one less level of complexity.

So, the integrals FI_1 should obey to the equation similar to one in (15). It has the following form

$$\left((x + a_1) \frac{d}{dx} - \bar{k}_1 \varepsilon \right) \text{FI}_1 = \text{less}^2 \text{ complicated diagrams} (\equiv \text{FI}_2). \quad (16)$$

Thus, we will have the set of equations for all Feynman integrals FI_n as

$$\left((x + a_n) \frac{d}{dx} - \bar{k}_n \varepsilon \right) \text{FI}_n = \text{less}^{n+1} \text{ complicated diagrams} (\equiv \text{FI}_{n+1}), \quad (17)$$

with the last integral FI_{n+1} contains only tadlopes.

Moreover, following to [29] we can recover the above set of the inhomogeneous equations as the homogeneous matrix equation

$$\frac{d}{dx} \widehat{FI} - \varepsilon \widehat{K} \widehat{FI} = 0, \quad (18)$$

for the vector

$$\widehat{FI} = \begin{pmatrix} \text{FI} \\ \text{FI}_1/\varepsilon \\ \dots \\ \text{FI}_n/\varepsilon^n \end{pmatrix},$$

where the matrix \widehat{K} contains the functions $\bar{k}_j/(x + a_j)$ as its elements. The form (18) is very popular now (see the recent report [30] and discussion therein)

Note that for the real calculations of FI_n it is convenient to do the replacement

$$\text{FI}_n \rightarrow \widetilde{\text{FI}}_n, \quad \text{FI}_n = \widetilde{\text{FI}}_n \overline{\text{FI}}_n,$$

³The index of the triangle is equal to the sum of powers of its propagators. The internal loop corresponds to the triangle with one propagator having the power equal to 0.

where the term $\overline{\text{FI}}_n$ obeys the corresponding homogeneous equation

$$\left((x + a_n) \frac{d}{dx} - \bar{k}_n \varepsilon \right) \overline{\text{FI}}_n = 0, \quad (19)$$

The replacement simplifies the above equation (17) to the following form

$$(x + a_n) \frac{d}{dx} \tilde{\text{FI}}_n = \tilde{\text{FI}}_{n+1} \frac{\overline{\text{FI}}_{n+1}}{\overline{\text{FI}}_n} \quad (20)$$

having the solution

$$\tilde{\text{FI}}_n(x) = \int_0^x \frac{dx_1}{x_1 + a_n} \tilde{\text{FI}}_{n+1}(x_1) \frac{\overline{\text{FI}}_{n+1}(x_1)}{\overline{\text{FI}}_n(x_1)} \quad (21)$$

Usually there are some cancellations in the ratio $\overline{\text{FI}}_{n+1}/\overline{\text{FI}}_n$ and sometimes it is equal to 1. In the last case, the equation (21) coincides with definition of Goncharov Polylogarithms (see [31] and references therein).

The series (9), (10) and (13) can be expressed as combination of the Nilson [32] and Remiddi-Vermaseren [33] polylogarithms with the weight $4 - N$ (see [10, 26]). More complicated case was considered in [34].

3 $\mathcal{N} = 4$ SYM

The ADs govern the Bjorken scaling violation for parton distributions (\equiv matrix elements of the twist-2 Wilson operators) in a framework of Quantum Chromodynamics (QCD).

The BFKL and Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) [35] equations resum, respectively, the most important contributions $\sim \alpha_s \ln(1/x_B)$ and $\sim \alpha_s \ln(Q^2/\Lambda^2)$ in different kinematical regions of the Bjorken variable x_B and the “mass” Q^2 of the virtual photon in the lepton-hadron DIS and, thus, they are the cornerstone in analyses of the experimental data from lepton-nucleon and nucleon-nucleon scattering processes. In the supersymmetric generalization of QCD the equations are simplified drastically [36]. In the $\mathcal{N} = 4$ SYM the eigenvalues of the AD matrix contain only one *universal* function with shifted arguments [37, 1].

1. The three-loop result ⁴ for the universal AD $\gamma_{uni}(j)$ for $\mathcal{N} = 4$ SYM is [6]

$$\gamma_{uni}(j) = \hat{a}\gamma_{uni}^{(0)}(j) + \hat{a}^2\gamma_{uni}^{(1)}(j) + \hat{a}^3\gamma_{uni}^{(2)}(j) + \dots, \quad \hat{a} = \frac{\alpha N_c}{4\pi}, \quad (22)$$

where

$$\frac{1}{4}\gamma_{uni}^{(0)}(j+2) = -S_1, \quad (23)$$

$$\frac{1}{8}\gamma_{uni}^{(1)}(j+2) = \left(S_3 + \bar{S}_{-3}\right) - 2\bar{S}_{-2,1} + 2S_1\left(S_2 + \bar{S}_{-2}\right), \quad (24)$$

$$\begin{aligned} \frac{1}{32}\gamma_{uni}^{(2)}(j+2) = & 2\bar{S}_{-3}S_2 - S_5 - 2\bar{S}_{-2}S_3 - 3\bar{S}_{-5} + 24\bar{S}_{-2,1,1,1} \\ & + 6\left(\bar{S}_{-4,1} + \bar{S}_{-3,2} + \bar{S}_{-2,3}\right) - 12\left(\bar{S}_{-3,1,1} + \bar{S}_{-2,1,2} + \bar{S}_{-2,2,1}\right) \\ & - \left(S_2 + 2S_1^2\right)\left(3\bar{S}_{-3} + S_3 - 2\bar{S}_{-2,1}\right) - S_1\left(8\bar{S}_{-4} + \bar{S}_{-2}^2\right. \\ & \left. + 4S_2\bar{S}_{-2} + 2S_2^2 + 3S_4 - 12\bar{S}_{-3,1} - 10\bar{S}_{-2,2} + 16\bar{S}_{-2,1,1}\right) \end{aligned} \quad (25)$$

with $S_{\pm a, \pm b, \pm c, \dots} \equiv S_{\pm a, \pm b, \pm c, \dots}(j)$ and

$$\bar{S}_{-a, b, c, \dots}(j) = (-1)^j S_{-a, b, c, \dots}(j) + S_{-a, b, c, \dots}(\infty) \left(1 - (-1)^j\right). \quad (26)$$

The expression (26) is the analytical continuation (to real and complex j) [38] of the harmonic sums $S_{-a, b, c, \dots}(j)$.

The results for $\gamma_{uni}^{(3)}(j)$ [39, 40], $\gamma_{uni}^{(4)}(j)$ [41] and $\gamma_{uni}^{(5)}(j)$ [42] can be obtained from the long-range asymptotic Bethe equations [43] for twist-two operators and the additional contribution of the wrapping corrections. The similar calculations for the twist-three ADs can be found in [44].

2. Similarly to the eqs. (11) and (14) let us to introduce the transcendental level i for the harmonic sums $S_{\pm a}(j)$ and Euler-Zagier constants $\zeta(\pm a)$ in the following way

$$S_{\pm a, \pm b, \pm c, \dots}(j) \sim \zeta(\pm a, \pm b, \pm c, \dots) \sim 1/j^i, \quad (i = a + b + c + \dots) \quad (27)$$

⁴ Note, that in accordance with Ref. [3] our normalization of $\gamma(j)$ contains the extra factor $-1/2$ in comparison with the standard normalization (see [1]) and differs by sign in comparison with one from Ref. [8].

Then, the basic functions $\gamma_{uni}^{(0)}(j)$, $\gamma_{uni}^{(1)}(j)$ and $\gamma_{uni}^{(2)}(j)$ are assumed to be of the types $\sim 1/j^i$ with the levels $i = 1$, $i = 3$ and $i = 5$, respectively. A violation of this property could be derived from contributions of the terms appearing at a given order from previous orders of the perturbation theory. Such contributions could be generated and/or removed by an appropriate finite renormalization and/or redefinition of the coupling constant. But these terms do not appear in the $\overline{\text{DR}}$ -scheme [45].

It is known, that at the first three orders of perturbation theory (with the SUSY relation for the QCD color factors $C_F = C_A = N_c$) the most complicated contributions (with $i = 1$, 3 and 5 , respectively) are the same as in QCD [8]. This property allows one to find the universal ADs $\gamma_{uni}^{(0)}(j)$, $\gamma_{uni}^{(1)}(j)$ and $\gamma_{uni}^{(2)}(j)$ without knowing all elements of the AD matrix [1], which was verified for $\gamma_{uni}^{(1)}(j)$ by the exact calculations in [5].

Note that in $\mathcal{N} = 4$ SYM the some partial cases of ADs are known also at the large couplings from string calculations and AdS/QFT correspondence [46]. We would like to note that if the property of the maximal transcendentality is existed at low coupling, then sometimes it appears at large couplings, too (see, for example, the results for the cusp AD at low [47] and large [48] couplings, both of which are based on the Beisert-Eden-Staudacher equation [49]). This is not correct, however, for Pomeron intercept, which results are lost the property of the maximal transcendentality at large couplings (see [6, 50, 51]). The reason of the difference in the results for the cusp AD and Pomeron intercept is not clear now. It needs additional investigations.

4 Conclusion

In the first part of this short review we presented the consideration of Feynman diagrams (mostly master integrals), which obey to the *transcendentality principle* (11), (12) and (14). Its application leads to the possibility to get the results for most of master integrals without direct calculations.

The second part contains the universal AD $\gamma_{uni}(j)$ for the $\mathcal{N} = 4$ SYM in the first three terms of perturbation theory. All the results have been obtained with using the *transcendentality principle* (27). At the first three orders, the universal ADs have been extracted directly from the corresponding QCD calculations. The results for four, five and six loops have been

obtained from the long-range asymptotic Bethe equations [43] together with some additional terms, so-called *wrapping corrections*, coming in agreement with Luscher approach.

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