

NON STANDARD ANALYSIS AS A FUNCTOR, AS LOCAL, AS ITERATED

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ABSTRACT. This note has several aims. Firstly, it portrays a non-standard analysis as a functor, namely a functor $*$ that maps any set A to the set $*A$ of its non-standard elements. That functor, from the category of sets to itself, is postulated to be an equivalence on the full subcategory of finite sets onto itself and to preserve finite projective limits (equivalently, to preserve finite products and equalizers). Secondly, “Local” non-standard analysis is introduced as a structure which we call lim-rim, in particular exact lim-rims. The interplay between these, and ultrafilters and ultrapowers, and also cardinality relations and notions depending on a cardinality such as saturation and what we call “confinement” and “exactness”, are investigated.

In particular, one constructs non-standard analyses, with “good” kinds of lim-rim. In these one may say that $*A$ – “the adjunction of all possible limits from A ” – plays a role analogous to that of the algebraic closure of a field – “the adjunction of all roots of polynomials”. Then in the same spirit as with the latter, one has uniqueness up to isomorphism, and also universality and homogeneity, provided one has enough General Continuum Hypothesis. The cardinality of $*A$ will be something like $2^{2^{|A|}}$ – the same as that of the set of ultrafilters in A , and one has a high degree of saturation.

Also, one notes that the functor $*$ can be applied again to $*A$, giving $**A$, $***A$, and so forth. In particular, we focus on the two different embeddings of $*A$ into $**A$ and prove some of their properties, with some applications.

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This note is a sequel to the less formal [Le]. Here I make things more precise, clarify and develop them, and add some material about iterated non-standard analysis. In particular, a non-standard analysis (which here gives “non-standard elements” to any set) is approached as a functor (a connection between non-standard analysis and category theory was given in [S]). “Local” non-standard analysis is introduced as a structure which I call lim-rim, in particular exact lim-rims. The interplay between these, and ultrafilters and ultrapowers, and also cardinality relations and notions depending on a cardinality such as saturation and what I call “confinement” and “exactness”, are investigated.

In the second section, dedicated to iterated non-standard analysis, the two embeddings ν and ${}^* \nu$ of ${}^* A$ into ${}^{**} A$ are focused upon. These embeddings are highly distinct. In the last paragraph a simple criterion (Prop. 29) that the values they take satisfy a given standard relation is given, which has some interesting consequences and applications.

0. NOTATIONS

Denote the cardinality of a set A by $|A|$. For a cardinality \mathfrak{m} , denote by \mathfrak{m}^+ the successor cardinality. Denote the power set of a set A (the set of all subsets of A) by $\mathcal{P}A$.

1. NON-STANDARD ANALYSIS

1.1. Non-Standard Analysis as a Functor.

Definition 1. Let \mathcal{SET} be the category of sets and maps between them, and let \mathcal{FIN} be its full subcategory of finite sets and maps between them. A **non-standard analysis** will be defined as a (covariant) functor ${}^* : \mathcal{SET} \rightarrow \mathcal{SET}$ which (i) maps \mathcal{FIN} to itself as a self-equivalence of \mathcal{FIN} ; (ii) preserves finite projective limits.

* of a set A will be denoted by ${}^* A$ and its elements will be referred to as the **non-standard elements** of A (thus, if A is, say, the set of integers, then the elements of ${}^* A$ will be called * integers). Similarly the image by * of a map f will be denoted by ${}^* f$.

In particular, for every a the set $\{a\}$ is a singleton, thus an object of \mathcal{FIN} , and * being an equivalence on \mathcal{FIN} , ${}^* \{a\}$ is also a singleton and its sole element will be denoted by $\nu(a)$ (ν from “new”). In many cases, we will tend to identify a with $\nu(a)$ (see below).

As for condition (ii), it is equivalent to * preserving finite products (equivalently products of two sets A and B , being the projective limit of the diagram A, B (no maps)) and equalizers $\{c \in C \mid f_1(c) = f_2(c)\}$ of two maps $f_1, f_2 : C \rightarrow D$, (i.e. the projective limit of the diagram $C \rightarrow D$ – the maps are f_1 and f_2).

This means that the images by * of the projections $A \times B \rightarrow A$, $A \times B \rightarrow B$ make ${}^*(A \times B)$ the product ${}^* A \times {}^* B$; and if E is the subset $\{c \in C \mid f_1(c) = f_2(c)\}$, i.e., with the inclusion map $E \rightarrow C$, the equalizer of f_1 and f_2 , then * of the inclusion map makes ${}^* E$ the equalizer of ${}^* f_1, {}^* f_2 : {}^* C \rightarrow {}^* D$. But since any subset of C is an equalizer, this means that * of any inclusion map $E \rightarrow C$ is injective, and can be viewed as an inclusion ${}^* E \subset {}^* C$, and in the above situation ${}^* E = \{{}^* c \in {}^* C \mid {}^* f_1({}^* c) = {}^* f_2({}^* c)\}$.

In particular, * of the diagonal of $A \times A$ – the equalizer of the two projections – can serve as the diagonal of ${}^* A \times {}^* A$.

Example 2. Of course, a trivial example for non-standard analysis is the identity functor, where ${}^*A = A$ and all non-standard elements are, in fact, standard – of the form $\nu(a)$.

To get a non-trivial example, Let S be a fixed set of indices and \mathcal{U} a fixed ultrafilter in S . For any set A we have the **ultrapower** $A_{\mathcal{U}} := A^S/\mathcal{U}$, defined as the power A^S modulo the equivalence relation: equality modulo \mathcal{U} .

We have here a covariant functor $A \mapsto A_{\mathcal{U}}$ (which depends on S and \mathcal{U}) from the category of sets to itself. The image $f_{\mathcal{U}}$ by the functor of a map $f : A \rightarrow B$ between sets will map $[\xi] \in A_{\mathcal{U}}$, where $\xi : S \rightarrow A$, to $[f \circ \xi] \in B_{\mathcal{U}}$.

This ultrapower functor satisfies (i) and (ii) of Definition 1. Indeed, (i) is immediate, because if F is finite then for any $\xi : S \rightarrow F$ there is a unique $a \in F$ such that $\xi^{-1}(\{a\}) \in \mathcal{U}$, and then ξ is, modulo \mathcal{U} , the constant a . As for (ii), this functor preserves finite products – $(A \times B)_{\mathcal{U}}$ is identified with $A_{\mathcal{U}} \times B_{\mathcal{U}}$ in an obvious manner, and if $f_1, f_2 : C \rightarrow D$, and $E = \{c \in C \mid f_1(c) = f_2(c)\}$ is the equalizer, then the equalizer of $(f_1)_{\mathcal{U}}, (f_2)_{\mathcal{U}} : C_{\mathcal{U}} \rightarrow D_{\mathcal{U}}$ is

$$\begin{aligned} & \{[\xi] \in C_{\mathcal{U}} \mid (f_1)_{\mathcal{U}}([\xi]) = (f_2)_{\mathcal{U}}([\xi])\} = \\ & = \{[\xi] \in C_{\mathcal{U}} \mid [f_1 \circ \xi] = [f_2 \circ \xi]\} = \\ & = \{[\xi] \in C_{\mathcal{U}} \mid \exists U \in \mathcal{U}, (f_1 \circ \xi)|_U = (f_2 \circ \xi)|_U\} = \\ & = \{[\xi] \in C_{\mathcal{U}} \mid \exists U \in \mathcal{U} \xi(U) \subset E\} = \\ & = \{[\xi] \in C_{\mathcal{U}} \mid \exists \eta : S \rightarrow E [\xi] = [\eta]\}, \end{aligned}$$

which is identified with $E_{\mathcal{U}}$. □

This means that taking ${}^*A :=$ the ultrapower $A_{\mathcal{U}}$, gives us a non-standard analysis, as per Definition 1.

1.2. Non-Standard Analysis as a Functor, continued. If A is a set and $a \in A$, there is an inclusion map $\{a\} \rightarrow A$, and its * is an injection ${}^*\{a\} \rightarrow {}^*A$, i.e. $\{\nu(a)\} \rightarrow {}^*A$, identifying $\nu(a)$ with an element of *A (Consequently, if $A \neq \emptyset$ then ${}^*A \neq \emptyset$) – if a belongs to some (standard) set – has some property – then $\nu(a)$ is one of the non-standard elements of the set – has the * property. These $\nu(a)$'s, with $a \in A$, might be referred to as the **standard elements** of A , but (if A is infinite) A may have other elements, not of the form $\nu(a)$, $a \in A$.

Similarly, $\nu(a_1, \dots, a_n)$ is identified with $(\nu(a_1), \dots, \nu(a_n))$, and if a_1, \dots, a_n satisfy some relation $R \subset A_1 \times \dots \times A_n$ then $\nu(a_1), \dots, \nu(a_n)$ satisfies *R .

We will sometimes loosely view any finite set F as a set of a kind of *truth-values* and any map between finite sets $f : F_1 \rightarrow F_2$ as a kind of *logical operation*, à la **AND**, **OR**, **NOT**. Then the equivalence that * induces on \mathcal{FLN} is used as an identification for these truth-values and logical operations. Maps $f : A_1 \times A_2 \times \dots \times A_n \rightarrow F$, where $A_i, i = 1 \dots n$ are sets and F is finite, are viewed as *predicates* on A_1, A_2, \dots, A_n . Composing them with maps between finite sets defines a kind of *logical operations among predicates*. But the fact that * preserves finite products and equalizers makes, e.g., a $\{\mathbf{True}, \mathbf{False}\}^2$ -valued predicate the same as a pair of $\{\mathbf{True}, \mathbf{False}\}$ -valued predicates, and if $F' \subset F$ then F' -valued predicates embed into F -valued ones. Therefore the $\{\mathbf{True}, \mathbf{False}\}$ -valued predicates determine all others, while applying * to predicates commutes with all “logical operations”, in other words, applying * to $\{\mathbf{True}, \mathbf{False}\}$ -valued predicates commutes with the propositional calculus operations **AND**, **OR**, **NOT** etc.

As said above, we identify $\nu(\mathbf{True}) = \mathbf{True}$, $\nu(\mathbf{False}) = \mathbf{False}$.

For any $\{\mathbf{True}, \mathbf{False}\}$ -valued predicate f , if E is the subset of $A_1 \times A_2 \times \dots \times A_n$ where it holds, then E is the equalizer of f with the constant map $A_1 \times A_2 \times \dots \times A_n \rightarrow \mathbf{True}$, the latter being the composition of $\{\mathbf{True}\} \rightarrow \{\mathbf{True}, \mathbf{False}\}$ with the sole map $A_1 \times A_2 \times \dots \times A_n \rightarrow$

$\{\mathbf{True}\}$. Hence its $*$ is the constant $*A_1 \times *A_2 \times \cdots \times *A_n \rightarrow \mathbf{True}$ and $*E$, as a subset of $*A_1 \times *A_2 \times \cdots \times *A_n$, will be its equalizer with $*f$, i.e. the set where $*f$ holds.

As for quantifiers, we shall show that the operation $*$ on $\{\mathbf{True}, \mathbf{False}\}$ -valued predicates commutes with them too. Since it commutes with **NOT**, it suffices to show that it commutes with \exists .

Given a predicate $f : A_1 \times A_2 \times \cdots \times A_n \rightarrow \{\mathbf{True}, \mathbf{False}\}$, to which we apply $(\exists x_1 \in A_1)$, we get the predicate $(\exists x_1 \in A_1)f$, and we have to prove that the $\{\mathbf{True}, \mathbf{False}\}$ -valued functions (“predicates”):

$$(1) \quad *[(\exists x_1 \in A_1) f(x_1, \dots, x_n)] = (\exists *x_1 \in *A_1) *f(*x_1, \dots, *x_n).$$

If $A_1 = \emptyset$ this is immediate. Thus suppose $A_1 \neq \emptyset$, consequently $*A_1 \neq \emptyset$.

Firstly, denote $g(x_2, \dots, x_n) := (\exists x_1 \in A_1)f(x_1, \dots, x_n)$. Then $A_1 \times \{g = \mathbf{True}\} \supset \{f = \mathbf{True}\}$. Hence $*A_1 \times \{*g = \mathbf{True}\} \supset \{*f = \mathbf{True}\}$, implying that $*A_1 \times \{*g = \mathbf{True}\}$, i.e. the product of $*A_1$ with the LHS of (1), contains the product of $*A_1$ with the RHS of (1). Since $*A_1 \neq \emptyset$, this implies LHS \supset RHS.

For the reverse inclusion we make use of a map j , defines on the subset $g = \mathbf{True}$ and mapping each such (x_2, \dots, x_n) to some $x_1 \in A_1$ such that $f(x_1, x_2, \dots, x_n)$ holds. (Note that if $n = 1$, $A_2 \times \cdots \times A_n$ is a singleton – a product of an empty family.) $*\{g = \mathbf{True}\}$ is the LHS of (1) and taking for any $(*x_2, \dots, *x_n) \in *\{g = \mathbf{True}\}$, $*x_1 := *j(*x_2, \dots, *x_n)$, proves that $*\{g = \mathbf{True}\}$ is contained in the RHS. \square

Therefore any non-standard analysis as per Definition 1 satisfies the *Transfer Principle*: predicates (in the usual sense, i.e. $\{\mathbf{True}, \mathbf{False}\}$ -valued), commute with all the logical operations of a first-order theory, where quantifiers taken over a set A should correspond to quantifiers taken over $*A$. In particular, a sentence which is a first-order logical combination of standard sets A is true if and only if the sentence which is the same first-order combination, each A replaced by $*A$, (hence each element a by $\nu(a)$), is true.

Since, by the above, inclusion of sets implies inclusion of the $*$ -sets,¹ we may define the $*$ of a class (e.g. of the relation \in between sets) as the direct limit of the $*$ of all sets contained in the class.

Note that often we shall use the same notation for a relation R and for $*R$, for example write $<$ for $*<$ or \in for $*\in$. By Transfer, any first-order property of R holds for $*R$. Thus, for example, if $*S$ is a $*$ set (i.e. a non-standard element of some family of sets), then $*S$ is determined by the set of $*x$ such that $*x * \in *S$, the latter set called the **scope** [MH] of $*S$ and sometimes denoted by $\hat{*S}$. Thus $*x \in \hat{*S} \Leftrightarrow *x * \in *S$. But sometimes we shall identify $*S$ with $\hat{*S}$, view $*\in$ as a particular case of \in , and thus view a $*$ set as a set. Yet, as is well known, not all sets of non-standard elements are of the form $\hat{*S}$ for some $*$ set $*S$. Sets of non-standard elements of the latter form are called **internal** and by our above identifications we may view internal sets as the same as $*$ sets. Sets of non-standard elements which are not internal will be called **external**.²

¹One may, say, argue thus: if C_1, C_2 are subsets of a set A , with inclusion maps $i_1 : C_1 \rightarrow A, i_2 : C_2 \rightarrow A$, then $C_1 \subset C_2$ if and only if there is an injection $j : C_1 \rightarrow C_2$ such that $i_1 = i_2 \circ j$.

²For example, by Transfer any bounded internal set of positive $*$ integers has a greatest element, and any positive $*$ integer less or equal than some standard n (i.e. $\nu(n)$) $*$ belongs to $*\{1, \dots, n\} = \{1, \dots, n\}$, i.e. is a standard $\nu(m)$, $m \leq n$. Therefore if $*\mathbb{N}$ contains any element which is not standard, then this element is bigger than all the standard positive integers, the set of these standard positive integers is bounded but has no greatest element, hence cannot be internal, i.e. is external.

Note, that for a set A ,

$$\forall x (x \in A \Leftrightarrow \exists y \in \{A\} x \in y)$$

is true, therefore

$$\forall^* x (^*x \in ^*A \Leftrightarrow \exists ^*y \in ^*\{A\} ^*x \in ^*y).$$

is true, where we took the $*$ of the relation \in between elements and sets. But $^*\{A\} = \{\nu(A)\}$, hence we find that as sets $^*A = \nu(A)$ (i.e. $^*A = \nu(\hat{A})$). But we shall keep using different notations: *A where we emphasize the set of (non-standard) elements, and $\nu(A)$ when we emphasize the single element.

1.3. Features of the Conventional Approach to Non-Standard Analysis. non-standard analysis was constructed by its founder, Abraham Robinson ([R]) as a non-standard model of part of Set Theory. In his and following treatments (see several treatises in the bibliography)³ the idea is to start with a “big” structure – a set V , with the restriction to it of the membership relation \in , which will be big enough so that usual mathematics could be done in it (should contain, e.g. \mathbb{N} (the natural numbers), \mathbb{R} (the real numbers), sets of them, sets of these etc.) V is to be provided with a “mirror” “non-standard” structure *V with a relation $^*\in$, viewed as an “interpretation” of the \in of V (Also, “interpret” “=” in V by “=” in *V), so that

- (1) V is embedded in *V – identified with a subset of *V . The members of V (thus considered as members of *V) are referred to as *standard* (yet by “non-standard” elements we mean any members of *V).
- (2) A *Transfer Principle* holds: any (in some constructions one has to say: “bounded”) first-order logical expression with members of V as constants which is a true sentence when quantifiers are interpreted relative to V will remain a true sentence if (the constants – members of V – are replaced by their identified images in *V and) \in is replaced by $^*\in$ and the quantifiers are interpreted relative to *V . In this sense *V is a model to the true in V sentences of the first-order theory with \in and all elements of V as constants.
- (3) By Transfer, for any standard set A , a standard $a \in V$ will be an $^*\in$ -member of it if and only if it is a usual member, but in *V A may have other, *non-standard* members. The set of all elements of *V which $^*\in$ -belong to A is denoted by *A . Again by Transfer, $^*(A \times B)$ is canonically identified with $^*A \times ^*B$, thus for relations, functions etc. we have the * -relation, * -function etc. among members of *V (which for standard elements coincides with the original relation, function etc.) When there is no danger of misunderstanding, one often uses the same symbol (e.g. $+$, $<$) for the * -relation etc. as for the original.
- (4) *V is indeed “non-standard” – it contains elements not in V , and one has features, usually some kind of “saturation” (see below), which will ensure, say, that there is an $^*n \in ^*V$ which $^*\in$ -belongs to the set \mathbb{N} of natural numbers, but such that for every standard $m \in \mathbb{N}$, $m \text{ } ^* < \text{ } ^*n$ (i.e. $(m, ^*n)$ $^*\in$ -belongs to the graph of $<$). Such *n 's are naturally referred to as “infinite” or “unlimited”⁴ and by Transfer they have reciprocals in $^*\mathbb{R}$, referred to as “infinitesimal”.

³In another direction, beginning with E. Nelson [N], one constructs a special Set Theory for non-standard analysis.

⁴In fact, as in footnote 2, for \mathbb{N} the existence of “infinite” members needs only the existence of some not standard members – one easily proves using Transfer that all * -members of a finite standard A are standard, hence any * -member of \mathbb{N} which is not standard is automatically bigger than all standard numbers.

The “philosophy” here is to develop the theory from these properties, paying less attention to the way such *V is constructed (or proved to exist). The usual construction is by an ultrapower or a modification of it. Anyway, one has to have a set V to make the construction, hence only subsets of it will have non-standard members.

1.4. Ultrafilters of Non-Standard Elements. Let us be given a non-standard analysis, as per Definition 1. For any set A , and any ${}^*x \in {}^*A$, and for any $S \subset A$, we have ${}^*S \subset {}^*A$, thus either ${}^*x \in {}^*S$ or *x is not in ${}^*S \Leftrightarrow$ it is in the complement ${}^*A \setminus {}^*S = {}^*(A \setminus S)$. Put otherwise: for any $\{\mathbf{True}, \mathbf{False}\}$ -valued predicate $f : A \rightarrow \{\mathbf{True}, \mathbf{False}\}$ we have ${}^*f : {}^*A \rightarrow {}^*\{\mathbf{True}, \mathbf{False}\} = \{\mathbf{True}, \mathbf{False}\}$ and ${}^*f({}^*x)$ is either \mathbf{True} or \mathbf{False} .

Hence any ${}^*x \in {}^*A$ induces a map $\{\mathbf{True}, \mathbf{False}\}^A \rightarrow \{\mathbf{True}, \mathbf{False}\}$, which is easily seen to be a Boolean-algebra homomorphism, i.e. an ultrafilter in A .

In fact, we may replace $\{\mathbf{True}, \mathbf{False}\}$ by any other finite set F , and view the set βA of the ultrafilters in A as the set of natural mappings between the functor from \mathcal{FIN} to \mathcal{SET} $F \mapsto F^A = \text{Hom}(A, F)$, and the inclusion functor $\mathcal{FIN} \rightarrow \mathcal{SET}$.

Then to a ${}^*x \in {}^*A$ corresponds the ultrafilter: the mapping $[f \mapsto {}^*f({}^*x)] : \text{Hom}(A, F) \rightarrow F$. This is clearly a natural mapping between the functors $A \mapsto {}^*A$ and $A \mapsto \beta A$.

Note, that although the functor $A \mapsto \beta A$ is a canonically defined functor on sets, which maps injections to injections, it does not preserve finite products (hence is not a non-standard analysis as per Definition 1). The projections $A \times B \rightarrow A$ and $A \times B \rightarrow B$ define onto mappings $\beta(A \times B) \rightarrow \beta A$ and $\beta(A \times B) \rightarrow \beta B$, hence induce a $\beta(A \times B) \rightarrow \beta A \times \beta B$, which is also onto, but is usually not one-to-one. Thus this functor does not satisfy (ii) of Definition 1 (while it satisfies (i)).

One may view a non-standard analysis as a quest to extend sets by adjoining all (or some) possible limits. This in close analogy in spirit to extending a field to its algebraic closure by adjoining all possible roots of algebraic equations. In the latter case one would like to simply define the adjoined elements as labeled by the irreducible polynomial they satisfy, but that is hampered by such facts as the polynomial whose roots are the sums of the roots of two irreducible polynomials being not irreducible.⁵ The algebraic closure thus contains conjugate elements with the same irreducible polynomial and for any such elements there is an automorphism of the algebraic closure exchanging them.

Similarly in our case we would like to label the adjoined “non-standard elements” by the ultrafilters on the original set. After all, these ultrafilters (and ultrafilters in finite products of sets) give all the possible ways properties (and relations) can conceivably hold or not. But that would not work precisely because the functor β does not preserve finite products. There is a naturally defined Cartesian product of filters, but the Cartesian product of two ultrafilters $\mathcal{U} \in \beta A, \mathcal{V} \in \beta B$ is in general not an ultrafilter – it is the intersection of all ultrafilters in $A \times B$ that map to $(\mathcal{U}, \mathcal{V})$ in the above mapping $\beta(A \times B) \rightarrow \beta A \times \beta B$.

We too will have a construction (§1.10) where there will be “conjugate” non-standard elements with the same ultrafilter and any such conjugate elements will be exchangeable by an automorphism, provided one has enough General Continuum Hypothesis (GCH), and then, in the same spirit as with algebraic closures, one has uniqueness (for fixed “basis” B – see below) up to isomorphism, and also universality and homogeneity.

⁵Which may be viewed as the “reason” for the existence of several conjugate roots: the polynomial $d(t)$ whose roots are the differences of the roots of the irreducible $f(t)$ has a factor t but also other irreducible factors, testifying to differences $\neq 0$.

1.5. Lim-Rims – Local Non-Standard Analysis. Given a non-standard analysis, one may focus on a particular set B and try to extract what the non-standard analysis says about this B (hence about its subsets).

The information telling which non-standard elements satisfy which relations is encoded in the ultrafilters on B^I, I finite. The non-standard analysis gives us a natural mapping $(^*B)^I \rightarrow \beta(B^I)$ between these two (contravariant) functors on I from \mathcal{FIN} to \mathcal{SET} . If for any set $I \in \mathcal{SET}$ we will define $\beta_c(A^I)$ as the set of ultrafilters on the *cylinder Boolean algebra of B^I* (the members of the cylinder Boolean algebra are the subsets of B^I that depend only on a finite number of coordinates. In case I is finite this is just the power set of B^I), then any natural $(^*B)^I \rightarrow \beta(B^I)$ on the finite sets I extends in a unique way to a natural $(^*B)^I \rightarrow \beta_c(B^I)$ on all sets I . Moreover, we can take $I = ^*B$ itself and then the image by the natural mapping of any element of $(^*B)^I$ will be known if we know the image in $\beta_c(B^{^*B})$ of the identity family $\in (^*B)^{^*B}$.

Definition 3. We say that a set E has the structure of a **lim-rim** over a set⁶ (basis) B if either of the equivalent following is given. (i) An ultrafilter \mathcal{L} on the cylinder Boolean algebra of B^E , which will be referred to as the **defining cylinder ultrafilter** of the lim-rim (then we shall also speak of the lim-rim (E, \mathcal{L}) over B). (ii) A natural mapping between the (contravariant) functors on sets $I: E^I$ and $\beta_c(B^I)$, where the latter denotes the set of ultrafilters in the cylinder Boolean algebra. It will be referred to as the **defining natural mapping** of the lim-rim. (Note that these functors and the natural mapping are defined for any I , yet they are determined by giving them for the finite I .)

Here, the defining cylinder ultrafilter \mathcal{L} in B^E can be recovered as the image, by the natural mapping, in $\beta_c(B^E)$, of the identity family in E^E . On the other hand, if \mathcal{L} is given, what it does is deciding, for any (finite) family $\eta \in E^I$ indexed by a finite I (pushing \mathcal{L} , using the map $B^E \rightarrow B^I$ induced by $\eta: I \rightarrow E$, to an ultrafilter in B^I) whether relations – subsets of B^I – hold or not, thus transferring such I -relations in B (= subsets of B^I) into I -relations in E , as a non-standard setting should. This new I -relation in E will be referred to as the *transfer* or $*$ of the original I -relation in B .

To put it otherwise: for an n -tuple (e_1, \dots, e_n) of elements of E , a relation $R \subset B^n$ $*$ -holds for (e_1, \dots, e_n) if and only if for $\psi \in B^E$, $(\psi_{e_1}, \dots, \psi_{e_n}) \in R$ holds modulo \mathcal{L} .

To conclude:

Proposition 4. Given a non-standard analysis as per Definition 1, it induces on any set B a lim-rim E over B , where $E = ^*B$.

And as said above, this lim-rim captures which relations about B , or its subsets, are satisfied by n -tuples of non-standard elements e_1, \dots, e_n . We may say that a lim-rim E over B defines a “local” non-standard analysis, with E the set of non-standard elements of B , completely satisfactory as a non-standard analysis as long as one restricts oneself only to B or its subsets.

Remark 5. There can be sub-lim-rims of a lim-rim E over B in two ways (or both combined): Firstly, any subset $E' \subset E$ has the structure of a lim-rim over B , and secondly for any $B' \subset B$ the set $^*B'$ of the $e \in E$ that satisfy the $*$ of “belongs to B' ” forms a lim-rim over B' . Instead of talking about sub-lim-rims one may talk about embeddings. Isomorphisms and automorphisms of lim-rims are defined as expected (as those which respect the ultrafilter \mathcal{L}).

⁶we shall usually have B infinite.

1.6. Confined Non-Standard Analysis, Separated and Rectified Lim-Rims.

Definition 6. Let \mathfrak{m} be an infinite cardinality. Given a non-standard analysis as per Definition 1, let A be a set and ${}^*x \in {}^*A$. We say that *x is **\mathfrak{m} -confined** or has **confinement \mathfrak{m}** if there exists a (standard) set $A' \subset A$ of cardinality $\leq \mathfrak{m}$ such that ${}^*x \in {}^*A'$. If \mathfrak{m} is a cardinality such that all non-standard *x 's in *A for all sets A are \mathfrak{m} -confined, we say that the non-standard analysis is **\mathfrak{m} -confined**, or that it has **confinement \mathfrak{m}** .

It is clear that for any infinite cardinality \mathfrak{m} , and any (standard) map $f : A \rightarrow B$, f maps non-standard elements of confinement \mathfrak{m} to non-standard elements of confinement \mathfrak{m} (if ${}^*x \in {}^*A'$ where $A' \subset A$ and $|A'| \leq \mathfrak{m}$, then ${}^*f({}^*x) \in {}^*f({}^*A')$. By Transfer ${}^*f({}^*A') = {}^*(f(A'))$ and $|f(A')| \leq \mathfrak{m}$).

Therefore if for each set A we define ${}^*A_{\mathfrak{m}} :=$ the set of ${}^*x \in {}^*A$ with confinement \mathfrak{m} , we get a functor $A \mapsto {}^*A_{\mathfrak{m}}$. That functor is a non-standard analysis. Indeed, for finite sets F every non-standard element is standard, thus * belongs to a standard singleton, hence is of all confinements. Therefore ${}^*F_{\mathfrak{m}} = {}^*F$; for a product $A \times B$, an $({}^*x, {}^*y) \in {}^*(A \times B) = {}^*A \times {}^*B$ is \mathfrak{m} -confined if and only if both *x and *y are so. Therefore ${}^*(A \times B)_{\mathfrak{m}} = {}^*A_{\mathfrak{m}} \times {}^*B_{\mathfrak{m}}$; and for equalizers the required assertion is clear.

We call the non-standard analysis $A \mapsto {}^*A_{\mathfrak{m}}$: the non-standard analysis $A \mapsto {}^*A$ **\mathfrak{m} -confined**, or the **\mathfrak{m} -confinement** of the non-standard analysis $A \mapsto {}^*A$.

Now suppose a non-standard analysis is given which has confinement \mathfrak{m} , and let B be some set of cardinality \mathfrak{m} . Suppose we know the lim-rim $E = {}^*B$ over B that A induces on B .

It seems evident that we can recover the non-standard elements *A of any set (or class) A using that lim-rim. Indeed, since the non-standard analysis has confinement \mathfrak{m} , any non-standard element *x of A belongs to a set of the form $f(B)$ for some (standard) $f : B \rightarrow A$, hence, by Transfer, is of the form ${}^*x = f(e)$ for some $e \in E = {}^*B$. We can take the pair (f, e) as a kind of “coordinates” for *x , and all we need is to determine when two pairs (f_1, e_1) (f_2, e_2) define the same *x and how ${}^*g : {}^*A \rightarrow {}^*A'$, for some $g : A \rightarrow A'$, is expressed using these “coordinates”. But these characterisations are straightforward, since Transfer must hold. In fact, we use the “template” lim-rim E over B to define the non-standard elements of any set by, in essence, “carrying” them over from B in some analogy with the way tangent vectors may be defined in any smooth manifold by “carrying” them over from \mathbb{R}^n .

By this way we can start from any lim-rim (E, \mathcal{L}) over B and define a $|B|$ -confined non-standard analysis which will be (equivalent to) the $|B|$ -confinement of any non-standard analysis which induces on B a lim-rim isomorphic to (E, \mathcal{L}) .

Still, we can achieve that goal more “neatly”.

Proposition 7. Let (E, \mathcal{L}) be a lim-rim over an infinite set (basis) B . For any set X , let *X be defined as the “cylindrical” ultrapower of X with respect to B^E and \mathcal{L} , namely, the set $\left(X^{B^E} \right)_c$ of all mappings from B^E to X which depend only on a finite set of coordinates in B^E (that set of coordinates depending on the mapping), factored by the equivalence relation of equivalence modulo \mathcal{L} . (And the * of mappings between sets $f : X \rightarrow Y$ defined by composing with a representant of *x).

Then: (i) $X \mapsto {}^*X$ is a $|B|$ -confined non-standard analysis. It is called: the ($|B|$ -confined) non-standard analysis **induced by the lim-rim**.

(ii) If $X \mapsto {}^*_0X$ is a $|B|$ -confined non-standard analysis which induces on B a lim-rim isomorphic to (E, \mathcal{L}) , then $X \mapsto {}^*_0X$ is equivalent to $X \mapsto {}^*X$ via the two following natural mappings:

$\alpha = \alpha_X : {}^*X \rightarrow {}^*X$: let $\phi : B^E \rightarrow X$ be a representant of ${}^*x \in {}^*X$. ϕ depends only on a finite set of coordinates e_1, \dots, e_n . Hence we may write for $\psi \in B^E$, $\phi(\psi) = f(\psi_{e_1}, \dots, \psi_{e_n})$, $f : B^n \rightarrow X$. We may assume $E = {}^*_0B$, and define $\alpha({}^*x) := \binom{{}^*f}{{}_0f}(e_1, \dots, e_n) \in {}^*_0X$.

$\gamma = \gamma_X : {}^*_0X \rightarrow {}^*X$: let ${}^*_0x \in {}^*_0X$. Since the non-standard analysis is $|B|$ -confined, there is an $f : B \rightarrow X$ such that ${}^*_0x \in {}^*_0(f(B)) = \binom{{}^*f}{{}_0f}(E)$. Thus ${}^*_0x = \binom{{}^*f}{{}_0f}(e)$ for some $e \in E$. Define $\gamma({}^*_0x)$ as $\psi \in B^E \mapsto f(\psi_e)$ modulo \mathcal{L} . This does not depend on the choice of f .

Proof. Clearly $X \mapsto {}^*X$ is a functor, and as in Example 2 one sees that it satisfies (i) and (ii) of Definition 1. Therefore it is a non-standard analysis. Every ${}^*x \in {}^*X$, with a representant $\phi : B^E \rightarrow X$ depending on the n coordinates e_1, \dots, e_n , $\phi(\psi) = f(\psi_{e_1}, \dots, \psi_{e_n})$, * belongs to $f(B^n)$, of cardinality $\leq |B|$. This proves (i).

$\gamma({}^*_0x)$ does not depend on the choice of f . Indeed, if ${}^*_0x = \binom{{}^*f_1}{{}_0f_1}(e_1) = \binom{{}^*f_2}{{}_0f_2}(e_2)$, then (e_1, e_2) * belongs to the set $\{(b_1, b_2) \in B \times B \mid f_1(b_1) = f_2(b_2)\}$. This is the * of a 2-relation in B , and by the way * is defined for a lim-rim, this means that for $\psi \in B^E$, (ψ_{e_1}, ψ_{e_2}) is in that set modulo \mathcal{L} , i.e. $f_1(\psi_{e_1}) = f_2(\psi_{e_2})$ modulo \mathcal{L} . This means that the two definitions for $\gamma({}^*_0x)$ coincide.

Clearly α and γ are natural mappings.

$\alpha \circ \gamma$ is the identity. Indeed, start from some ${}^*_0x \in {}^*_0X$. ${}^*x := \gamma({}^*_0x)$ is $\psi \in B^E \mapsto f(\psi_e)$ modulo \mathcal{L} , where $f : B \rightarrow X$ is such that ${}^*_0x = \binom{{}^*f}{{}_0f}(e)$. Thus $\psi \in B^E \mapsto f(\psi_e)$ is a representant of *x , and $\alpha(\gamma({}^*_0x)) = \alpha({}^*x) := \binom{{}^*f}{{}_0f}(e) = {}^*_0x$.

$\gamma \circ \alpha$ is the identity. Indeed, start from some ${}^*x \in {}^*X$ represented by $\phi : B^E \rightarrow X$ which depends on the coordinates e_1, \dots, e_n , i.e. for $\psi \in B^E$, $\phi(\psi) = f(\psi_{e_1}, \dots, \psi_{e_n})$. Then ${}^*_0x := \alpha({}^*x) := \binom{{}^*f}{{}_0f}(e_1, \dots, e_n)$. But by the $|B|$ -confinement there are $g : B \rightarrow X$ and $e \in E$ such that ${}^*_0x = \binom{{}^*g}{{}_0g}(e)$. Then (e, e_1, \dots, e_n) * belongs to the set $\{(b, b_1, \dots, b_n) \in B^{n+1} \mid g(b) = f(b_1, \dots, b_n)\}$, meaning that for $\psi \in B^E$, $(\psi_e, \psi_{e_1}, \dots, \psi_{e_n})$ is in that set modulo \mathcal{L} , i.e. $g(\psi_e) = f(\psi_{e_1}, \dots, \psi_{e_n})$ modulo \mathcal{L} . Thus, *x is represented also by $\psi \mapsto g(\psi_e)$, hence ${}^*x = \gamma({}^*_0x) = \gamma(\alpha({}^*x))$. \square

Corollary 8. Let (E, \mathcal{L}) be a lim-rim over an infinite B . Then TFAE: (i) There exists a non-standard analysis which induces on B a lim-rim isomorphic to E . (ii) The $|B|$ -confined non-standard analysis induced by the lim-rim E induces on B a lim-rim isomorphic to E .

If these hold, we say that the lim-rim is **rectified**.

If these hold, one may take as an isomorphism the map $\gamma_B : E \rightarrow {}^*B$ (notations as in Prop. 7).

For any (E, \mathcal{L}) , γ_B can be characterized as mapping $e \in E$ to the projection $\psi \in B^E \mapsto \psi_e$ modulo \mathcal{L} .

Proof. If there exists a non-standard analysis $A \mapsto {}^*_0X$ inducing on B a lim-rim isomorphic to E , then, by taking its $|B|$ -confinement, we may assume it is $|B|$ -confined, and then by Prop. 7 it is equivalent to the $|B|$ -confined non-standard analysis induced by E , hence the latter too induces on B a lim-rim isomorphic to E .

In this case we may take as the isomorphism the mapping γ_B from Prop. 7.

For any (E, \mathcal{L}) , in the definition of γ_B we identify ${}^*_0B = E$ and take $f = \mathbf{1}_B$ (and ${}^*_0x = e$). Then $\gamma_B(e) = (\psi \mapsto \psi_e)$ modulo \mathcal{L} . \square

Definition 9. A lim-rim E over B is called **separated** if for $e_1, e_2 \in E$, the set $\{\psi \in B^E \mid \psi_{e_1} = \psi_{e_2}\}$ is in \mathcal{L} only if $e_1 = e_2$.

Remark 10. For any lim-rim, the relation “ $\{\psi \in B^E \mid \psi_{e_1} = \psi_{e_2}\}$ is in \mathcal{L} ” is an equivalence relation in E and factoring E by it gives a separated lim-rim.

The lim-rim E being separated just says that the map γ_B mentioned in Cor. 8: $e \mapsto \left((\psi \mapsto \psi_e) \text{ mod } \mathcal{L} \right)$ is one-to-one. Therefore a rectified lim-rim is separated. (Or alternatively, the fact that the $*$ of a diagonal is a diagonal means that the lim-rim that a non-standard analysis induces on a set B must be separated.)

For any lim-rim E over B , the lim-rim of B in the $|B|$ -confined non-standard analysis induced by it is called the **rectification** of E . Cor. 8 can be phrased as saying that E is rectified if and only if it is isomorphic to its rectification, and then as an isomorphism we can take the map $\gamma_B : E \rightarrow *B$ mentioned there.

Proposition 11. Let (E, \mathcal{L}) be a lim-rim over an infinite B , and suppose E separated (then γ_B is injective. notations as above). Then γ_B is an isomorphism of lim-rims over B from E to $\gamma_B(E) \subset *B$.

Proof. Since γ_B is a bijection between E and $\gamma_B(E)$, all we need is to prove it preserves the ultrafilters on the cylinder Boolean algebras of B^E and $B^{\gamma_B(E)}$ which define the lim-rims.

Now, the set (here $h : B^n \rightarrow B$ is a function and $0 \in B$ is some fixed (standard) element)

$$\left\{ \phi \in B^{\gamma_B(E)} \mid h\left(\phi_{\gamma_B(e_1)}, \dots, \phi_{\gamma_B(e_n)}\right) = 0 \right\}$$

belongs to the ultrafilter of $\gamma_B(E)$ if in $X \mapsto *X =$ the $|B|$ -confined non-standard analysis induced by the lim-rim E ,

$$*h\left(\gamma_B(e_1), \dots, \gamma_B(e_n)\right) = 0 (= \nu(0)),$$

that is, using the characterisation of γ_B in Cor. 8 (here ψ varies over B^E),

$$*h\left(\left(\psi \mapsto \psi_{e_1}\right) \text{ mod } \mathcal{L}, \dots, \left(\psi \mapsto \psi_{e_n}\right) \text{ mod } \mathcal{L}\right) = (\psi \mapsto 0) \text{ mod } \mathcal{L},$$

which means

$$\psi \mapsto h(\psi_{e_1}, \dots, \psi_{e_1}) = (\psi \mapsto 0) \text{ mod } \mathcal{L},$$

i.e.,

$$h(\psi_{e_1}, \dots, \psi_{e_1}) = 0 \text{ mod } \mathcal{L},$$

which just says that

$$\left\{ \psi \in B^E \mid h(\psi_{e_1}, \dots, \psi_{e_n}) = 0 \right\}$$

belongs to \mathcal{L} . □

1.7. Exact Lim-Rims. Abraham Robinson, the founder of non-standard analysis [R] did not, of course, wish that the non-standard analysis will be just the trivial $*A = A$. His requirement that the non-standard analysis be an *enlargement* can be phrased, on the level of lim-rims, as the requirement that for any ultrafilter \mathcal{U} on B (or also on B^I , I finite) there will be an element $*x \in *B$ (or in $(*B)^I$) which is mapped to \mathcal{U} by the natural mapping that defines the lim-rim. But one may require slightly more:

Definition 12. A lim-rim E over an infinite B is called **exact** if for any finite I, J and for any $\xi : I \rightarrow J$, the diagram expressing the naturality of the natural mapping between the (contravariant) functors $\bullet \rightarrow E^\bullet$ and $\beta(B^\bullet)$ which defines the lim-rim, is “exact”, in the sense that a member of E^I and a member of $\beta(B^J)$ which map to the same member of $\beta(B^I)$ both come from some same member of E^J .

Note, that for this to hold it suffices that it holds for inclusions “adding one element” $I \rightarrow I \cup \{i\}$ and for the map $\{1, 2\} \rightarrow \{1\}$ (the latter guarantees, as easily seen, that the $*$ of the relation of equality in B will be the relation of equality in E – i.e. that E be separated).

Note that for $I = \emptyset$ both E^\emptyset and $\beta(B^\emptyset)$ are singletons, $\{\emptyset\}$ if you wish, with subsets naturally viewed as the truth-values $\mathbf{True} = \{\emptyset\}$, $\mathbf{False} = \emptyset$ which the natural mapping always preserves.

If E is exact then, in particular, applying the exactness of the diagram for the map $\emptyset \rightarrow \{1\}$ (or $\emptyset \rightarrow I$, I finite) we have that for every ultrafilter \mathcal{U} in B there is an element $e \in E$ whose ultrafilter, i.e. whose image in the natural mapping $E \rightarrow \beta B$, is \mathcal{U} , that is, e $*$ belongs to all members of \mathcal{U} (and of course does not $*$ belong to non-members of \mathcal{U} , which are the complements of members).⁷ If \mathcal{U} is fixed (principal) – is the family of all subsets containing some $a \in B$ – than e is unique and is the element of E to be identified with a (i.e. $e = \nu(a)$). If \mathcal{U} is free (non-principal), however, one proves that (if E is exact) e is never unique and we have “conjugate” non-standard elements with the same ultrafilter which, as we shall see below, will be exchangeable by an automorphism with a favorable choice of the lim-rim.

Remark 13. We may express the exactness of the diagram for a map $I \rightarrow I \cup \{i\}$ as follows: For every family $\eta : I \rightarrow E$ of elements of E , and any ultrafilter \mathcal{U} in $B^{I \cup \{i\}}$ which projects on B^I to the ultrafilter pulled-back from \mathcal{L} by η , one can find an element of E as the image of i so that the pull-back of \mathcal{L} by the extended η will be \mathcal{U} .

This implies that for exact lim-rims, $*$ of relations will commute with projections $B^{I \cup \{i\}} \rightarrow B^I$ ($i \notin I$), i.e. with applying \exists . Of course, for any lim-rim, $*$ of relations commutes with Propositional Calculus operations. Thus in an exact lim-rim it commutes with all first-order logical operations. And one has a *Transfer Principle*: Any true sentence made as a first-order logical combination of relations on B will turn, by transferring each of the argument relations, into a sentence true for E .

Proposition 14. Any exact lim-rim (E, \mathcal{L}) over B is rectified.

Proof. By Prop. 11 all we need to prove is that if E is exact the map γ_B mentioned there is onto.

Recall that γ_B is a map from E to $*B =$ the set of cylindrical maps $\Psi : B^E \rightarrow B$ modulo \mathcal{L} , which maps $e \in E$ to $\left((\psi \mapsto \psi_e) \text{ mod } \mathcal{L} \right) \in *B$.

Now let $*b \in *B$ be some cylindrical map $\Psi : B^E \rightarrow B$ modulo \mathcal{L} . Then

$$\Psi : \psi \in B^E \mapsto h(\psi_{e_1}, \dots, \psi_{e_n}) \in B,$$

where $e_1, \dots, e_n \in E$ and h is some $h : B^n \rightarrow B$.

Let \mathcal{U} be the push of \mathcal{L} by the map

$$\Phi_1 : \psi \mapsto \left(\psi_{e_1}, \dots, \psi_{e_n}, h(\psi_{e_1}, \dots, \psi_{e_n}) \right)$$

to an ultrafilter in B^{n+1} . Then E being exact implies, by Remark 13, that there is an $e_0 \in E$ such that the push of \mathcal{L} by the map

$$\Phi_0 : \psi \mapsto \left(\psi_{e_1}, \dots, \psi_{e_n}, \psi_{e_0} \right)$$

⁷This does not hold, in general, for non-standard analysis constructed by an ultrapower (if the power in the ultrapower is countable, a countable set will have only 2^{\aleph_0} non-standard members but has $2^{2^{\aleph_0}}$ ultrafilters.)

is the same ultrafilter \mathcal{U} . Let

$$T := \{(b_1, \dots, b_n, b_0) \in B^n \mid b_0 = h(b_1, \dots, b_n)\}.$$

Then $\Phi_1^{-1}(T)$ is the whole B^E , hence $T \in \mathcal{U}$, but that means that $\Phi_0^{-1}(T) \in \mathcal{L}$, which just says that

$$\Psi(\psi) = h(\psi_{e_1}, \dots, \psi_{e_n}) = \psi_{e_0} \pmod{\mathcal{L}}.$$

Thus $*b = \gamma_B(e_0)$. □

1.8. **m-Exactness, m-Saturation.**

Definition 15. *Let \mathfrak{m} is an infinite cardinal.*

*A lim-rim E over an (infinite) set B is called **m-exact** if it is separated and exactness of the diagram of the natural mapping, as in Definition 12, holds for the inclusion $I \rightarrow I \cup \{i\}$ for any $|I| < \mathfrak{m}$.*

As in Remark 13, this means: For any I such that $|I| < \mathfrak{m}$, every family $\eta : I \rightarrow E$ of elements of E , and any ultrafilter \mathcal{U} in the cylinders of $B^{I \cup \{i\}}$ which projects on B^I to the ultrafilter pulled-back from \mathcal{L} by η , one can find an element of E as the image of i so that the pull-back of \mathcal{L} by the extended η will be \mathcal{U} .

Remark 16. *By a transfinite process of adding elements i , the **m-exactness** of E will guarantee exactness of the diagram of the natural mappings for any inclusion $I \rightarrow M$ for $|I| < \mathfrak{m}$, $|M| \leq \mathfrak{m}$.*

In particular, by the definition, a lim-rim is exact if and only if it is separated and \aleph_0 -exact.

Proposition 17. *Let \mathfrak{m} is an infinite cardinal.*

*Let E be an **m-exact** lim-rim over an (infinite) set B . Let $B' \subset B$, B' also infinite. Let $E' = *B'$ be the set of members of E that $*$ belong to B' . Then the sub-lim-rim E' over B' is also **m-exact**.*

Proof. Let, as in Def. 15, I satisfy $|I| < \mathfrak{m}$, $\eta : I \rightarrow E'$ and \mathcal{V} an ultrafilter in the cylinders of $(B')^{I \cup \{i\}}$ which projects on $(B')^I$ to the ultrafilter pulled-back by η from the defining cylinder ultrafilter \mathcal{L}' of E' .

Note, that if \mathcal{L} was the defining cylinder ultrafilter of E , then for any finite set $e_1, \dots, e_n \in E'$, the preimage of $(B')^n \subset B^n$ by $B^I \rightarrow B^{\{e_1, \dots, e_n\}}$ is a member of \mathcal{L} and \mathcal{L}' obtains from \mathcal{L} by intersecting its members with these preimages. Thus, the inclusion $(B')^{I \cup \{i\}} \rightarrow B^{I \cup \{i\}}$ maps \mathcal{V} into an ultrafilter in the cylinders of $B^{I \cup \{i\}}$ (which contains all primages of $(B')^n$ by $B^I \rightarrow B^{\{\iota_1, \dots, \iota_n\}}$ for $\iota_1, \dots, \iota_n \in I \cup \{i\}$) and which will project on B^I to the ultrafilter pulled-back by η from \mathcal{L} .

Hence, by the **m-exactness** of E , there is an element of E as $\eta(i)$ so that the pull-back of \mathcal{L} by the extended η is \mathcal{U} . Since \mathcal{U} contains the preimage of B' by the projection $B^{I \cup \{i\}} \rightarrow B^{\{i\}}$, $\eta(0)$ belongs to $*B' = E'$. Thus the extended η maps $I \cup \{i\}$ to E' . And the way \mathcal{U} is constructed from \mathcal{V} and \mathcal{L}' is constructed from \mathcal{L} means that the pull-back of \mathcal{L}' by the extended η is \mathcal{V} , as required. □

m-exactness is related to the property of a non-standard analysis being **m-saturated**.

Recall (see end of §1.2) the distinction between **internal** and **external** sets of non-standard elements: For some (standard) set A , a subset of $*A$ is called internal if it is of the form $*\hat{X} := \{*y \in *A \mid *y \in *X\}$ for some $*X \in *\mathcal{P}A$ where $\mathcal{P}A$ is the power set. A subset of $*A$ which is not internal is called external.

Remark 18. *An alternative characterization for an internal set: an internal subset of *A is any set of the form: the section parallel to ${}^*A, {}^*R[{}^*y]$ of *R through some element ${}^*y \in {}^*C$ for some (standard) set C and some (standard) relation $R \subset C \times A$.*

*Indeed, by Transfer, ${}^*R[{}^*y]$ is just the set of non-standard * members of the image *X of *y by the * of the map $y \in C \mapsto R[y] \in \mathcal{P}A$. On the other hand, if ${}^*X \in {}^*\mathcal{P}A$, then the set of its * members is the section parallel to *A through *X of the * of the relation $y \in X$ in $A \times \mathcal{P}A$.*

Definition 19. *Let \mathfrak{m} is an infinite cardinal.*

*A non-standard analysis is called **\mathfrak{m} -saturated inside a (standard) set A** , if a family of cardinality $< \mathfrak{m}$ of internal subsets of *A has non-empty intersection provided any finite subfamily has non-empty intersection. If a non-standard analysis is **\mathfrak{m} -saturated inside every (standard) set**, then it is called **\mathfrak{m} -saturated**.*

Proposition 20. *Let \mathfrak{m} is an infinite cardinal.*

(i) *Suppose a lim-rim E over an infinite B is \mathfrak{m} -exact. Then the $|B|$ -confined non-standard analysis induced by it is $\min(\mathfrak{m}, |B|^+)$ -saturated, while it is \mathfrak{m} -saturated inside every (standard) set of cardinality $\leq |B|$.*

(ii) *If a non-standard analysis, as per Definition 1, is \mathfrak{m} -saturated, and B is an infinite (standard) set, then the lim-rim which the non-standard analysis induces over B is \mathfrak{m}_1 -exact if $\mathfrak{m}' < \mathfrak{m}_1 \Rightarrow 2^{|B|} \cdot \mathfrak{m}' < \mathfrak{m}$.*

Proof. (i) Note first, that since the lim-rim E is \mathfrak{m} -exact, hence exact, it is rectified. Therefore we may identify ${}^*B = E$.

Let us be given a family of cardinality $< \mathfrak{m}$ of internal subsets of the * of some standard set C , where every finite subfamily has non-empty intersection.

Every member of the family of internal sets is the scope $\hat{*}X_\iota$ of some $*X_\iota \in {}^*\mathcal{P}C$, and, the non-standard analysis being $|B|$ -confined, there is a standard subset $Y_\iota \subset \mathcal{P}C$ of cardinality $\leq |B|$ such that $*X \in *Y_\iota$. Denoting by R_ι the relation \in in $C \times Y_\iota$, $\hat{*}X_\iota$ is the section parallel to $*C$ of $*R_\iota$ through $*X_\iota$. As $|Y_\iota| \leq |B|$, we may replace the Y_ι 's with subsets of B , shifting the R_ι 's accordingly. Then the $*X_\iota$'s belong to $E = *B$, and they form an element of E^I for the set I of the ι 's, where $|I| < \mathfrak{m}$.

Now we distinguish between the two cases. If $|C| \leq |B|$, we may assume $C \subset B$. If we do not have $|C| \leq |B|$, replace \mathfrak{m} by $\min(\mathfrak{m}, |B|^+)$, that is, assume $\mathfrak{m} \leq |B|^+$. Then the cardinality of the family of internal sets is $\leq |B|$ and if we pick an element from every intersection of a finite subfamily, we get at most $|B|$ elements which, by $|B|$ -confinement, all belong to some $*C'$ for some (standard) $C' \subset C$ with $|C'| \leq |B|$. And we proceed with $\hat{*}X_\iota \cap *C'$ instead of $\hat{*}X_\iota$ as in the first case.

So we assume $C \subset B$; writing $B = B^{\{0\}}$, every R_ι is a relation = subset of $B^{\{0,\iota\}}$. Which can be viewed as a ‘‘cylinder’’ in $B^{\{0\} \cup I}$. We wish to have an ultrafilter \mathcal{U} in the cylinders of $B^{\{0\} \cup I}$ which contains these R_ι 's and also all sets which depend only on a finite number of coordinates in I , that the $*X_\iota$ satisfy. The fact that \mathcal{U} exists, i.e. that any finite number of these intersect, follows from the fact that their * intersect because any finite subfamily of the $*R_\iota[{}^*X_\iota]$ intersects. Now, \mathfrak{m} -exactness finishes the task: it says that there is a $*X_0 \in E$ such that the push of \mathcal{L} to $B^{\{0\} \cup I}$ is \mathcal{U} , that is, the family $({}^*X_0, ({}^*X_\iota))$ * belongs to all members of \mathcal{U} , meaning that $*X_0$ is in the intersection of all the members of the family of internal sets.

(ii) Denote $E = *B$. Let I be such that $|I| = \mathfrak{m}' < \mathfrak{m}_1$, let $\eta : I \rightarrow E$ be a family of elements of E , and \mathcal{U} an ultrafilter in the cylinders of $B^{I \cup \{0\}}$ which projects on B^I to the ultrafilter \mathcal{U}' pulled-back from \mathcal{L} by η , and one wishes to find an element of E as $\eta(0)$ – the image of 0, so that the pull-back of \mathcal{L} by the extended η will be \mathcal{U} .

To satisfy this, $\eta(0)$, together with the $\eta(\iota)$, $\iota \in I$, must \ast belong to all the members U of \mathcal{U} . Belonging to each such U requires $\eta(0)$ to belong to a certain internal subset of $\ast B = E$. To belong to any finite number of these internal subsets, is to satisfy the conjunction of these $\ast U$'s with a finite number of $\eta(\iota)$'s, $\eta_{\iota_1}, \dots, \eta_{\iota_n}$ as parameters. The set of these parameters belongs to the \ast of a any member U' of \mathcal{U}' . Hence, Transfer will tell us that there is such an $\eta(0)$, if we can find a member U' of \mathcal{U}' so that for any “standard” “parameters” $b_{\iota_1}, \dots, b_{\iota_n} \in B$ there is a $b_{\iota_0} \in B$ which together with them belong to all the (finite number of) U 's. But for this just take as U' the projection on B^I of the intersection of these U 's.

Thus, every finite number of these internal subsets intersect.

Also, the cardinality of \mathcal{U} is at most the cardinality of the set of all “cylinders” in B^I , which is $\leq 2^{|B|^I} \cdot \mathfrak{m}' < \mathfrak{m}$.

Hence, by saturation, all requirements for $\eta(0)$ can be fulfilled. \square

1.9. Regular Lim-Rims, Lim-Rims with Homogeneity and Universality.

Definition 21. A lim-rim E over B is called **regular** if it is $|E|$ -exact.

A regular lim-rim is *universal*, in the sense that any embedding in E of a sub-lim-rim M' of cardinality $< |E|$ of a separated lim-rim M over B of cardinality $|M| \leq |E|$ can be extended to an embedding of all M .

Indeed, the embedding of M' can be viewed as a family of elements of E so that the pull-back of the defining cylinder ultrafilter in B^E is the defining cylinder ultrafilter in $B^{M'}$, which is the projection of the defining cylinder ultrafilter in B^M , and one uses Remark 16. (Separation will guarantee that distinct elements map to distinct elements, so one really has a one-to-one embedding)

If, instead of the argument of Remark 16 one uses a transfinite back-and-forth construction, i.e. adds an element i on one side of the embedding, then an element on the other, then again on the first side etc., one concludes that a regular lim-rim is *homogeneous* – any isomorphism between two sub-lim-rims of cardinality $< |E|$ in two regular lim-rims E over B with the same cardinality $|E|$, can be extended to an isomorphism between the E 's. In particular (if we start from the empty isomorphism) any two such lim-rims E are isomorphic. Also (starting from singletons) any two members of E which map to the same ultrafilter in βB are exchangeable by an automorphism of E .

Thus one might say that there is, up to isomorphism, at most one regular lim-rim over B of any specific cardinality.

Prop. 17 implies

Proposition 22. Let E be a regular lim-rim over an (infinite) set B . Let $B' \subset B$, B' also infinite. Let $E' = \ast B'$ be the set of members of E that \ast belong to B' . Then the sub-lim-rim E' over B' is also regular.

Indeed, by Prop. 17 E' is $|E|$ -exact, *a fortiori* $|E'|$ -exact.

1.10. Cardinalities.

Proposition 23. Suppose a non-standard analysis has confinement \mathfrak{m} and saturation \mathfrak{m}' . Then $\mathfrak{m}' \leq \mathfrak{m}^+$.

Proof. Take a (standard) set A of cardinality \mathfrak{m}^+ , and well-order it by the set of all ordinals of cardinality $\leq \mathfrak{m}$. Take the standard subsets $[a, \rightarrow [$ of A , and the corresponding internal sets of their \ast members. Any finite number of these intersect, and there are \mathfrak{m}^+ of them. If we had saturation $(\mathfrak{m}^+)^+$ there would be a \ast member $\ast a$ of A which belongs to all of them, hence is bigger than all standard $a \in A$, and thus cannot \ast belong to any standard set $A' \subset A$ of

cardinality $\leq \mathfrak{m}$ since any such set is bounded by some (standard) $a \in A$. Thus we would not have confinement \mathfrak{m} . \square

If a lim-rim E over B is exact, then, as we saw in §1.7, for every ultrafilter \mathcal{U} in B there is a ${}^*b \in E$ that maps to \mathcal{U} in $E \rightarrow \beta B$. Since there are $2^{2^{|B|}}$ ultrafilters in B (see Appendix A), we have

Proposition 24. *If a lim-rim E over B is exact, then $|E| \geq 2^{2^{|B|}}$.* \square

Proposition 25. *Let B be an infinite set of cardinality \mathfrak{b} . Then there exists a lim-rim E over B such that: (i) $|E| = 2^{2^{\mathfrak{b}}}$; (ii) The lim-rim E is $(2^{\mathfrak{b}})^+$ -exact.*

(This does not require any Generalized Continuum Hypothesis (GCH). But if we have GCH, at least the case $2^{2^{\mathfrak{b}}} = (2^{\mathfrak{b}})^+$ of GCH, we conclude that E is regular – is the regular lim-rim over B with cardinality $2^{2^{\mathfrak{b}}}$).

By Prop. 20 we will have that the $|B|$ -confined non-standard analysis induced by this lim-rim is \mathfrak{b}^+ -saturated, and is $(2^{\mathfrak{b}})^+$ -saturated inside a (standard) set of cardinality $\leq \mathfrak{b}$.

Proof. Let Ω be the set of ordinals of cardinality $\leq 2^{\mathfrak{b}}$. It is the smallest ordinal of cardinality $(2^{\mathfrak{b}})^+$.

We shall construct E , together with the defining cylinder ultrafilter \mathcal{L} , in steps indexed by Ω , always adding $e \in E$'s so that for a family $\eta : I \subset 2^B \rightarrow$ the set E' of previously constructed e 's and for a cylinder ultrafilter \mathcal{U} in $B^{I \cup \{0\}}$ whose projection on B^I is the ultrafilter pulled-back from \mathcal{L} by η , the added e will serve as $\eta(0)$ so that \mathcal{U} is the ultrafilter pulled-back by the extended η . To this end we appropriately choose the ultrafilter \mathcal{L} in the cylinders of $B^{E' \cup \{e\}}$, of course one that projects on $B^{E'}$ to the previously constructed \mathcal{L} .

For any $\iota \in \Omega$, supposing we already have E_ι and \mathcal{L}_ι , we shall take care simultaneously of all possible η into $E' = E_\iota$ and \mathcal{U} 's, adding one e for each of them, to get $E_{\iota+1}$. If we assume as an induction hypothesis that $|E_\iota| \leq 2^{2^{\mathfrak{b}}}$, then there are at most $(2^{2^{\mathfrak{b}}})^{2^{\mathfrak{b}}} = 2^{2^{\mathfrak{b}}}$ such η 's, and for each $I \subset 2^B$ the cardinality of the cylinder Boolean algebra of B^I is $|I| \cdot 2^{\mathfrak{b}} \leq 2^{\mathfrak{b}}$, hence the cardinality of all possible cylinder ultrafilters, these being subsets of this Boolean algebra, is $\leq 2^{2^{\mathfrak{b}}}$. Thus there are at most $2^{2^{\mathfrak{b}}}$ pairs η, \mathcal{U} to take care of, hence $2^{2^{\mathfrak{b}}}$ added e 's, and also $|E_{\iota+1}| \leq 2^{2^{\mathfrak{b}}}$.

For limit ordinals $\iota < \Omega$ (and, in fact, also for Ω itself), we just take as E_ι the union of everything constructed at $\iota' < \iota$ (a union of at most $2^{\mathfrak{b}}$ terms – $(2^{\mathfrak{b}})^+$ for Ω) and as \mathcal{L} the union of the \mathcal{L} 's – note that we are always considering cylinders, which depend only on a finite number of coordinates. Clearly we will preserve the fact $|E_\iota| \leq 2^{2^{\mathfrak{b}}}$.

The final E will be E_Ω , after we had made it *separated* by taking its factor set with respect to the equivalence relation on e_1, e_2 : $\{\psi \mid \psi_{e_1} = \psi_{e_2}\} \in \mathcal{L}$. (i) will be satisfied. (By the above $|E| \leq 2^{2^{\mathfrak{b}}}$ the inverse inequality is Prop. 24). (ii) will be satisfied too, since every η for E_Ω concerns at most $2^{\mathfrak{b}}$ elements e , constructed at $2^{\mathfrak{b}}$ ι 's, the set of these ι 's is bounded in Ω by some ι' , and this η has been dealt with – received its $\eta(0)$ – already in $E_{\iota'}$.

We still have to prove that we can construct the cylinder ultrafilter $\mathcal{L}_{\iota+1}$ when we construct $E_{\iota+1}$ above. $\mathcal{L}_{\iota+1}$ must: (a) Project to \mathcal{L}_ι on B^{E_ι} . That is, contain all cylinders, depending on a finite number of coordinates in E_ι , that belong to \mathcal{L}_ι . (b) For each e constructed in the $\iota + 1$ step for some η and \mathcal{U} , it must be pulled back by the extended η to \mathcal{U} . That is, for every $U \in \mathcal{U}$ which depends, as a subset of B^{n+1} , on the coordinates $0, i_1, \dots, i_n$, with $\eta(i_j) = e_j \in E_\iota$, $j = 1, \dots, n$ it must contain the set

$$(2) \quad \{\psi \in B^{E_{\iota+1}} \mid (\psi_e, \psi_{e_1}, \dots, \psi_{e_n}) \in U\}.$$

To prove such a cylinder ultrafilter $\mathcal{L}_{\iota+1}$ exists, all we need to show is that any finite set of these sets, which it must contain, intersect. That is, we must prove that for any cylinder V

in \mathcal{L}_i , which is of the form (here $e_1, \dots, e_n \in E_i$, $V \subset B^n$):

$$(3) \quad \{\psi \in B^{E_i+1} \mid (\psi_{e_1}, \dots, \psi_{e_n}) \in V\},$$

and any finite number of e 's and sets as in (2), there is a ψ which belongs to all of them. For the latter, we will be able to find a ψ_e if $(\psi_{e_1}, \dots, \psi_{e_n})$ belongs to the projection U' of U on $B^n = B^{\{i_1, \dots, i_n\}}$. This projection is, by assumption, in the cylinder ultrafilter pulled-back from \mathcal{L}_i by the unextended η . This means that the set of ψ satisfying this is a cylinder in \mathcal{L}_i , as in (3). Hence we are left with a finite number of conditions of the form (3), for cylinders in \mathcal{L}_i which intersect, \mathcal{L}_i being a cylinder ultrafilter. \square

Proposition 26. *Let \mathfrak{b} be an infinite cardinality.*

*Let us work in a non-standard analysis with confinement \mathfrak{b} , saturation \mathfrak{b}^+ and saturation $(2^{\mathfrak{b}})^+$ inside (standard) sets of cardinality $\leq \mathfrak{b}$, and in which for $|B| = \mathfrak{b}$ we have $|{}^*B| = 2^{2^{\mathfrak{b}}}$. Then*

- (i) *Let B be a set of cardinality \mathfrak{b} . Then for every internal subset $\beta \subset {}^*B$ which contains all standard elements of B , and for every ultrafilter \mathcal{U} on B , $\exists {}^*x \in \beta$ whose ultrafilter is \mathcal{U} .*

*Consequently, $|\beta| = 2^{2^{\mathfrak{b}}}$ (recall that $|{}^*B| = 2^{2^{\mathfrak{b}}}$).*

- (ii) *If α is internal, then either α is finite or \exists an internal one-one mapping from an internal β as in (i) to α , hence $|\alpha| \geq 2^{2^{\mathfrak{b}}}$.*
- (iii) *If α is * finite (i.e. a * member of a standard family of finite sets) then $|\alpha|$ is finite or $2^{2^{\mathfrak{b}}}$.*

Proof. (i) We use $(2^{\mathfrak{b}})^+$ -saturation (which holds when working inside B): for any finite family of members of $\mathcal{U} \exists {}^*x \in \beta$ belonging to all of them (indeed a standard one), and by saturation we are done.

(ii) Here we use \mathfrak{b}^+ -saturation: If α is not finite, then for every finite set F of standard members of $B \exists$ an internal set $\beta_F \subset {}^*B$ containing all members of F and an internal one-one mapping $\beta_F \rightarrow \alpha$ (take $\beta_F = F$), hence by saturation \exists a β suitable for all standard members of B .

(iii) Suppose α not finite. By (ii) $|\alpha| \geq 2^{2^{\mathfrak{b}}}$. On the other hand, by \mathfrak{b}^+ -confinement α * belongs to a standard set S of cardinality $\leq \mathfrak{b}$ whose members are finite sets. Then $|\cup S| \leq \mathfrak{b}$, thus $|{}^*(\cup S)| \leq 2^{2^{\mathfrak{b}}}$, while $\alpha \subset {}^*(\cup S)$. \square

From Prop. 25, Prop. 26 and Prop. 22 we get

Corollary 27. *Let E be a lim-rim over B as in Prop. 25. Let B' be an infinite subset of B . Suppose we have the case $2^{2^{|B|}} = \left(2^{|B|}\right)^+$ of GCH. Let $E' = {}^*B' \subset E$. Then the lim-rim E' over B' is the regular lim-rim of cardinality $2^{2^{|B|}}$ over B' .*

2. ITERATED NON-STANDARD ANALYSIS

2.1. The Embeddings ν and ${}^*\nu$. Fix a non-standard analysis as per Definition 1. For every set A , we have *A , but, of course, *A being again a set, we have ${}^{**}A$, also ${}^{***}A$ etc. The non-standard analysis defines, for every set A , an injection $\nu : A \rightarrow {}^*A$ mapping $a \in A \mapsto \nu(a) \in {}^*A$. Therefore we have (again an injection, by Transfer) ${}^*\nu : {}^*A \rightarrow {}^{**}A$. Note that we still have $\nu : {}^*A \rightarrow {}^{**}A$.

Both ν and ${}^*\nu$ are injective. Thus both “embed” *A in ${}^{**}A$. Yet they are highly different, as we shall see.

ν , of course, satisfies Transfer. But by Transferring Transfer we find that also ${}^*\nu$ satisfies a kind of Transfer. One just has to be careful with quantifiers – on which set to take them.

Let us note that

Proposition 28. *For any (standard) set A , and any (standard) $a \in A$, the $**A$ -elements $\nu(\nu(a))$ and $(*\nu)(\nu(a))$ are the same.*

Proof. For any map $f : A \rightarrow B$, $*f : *A \rightarrow *B$ does to the ν of elements of A the same as f does to the elements. This means $*f(\nu(a)) = \nu(f(a))$, $a \in A$. Now take $B = *A$ and $f : A \rightarrow B$ to be $\nu : A \rightarrow *A$. \square

And one easily sees that $a \mapsto \nu(\nu(a))$ and $a \mapsto (*\nu)(\nu(a))$ “transfer” relations and functions in the same way.

To clarify the relationship between these ν and $*\nu$ let us consider, as an example, the case $A = \mathbb{R}$, with its subset \mathbb{N} .

2.2. Non-Standard Analysis and Iterated Non-Standard Analysis in \mathbb{R} and \mathbb{N} . We place ourself in a non-standard analysis which is \mathbb{N}_0^+ -saturated, and look at the non-standard analysis inside \mathbb{R} , in particular inside its subset \mathbb{N} .

\mathbb{N}_0^+ -saturation implies that for a descending sequence $*a_n$, (and an ascending sequence $*b_n$) of $*$ members of \mathbb{R} (or any other linearly ordered set), such that $*a_n > *b_m \forall n, m$ there is a $*$ member $*c$ of \mathbb{R} such that $*a_n > *c$ (and $*c > *b_m$) $\forall n, m$. In particular, there are $*$ reals, also $*$ natural numbers, that are bigger than all finite standard numbers. These are sometimes called (positive) *infinite* or *unlimited*, while a $*$ real whose absolute value is not unlimited will be called *limited*. While for any $n \in \mathbb{N}$ the set $] \leftarrow, n]$ in \mathbb{N} is finite, therefore all its $*$ members are standard, and consequently all $*$ natural numbers are either standard or unlimited, there are $*$ real numbers > 0 but smaller than any standard positive real. As is well-known, these are called (positive) *infinitesimal* $*$ reals (and a positive or negative $*$ real is called infinitesimal if its absolute value is infinitesimal). As is well-known, and easily proved using the completeness of \mathbb{R} (say, the existence of suprema and infima), for any *limited* $*a \in *R$ there is a unique standard $\text{st}(*a) \in \mathbb{R}$ so that $*a - \text{st}(*a)$ is infinitesimal. Two $*$ reals whose difference is infinitesimal are called *near*.

Now, as said above, $*R$ also has its $*$ members, forming the set $**R$ which may be referred to as the $**$ reals. And there is a $\nu : *R \rightarrow **R$ which maps each $*$ real to the $**$ real to be identified with it, and since there is $\nu : \mathbb{R} \rightarrow *R$, it gives rise to $*\nu : *R \rightarrow **R$. As said above, these are two “embeddings” of $*R$ into $**R$.

Let’s take a look at the set of positive (thus nonzero) infinitesimal $*$ reals. Note that, by the above, for any ascending (resp. descending) sequence of them there is a positive infinitesimal further bigger (resp. smaller) than all of them. The set of infinitesimals is external – if it were internal it would have a $*$ supremum which is impossible. It has $*$ members, the $*$ infinitesimals – they are $**$ reals. In particular, since for any set A ν maps A into $*A$, ν of infinitesimals are $*$ infinitesimals.

Now, since the positive infinitesimals are the positive $*$ reals that are smaller than $\nu(r)$ for all reals $r > 0$, we have by Transfer: the positive $*$ infinitesimals are the positive $**$ reals smaller than $*\nu(*r)$ for all $*$ reals $*r > 0$. But note that we can take here as $*r$ a positive infinitesimal! Thus we have:

All positive $*$ infinitesimals, which include the ν of positive infinitesimals, are smaller than all $*\nu$ of positive infinitesimals!

Turning to the other end, there are the positive $*$ unlimited $**$ reals, which include the ν of unlimited $*$ reals. They are characterized as bigger than $*\nu(*r)$ for all $*r \in *R$. Thus they are bigger than the $*\nu$ of unlimited $*$ reals.

Consider, in particular, $**$ natural numbers. We saw that the $*$ atural numbers are partitioned into two sets: $\nu(\mathbb{N})$ and the unlimited $*$ natural numbers, the latter all bigger than the former. This gives by Transfer: the $**$ natural numbers are partitioned into $*\nu(*\mathbb{N})$ and the $*$ unlimited $**$ natural numbers, the latter all bigger than the former. The latter include the ν of unlimited $*$ natural numbers. The former are, of course, partitioned into $\nu(\nu(\mathbb{N})) = (*\nu)(\nu(\mathbb{N}))$, and the $*\nu$ of the unlimited, the latter all bigger than the former. To repeat: $**\mathbb{N}$ is partitioned into three parts: $\nu(\nu(\mathbb{N})) = (*\nu)(\nu(\mathbb{N}))$, all smaller than $*\nu$ of the unlimited, these all smaller than the $*$ unlimited, which include the ν of the unlimited.

If we assume the non-standard analysis is \aleph_0 -confined (or more generally, is \mathfrak{m} -confined and \mathfrak{m}^+ -saturated for some infinite cardinal \mathfrak{m}), we will have: every positive $*$ infinitesimal $**\varepsilon$ belongs to a countable set of infinitesimals, the latter, by \aleph_0^+ -saturation, as said above, has a positive infinitesimal upper (resp. lower) bound. Hence (the ν value) of the bound is bigger (resp. smaller) also than $**\varepsilon$. Thus “being less than (the ν -value of) a positive infinitesimal” is the same as “being less than a positive $*$ infinitesimal”. Similarly, in $**\mathbb{R}$ or in $**\mathbb{N}$, every $*$ unlimited $**$ number has an upper (resp. lower) bound which is ν of an unlimited $*$ number.

We had the (equivalence) relation in $*\mathbb{R}$ of “being near”, i.e. having an infinitesimal difference. In fact, it is an intersection $\bigcap_n *U_n$ of the $*$'s of a decreasing sequence of standard relations $U_n \subset \mathbb{R} \times \mathbb{R}$, in this case a basis to the uniformity in \mathbb{R} . We had: the equivalence class of a limited $*a \in *\mathbb{R}$ contains a unique standard $\nu(\text{st}(*a))$, $\text{st}(*a)$ called the *standard part* of $*a$. Moving from such a relation to a relation in $**\mathbb{R}$, we have two options:

The “strong” (equivalence) relation $*(\bigcap_n *U_n)$ – the difference being a $*$ infinitesimal. The existence of the standard part implies by Transfer: the equivalence class of a $*$ limited $**a \in **\mathbb{R}$ contains a unique element $*\nu(*\text{st}(**a)) \in *\nu(*\mathbb{R})$.

The “weak” (equivalence) relation $\bigcap_n **U_n$, in our case saying that the difference being less than any $\nu(\nu(a))$, $a > 0, a \in \mathbb{R}$. Lets call such $**$ reals “*weakly infinitesimal*”. These are the elements equivalent to 0, and they include all $*\nu$ of infinitesimals. In fact, any $**a \in **\mathbb{R}$ with standard bounds is thus “weakly” equivalent to $\nu(\nu(\text{st}(*\text{st}(**a))))$, which is the only standard element in its equivalence class.

(Note that we could start with only the rational numbers \mathbb{Q} , consider $*\mathbb{Q}$, define infinitesimals and being near as above, and then recover \mathbb{R} as the set of equivalence classes of *limited* $*$ rationals with respect to nearness. Transferring, $*\mathbb{R}$ will be identified with the set of equivalence classes of $*$ limited $**$ rationals with respect to $*$ nearness, i.e. having a $*$ infinitesimal difference.)

2.3. A Criterion for ν and $*\nu$ of $*$ Elements to Satisfy Some Standard Relation, and Applications.

Return to general (standard) sets.

Proposition 29. *Let A and B be (standard) sets and let $R \subset A \times B$ be a (standard) relation between elements of A and elements of B . Let $*a \in *A$ and $*b \in *B$. Then a necessary and sufficient condition that $*\nu(*b)$ is in the relation R with $\nu(*a)$ is*

$$*b \text{ belongs to the set of (standard) } b \in B \text{ that are in the relation } R \text{ with } *a \quad (\star)$$

Proof. Indeed, (\star) holds if and only if $*b$ satisfies the $*$ of the property of $b \in B$: “ $\nu(b)$ is in the relation R with $*a$ ”. By Transferring, this means that it satisfies: “ $*\nu(*b)$ is in the relation R with $\nu(*a)$ ”. \square

Some examples:

- Take R as the relation of *inequality* in A . Thus to have ${}^*\nu({}^*b) \neq \nu({}^*a)$, *b must * belong to the set of $b \in A$ unequal to *a , this set being the whole A unless *a is standard. For standard ${}^*a = \nu(a)$, $a \in A$, ${}^*\nu({}^*b)$ can equal $\nu({}^*a) = {}^*\nu(\nu(a))$ only if ${}^*b = \nu(a)$. Therefore:

The ${}^*\nu$ of a *A -member can equal the ν of a *A -member only if both are standard and equal.

- Let A be partially ordered, and take as R the relation $a \geq b$. (\star) says that *b * belongs to the cone of the standard minorants of a . Thus if *a is “infinite” in the sense of majorizing all standard elements, then the latter cone is everything so we always have ${}^*\nu({}^*b) \leq \nu({}^*a)$ (similarly to what we saw in \mathbb{R} and \mathbb{N} in the previous subsection).
- Let A be a (standard) set and B a set of subsets of A , with the relation $a \in b$. Then $\nu({}^*a) \in {}^*\nu({}^*b)$ (here we identify a * set with its scope – see §1.2) if and only if *b is a * member of the family of standard sets * containing *a , and ${}^*\nu({}^*a) \in \nu(b)$ if and only if *a is a * member of the set of standard members of *b .
- Let X be a Banach (resp. Hilbert) space. Then on *X there is a ${}^*\mathbb{R}$ -valued norm (resp. inner product and norm). In a well-known construction, one takes the space of elements of *X with *limited* norm and quotient it by the subspace of infinitesimal vectors (i.e. those with infinitesimal norm), (and denote equivalence classes modulo the infinitesimals by $[]$), to get an ordinary Banach (resp. Hilbert) space, called the *non-standard hull* of X . Denote it by $\mathcal{H}(X)$. It is complete, provided we have \aleph_0^+ -saturation.⁸ The $\nu : H \rightarrow {}^*H$, followed by the factoring by the infinitesimals, defines a (norm-preserving) embedding of X into $\mathcal{H}(X)$.

(In general, this embedding is not onto – $\mathcal{H}(X)$ has much more elements than X . For example, if $X = \ell^p$ then $\mathcal{H}(X)$ consists of equivalence classes of “sequences” defined on ${}^*\mathbb{N}$, and a sequence may have infinitesimal entries but finite nonzero norm, or can be supported on unlimited * numbers, and then it is not in the image of X .)

As for ${}^{**}X$, we have here, similarly to what we saw in \mathbb{R} , the * infinitesimal vectors – vectors with * infinitesimal norm, which are part of the ** vectors with “weakly infinitesimal” norm. Call the latter “weakly infinitesimal” ** vectors. Factoring the space of ** vectors whose norm is bounded by a ($\nu \circ \nu$ of a) standard number by the “weakly infinitesimal” ones one again gets an ordinary Banach space (resp. Hilbert space) $\mathcal{H}^{(2)}(X)$.

Now, $\nu, {}^*\nu : {}^*X \rightarrow {}^{**}X$ will define two embeddings of $\mathcal{H}(X)$ into $\mathcal{H}^{(2)}(X)$, which by abuse of language we also refer to as ν and ${}^*\nu$.

To see an example of what Prop. 29 would say here, take X to be a Hilbert space. Suppose $[{}^*a] \in \mathcal{H}(X)$ and $[{}^*b]$ belong to the orthogonal complement of X in $\mathcal{H}(X)$. Then for any (standard) $\varepsilon > 0$ the set of standard vectors b with $|\langle b, {}^*a \rangle| < \varepsilon$ is everything. So by Prop. 29 $|\langle {}^*\nu({}^*b), \nu({}^*a) \rangle| < \varepsilon$. Consequently:

If $[{}^*a] \in \mathcal{H}(X)$ and $[{}^*b]$ belong to the orthogonal complement of X in $\mathcal{H}(X)$ then $[{}^*\nu({}^*b)]$ is always orthogonal to $[\nu({}^*a)]$. (In particular, $[{}^*\nu({}^*b)]$ is orthogonal to $[\nu({}^*b)]$.)

Another application of Prop. 29,

⁸Indeed, Suppose $[{}^*x_n]$ is a Cauchy sequence in $\mathcal{H}(X)$. For any $m \in \mathbb{N}$, choose a ball $B_m = \{{}^*x \in {}^*X \mid \|{}^*x - {}^*a\| < 1/m\}$ of radius $1/m$ which contains all *x_n from some n on. The B_m are internal sets and any finite number of them intersect (since they all contains all *x_n from some n onward). By \aleph_0^+ -Saturation they all intersect, and if *x_0 is a member of the intersection, then clearly $[{}^*x_n] \rightarrow [{}^*x_0]$.

Proposition 30. *Assume the non-standard analysis is \aleph_0^+ -saturated.*

Let A be linearly ordered. (i.e. every $a, b \in A$ satisfy $a \geq b$ or $b \geq a$). Then a necessary and sufficient condition for A to be well-ordered is that

$$\text{For every } {}^*a \in {}^*A, {}^*\nu({}^*a) \leq \nu({}^*a).$$

Proof. Suppose A is well-ordered and ${}^*a \in {}^*A$. Let c be the least (standard) element of A satisfying $c > {}^*a$ – if there is no c with $c > {}^*a$ denote $c = \infty$. Then *a belongs to the set of standard elements $< c$, which is the set of standard elements $\leq {}^*a$, hence by Prop. 29 ${}^*\nu({}^*a) \leq \nu({}^*a)$.

Suppose now that A is not well-ordered, thus contains a strictly descending sequence (a_n) . By \aleph_0^+ -saturation, there is a ${}^*a \in {}^*\{a_n\}$ smaller than all the a_n . The set of (standard) members of $\{a_n\}$ that are $> {}^*a$ is the whole $\{a_n\}$, to which *a certainly belongs. Hence, by Prop. 29, ${}^*\nu({}^*a) > \nu({}^*a)$ and the condition is not satisfied. \square

This gives an immediate (non-standard) proof to the fact that if some (standard) operation \circ in A satisfies

$$a' \geq a, b' \geq b \Rightarrow a' \circ a \geq b' \circ b$$

then if $S, S' \subset A$ are well-ordered then $S \circ S'$ is also well-ordered.

Similar things can be proved for *partial* orders.

APPENDIX A. A PROOF THAT AN INFINITE SET B HAS $2^{2^{|B|}}$ ULTRAFILTERS

For completeness, we give a proof for the well known

Proposition 31. *The cardinality of the set βB of ultrafilters in an infinite set B is $2^{2^{|B|}}$.*

Proof. Firstly, $|\beta B| \leq 2^{2^{|B|}}$ since $\beta B \subset \mathcal{P}\mathcal{P}B$.

To prove the inverse inequality, it is enough to find a compact Hausdorff topological space X of cardinality $2^{2^{|B|}}$ with a dense subset of cardinality $|B|$. Indeed, every point of X is a limit of an ultrafilter in the dense subset, and an ultrafilter has a unique limit, hence there must be at least $2^{2^{|B|}}$ such ultrafilters.

We proceed to find such an X .

Denote by $\mathbf{2} = \{0, 1\}$ the field, thus additive group, with two elements, endowed with the discrete topology.

Consider the group $\mathbf{2}^B$. By viewing the members of $\mathbf{2}^B$ as characteristic functions of subsets of B , we can identify $\mathbf{2}^B = \mathcal{P}B$. $\mathbf{2}^B$ is a linear vector space over the field $\mathbf{2}$. As such, it has a (Hamel) basis W , and since the vector space is the set of sums of finite subsets of W , we have $|W| = |\mathbf{2}^B| = 2^{|B|}$.

Let H be the group of all homomorphisms $h : \mathbf{2}^B \rightarrow \mathbf{2}$. Since such a homomorphism is uniquely determined by giving its values on the members of W , which may be given arbitrarily, we may identify $H = \mathbf{2}^W$. The product topology here, which is also the weakest topology such that all evaluation maps at elements of $\mathbf{2}^B$ are continuous (hence does not depend on the choice of basis W), makes H a compact (Hausdorff) topological group of cardinality $2^{2^{|B|}}$.

B can be viewed as a subset of H , identifying every $b \in B$ with the homomorphism $h : \mathbf{2}^B \rightarrow \mathbf{2}$ of evaluation at b . Let $\langle B \rangle$ be the subgroup (also $\mathbf{2}$ -linear subspace) of H generated by this B . Since $\langle B \rangle$ is just the set of sums of finite subsets of B , we have $|\langle B \rangle| = |B|$.

Thus, we shall be finished if we prove $\langle B \rangle$ is dense in H .

Let $\overline{\langle B \rangle}$ be the closure of $\langle B \rangle$ in H , and suppose $\overline{\langle B \rangle} \neq H$. Then there is a point $h_0 \in H \setminus \overline{\langle B \rangle}$ and a neighborhood of h_0 disjoint from $\overline{\langle B \rangle}$. Thus there are $w_1, \dots, w_n \in W$ so that no element of $\overline{\langle B \rangle}$, in particular no element $\sum_i b_i \in \langle B \rangle$, $(b_i)_i$ a finite subset of B , satisfies $\sum_i w_1(b_i) = h_0(w_1), \dots, \sum_i w_n(b_i) = h_0(w_n)$. This means that the set of all

$\sum_{\iota} (w_1(b_{\iota}), \dots, w_n(b_{\iota})) \in \mathbf{2}^n$ for all finite subsets $(b_{\iota})_{\iota}$, is not the whole of $\mathbf{2}^n$. Since this set is a linear subspace of the n -dimensional vector space $\mathbf{2}^n$ over $\mathbf{2}$, and it is not the whole space, there is a non-zero linear functional which vanishes on it. Thus, there are $\varepsilon_1, \dots, \varepsilon_n \in \mathbf{2}$, not all of them 0, so that $\sum \varepsilon_j w_j(b) = 0$ for all $b \in B$, that is $\sum \varepsilon_j w_j = 0$, contradicting the linear independence of the basis W . \square

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