

Multi-stage waveform Relaxation and Multisplitting Methods for Differential Algebraic Systems

Jürgen Geiser¹

Ruhr University of Bochum, Department of Electrical Engineering and Information Technology, Universitätsstraße 150, D-44801 Bochum, Germany,
juergen.geiser@ruhr-uni-bochum.de

Abstract. We are motivated to solve differential algebraic equations with new multi-stage and multisplitting methods. The multi-stage strategy of the waveform relaxation (WR) methods are given with outer and inner iterations. While the outer iterations decouple the initial value problem of differential algebraic equations (DAEs) in the form of $A \frac{dy(t)}{dt} + By(t) = f(t)$ to $M_A \frac{dy^{k+1}(t)}{dt} + M_1 y^{k+1}(t) = N_1 y^k(t) + N_A \frac{dy^k(t)}{dt} + f(t)$, where $A = M_A - N_A$, $B = M_1 - N_1$. The inner iterations decouple further $M_1 = M_2 - N_2$ and $M_2 = M_3 - N_3$ with additional iterative processes, such that we result to invert simpler matrices and accelerate the solver process. The multisplitting method use additional a decomposition of the outer iterative process with parallel algorithms, based on the partition of unity, such that we could improve the solver method. We discuss the different algorithms and present a first experiment based on a DAE system.

Keywords. Numerical Analysis, Multi-level Waveform relaxation methods, Multi-splitting methods, Differential-Algebraic equations.

AMS subject classifications 35K28, 35K27, 35K20, 47D60, 47H20, 65M12, 65M55.

1 Introduction

We are motivated to accelerate solver methods for differential-algebraic equations (DAEs). Many mathematical methods are based on such combination of differential and algebraic equations, e.g., simulations of the power systems, constrained mechanical systems, singular perturbations, see [6] and [2].

We start with respect to assume, that the considered partial differential-algebraic equations (PDAEs) can be semi-discretized and written into an initial value problem of differential algebraic equations (DAEs) in the form:

$$A \frac{dy(t)}{dt} + By(t) = f(t), \quad t \in [t_0, T], \quad y(t_0) = y_0, \quad (1)$$

where $A \in \mathbb{C}^{m \times m}$ is a singular and $B \in \mathbb{C}^{m \times m}$ is a non-singular complex matrix with rank m and $f(t) : [t_0, T] \rightarrow \mathbb{C}^m$ is a sufficient smooth right hand side.

For solving such problem initial value problems, waveform-relaxation (WR) methods are developed and investigated by many authors, see [3], [5] and [7].

The main idea is to decompose the partitions of large systems into iteratively coupled smaller subsystems and solve such subsystems independently over the integration intervals (also called time-windows), see [7]. For the performance of the algorithms, since recent years two-stage strategy is introduced in WR methods, means a first splitting in blocks for pure parallel splitting. For each processor, we apply additional a splitting, i.e., an inner splitting instead of a direct method, see [1].

We propose additional a multistage splitting, means, we assume to split additional the inner splitting such that we could also preform the inner splitting with parallel splitting, while only the last inner splitting is done serial.

The new class of multi-stage waveform-relaxation (MSWR) methods are discussed in the following, with respect to the three-stage waveform-relaxation (TH-SWS) method.

For simplification, we deal with

$$\frac{dy(t)}{dt} + By(t) = f(t), \quad t \in [t_0, T], y(t_0) = y_0, \quad (2)$$

where $B = M_1 - N_1$ and the outer iteration is obtained as

$$\frac{dy^{k+1}(t)}{dt} + M_1 y^{k+1}(t) = N_1 y^k(t) + f(t), \quad (3)$$

$$y^{k+1}(t_0) = y_0, \quad k = 1, 2, \dots, \quad (4)$$

where $B = M_1 - N_1$, then we apply the first inner iteration:

$$\frac{dz^{\nu+1}(t)}{dt} + M_2 z^{\nu+1}(t) = N_2 z^\nu(t) + N_1 y^k(t) + f(t), \quad (5)$$

$$z^{\nu+1}(t_0) = y_0^k, \quad k = 1, 2, \dots, \nu = 0, \dots, \nu_k - 1, \quad (6)$$

where $M_1 = M_2 - N_2$ and we obtain $y^{k+1} = z^{\nu_k}$.

The last or second inner iteration is given as:

$$\frac{d\tilde{z}^{\mu+1}(t)}{dt} + M_3 \tilde{z}^{\mu+1}(t) = N_3 \tilde{z}^\mu(t) + N_2 z^\nu(t) + N_1 y^k(t) + f(t), \quad (7)$$

$$\tilde{z}^{\mu+1}(t_0) = z^{\nu+1}(t_0) = y_0^k, \quad (8)$$

$$k = 1, 2, \dots, \nu = 0, \dots, \nu_k - 1, \mu = 0, \dots, \mu_{\nu_k} - 1,$$

where $M_2 = M_3 - N_3$ and we obtain $y^{k+1} = z^{\nu_k} = \tilde{z}^{\mu_{\nu_k}}$. We have at least μ_{ν_k} inner first iterations within the inner second iteration ν_k and within the $k + 1$ outer iterations. Means we deal with three-stage iterative methods.

For more flexibility in the approaches, we apply a multisplitting method, which is based on the partition of the unit, see [4].

We decompose the solution into several units, means we have:

$$y^{k+1} = \sum_{p=1}^L E_p y^{p,k+1}, \quad (9)$$

$$\sum_{p=1}^L E_p = I, \quad (10)$$

where $I \in \mathbb{C}^{m \times m}$ is the unit matrix and E_p are diagonal and the diagonal entries are given as $E_{p,ii} \geq 0$ for $i = 1, \dots, m$.

We can solve L independent waveform relaxation schemes in parallel, while the synchronization or update is done with the Equation (9).

The paper is outlined as following. In Section 2, we discuss the different hierarchy of solver methods. The numerical experiments are presented in Section 3. The conclusions are done in Section 4.

2 Hierarchy of Solver schemes

In the following, we deal with the following hierarchy of solver schemes:

1. One-stage WR schemes,
2. Two-stage WR schemes,
3. Three-stage WR schemes,
4. Multisplitting WR schemes: Jacobian-, Gauss-Seidel-Types,

where, we simplify the inversion of the matrices between the one-stage to the three-stage method, means the simplification of the inversion is done via additional inner iterative stages. Further the multisplitting approach allows to be more flexible in the parallelization of a multi-stage method.

2.1 One-stage WR method

In the following, we discuss the one-stage WR method.

1. We have the following WR method (in parallel):

$$\begin{aligned} (M_A + hM_1)y_{n+1}^{k+1} &= h(N_1 + \frac{1}{h}N_A)y_{n+1}^k + M_A y_n^{k+1} \\ &- N_A y_n^k + h f_{n+1}, \\ y_0^{k+1} &= y_0, \quad k = 0, 1, \dots, K, \quad n = 0, 1, 2, \dots, J, \end{aligned} \quad (11)$$

2. We have the following WR method (in serial):

$$\begin{aligned} (M_A + hM_1)y_{n+1}^{k+1} &= h(N_1 + \frac{1}{h}N_A)y_{n+1}^k + M_A y_n \\ &- N_A y_n + h f_{n+1}, \\ y_{n+1}^0 &= y_n, \quad k = 0, 1, \dots, K, \quad n = 0, 1, 2, \dots, J, \end{aligned} \quad (12)$$

where we apply the algorithm with $p = 50, q = 6, J = 20, h = 0.1$, here we have $\Delta x = 1.0$ and $\frac{D}{\Delta x^2} = 1.0$.

We have 2 possible stopping criteria:

(a) Error bound:

We have an stopping error norm with $\|y_{n+1}^{k+1} - y_{n+1}^k\| \leq 10^{-3}$.

(b) Fix number of outer-iterative steps: $K = 20$.

2.2 Two-Stage WR method

The two-stage WR method is given as:

1. We have the following Two-Stage WR method (in parallel):

$$\begin{aligned} (M_A + hM_2)z_{n+1}^{\nu+1} &= hN_2z_{n+1}^\nu + h(N_1 + \frac{1}{h}N_A)y_{n+1}^k + M_Az_n^{\nu+1} \quad (13) \\ &- N_Ay_n^k + hf_{n+1}, \\ z_0^{\nu+1}(t_0) &= y^k(t_0) = y_0, \quad k = 0, 1, \dots, K, \\ \nu &= 0, 1, \dots, \nu_k, \quad n = 0, 1, 2, \dots, J. \end{aligned}$$

2. We have the following Two-Stage WR method (in serial):

$$\begin{aligned} (M_A + hM_2)z_{n+1}^{\nu+1} &= hN_2z_{n+1}^\nu + h(N_1 + \frac{1}{h}N_A)y_{n+1}^k + M_Ay_n \quad (14) \\ &- N_Ay_n + hf_{n+1}, \\ z_{n+1}^0(t_n) &= y_{n+1}^k, \quad \nu = 0, 1, \dots, \nu_k, \text{ inner iteration} \\ y_{n+1}^{k+1} &= z_{n+1}^{\nu_k}, \quad k = 0, 1, \dots, K, \text{ outer iteration} \\ z_{n+1}^0(t_n) &= y_n, \text{ initialization} \\ n &= 0, 1, 2, \dots, J, . \end{aligned}$$

Two-Stage WR algorithm (serial) 1 is the given as:

Given the initial vector $y_0 = y(0)$,

$z_{n+1}^0(0) = y_0$,

for $n = 0, 1, \dots, J$ **do**

for $k = 0, 1, \dots, K$ **do**

for $\nu = 0, 1, \dots, \nu_k$ **do**

$$\begin{aligned} & (M_A + hM_2)z_{n+1}^{\nu+1} = \\ & hN_2z_{n+1}^\nu + h(N_1 + \frac{1}{h}N_A)y_{n+1}^k + M_Ay_n - N_Ay_n + hf_{n+1}, \end{aligned}$$

end

$$y_{n+1}^{k+1} = z_{n+1}^{\nu_k},$$

$$z_{n+1}^0 = y_{n+1}^{k+1},$$

end

$$y_{n+1} = z_{n+1}^{\nu_K},$$

$$z_{n+1}^0(t_{n+1}) = y_{n+1},$$

end

Algorithm 1: Two-Stage WR algorithm (serial)

where we apply the algorithm with $p = 50, q = 6, J = 20, h = 0.1$. Here we have $\Delta x = 1.0$ and $\frac{D}{\Delta x^2} = 1.0$.

We have 2 possible stopping criteria:

(a) Error bound:

We have an stopping error norm :
 for the outer iteration with $\|y_{n+1}^{k+1} - y_{n+1}^k\| \leq 10^{-3}$,
 and for the inner iteration $\|z_{n+1}^{\nu+1} - z_{n+1}^\nu\| \leq 10^{-3}$

(b) Fix number of outer-iterative steps: $K = 5$ and inner iterative steps $\nu_k = 4$.

2.3 Three-Stage WR method

The three-stage WR method is given as:

1. We have the following Three-Stage WR method (in parallel):

$$\begin{aligned} (M_A + hM_3)\tilde{z}_{n+1}^{\mu+1} &= hN_3\tilde{z}_{n+1}^\mu + hN_2z_{n+1}^\nu + h(N_1 + \frac{1}{h}N_A)y_{n+1}^k \\ &+ M_A\tilde{z}_n^{\mu+1} - N_Ay_n^k + hf_{n+1}, \\ \tilde{z}_0^{\mu+1}(t_0) &= z_0^{\nu+1}(t_0) = y^k(t_0) = y_0, \quad k = 0, 1, \dots, K, \\ \nu &= 0, 1, \dots, \nu_k, \mu = 0, 1, \dots, \mu_{\nu_k}, \quad n = 0, 1, 2, \dots, J. \end{aligned}$$

2. We have the following Two-Stage WR method (in serial):

$$\begin{aligned} (M_A + hM_3)\tilde{z}_{n+1}^{\mu+1} &= hN_3\tilde{z}_{n+1}^\mu + hN_2z_{n+1}^\nu + h(N_1 + \frac{1}{h}N_A)y_{n+1}^k \\ &+ M_Ay_n - N_Ay_n + hf_{n+1}, \\ \tilde{z}_{n+1}^0(t_n) &= z_{n+1}^{\nu_k}, \quad \mu = 0, 1, \dots, \mu_{\nu_k}, \text{ first inner iteration} \\ z_{n+1}^0(t_n) &= y_{n+1}^k, \quad \nu = 0, 1, \dots, \nu_k, \text{ second inner iteration} \\ y_{n+1}^{k+1} &= \tilde{z}_{n+1}^{\mu_{\nu_k}}, \quad k = 0, 1, \dots, K, \text{ outer iteration} \\ \tilde{z}_{n+1}^0(t_n) &= y_n, \text{ initialization} \\ n &= 0, 1, 2, \dots, J, . \end{aligned}$$

Three-Stage WR algorithm (serial) 2 is the given as:

```

Given the initial vector  $y_0 = y(0)$  ,
 $z_{n+1}^0(0) = y_0$ ,
for  $n = 0, 1, \dots, J$  do
  for  $k = 0, 1, \dots, K$  do
    for  $\nu = 0, 1, \dots, \nu_k$  do
      for  $\mu = 0, 1, \dots, \mu_{\nu_k}$  do
         $(M_A + hM_3)\tilde{z}_{n+1}^{\mu+1} = hN_3\tilde{z}_{n+1}^\mu + hN_2z_{n+1}^\nu + h(N_1 +$ 
         $\frac{1}{h}N_A)y_{n+1}^k + M_Ay_n - N_Ay_n + hf_{n+1}$ ,
      end
       $z_{n+1}^{\nu+1} = \tilde{z}_{n+1}^{\mu_{\nu_k}}$ ,
       $z_{n+1}^0 = z_{n+1}^{\nu+1}$ ,
    end
     $y_{n+1}^{k+1} = z_{n+1}^{\nu_k}$ ,
     $z_{n+1}^0 = y_{n+1}^{k+1}$ ,
  end
   $y_{n+1} = z_{n+1}^{\nu_K}$ ,
   $z_{n+1}^0(t_{n+1}) = y_{n+1}$ ,
end

```

Algorithm 2: Three-Stage WR algorithm (serial)

where we apply the algorithm with $p = 50, q = 6, J = 20, h = 0.1$. Further we have $\Delta x = 1.0$ and $\frac{D}{\Delta x^2} = 1.0$.

We have 2 possible stopping criteria:

(a) Error bound:

We have an stopping error norm :

for the outer iteration with $\|y_{n+1}^{k+1} - y_{n+1}^k\| \leq 10^{-3}$,

and for the inner iteration $\|z_{n+1}^{\nu+1} - z_{n+1}^\nu\| \leq 10^{-3}$

and for the second inner iteration $\|\tilde{z}_{n+1}^{\mu+1} - \tilde{z}_{n+1}^\mu\| \leq 10^{-3}$

(b) Fix number of outer-iterative steps: $K = 5$ and inner iterative steps $\nu_k = 2, \mu_{\nu_k} = 2$.

2.4 Multisplitting WR method

We have the following Multi-splitting WR method (in serial/parallel):

$$(M_{A_l} + hM_{1,l})y_{n+1}^{l,k+1} = h(N_{1,l} + \frac{1}{h}N_{A_l}) \left(\sum_{m=1}^L E_{l,m}y_{n+1}^{l,k} \right) + M_{A_l}y_n \quad (15)$$

$$-N_{A_l}y_n + hf_{n+1},$$

$$y_{n+1}^{l,0} = y_n, \quad l = 1, \dots, L, \quad k = 0, 1, \dots, K, \quad n = 0, 1, 2, \dots, J,$$

where we apply the algorithm with $p = 50, q = 6, J = 20, h = 0.1$ and the error norm $\|y_{n+1}^{k+1} - y_{n+1}^k\| \leq 10^{-3}$, here we have $\Delta x = 1.0$ and $\frac{D}{\Delta x^2} = 1.0$. Further L are the number of the processors.

Without losing the generality of the method, we concentrate on the following to $L = 2$.

Jacobian-Method The first processor is computing:

$$\begin{aligned} (M_{A_1} + hM_{1,1})y_{n+1}^{1,k+1} &= h(N_{1,1} + \frac{1}{h}N_{A_1}) \left(E_{1,1}y_{n+1}^{1,k} + E_{1,2}y_{n+1}^{2,k} \right) + M_A y_n \\ &- N_A y_n + h f_{n+1}, \\ y_{n+1}^{1,0} &= y_n, \quad l = 1, \dots, L, \quad k = 0, 1, \dots, K, \quad n = 0, 1, 2, \dots, J, \end{aligned} \quad (16)$$

The second processor is computing:

$$\begin{aligned} (M_{A_2} + hM_{1,2})y_{n+1}^{2,k+1} &= h(N_{1,2} + \frac{1}{h}N_{A_2}) \left(E_{2,1}y_{n+1}^{1,k} + E_{2,2}y_{n+1}^{2,k} \right) + M_A y_n \\ &- N_A y_n + h f_{n+1}, \\ y_{n+1}^{2,0} &= y_n, \quad l = 1, \dots, L, \quad k = 0, 1, \dots, K, \quad n = 0, 1, 2, \dots, J. \end{aligned} \quad (17)$$

where we decide if we have to switch of the mixing means: means if we have fulfilled:

$$\| (E_{1,1}y_{n+1}^{1,k} + E_{1,2}y_{n+1}^{2,k}) - y_{n+1}^{1,k-1} \| \leq \| y_{n+1}^{1,k} - y_{n+1}^{1,k-1} \| \quad (18)$$

$$\| (E_{2,1}y_{n+1}^{1,k} + E_{2,2}y_{n+1}^{2,k}) - y_{n+1}^{2,k-1} \| \leq \| y_{n+1}^{2,k} - y_{n+1}^{2,k-1} \| \quad (19)$$

we do not switch of the mixing, but if the mixing has a larger error we have:

$$y_{n+1}^k = y_{n+1}^{1,k}, \quad (20)$$

Remark 1. The multisplitting is switched off, if one partial solution is much more accurate, than the other partial solution. Then we only apply the best approximation.

Gauss-Seidel-Method (decoupled version, serial with 2 processors) In this version, we apply the well-known standard Gauss-Seidel method, which has the drawback of the serial treatment with the results.

The first processor is computing:

$$\begin{aligned} (M_{A_1} + hM_{1,1})y_{n+1}^{1,k+1} &= h(N_{1,1} + \frac{1}{h}N_{A_1}) \left(E_{1,1}y_{n+1}^{1,k} + E_{1,2}y_{n+1}^{2,k} \right) + M_A y_n \\ &- N_A y_n + h f_{n+1}, \\ y_{n+1}^{1,0} &= y_n, \quad l = 1, \dots, L, \quad k = 0, 1, \dots, K, \quad n = 0, 1, 2, \dots, J, \end{aligned} \quad (21)$$

Remark 2. Here, we can apply the result of the first processor, if we assume, that part is much more faster to the second processor.

The second processor is computing:

$$\begin{aligned} (M_{A_2} + hM_{1,2})y_{n+1}^{2,k+1} &= h(N_{1,2} + \frac{1}{h}N_{A_2}) \left(E_{2,1}y_{n+1}^{1,k+1} + E_{2,2}y_{n+1}^{2,k} \right) + M_A y_n \\ &\quad - N_A y_n + h f_{n+1}, \end{aligned} \quad (22)$$

$$y_{n+1}^{2,0} = y_n, \quad l = 1, \dots, L, \quad k = 0, 1, \dots, K, \quad n = 0, 1, 2, \dots, J.$$

where we decide if we have to switch of the mixing means: means if we have fulfilled:

$$\| (E_{1,1}y_{n+1}^{1,k} + E_{1,2}y_{n+1}^{2,k}) - y_{n+1}^{1,k-1} \| \leq \| y_{n+1}^{1,k} - y_{n+1}^{1,k-1} \| \quad (23)$$

$$\| (E_{2,1}y_{n+1}^{1,k+1} + E_{2,2}y_{n+1}^{2,k}) - y_{n+1}^{2,k-1} \| \leq \| y_{n+1}^{2,k} - y_{n+1}^{2,k-1} \| \quad (24)$$

we do not switch of the mixing, but if the mixing has a larger error we have:

$$y_{n+1}^k = y_{n+1}^{1,k}, \quad (25)$$

Remark 3. The multisplitting is switched off, if one partial solution is much more accurate, than the other partial solution. Otherwise, we apply the mixture of the results based on the multisplitting method.

Gauss-Seidel-Method (decoupled version) The first processors compute:

$$\begin{aligned} &\left((M_{A_1} + hM_{1,1}) - h(N_{1,1} + \frac{1}{h}N_{A_1})E_{1,1} \right) y_{n+1}^{1,k+1} \\ &= h(N_{1,1} + \frac{1}{h}N_{A_1}) \left(E_{1,2}y_{n+1}^{2,k} \right) + M_A y_n - N_A y_n + h f_{n+1}, \end{aligned} \quad (26)$$

$$y_{n+1}^{1,0} = y_n, \quad l = 1, \dots, L, \quad k = 0, 1, \dots, K, \quad n = 0, 1, 2, \dots, J,$$

The second processors compute:

$$\begin{aligned} &\left((M_{A_2} + hM_{1,2}) - h(N_{1,2} + \frac{1}{h}N_{A_2})E_{2,2} \right) y_{n+1}^{2,k+1} \\ &= h(N_{1,2} + \frac{1}{h}N_{A_2}) \left(E_{2,1}y_{n+1}^{1,k} \right) + M_A y_n - N_A y_n + h f_{n+1}, \end{aligned} \quad (27)$$

$$y_{n+1}^{2,0} = y_n, \quad l = 1, \dots, L, \quad k = 0, 1, \dots, K, \quad n = 0, 1, 2, \dots, J.$$

Gauss-Seidel-Method (coupled version) Here, we apply the coupled version of the GS method, which means one processor is faster with the computation and the other processor can profit from the improved computations.

The processors compute:

$$\begin{aligned} & \left((M_{A_1} + hM_{1,1}) - h(N_{1,1} + \frac{1}{h}N_{A_1})E_{1,1} \right) y_{n+1}^{1,k+1} \\ &= h(N_{1,1} + \frac{1}{h}N_{A_1}) \left(E_{1,2}\tilde{y}_{n+1}^{2,k+1} \right) + M_A y_n - N_A y_n + h f_{n+1}, \end{aligned} \quad (28)$$

$$\begin{aligned} & \left((M_{A_2} + hM_{1,2}) - h(N_{1,2} + \frac{1}{h}N_{A_2})E_{2,2} \right) y_{n+1}^{2,k+1} \\ &= h(N_{1,2} + \frac{1}{h}N_{A_2}) \left(E_{2,1}\tilde{y}_{n+1}^{1,k+1} \right) + M_A y_n - N_A y_n + h f_{n+1}, \end{aligned} \quad (29)$$

$$\begin{aligned} y_{n+1}^{1,0} &= y_n, \quad l = 1, \dots, L, \quad k = 0, 1, \dots, K, \quad n = 0, 1, 2, \dots, J, \\ y_{n+1}^{2,0} &= y_n, \quad l = 1, \dots, L, \quad k = 0, 1, \dots, K, \quad n = 0, 1, 2, \dots, J. \end{aligned}$$

where we have two cases:

- If Processor 1 is faster than Processor 2:

$$\tilde{y}_{n+1}^{1,k+1} = y_{n+1}^{1,k+1}, \quad (30)$$

$$\tilde{y}_{n+1}^{2,k+1} = y_{n+1}^{2,k}, \quad (31)$$

- If Processor 2 is faster than Processor 1:

$$\tilde{y}_{n+1}^{1,k+1} = y_{n+1}^{1,k}, \quad (32)$$

$$\tilde{y}_{n+1}^{2,k+1} = y_{n+1}^{2,k+1}. \quad (33)$$

Remark 4. For all multisplitting methods, we can also extend the one-stage waveform-relaxation method to a multi-stage waveform-relaxation method.

3 Numerical Experiments

In a first experiment, we apply a partial differential algebraic equation (PDAE), which combines partial differential and algebraic equations.

We choose an experiment, which is based on the following two equations:

$$\partial_t c_1 + \nabla \cdot \mathbf{F} c_1 = f_1(t), \quad \text{in } \Omega \times [0, t], \quad (34)$$

$$\nabla \cdot \mathbf{F} c_2 = f_2(t), \quad \text{in } \Omega \times [0, t], \quad (35)$$

$$\mathbf{F} = -D\nabla, \quad (36)$$

and we have the following DAE problem:

$$A\partial_t c + Bc = f(t), \quad \text{in } [0, t], \quad (37)$$

The analytical solution is given as:

$$y = [\cos(t), \sin(t), t, \cos(t), \sin(t), t, \dots, \cos(t), \sin(t), t] \in \mathbb{R}^m, \quad (38)$$

and we have to calculate $f(t)$ as:

$$f(t) = A\partial_t y(t) + By(t), \text{ in } [0, t], \quad (39)$$

where y is the analytical solution.

We apply the error in L_2 or L_{max} -norm means:

$$err_{L_2}(t) = \frac{1}{\Delta x} \left(\sum_{i=1}^I (y_{ana}(x_i, t) - y_{num}(x_i, t))^2 \right)^{1/2}, \quad (40)$$

$$err_{max}(t) = \max_{i=1}^I \|y_{ana}(x_i, t) - y_{num}(x_i, t)\|. \quad (41)$$

In the following we deal with the semidiscretized equation given with the matrices:

$$A = \begin{pmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & 0 \\ & & & & 0 \end{pmatrix} \in \mathbb{R}^{m \times m}, \quad (42)$$

where $I, 0 \in \mathbb{R}^{p \times p}$

We have the following two operators for the splitting method:

$$B_1 = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix} \in \mathbb{R}^{p \times p} \quad (43)$$

$$B = \frac{D}{\Delta x^2} \cdot \begin{pmatrix} B_1 & -I & & & \\ -I & B_1 & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & B_1 & -I \\ & & & -I & B_1 \end{pmatrix} \in \mathbb{R}^{m \times m} \quad (44)$$

with $pq = m$, where we assume $\frac{D}{\Delta x^2} = 1$.

Means A, B are $m \times m$ block-matrices.

We have the following splitting:

$$N_A = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \frac{1}{100}I & \\ & & & & 0 \end{pmatrix}, \quad (45)$$

$$N_1 = \frac{D}{\Delta x^2} \cdot \begin{pmatrix} 2I & I & & & \\ I & 2I & I & & \\ & \ddots & \ddots & \ddots & \\ & & & I & 2I & I \\ & & & & I & 2I \end{pmatrix} \in \mathbb{R}^{m \times m} \quad (46)$$

$$M_2 = \frac{D}{\Delta x^2} \cdot \begin{pmatrix} 8I & 0 & & & \\ 0 & 8I & I & & \\ & \ddots & \ddots & \ddots & \\ & & & 0 & 8I & 0 \\ & & & & 0 & 8I \end{pmatrix} \in \mathbb{R}^{m \times m} \quad (47)$$

where $I, 0 \in \mathbb{R}^{p \times p}$

$$M_3 = \frac{D}{\Delta x^2} \cdot \begin{pmatrix} 10I & 0 & & & \\ 0 & 10I & I & & \\ & \ddots & \ddots & \ddots & \\ & & & 0 & 10I & 0 \\ & & & & 0 & 10I \end{pmatrix} \in \mathbb{R}^{m \times m} \quad (48)$$

where $I, 0 \in \mathbb{R}^{p \times p}$

We have the following operators:

$$M_A = A + N_A, \quad (49)$$

$$M_1 = B + N_1, \quad (50)$$

$$N_2 = M_2 - M_1, \quad (51)$$

$$N_3 = M_3 - M_2, \quad (52)$$

Further we have the following matrices:

$$N_{A_1} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \frac{1}{100}I & \\ & & & & 0 \end{pmatrix}, N_{A_2} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \frac{1}{100}I \end{pmatrix} \in \mathbb{R}^{m \times m}, \quad (53)$$

$$N_{1,1} = \frac{D}{\Delta x^2} \cdot \begin{pmatrix} 2I & I & & & \\ I & 2I & I & & \\ & \ddots & \ddots & \ddots & \\ & & & I & 2I & I \\ & & & & I & 2I \end{pmatrix} \in \mathbb{R}^{m \times m}, \quad (54)$$

$$N_{1,2} = \frac{D}{\Delta x^2} \cdot \begin{pmatrix} 3I & I & & & \\ I & 3I & I & & \\ & \ddots & \ddots & \ddots & \\ & & & I & 3I & I \\ & & & & I & 3I \end{pmatrix} \in \mathbb{R}^{m \times m} \quad (55)$$

and the overlapping matrices are given as (where we deal with a symmetric overlap, means in both directions of the diagonal matrix):

1. Overlap is between block-matrices at $m/2$ and $m/2 + 1$):

$$E_{1,1} = \begin{pmatrix} I & & & & \\ & \ddots & & & \\ & & I & & \\ & & & \alpha_1 I & \\ & & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}, \quad (56)$$

$$E_{1,2} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \alpha_2 I & \\ & & & & I \\ & & & & & \ddots \\ & & & & & & I \end{pmatrix} \in \mathbb{R}^{m \times m}, \quad (57)$$

Remark 7. We compared the errors between the multi-level WR and the Multi-splitting WR with Jacobian and Gauss-Seidel types. For the multi-level methods, we present the benefit in the higher level methods, while we only invert smaller matrices. The highest accuracy is given with the one-level method and the MS-Gauss-Seidel method, while the inversion matrix has the largest amount of information, but the methods are at least very expensive.

We could also improve the accuracy of the MS methods based on the different overlapping means for $o = 1$, we have only one overlap, while $o = m/2 - 1$ has the largest overlap. The balance and optimal values are between.

We apply the numerical example and obtain for the one-stage, two-stage and three-stage method the following results in Figure 1.

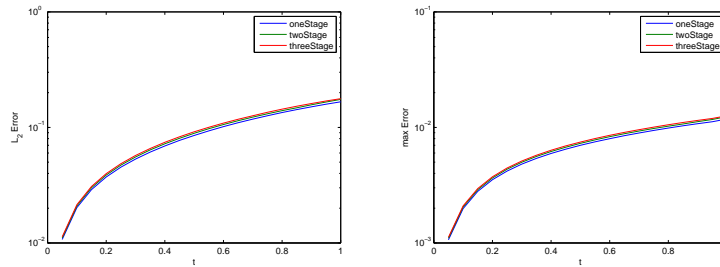


Fig. 1. The errors between the exact and numerical scheme of the one-, two- and three-stage method is given (left hand side: L_2 -errors, right hand side: L_∞ -error).

Remark 8. We obtain the same accuracy of the three and two-stage method as for the one-stage method. This means, that we can reduce the computational amount of work and received the same accurate result.

We apply the numerical example and obtain for the Multisplitting method the following results in Figure 2.

Remark 9. We obtain the same accuracy of the three and two-stage method as for the one-stage method. This means, that we can reduce the computational amount of work and received the same accurate result. The multisplitting method has also the same accuracy as the different multi-stage methods, here, we have the benefit of the parallel versions.

4 Conclusions and Discussions

We discuss multi-stage waveform-relaxation methods and multisplitting methods for differential algebraic equations. While the multi-stage waveform-relaxation

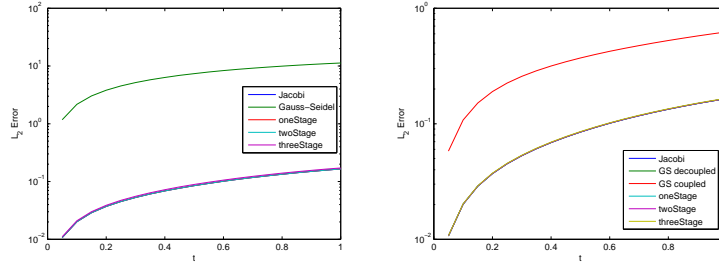


Fig. 2. The errors between the exact and numerical scheme of the one-stage, two-stage, three-stage, Jacobi- and GS Multisplitting method is presented (left hand side: one-stage, two-stage, three-stage, Jacobi-, uncoupled GS-method, right hand side: one-stage, two-stage, three-stage, Jacobi-, uncoupled and coupled GS-methods).

methods can reduce their computational work with simplifying the inverse matrices, the multisplitting methods have their benefits in parallelizing their procedure. We test the ideas in a first partial differential algebraic equation and see the benefit in the multi-stage waveform-relaxation method. In future, we will discuss the numerical analysis of the different methods and present more numerical examples.

References

1. W. Bao and Y. Song. *Two stage waveform relaxation method for the initial value problems of differential-algebraic equations*. Journal of Computational and Applied Mathematics, 236:1123-1136, 2011.
2. K. Brenan, S. Campbell, L. Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*. North-Holland, New York, 1989.
3. B. Leimkuhler, U. Miekkala and O. Nevanlinna. *Waveform relaxation for linear RC-circuits*. Impact Comput. Sci. Eng., 3:123-145, 1991.
4. D.P. O’Leary and R.E. White. *Multi-splittings of matrices and parallel solution of linear systems*. SIAM J. Algebraic Discrete Methods, 6:630-640, 1985.
5. E. Lelarsmee, A.E. Ruehli, and A.L. Sangiovanni-Vincentelli. *The waveform relaxation method for time-domain analysis of large scale integrated circuits*. IEEE Trans. Comput.-Aided Des., 1(3):131-145, 1982.
6. B. Stott. *Power system dynamic response calculations*. In Proceedings of the IEEE, 67:219-241, 1979.
7. S. Vandewalle. *Parallel Multigrid Waveform Relaxation for Parabolic Problems*. Teubner Skripten zur Numerik, B.G. Teubner Stuttgart, 1993.