G-marked moduli spaces

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Abstract

The aim of this paper is to investigate the closed subschemes of the moduli spaces corresponding to projective varieties which admit an effective action by a given finite group G. To achieve this, we introduce the moduli fuctor M_h^G of G-marked Gorenstein canonical models with Hilbert polynomial h, and prove the existence of $\mathfrak{M}_h[G]$, the coarse moduli scheme for M_h^G. Then we show that $\mathfrak{M}_h[G]$ has a proper and finite morphism to $\mathfrak{M}_h$ so that its image $\mathfrak{M}_h(G)$ is a closed subscheme.

1 Introduction

The moduli theory of algebraic varieties was motivated by the attempt to fully understand Riemann's assertion in [Rie57] that the isomorphism classes of Riemann surfaces of genus $g > 1$ depend on $(3g-3)$ parameters (called "moduli"). The modern approach to moduli problems via functors was developed by Grothendieck and Mumford (cf. [MF82]), and later by Gieseker, Kollár, Viehweg, et al (cf. [Gie77],[Kol13],[Vie95]). The idea is to define a moduli functor for the given moduli problem and study the representability of the moduli functor via an algebraic variety or some other geometric object. For instance, in the case of smooth projective curves of genus $g \geq 2$, we consider the (contravariant) functor $\mathcal{M}_g$ from the category of schemes to the category of sets, such that

(1) For any scheme T, $\mathcal{M}_g(T)$ consists of the T-isomorphism classes of flat projective families of curves of genus g over the base T.

(2) Given a morphism $f : S \to T$, $\mathcal{M}_g(f) : \mathcal{M}_g(T) \to \mathcal{M}_g(S)$ is the map associated to the pull back.

It has been showed by Mumford that there exists a quasi-projective *coarse moduli scheme* $\mathfrak{M}_g$ for the functor $\mathcal{M}_g$ (cf.[Mum62]), in the following sense:
there exists a natural transformation $\eta : \mathcal{M}_g \to \mathrm{Hom}(-, \mathfrak{M}_g)$, such that $\eta_{\mathbf{Spec}(\mathbb{C})}$: $\mathcal{M}_g(\mathbf{Spec}(\mathbb{C})) \to \mathrm{Hom}(\mathbf{Spec}(\mathbb{C}), \mathfrak{M}_g)$ is bijective and η is universal among such natural transformations. This means that the closed points of $\mathfrak{M}_g$ are in one to one correspondence with the isomorphism classes of curves of genus g and given a family $\mathfrak{X} \to T$ of curves of genus g, we have a morphism (induced by η) from T to $\mathfrak{M}_g$ such that any (closed) point $t \in T$ is mapped to $[\mathfrak{X}_t]$ in $\mathfrak{M}_g$.
The definitions are the same in higher dimensions, if one replaces curves of genus $g \geq 2$ by Gorenstein varieties with ample canonical classes. The existence of a coarse moduli space is then more difficult to prove, we refer to [Vie95] and [Kol13] for further discussions.
For several purposes, it is important to generalize the method to moduli problems of varieties which admit an effective action by a given finite group G. Here we consider the concept of a *G-marked variety*, which is a triple (X,G,α) such that X is a projective variety and $\alpha : G \times X \to X$ is a faithful action. The isomorphsims between G-marked varieties are G-equivariant isomorphisms (for more details, see definition 2.1). In similarity to the case of $\mathcal{M}_g$, we study in this article the moduli functor M_h^G of G-marked Gorenstein canonical models with Hilbert polynomial h such that, for any scheme T, $\mathsf{M}_h^G(T)$ is the set of T-isomorphism classes of G-marked flat families of Gorenstein canonical models with Hilbert polynomial h over the base T and, given a morphism $f : S \to T$, $\mathsf{M}_h^G(f)$ is the map associated to the set of pull-backs (cf. 2.4).
We refer to the recently published survey article [Cat15], Section 10, for some applications in the case of algebraic curves and surfaces; there the author discusses several topics on the theory of G-marked curves and sketches the construction of the moduli space of G-marked canonical models of surfaces.
The main theorem of this article is the following:

Theorem 1.1. *Given a finite group G and a Hilbert polynomial $h \in \mathbb{Q}[t]$, there exists a quasi-projective coarse moduli scheme $\mathfrak{M}_h[G]$ for M_h^G, the moduli functor of G-marked Gorenstein canonical models with Hilbert polynomial h.*

The structure of this paper is as follows:
In section 2 we introduce the definition of "G-marked varieties" and the moduli

problem associated to it by defining the moduli functor M_h^G for a given group G and Hilbert polynomial h.
In section 3 we study two basic properties (boundedness and local closedness) of the moduli functor M_h^G.
Recall that a moduli functor of varieties M is called *bounded* if the objects in $\mathsf{M}(\mathbf{Spec}(\mathbb{C}))$ are parameterized by a finite number of families (For a stronger definition, see [Kov09], Definition 5.1). In this article we show in (3.20) that M_h^G is bounded by a family $U_{N,h'}^G \to H_{N,h'}^G$ over an appropriate subscheme of a Hilbert scheme.
However the versal family $U_{N,h'}^G \to H_{N,h'}^G$ that we get in (3.20) may not belong to $\mathsf{M}_h^G(H_{N,h'}^G)$, i.e., not every fibre of the family is a G-marked canonical model. Here comes the problem of local closedness: roughly speaking, a moduli functor M of varieties is called *locally closed* if for any flat projective family $\mathfrak{X} \to T$, the subset $\{t \in T | \mathfrak{X}_t \in \mathsf{M}(\mathbf{Spec}(\mathbb{C}))\}$ is locally closed in T (for a precise definition of local closedness, see [Kov09], 5.C). We solve this problem in (3.22) by taking a locally closed subscheme $\bar{H}_{N,h'}^G$ of $H_{N,h'}^G$ and considering the restriction of $U_{N,h'}^G \to H_{N,h'}^G$ to $\bar{H}_{N,h'}^G$.
In Section 4 we first apply Geometric Invariant Theory, obtaining the quotient $\mathfrak{M}_h[G]$ of $\bar{H}_{N,h'}^G$ by some reductive groups. Then we prove that $\mathfrak{M}_h[G]$ is the coarse moduli scheme for our moduli fuctor M_h^G.

2 G-marked varieties

In this article we work over the complex field $\mathbb{C}$. By a "scheme" we mean a separated scheme of finite type over $\mathbb{C}$, a point in a scheme is assumed to be a closed point. Moreover, G shall always denote a finite group.

Definition 2.1 ([Cat15], definition 181)**.** (1) A *G-marked (projective) variety (resp. scheme)* is a triple (X, G, ρ) where X is a projective variety (resp. scheme) and $\rho : G \to \mathrm{Aut}(X)$ is an injective homomorphism. Or equivalently, it is a triple (X, G, α) where $\alpha : X \times G \to X$ is a faithful action of G on X.
(2) A morphism f between two G-marked varieties (X, G, ρ) and (X', G, ρ') is a G-equivariant morphism $f : X \to X'$, i.e., $\forall g \in G, f \circ \rho(g) = \rho'(g) \circ f$.
(3) A family of G-marked varieties (resp. schemes) is a triple $((p : \mathfrak{X} \to T), G, \rho)$,

where G acts faithfully on $\mathfrak{X}$ via an injective homomorphsim $\rho : G \to \mathrm{Aut}(\mathfrak{X})$ and trivially on T, p is flat, projective and G-equivariant and $\forall t \in T$, the induced triple $(\mathfrak{X}_t, G, \rho_t)$ is a G-marked variety (resp. scheme).
(4) A morphism between two G-marked families $((p : \mathfrak{X} \to T), G, \rho)$ and $((p' : \mathfrak{X}' \to T'), G, \rho')$ is a commutative diagram:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\tilde{f}} & \mathfrak{X}' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ T & \xrightarrow{f} & T' \end{array}$$

where $\tilde{f} : \mathfrak{X} \to \mathfrak{X}'$ is a G-equivariant morphism.
(5) Let $((p : \mathfrak{X} \to T), G, \rho)$ be a G-marked family and $f : S \to T$ a morphism. Denote by $\mathfrak{X}_S$ (or $f^*\mathfrak{X}$) the fiber product of f and p, ρ induces a G-action ρ_S (or $f^*\rho$) on $\mathfrak{X}_S$ such that $((p_S : \mathfrak{X}_S \to S), G, \rho_S) =: f^*((p : \mathfrak{X} \to T), G, \rho)$ is again a G-marked family.

Remark 2.2. Observe that, given a flat family of varieties $\mathfrak{X} \to T$ with a group G acting on each fiber, we do not yet have a G-marked family, i.e., we may not find an action of G on $\mathfrak{X}$. For any point $t \in T$, we can find a suitable analytic neighborhood D such that the action of G on $\mathfrak{X}_t$ can be extended to an action on $\mathfrak{X}|_D \to D$. However if one wants to extend the action to the whole family, there comes the problem of monodromy: for another point $t' \in T$, the extensions along two different paths connecting t and t' may not result in same actions on $\mathfrak{X}_{t'}$.

Definition 2.3. A normal projective variety X is called a *canonical model* if X has canonical singularities (cf. [Rei87]) and K_X is ample.

Definition 2.4. Denote by $\mathfrak{Sch}$ the category of schemes (over $\mathbb{C}$). The *moduli functor of G-marked Gorenstein canonical models* with Hilbert polynomial $h \in \mathbb{Q}[t]$ is a contravariant functor:

$$\mathsf{M}_h^G : \mathfrak{Sch} \to \mathfrak{Sets}, \ \textit{such that}$$

(1) For any scheme T,

$$\begin{aligned} \mathsf{M}_h^G(T) := \{((p : \mathfrak{X} \to T), G, \rho) |\ & p \text{ is flat and projective, all fibres of p} \\ & \text{are canonical models, } \omega_{\mathfrak{X}/T} \text{ is invertible,} \\ & \forall t \in T, \forall k \in \mathbb{N}, \chi(\mathfrak{X}_t, \omega_{\mathfrak{X}_t}^k) = h(k)\} / \simeq \end{aligned}$$

where "$\simeq$" is the equivalence relation given by the isomorphisms of G-marked families over T (i.e., in the commutative diagram of 2.1 (4), take $T' = T$ and $f = id_T$).
(2) Given $f \in \mathrm{Hom}(S,T)$, $\mathsf{M}_h^G(f) : \mathsf{M}_h^G(T) \to \mathsf{M}_h^G(S)$ is the map associated to the pull back, i.e.,

$$[((p : \mathfrak{X} \to T), G, \rho)] \mapsto [((p_S : \mathfrak{X}_S \to S), G, \rho_S)].$$

Remark 2.5. In this article, whenever we write $((\mathfrak{X} \to T), G, \rho) \in \mathsf{M}_h^G(T)$, we mean choosing a representative $((\mathfrak{X} \to T), G, \rho)$ from the isomorphism class $[((\mathfrak{X} \to T), G, \rho)] \in \mathsf{M}_h^G(T)$.

In the case where G is trivial, we denote by M_h the corresponding functor.

3 Basic properties of M_h^G

In this section we study two important properties of the moduli functor M_h^G: boundedness and local closedness. The main results are (3.19), (3.20) for boundedness and (3.22) for local closedness.
In the case where G is trivial boundedness is already known (cf. [Kar00], [Mat86]). However we can not apply it directly to the general case since we have an action by G. Here we introduce the notion of "bundle of G-frames" to solve this problem.
Let Y be a scheme and $\mathcal{E}$ a locally free sheaf of rank n on Y. Set

$$\mathbb{V}(\mathcal{E}) := \mathbf{Spec}_Y Sym(\mathcal{E}^\vee),$$

the geometric vector bundle associated to $\mathcal{E}$ over Y (cf. [Har77], Exercise II.5.18).

Definition 3.1 (Frame Bundle)**.** Let $\mathcal{E}$ be a locally free sheaf of rank n on a scheme Y. In this article we call (what is in bundle theory called) the principal bundle associated to $\mathbb{V}(\mathcal{E})$ the *frame bundle* $\mathcal{F}(\mathcal{E})$ of $\mathcal{E}$ over Y. For any $y \in Y$, the fibre $\mathcal{F}(\mathcal{E})_y$ over y is called the set of frames (i.e., bases) for the vector space $\mathcal{E} \otimes \mathbb{C}(y)$.
Hence $\mathcal{F}(\mathcal{E})$ is the open subscheme of $\mathbb{V}(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^n, \mathcal{E}))$ such that, $\forall y \in Y$, the fibre $\mathcal{F}(\mathcal{E})_y$ corresponds to the invertible homomorphisms. We denote a point in $\mathcal{F}(\mathcal{E})$ as a pair (y, ψ), where y is a point in Y and $\psi : \mathbb{C}^n \to \mathcal{E} \otimes \mathbb{C}(y)$ is an isomorphism of $\mathbb{C}$-vector spaces.

Proposition 3.2. *Let $\mathcal{E}$ be a locally free sheaf of rank n on a scheme Y and $p : \mathcal{F}(\mathcal{E}) \to Y$ the natural projection. There exists a tautological isomorphism $\phi_{\mathcal{E}} : \mathcal{O}^n_{\mathcal{F}(\mathcal{E})} \to p^*\mathcal{E}$ of sheaves on $\mathcal{F}(\mathcal{E})$ such that, for any point $z := (y, \psi) \in \mathcal{F}(\mathcal{E})$, $\phi_{\mathcal{E}}|_{\{z\}} = \psi$ via the isomorphism $Hom(\mathbb{C}^n, p^*\mathcal{E} \otimes \mathbb{C}(z)) \simeq Hom(\mathbb{C}^n, \mathcal{E} \otimes \mathbb{C}(y))$.*

Proof. This proposition is well known (the idea is similar to that of [Gro58]). Observe that $p^*\mathcal{E}$ has n global sections $s_1(\mathcal{E}), ..., s_n(\mathcal{E})$ such that for any $z = (y, \psi) \in \mathcal{F}(\mathcal{E})$, $s_i(\mathcal{E}) \otimes \mathbb{C}(z) = \psi(e_i)$, where $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{C}^n$ and we identify $p^*\mathcal{E} \otimes \mathbb{C}(z)$ with $\mathcal{E} \otimes \mathbb{C}(y)$. Then the universal basis morphism $\phi_{\mathcal{E}} := (s_1(\mathcal{E}), ..., s_n(\mathcal{E})) : \mathcal{O}^n_{\mathcal{F}(\mathcal{E})} \to p^*\mathcal{E}$ is an isomorphism of locally free sheaves. □

Remark 3.3. The set of sections $\{s_i(\mathcal{E})\}_{i=1}^n$ (or equivalently, the isomorphism $\phi_{\mathcal{E}}$) satisfy the following properties:
(1) Any morphism $f : X \to Y$ induces a morphism $f_{\mathcal{F}} : \mathcal{F}(f^*\mathcal{E}) \to \mathcal{F}(\mathcal{E})$, then we have that $f^*_{\mathcal{F}}(s_i(\mathcal{E})) = s_i(f^*\mathcal{E})$.
(2) Given an isomorphism $l : \mathcal{E}_1 \to \mathcal{E}_2$ of locally free sheaves on Y, then we obtain an induced isomorphism $l_{\mathcal{F}} : \mathcal{F}(\mathcal{E}_1) \to \mathcal{F}(\mathcal{E}_2)$ commuting with the projections $p_j : \mathcal{F}(\mathcal{E}_j) \to Y$, $j = 1, 2$. We have that $l^*_{\mathcal{F}}(\phi_{\mathcal{E}_2}) = p_1^*(l) \circ \phi_{\mathcal{E}_1}$.

Definition 3.4. Let $\mathcal{E}$ be a locally free sheaf of rank n on a scheme Y: we say that a group G acts *faithfully and linearly* on $\mathcal{E}$ if
(1) The action is given by an injective homomorphism $\rho : G \hookrightarrow \mathrm{Aut}_{\mathcal{O}_Y}(\mathcal{E})$;
(2) $\forall y \in Y$, the induced action ρ_y is a faithful representation of G on $\mathbb{C}^n$.
In this case we call the pair $(\mathcal{E}, \rho)$ a locally free G-sheaf.

Definition 3.5. (1) Given $\phi \in \mathrm{Aut}(Y)$, let $\Gamma_\phi : Y \to Y \times Y$ be the graph map of ϕ. The *fixpoints scheme* of ϕ (denoted by $\mathrm{Fix}(\phi)$) is the (scheme-theoretic) inverse image of Δ by Γ_ϕ, where Δ is the diagonal subscheme of $Y \times Y$.
(2) Given an action of G on Y, the *fixpoints scheme* of G on Y is:

$$Y^G := \cap_{g \in G} Fix(\phi_g),$$

where $\phi_g : Y \to Y, y \mapsto gy$.

Remark 3.6. (1) Let $f : X \to Y$ be a G-equivariant morphism between two schemes on which G acts. Then there is a natural restriction morphism $f|_{X^G} : X^G \to Y^G$.
(2) Let G act on Y; then for any subgroup H of G there is an induced $C(H)$-action on Y^H, where $C(H)$ is the centralizer group of H in G.

Definition 3.7. Let $(\mathcal{E}, \rho)$ be a locally free G-sheaf of rank n on Y. Given a faithful linear representation $\beta : G \to GL(n, \mathbb{C})$, we define an action (β, ρ) of G on $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^n, \mathcal{E})$: $\forall g \in G$, open subset $U \subset Y, \phi \in \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^n, \mathcal{E})(U)$ *and* $s \in \mathcal{O}_Y^n(U)$; $(g\phi)(s) := \rho(g)\phi(\beta(g^{-1})s)$. The action (β, ρ) restricts naturally to $\mathcal{F}(\mathcal{E})$, we denote by $\mathcal{F}(\mathcal{E}, G, \rho; \beta)$ the corresponding fixpoints scheme: it is called the *bundle of G-frames of $\mathcal{E}$ associated to the action ρ with respect to β.*

Remark 3.8. (1) Denoting by $C(G, \beta)$ the centralizer group of $\beta(G)$ in $GL(n, \mathbb{C})$, an easy observation is that $\forall y \in Y$, the fiber $\mathcal{F}(\mathcal{E}, G, \rho; \beta)_y$ corresponds to the set of G-equivariant isomorphisms between the G-linear representations β and ρ_y. Therefore we have that either $\mathcal{F}(\mathcal{E}, G, \rho; \beta)_y = \emptyset$, or $\mathcal{F}(\mathcal{E}, G, \rho; \beta)_y \simeq C(G, \beta)$.
(2) If $\beta, \beta' : G \to GL(n, \mathbb{C})$ are equivalent representations (i.e., there exists $g \in GL(n, \mathbb{C})$ such that $\beta' = g\beta g^{-1}$), then we have that $\mathcal{F}(\mathcal{E}, G, \rho; \beta) \simeq \mathcal{F}(\mathcal{E}, G, \rho; \beta')$.

Observe that if Y is connected and there exists $y \in Y$ such that $\mathcal{F}(\mathcal{E}, G, \rho; \beta)_y \simeq C(G, \beta)$, then $\mathcal{F}(\mathcal{E}, G, \rho; \beta)_{y'} \simeq C(G, \beta)$ for all $y' \in Y$ (See [Cat13], Prop 37), hence we have the following definition:

Definition 3.9. Let Y be a connected scheme and $(\mathcal{E}, \rho)$ a locally free G-sheaf of rank n on Y. We say that $(\mathcal{E}, \rho)$ (or $\mathcal{E}$ if the action is clear from the context) has *decomposition type* β, where $\beta : G \to \mathbb{C}^n$ is a faithful representation, if there exists $y \in Y$, such that $\rho_y \simeq \beta$.

Definition 3.10 (Bundle of G-frames)**.** Let $(\mathcal{E}, \rho)$ be a locally free G-sheaf of rank n on a scheme Y. We define the *bundle of G-frames of $\mathcal{E}$ associated to* ρ, denoted by $\mathcal{F}(\mathcal{E}, G, \rho)$ (or $\mathcal{F}(\mathcal{E}, G)$ when ρ is clear from the context), as follows:
If Y is connected and $\mathcal{E}$ has decomposition type β, then $\mathcal{F}(\mathcal{E}, G, \rho) := \mathcal{F}(\mathcal{E}, G, \rho; \beta)$. In general $\mathcal{F}(\mathcal{E}, G, \rho)$ is the (disjoint) union of the bundles of G-frames of $\mathcal{E}$ restricted to each connected component.

Remark 3.11. Since we can vary β in its equivalence class, we see from (3.8) that $\mathcal{F}(\mathcal{E}, G)$ is unique up to isomorphisms.

Definition 3.12. Let $(\mathcal{E}, \rho)$ be a free G-sheaf of rank n on a scheme Y. The action is said to be *defined over* $\mathbb{C}$ if there exists a G-equivariant isomorphism $\phi : (\mathcal{O}_Y^n, \beta) \to (\mathcal{E}, \rho)$, where $\beta : G \to GL(n, \mathbb{C})$ is a faithful representation.

Proposition 3.13. *Let $(\mathcal{E}, \rho)$ be a locally free G-sheaf of rank n on a connected scheme Y with decomposition type β. The projection $p : \mathcal{F}(\mathcal{E}, G) \to Y$ induces an action $p^*\rho$ on $p^*\mathcal{E}$. Then $(p^*\mathcal{E}, p^*\rho)$ is defined over $\mathbb{C}$: the morphism $\phi_{\mathcal{E},G} := \phi_{\mathcal{E}}|_{\mathcal{F}(\mathcal{E},G)} : (\mathcal{O}^n_{\mathcal{F}(\mathcal{E},G)}, \beta) \to (p^*\mathcal{E}, p^*\rho)$ is a G-equivariant isomorphism, where $\phi_{\mathcal{E}}$ is the universal basis morphism defined in (3.2.*

Proof. It is clear that $\phi_{\mathcal{E},G}$ is an isomorphism of sheaves, what remains to show is that $\phi_{\mathcal{E},G}$ is G-equivariant. Since $\phi_{\mathcal{E},G}$ is an isomorphism of locally free sheaves, it suffices to show that $\forall (y, \psi) \in \mathcal{F}(\mathcal{E}, G)$, $\phi_{\mathcal{E},G}|_{\{(y,\psi)\}}$ is G-equivariant. By our construction in (3.2), we have that $p^{-1}(y) \subset \mathbb{V}(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}^n_Y, \mathcal{E}))^G_y \simeq \mathrm{Hom}(\mathbb{C}^n, \mathcal{E} \otimes \mathbb{C}(y))^G$, where the G-action is (β, ρ_y). Under this isomorphism the point (y, ψ) corresponds exactly to $\phi_{\mathcal{E},G}|_{\{(y,\psi)\}}$, hence $\phi_{\mathcal{E},G}|_{\{(y,\psi)\}}$ is G-equivariant. □

Remark 3.14. Given a locally free G-sheaf $(\mathcal{E}, \rho)$ of rank n on Y, set $s_i(\mathcal{E}, G) := s_i(\mathcal{E})|_{\mathcal{F}(\mathcal{E},G)}$, then $\{s_i(\mathcal{E}, G)\}$ and $\phi_{\mathcal{E},G}$ have similar properties as $\{s_i(\mathcal{E})\}$ and $\phi_{\mathcal{E}}$ have in (3.3).

Proposition 3.15. *Assume that Y is connected and $(\mathcal{E}, \rho)$ is a locally free G-sheaf of rank n on Y with decomposition type β. Then there is a natural $C(G, \beta)$-action on $\mathcal{F}(\mathcal{E}, G)$ and Y is a categorical quotient of $\mathcal{F}(\mathcal{E}, G)$ by $C(G, \beta)$.*

Proof. To see the $C(G, \beta)$-action, it suffices to notice that the actions β and ρ on $\mathcal{F}(\mathcal{E})$ commute, i.e., $\forall g \in G$, $\beta(g)\rho(g) = \rho(g)\beta(g)$ as elements in $\mathrm{Aut}(\mathcal{F}(\mathcal{E}))$.
From the construction of $\mathcal{F}(\mathcal{E}, G)$, one observes that the projection $p : \mathcal{F}(\mathcal{E}, G) \to Y$ is affine and $C(G, \beta)$-equivariant, therefore we may assume that $Y, \mathcal{F}(\mathcal{E}, G)$ are affine schemes and A (resp. B) is the coordinate ring of Y (resp. $\mathcal{F}(\mathcal{E}, G)$). Since p is surjective and $C(G, \beta)$-equivariant, we have that $A \subset B^{C(G,\beta)} \subset B$. Noting that B is a finitely generated $\mathbb{C}$-algebra and $C(G, \beta)$ is a reductive group (cf. 3.17), we conclude that $B^{C(G,\beta)}$ is a finitely generated $\mathbb{C}$-algebra and $\mathbf{Spec}B^{C(G,\beta)}$ is the universal categorical quotient of $\mathcal{F}(\mathcal{E}, G)$ by $C(G, \beta)$ (see [MF82], p.27). Now since every fibre of p is a closed $C(G, \beta)$-orbit (in fact isomorphic to $C(G, \beta)$), which must map to a point in $\mathbf{Spec}B^{C(G,\beta)}$, for dimensional reasons we conclude that $B^{C(G,\beta)}$ is a finite A-module. For any maximal ideal m of A, by the definition of a universal categorical quotient (cf. [MF82], p.4) we see that $\mathbf{Spec}(B^{C(G,\beta)} \otimes_A \mathbb{C}(m))$ is the categorical quotient of $p^{-1}(\mathbf{Spec}(\mathbb{C}(m))) \simeq C(G, \beta)$ by $C(G, \beta)$, hence $B^{C(G,\beta)} \otimes_A \mathbb{C}(m) = \mathbb{C}$, which implies that $(B^{C(G,\beta)}/A) \otimes_A \mathbb{C}(m) = 0$. By Nakayama's lemma, we have that $(B^{C(G,\beta)}/A)_m = 0$, which implies that $A = B^{C(G,\beta)}$. □

Before stating the Boundedness theorem, let's first recall the action of general linear groups on Hilbert schemes (cf. [Vie95], Section 7.1).
Denote by $H_{n,h}$ the Hilbert scheme of the closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial h and by $U_{n,h} \subset H_{n,h} \times \mathbb{P}^n$ the universal family. Let $\Phi : GL(n+1, \mathbb{C}) \times \mathbb{P}^n \to \mathbb{P}^n$ be the natural action, so that there is an action $\Psi : GL(n+1) \times H_{n,h} \to H_{n,h}$, such that $\forall g \in GL(n+1, \mathbb{C})$, $U_{n,h}$ is invariant under the morphism $\Psi_g \times \Phi_g$.

Given a (finite) group G, a faithful representation of G on $V := \mathbb{C}^{n+1}$ is given by an injective homomorphism $\beta : G \to GL(n+1, \mathbb{C})$, or equivalently, by a decomposition $V = \bigoplus_{\rho \in Irr(G)} W_\rho^{n(\rho)}$. Two representations are equivalent (i.e. the images of G are conjugate as subgroups of $GL(n+1, \mathbb{C})$) if and only if they have the same decomposition type (cf. [Ser77], Chap.2), hence the set of equivalence classes $\mathcal{B}_n$ of G-representations on V is finite.

Definition 3.16. Let $\beta : G \to GL(n+1, \mathbb{C})$ be an injective homomorphism, it induces an action $\Psi|_{\beta(G)}$ of G on $H_{n,h}$. Define $H_{n,h}^{G,\beta}$ to be the fixpoints scheme of $\beta(G)$ on $H_{n,h}$. Denote by $U_{n,h}^{G,\beta}$ the restriction of $U_{n,h}$ from $H_{n,h}$ to $H_{n,h}^{G,\beta}$.

Remark 3.17. (1) We have already seen that $C(G, \beta)$, the centralizer group of $\beta(G)$ in $GL(n+1, \mathbb{C})$, has a natural action on $H_{n,h}^{G,\beta}$ (cf. remark 3.6). By Schur's Lemma one obtains that $C(G, \beta) \simeq \Pi_{\rho \in Irr(G)} GL(n(\rho), \mathbb{C})$, hence $C(G, \beta)$ is reductive.
(2) Let β, β' be two equivalent representations, such that $\beta' = g\beta g^{-1}$ for some $g \in GL(n+1, \mathbb{C})$, then $H_{n,h}^{G,\beta}$ is isomorphic to $H_{n,h}^{G,\beta'}$ via Ψ_g as subschemes of $H_{n,h}$.
(3) Since $U_{n,h}^{G,\beta}$ (as a subscheme of $H_{n,h}^{G,\beta} \times \mathbb{P}^n$) is invariant under the morphism $id \times (\Phi|_{\beta(G)})$, we obtain a G-marked family $((p_\beta : U_{n,h}^{G,\beta} \to H_{n,h}^{G,\beta}), G, \beta)$.

Definition 3.18. Let V be a $\mathbb{C}$-vector space of dimension $n+1$. Denoting by $\mathcal{B}_n$ the set of equivalence classes of linear representations of G on V, we pick one representative in each equivalence class of $\mathcal{B}_n$ and define:

$$((p : U_{n,h}^G \to H_{n,h}^G), G, \mathcal{B}_n) := \bigsqcup_{[\beta] \in \mathcal{B}_n} ((p_\beta : U_{n,h}^{G,\beta} \to H_{n,h}^{G,\beta}), G, \beta),$$

where "$\bigsqcup$" means a disjoint union.
Note that two different choices of representatives result in isomorphic families.

By Matsusaka's big theorem ([Mat86], Theorem 2.4), there exists an integer k_0 such that $\forall X \in \mathsf{M}_h(\mathbf{Spec}\mathbb{C})$, $\omega_X^{k_0}$ is very ample and has vanishing higher cohomology groups, we fix one such k_0 for the rest of the article (we refer to [Siu93], [Dem96] and [Siu02] for an effective bound on k_0). Let $(p : \mathfrak{X} \to T) \in \mathsf{M}_h(T)$; by "Cohomology and Base change" (cf. [Mum70], II.5), $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ is a locally free sheaf of rank $h(k_0)$. Moreover we have that $p^*p_*(\omega^{k_0}_{\mathfrak{X}/T}) \twoheadrightarrow \omega^{k_0}_{\mathfrak{X}/T}$, which induces a T-embedding $\mathfrak{X} \hookrightarrow \mathbb{P}(p_*(\omega^{k_0}_{\mathfrak{X}/T}))$ such that $\omega^{k_0}_{\mathfrak{X}/T} \simeq \mathcal{O}_{\mathbb{P}(p_*(\omega^{k_0}_{\mathfrak{X}/T}))}(1)$ (cf. [Har77], II.7.12). Assume in addition that $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ is trivial, the T-embedding becomes $\mathfrak{X} \hookrightarrow T \times \mathbb{P}^N$ $(N := h(k_0) - 1)$. Set $h'(k) := h(k_0 k)$, then there exists a morphsim $f : T \to H_{N,h'}$ such that $\mathfrak{X} \simeq f^* U_{N,h'}$. Now taking the group action into account, we have the following:

Proposition 3.19 (Boundedness)**.** *Given $((p : \mathfrak{X} \to T), G, \rho) \in \mathsf{M}^G_h(T)$, denote by $\bar{\rho}$ the induced action of G on $p_*(\omega^{k_0}_{\mathfrak{X}/T})$. Assume that $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ is trivial and $\bar{\rho}$ is defined over $\mathbb{C}$, then there exists $f : T \to H^G_{N,h'}$, such that $((\mathfrak{X} \to T), G, \rho) \simeq f^*((U^G_{N,h'} \to H^G_{N,h'}), G, \mathcal{B}_N)$, and $\omega^{k_0}_{\mathfrak{X}/T} \simeq \mathcal{O}_{T\times\mathbb{P}^N}(1)|_{\mathfrak{X}}$.*

Proof. It suffices to prove the statement on each connected component of T, hence we may assume that T is connected and $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ has decomposition type β. The action $\bar{\rho}$ on $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ induces an action of G on $\mathbf{Proj}_T(p_*(\omega^{k_0}_{\mathfrak{X}/T})) = T \times \mathbb{P}^N$ such that the embedding $i : \mathfrak{X} \to T \times \mathbb{P}^N$ is G-equivariant. Since by assumption $\bar{\rho}$ is defined over $\mathbb{C}$, we may require that the action of G on $T \times \mathbb{P}^N$ is given by $\pi_2^*(\beta)$, where $\pi_2 : T \times \mathbb{P}^N \to \mathbb{P}^N$ is the projection onto the second factor. Now by the universal property of the Hilbert scheme, there exists $f : T \to H_{N,h'}$, such that $i(\mathfrak{X}) = (f \times Id_{\mathbb{P}^N})^* U_{N,h'}$. To complete the proof, it remains to show that f factors through $H^{G,\beta}_{N,h'}$, which is equivalent to the property that $\forall g \in G$, $\Psi_{\beta(g)} \circ f = f$; again by the universal property of the Hilbert scheme, this is equivalent to showing that $\forall g \in G$, $((\Psi_{\beta(g)} \circ f) \times id_{\mathbb{P}^N})^* U_{N,h'} = i(\mathfrak{X})$. However we have that

$$((\Psi_{\beta(g)} \circ f) \times id_{\mathbb{P}^N})^* U_{N,h'} = (f \times id_{\mathbb{P}^N})^* (\Psi_{\beta(g)} \times id_{\mathbb{P}^N})^* U_{N,h'}$$

$$= (f \times id_{\mathbb{P}^N})^* (id_{U_{N,h'}} \times \Phi_{\beta(g)^{-1}})^* U_{N,h'} = (id_T \times \Phi_{\beta(g)^{-1}})^* (f \times id_{\mathbb{P}^N})^* U_{N,h'}$$

$$= (id_T \times \Phi_{\beta(g)^{-1}})^* (i(\mathfrak{X})),$$

which is simply $i(\mathfrak{X})$ as the embedding $i : \mathfrak{X} \to T \times \mathbb{P}^N$ is G-equivariant. $\square$

Combining (3.13) with (3.19), we have the following corollary:

Corollary 3.20. *For any scheme T and $((p : \mathfrak{X} \to T), G, \rho) \in \mathsf{M}^G_h(T)$, let $q : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \to T$ be the bundle of G-frames of $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ over T. Then the* *isomorphism* $\phi_{p_*(\omega^{k_0}_{\mathfrak{X}/T}),G}$ *induces a morphism* $f_{\mathfrak{X}/T,k_0,G} : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \to H^G_{N,h'}$, *such that* $\mathsf{M}^G_h(q)((\mathfrak{X} \to T), G, \rho) \simeq f^*_{\mathfrak{X}/T,k_0,G}((U^G_{N,h'} \to H^G_{N,h'}), G, \mathcal{B}_N)$*, where* $N := h(k_0) - 1$*,* $h'(k) := h(k_0 k)$*.*

Remark 3.21. Given an isomorphism $((p : \mathfrak{X}_1 \to T), G, \rho_1) \simeq ((p : \mathfrak{X}_2 \to T), G, \rho_2)$, we have an induced isomorphsim $l : p_*(\omega^{k_0}_{\mathfrak{X}_1/T}) \to p_*(\omega^{k_0}_{\mathfrak{X}_2/T})$ of G-sheaves on T. Both $p_*(\omega^{k_0}_{\mathfrak{X}_1/T})$ and $p_*(\omega^{k_0}_{\mathfrak{X}_1/T})$ have decomposition type β. Then l induces a $C(G,\beta)$-equivariant isomorphism: $l_{\mathcal{F}} : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}_1/T}), G) \to \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}_2/T}), G)$. From 3.3, 3.14 and the proof of 3.19 we have that $f_{\mathfrak{X}_1/T,k_0,G} = f_{\mathfrak{X}_2/T,k_0,G} \circ l_{\mathcal{F}}$.

We have already proven that M^G_h is bounded (in the sense of 3.20). However in general $H^G_{N,h'}$ may not be a parametrizing space for M^G_h, i.e., some fibre of $((U^G_{N,h'} \to H^G_{N,h'}), G, \mathcal{B}_N)$ may not be a canonical model. We will see that the set of points in $H^G_{N,h'}$ over which the fibre is a Gorenstein canonical model forms a locally closed subscheme $\bar{H}^G_{N,h'}$. In general such problems correspond to studying the local closedness of the moduli functor (we refer to [Kov09], 5.C, for more details).
Here we do not state a general "G-version" of local closedness, but only consider the case of Hilbert schemes.

Proposition 3.22. *Using the same notations as in (3.19), there exists a locally closed subscheme* $\bar{H}^G_{N,h'}$ *of* $H^G_{N,h'}$*, satisfying the following conditions:*
(1) $((\bar{U}^G_{N,h'} \to \bar{H}^G_{N,h'}), G, \mathcal{B}_N) := ((U^G_{N,h'} \to H^G_{N,h'}), G, \mathcal{B}_N)|_{\bar{H}^G_{N,h'}} \in \mathsf{M}^G_h(\bar{H}^G_{N,h'})$
(2) The morphism f *that we get in (3.19) factors through* $\bar{H}^G_{N,h'}$*.*

Proof. In the case where G is trivial the existence of $\bar{H}_{N,h'}$ follows from the facts that the subset $\{x \in H_{N,h'} |\ (\omega^{k_0}_{U_{N,h'}})_x \simeq (\mathcal{O}_{\mathbb{P}^N}(1)|_{U_{N,h'}})_x\}$ is closed in $H_{N,h'}$ (cf.[Mum70], II.5, Corollary 6) and being canonical and Gorenstein is an open property (cf.[Elk81]).
In general we set $\bar{H}^{G,\beta}_{N,h'} := \bar{H}_{N,h'} \bigcap H^{G,\beta}_{N,h'}$ and $\bar{H}^G_{N,h'} := \bigsqcup \bar{H}^{G,\beta}_{N,h'}$. For condition (1), the fact that $\bar{U}_{N,h'} \to \bar{H}_{N,h'} \in \mathsf{M}_h(\bar{H}_{N,h'})$ implies that $\bar{U}^G_{N,h'} \to \bar{H}^G_{N,h'} \in \mathsf{M}_h(\bar{H}^G_{N,h'})$, now taking the action of G into account, we have that $((\bar{U}^G_{N,h'} \to \bar{H}^G_{N,h'}), G, \mathcal{B}_N) \in \mathsf{M}^G_h(\bar{H}^G_{N,h'})$. Condition (2) is satisfied for similar reasons. $\square$

Remark 3.23. (1) Given $(X_1, G, \rho_1), (X_2, G, \rho_2) \in \mathsf{M}^G_h(\mathbf{Spec}\mathbb{C})$, such that $H^0(\omega^{k_0}_{X_1})$ and $H^0(\omega^{k_0}_{X_2})$ have the same decomposition type β, by (3.22) there exist $f_i : \mathbf{Spec}(\mathbb{C}) \to \bar{H}^{G,\beta}_{N,h'}$, $(X_i, G, \rho_i) \simeq \mathsf{M}^G_h(f_i)((\bar{U}^{G,\beta}_{N,h'} \to \bar{H}^{G,\beta}_{N,h'}), G, \beta)$ for $i = 1, 2$.

From the proof of 3.19 we see that X_1, X_2 are isomorphic as G-marked varieties $\Longleftrightarrow$ $\exists g \in C(G,\beta)$ such that $f_1(\mathbf{Spec}(\mathbb{C})) = \Psi_g f_2(\mathbf{Spec}(\mathbb{C}))$.
(2) Following the notations in (3.20), assume that T is connected and $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ has decomposition type β, denote by Ψ' the action of $C(G,\beta)$ on $\mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G)$, from the proof in (3.13) we see that $\forall g \in C(G,\beta)$, $\Psi'_g \times \Phi_g$ leaves $q^*\mathfrak{X} = f^*(\bar{U}^{G,\beta}_{N,h'})$ invariant as a subscheme of $\mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \times \mathbb{P}^N$, i.e. $(\Psi'_g \times \Phi_g) f^*(\bar{U}^{G,\beta}_{N,h'}) = f^*(\bar{U}^{G,\beta}_{N,h'})$. This implies that

$$(\Psi'_g \times id) f^*(\bar{U}^{G,\beta}_{N,h'}) = f^*((id \times \Phi_{g^{-1}})(\bar{U}^{G,\beta}_{N,h'})) = f^*((\Psi_g \times id)(\bar{U}^{G,\beta}_{N,h'})).$$

Therefore we conclude that the map we get in (3.20), $f : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \to \bar{H}^{G,\beta}_{N,h'}$, is $C(G,\beta)$-equivariant.

4 The Construction of $\mathfrak{M}_h[G]$

In section 3 we have obtained a parametrizing space $\bar{H}^G_{N,h'}$ for the moduli functor M^G_h, now we will construct $\mathfrak{M}_h[G]$ as a quotient space of $\bar{H}^G_{N,h'}$ and show that it is exactly the coarse moduli scheme for M^G_h.
In (3.17) we have seen that the group $C(G,\beta)$ acts on $H^{G,\beta}_{N,h'}$: it is clear that the subscheme $\bar{H}^{G,\beta}_{N,h'}$ is invariant under this action. The first goal of this section is to show that the quotient $\bar{H}^{G,\beta}_{N,h'}/C(G,\beta)$ exists (as a scheme).
Set $SC(G,\beta) := SL(N+1,\mathbb{C}) \bigcap C(G,\beta)$, it is easy to see that $\bar{H}^{G,\beta}_{N,h'}/C(G,\beta) \simeq \bar{H}^{G,\beta}_{N,h'}/SC(G,\beta)$ (if one of them exists). Therefore from now on we consider $\bar{H}^{G,\beta}_{N,h'}/SC(G,\beta)$ instead. (It is not difficult to show that $SC(G,\beta)$ is reductive.)

Lemma 4.1. *$SC(G,\beta)$ acts properly on $\bar{H}^{G,\beta}_{N,h'}$ and $\forall x \in \bar{H}^{G,\beta}_{N,h'}$, the stabilizer subgroup $Stab(x)$ is finite.*

Proof. In the case where G is trivial the lemma is known by studying the separatedness of the corresponding functor (cf. [Vie95], 7.6, 8.21; [Kov09], 5.D). Now since $SC(G,\beta)$ is a closed subgroup of $SL(N+1,\mathbb{C})$ and $\bar{H}^{G,\beta}_{N,h'}$ is a closed subscheme of $\bar{H}_{N,h'}$ which stays invariant under the action of $SC(G,\beta)$, the lemma follows immediately. □

In order to apply Geometric Invariant theory, we have to find a $SC(G,\beta)$-linearized invertible sheaf on $\bar{H}^{G,\beta}_{N,h'}$ and verify some stability conditions (cf. [MF82], Chap.1). Let's first look at the case where G is trivial, let $p : \bar{U}_{N,h'} \to \bar{H}_{N,h'}$ be the universal family, define $\lambda_{k_0} := \det(p_*(\omega^{k_0}_{\bar{U}_{N,h'}/\bar{H}_{N,h'}}))$. A result of Viehweg (see [Vie95] 7.17) states that λ_{k_0} admits an $SL(N+1,\mathbb{C})$-linearization and $\bar{H}_{N,h'} = (\bar{H}_{N,h'})^s(\lambda_{k_0})$, where $(\bar{H}_{N,h'})^s(\lambda_{k_0})$ denotes the set of $SL(N+1,\mathbb{C})$-stable points with respect to λ_{k_0}. Then it is easy to get the following proposition:

Proposition 4.2. *There exists a geometric quotient* $(\mathfrak{M}^{G,\beta}_{k_0,h}, \pi_\beta)$ *of* $\bar{H}^{G,\beta}_{N,h'}$ *by* $SC(G,\beta)$*, moreover:*
(1) The quotient map $\pi_\beta : \bar{H}^{G,\beta}_{N,h'} \to \mathfrak{M}^{G,\beta}_{k_0,h}$ *is an affine morphism.*
(2) There exists an ample invertible sheaf $\mathcal{L}$ *on* $\mathfrak{M}^{G,\beta}_{k_0,h}$ *such that* $\pi^*_\beta\mathcal{L} \simeq (\lambda^{G,\beta}_{k_0})^n$ *for some* $n > 0$*, where* $\lambda^{G,\beta}_{k_0} := \det(p_*(\omega^{k_0}_{\bar{U}^{G,\beta}_{N,h'}/\bar{H}^{G,\beta}_{N,h'}}))$.

Proof. First note that $\omega^{k_0}_{\bar{U}_{N,h'}/\bar{H}_{N,h'}}|_{\bar{U}^{G,\beta}_{N,h'}} \simeq \omega^{k_0}_{\bar{U}^{G,\beta}_{N,h'}/\bar{H}^{G,\beta}_{N,h'}}$ (cf. [HK04], Lemma 2.6), then by applying "cohomology and base change", we have that $\lambda^{G,\beta}_{k_0} = \lambda_{k_0}|_{\bar{H}^{G,\beta}_{N,h'}}$. Since $\bar{H}^{G,\beta}_{N,h'}$ (as a subscheme of $\bar{H}_{N,h'}$) is invariant under the $SC(G,\beta)$-action, the $SL(N+1,\mathbb{C})$-linearization of λ_{k_0} induces a natural $SC(G,\beta)$-linearization of $\lambda^{G,\beta}_{k_0}$. By Lemma 4.1, we have that $SL(N+1,\mathbb{C})$ acts properly on $\bar{H}_{N,h'}$ and $SC(G,\beta)$ acts properly on $\bar{H}^{G,\beta}_{N,h'}$. Noting that a one-parameter subgroup $\mu : \mathbb{C}^* \to SC(G,\beta)$ is also a subgroup of $SL(N+1,\mathbb{C})$ and that $\bar{H}^{G,\beta}_{N,h'}$ is closed in $H^{G,\beta}_{N,h'}$, we see that for any $x \in \bar{H}^{G,\beta}_{N,h'}$, $\lim_{t\to 0}(\mu(t)x)$ exists in $\bar{H}^{G,\beta}_{N,h'}$ if and only if it exists in $H^{G,\beta}_{N,h'}$. Now by applying the Hilbert-Mumford criterion (cf. [MF82], Theorem 2.1), we have that $(\bar{H}_{N,h'})^s(\lambda_{k_0}) = \bar{H}_{N,h'} \Rightarrow (\bar{H}^{G,\beta}_{N,h'})^s(\lambda^{G,\beta}_{k_0}) = \bar{H}^{G,\beta}_{N,h'}$. Then the proposition is just a result of the standard GIT methods (cf. [MF82], Theorem 1.10). $\square$

We are ready to prove the main theorem (1.1):

Proof of (1.1). Set $\mathfrak{M}_h[G] := \bigsqcup_{[\beta]\in\mathcal{B}_N} \mathfrak{M}^{G,\beta}_{k_0,h}$, and note that if $\mathsf{M}^G_h(\mathbf{Spec}\mathbb{C}) = \emptyset$ then $\mathfrak{M}_h[G] = \emptyset$.
Let us make the following convention: $\forall$ natural transformation $\theta : \mathsf{M}^G_h \to \mathrm{Hom}(-,Q)$, $\forall$ scheme T and $[((p : \mathfrak{X} \to T), G, \rho)] \in \mathsf{M}^G_h(T)$, we write $\theta_T(\mathfrak{X})$ or simply $\theta(\mathfrak{X})$ as an abbreviation for $\theta_T([((p : \mathfrak{X} \to T), G, \rho)])$.
Step 1. Construction of a natural transformation $\eta : \mathsf{M}^G_h \to \mathrm{Hom}(-, \mathfrak{M}_h[G])$:
Let T be a scheme and $((p : \mathfrak{X} \to T), G, \rho) \in \mathsf{M}^G_h(T)$. It suffices to define η on each connected components of T, hence we assume furthermore that T is connected. We have the bundle of G-frames of $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ over T, $q : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \to T$.

By (3.20) and (3.22) there exists a morphsim $f_{\mathfrak{X}/T,k_0,G}:\mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}),G)\to \bar{H}^{G,\beta}_{N,h'}$, such that $\mathsf{M}^G_h(q)((\mathfrak{X}\to T),G,\rho)\simeq \mathsf{M}^G_h(f_{\mathfrak{X}/T,k_0,G})((\bar{U}^{G,\beta}_{N,h'}\to \bar{H}^{G,\beta}_{N,h'}),G,\beta)$ for some $[\beta]\in\mathcal{B}_N$. Set $\bar{f}_{\mathfrak{X}/T,k_0,G}:=\pi_\beta\circ f_{\mathfrak{X}/T,k_0,G}:\mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}),G)\to\mathfrak{M}^{G,\beta}_{k_0,h}$, by (3.23-2) we see that $\bar{f}_{\mathfrak{X}/T,k_0,G}$ is $C(G,\beta)$-equivariant (we take the trivial action on $\mathfrak{M}^{G,\beta}_{k_0,h}$), applying (3.15) we get a (unique) morphsim $\eta_T(\mathfrak{X}):T\to\mathfrak{M}_h[G]$ such that $\bar{f}_{\mathfrak{X}/T,k_0,G}=\eta_T(\mathfrak{X})\circ q$. Note that by 3.21 $\eta_T(\mathfrak{X})$ is independent of the representative family $((p:\mathfrak{X}\to T),G,\rho)$ that we choose, hence well defined.
In order to show that η is a natural transformation, let $l\in\mathrm{Hom}(S,T)$ and $((p:\mathfrak{X}\to T),G,\rho)\in\mathsf{M}^G_h(T)$, it suffices to show that $\eta_S(\mathfrak{X}_S)=\eta_T(\mathfrak{X})\circ l$. Without loss of generality we assume that S and T are connected and $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ has decomposition type β, now consider the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{F}((p_S)_*(\omega^{k_0}_{\mathfrak{X}_S/S}),G) & \xrightarrow{\tilde{l}} & \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}),G)\\
\Big\downarrow{\scriptstyle q_S} & & \Big\downarrow{\scriptstyle q}\\
S & \xrightarrow{l} & T
\end{array}$$

From (3.3-1) and (3.14) we see that $\bar{f}_{\mathfrak{X}_S/S,k_0,G}=\bar{f}_{\mathfrak{X}/T,k_0,G}\circ\tilde{l}$, then note that $\bar{f}_{\mathfrak{X}_S/S,k_0,G}$, $\bar{f}_{\mathfrak{X}/T,k_0,G}$ and $\tilde{l}$ are all $C(G,\beta)$-equivariant, therefore we get that $\eta_S(\mathfrak{X}_S)=\eta_T(\mathfrak{X})\circ l$ by (3.15).
Step 2. $\mathfrak{M}_h[G]$ is the coarse moduli scheme for M^G_h:
(1) $\eta_{\mathbf{Spec}\mathbb{C}}$ induces a one-to-one correspondence between $\mathsf{M}^G_h(\mathbf{Spec}\mathbb{C})$ and the set of (closed) points of $\mathfrak{M}_h[G]$.
Surjectivity follows from (3.22), injectivity follows from (3.23-1)
(2) The universal property of η.
Let $\theta:\mathsf{M}^G_h\to\mathrm{Hom}(-,Q)$ be another natural transformation. We will show that there exists a unique morphism $\gamma:\mathfrak{M}_h[G]\to Q$, such that $\theta=\mathrm{Hom}(\gamma)\circ\eta$. For any $[\beta]\in\mathcal{B}_N$, consider the universal family $((\bar{U}^{G,\beta}_{N,h'}\to\bar{H}^{G,\beta}_{N,h'}),G,\beta)\in\mathsf{M}^G_h(\bar{H}^{G,\beta}_{N,h'})$. It induces a morphism $\theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'}):\bar{H}^{G,\beta}_{N,h'}\to Q$. For any $g\in C(G,\beta)$, we have that

$$(\Psi_g\times id_{\mathbb{P}^N})((\bar{U}^{G,\beta}_{N,h'}\to\bar{H}^{G,\beta}_{N,h'}),G,\beta)=(id_{\bar{H}^{G,\beta}_{N,h'}}\times\Phi_{g^{-1}})((\bar{U}^{G,\beta}_{N,h'}\to\bar{H}^{G,\beta}_{N,h'}),G,\beta)$$

as subschemes of $\bar{H}^{G,\beta}_{N,h'}\times\mathbb{P}^N$, noting that the right hand side is isomorphic to $((\bar{U}^{G,\beta}_{N,h'}\to\bar{H}^{G,\beta}_{N,h'}),G,\beta)$ as G-mark families, we see that

$$\theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'})=\theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'})\circ\Psi_g,$$

which implies that $\theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'})$ is $C(G,\beta)$-equivariant (we take the trivial action on Q), hence it induces a (unique) morphism $\gamma_\beta : \mathfrak{M}^{G,\beta}_{k_0,h} \to Q$ such that $\theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'}) = \gamma_\beta \circ \eta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'})$. Now we can define $\gamma : \mathfrak{M}_h[G] \to Q$, such that the restriction of γ to each $\mathfrak{M}^{G,\beta}_{k_0,h}$ is γ_β.
Given $((p : \mathfrak{X} \to T), G, \rho) \in \mathsf{M}^G_h(T)$, let $q : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \to T$ be the bundle of G-frames of $p_*(\omega^{k_0}_{\mathfrak{X}/T})$, we assume again that T is connected and $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ has decomposition type β. By (3.20) and (3.22) there exists $f_{\mathfrak{X}/T,k_0,G} : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \to \bar{H}^{G,\beta}_{N,h'}$ such that

$$\mathsf{M}^G_h(q)((\mathfrak{X} \to T), G, \rho) \simeq \mathsf{M}^G_h(f_{\mathfrak{X}/T,k_0,G})((\bar{U}^{G,\beta}_{N,h'} \to \bar{H}^{G,\beta}_{N,h'}), G, \beta),$$

hence we have that

$$\theta(q^*\mathfrak{X}) = \theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'}) \circ f_{\mathfrak{X}/T,k_0,G} = \gamma_\beta \circ \eta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'}) \circ f_{\mathfrak{X}/T,k_0,G} = \gamma_\beta \circ \eta(q^*\mathfrak{X}),$$

where the first and third equalities hold since θ, η are natural transformation, the second equality holds from the construction of γ_β. Finally the fact that $f_{\mathfrak{X}/T,k_0,G}$ and $\theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'})$ are $C(G,\beta)$-equivariant $\Rightarrow$ $\theta(q^*\mathfrak{X})$ is also $C(G,\beta)$-equivariant. By (3.15) $\exists!\ l \in \mathrm{Hom}(T,Q)$ such that $\theta(q^*\mathfrak{X}) = l \circ q$, which implies that $\theta_T(\mathfrak{X}) = l = \gamma_\beta \circ \eta_T(\mathfrak{X})$. ☐

As an application of our results, we show that the locus $\mathfrak{M}_h(G)$ inside $\mathfrak{M}_h$ of the varieties which admit an effective action by the group G is closed. This has been proven in [Cat83], theorem 1.8 for the case of surfaces, the idea there generalizes naturally to the higher dimensional case.
Given a faithful representation $\beta : G \to GL(N+1,\mathbb{C})$, we have a natural inclusion $i_\beta : \bar{H}^{G,\beta}_{N,h'} \subset \bar{H}_{N,h'}$. Note that the restriction of the quotient map $\pi : \bar{H}_{N,h'} \to \mathfrak{M}_h$ to $\bar{H}^{G,\beta}_{N,h'}$ is $SC(G,\beta)$-equivariant, hence we obtain a morphism $u^{G,\beta}_{k_0,h} : \mathfrak{M}^{G,\beta}_{k_0,h} \to \mathfrak{M}_h$. We define a morphism $u^G_h : \mathfrak{M}_h[G] \to \mathfrak{M}_h$ such that $u^G_h|_{\mathfrak{M}^{G,\beta}_{k_0,h}} = u^{G,\beta}_{k_0,h}$. We denote by $\mathfrak{M}_h(G)$ the (scheme-theoretic) image of u^G_h in $\mathfrak{M}_h$. Then we can interpret the problem into showing that u^G_h maps $\mathfrak{M}_h[G]$ surjectively onto $\mathfrak{M}_h(G)$.

Corollary 4.3. *The morphism $u^G_h : \mathfrak{M}_h[G] \to \mathfrak{M}_h$ is finite and maps $\mathfrak{M}_h[G]$ surjectively onto $\mathfrak{M}_h(G)$; $\mathfrak{M}_h(G)$ is a closed subscheme of $\mathfrak{M}_h$.*

Proof. It is easy to see that u^G_h is quasi-finite: given a point $[X] \in \mathfrak{M}_h$, since $\mathrm{Aut}(X)$ is finite, then the set of injective homomorphisms $\rho : G \to \mathrm{Aut}(X)$ is

finite, hence $(u_h^G)^{-1}([X])$, which corresponds to the set of isomorphism classes of G-markings on X, is also finite.
For the remaining statements, it suffices to show that u_h^G is proper, which is equivalent to showing that $u_{k_0,h}^{G,\beta} : \mathfrak{M}_{k_0,h}^{G,\beta} \to \mathfrak{M}_h$ is proper for each $[\beta] \in \mathcal{B}_N$. Applying the valuative criterion of properness, we have to prove that for every pointed curve (C, O) (not necessarily complete) and for any commutative diagram

$$\begin{array}{ccc} C^\star & \xrightarrow{f'} & \mathfrak{M}_{k_0,h}^{G,\beta} \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle u_{k_0,h}^{G,\beta}} \\ C & \xrightarrow{f} & \mathfrak{M}_h \end{array}$$

where $C^\star := C - \{O\}$, there exists a unique $l : C \to \mathfrak{M}_{k_0,h}^{G,\beta}$ making the whole diagram commute.
By GIT we know that $\mathfrak{M}_{k_0,h}^{G,\beta}$ is quasi-projective and hence separated, therefore the uniqueness of l is clear. For the existence of l, since $\pi_\beta : \bar{H}_{N,h'}^{G,\beta} \to \mathfrak{M}_{k_0,h}^{G,\beta}$ is a quotient map of quasi-projective schemes, it suffices to show that there exist a finite morphism $v : (B, O') \to (C, O)$ and a morphism $l' : B \to \bar{H}_{N,h'}^{G,\beta}$ such that

$$(*)\ u_{k_0,h}^{G,\beta} \circ \pi_\beta \circ l' = f \circ v \textit{ and } \pi_\beta \circ (l'|_{B^\star}) = f' \circ (v|_{B^\star}),$$

where $B^\star := B - \{O'\}$.
Considering the quotient map $\pi : \bar{H}_{N,h'} \to \mathfrak{M}_h$, we can assume without loss of generality that we have a morphism $m : C \to \bar{H}_{N,h'}$, such that $f = \pi \circ m$. Then we obtain a family $(m^*(\bar{U}_{N,h'}) := \mathfrak{X} \to C) \in \mathsf{M}_h(C)$ such that $\mathfrak{X} \subset C \times \mathbb{P}^N$. The idea of constructing the morphism $v : (B, O') \to (C, O)$ is similar to that of 3.13. We consider first the subspace

$$Z := \{(t, A(t)|A(t)\mathfrak{X}_t \textit{ corresponds to a point in } \bar{H}_{N,h'}^{G,\beta}\} \subset C \times GL(N+1, \mathbb{C}).$$

By assumption we see that $p_1 : Z - p_1^{-1}(O) \to C^\star$ is surjective, where $p_1 : T \times GL(n+1, \mathbb{C}) \to T$ is the projection onto the first factor, hence we can find a curve B' inside Z, such that $p_1|_{B'} : B' \to C^\star$ is surjective. For similar reasons as in 3.13, we get a G-marked family $((p_1|_{B'})^*\mathfrak{X}^\star \to B'), G, \beta)$, where $\mathfrak{X}^\star := \mathfrak{X} - \mathfrak{X}_O$. After possibly taking the normalization of B', we can extend the morphism $p_1|_{B'}$ to a morphism $v : (B, O') \to (C, O)$ and we see that $(((v|_{B^\star})^*\mathfrak{X}^\star \to B^\star), G, \beta)$ is a G-marked family, where $B^\star := B - \{O'\}$.

We claim that the action of G on $(v|_{B^\star})^*\mathfrak{X}^\star \to B^\star$ can be extended to an action on $v^*\mathfrak{X} := \mathfrak{X}' \to B$. Since $\omega^{k_0}_{\mathfrak{X}'/T}$ induces an embedding $\mathfrak{X}' \to B \times \mathbb{P}^N$, we see that the claim is equivalent to that the action of G on $\Gamma(\mathfrak{X}' - \mathfrak{X}'_{O'}, \omega^{k_0}_{\mathfrak{X}'/T})$ extends to an action on $\Gamma(\mathfrak{X}', \omega^{k_0}_{\mathfrak{X}'/T})$. Noting that $\mathfrak{X}'$ is normal (since $\mathfrak{X}' \to T$ is a Gorenstein fibration of canonical models over a smooth base) and $\mathfrak{X}'_{O'}$ has codimension≥ 2 in $\mathfrak{X}'$, the restriction map

$$\Gamma(\mathfrak{X}', \omega^{k_0}_{\mathfrak{X}'/T}) \to \Gamma(\mathfrak{X}' - \mathfrak{X}'_{O'}, \omega^{k_0}_{\mathfrak{X}'/T})$$

is in fact an isomorphsim, hence the action of G on $\Gamma(\mathfrak{X}' - \mathfrak{X}'_{O'}, \omega^{k_0}_{\mathfrak{X}'/T})$ extends naturally to an action on $\Gamma(\mathfrak{X}', \omega^{k_0}_{\mathfrak{X}'/T})$.
Now we have a G-marked family $((\mathfrak{X}' \to B), G, \beta)$, by 3.19 we obtain a morphism $l' : B \to \bar{H}^{G,\beta}_{N,h'}$, it is easy to check that l' satisfies $(*)$. $\square$

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