A fermionic approach to tunneling through junctions of multiple quantum wires

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Junctions of multiple one-dimensional quantum wires of interacting electrons have received considerable theoretical attention as a basic constituent of quantum circuits. While results have been obtained on these models using bosonization and Density Matrix Renormalization Group (DMRG) methods, another powerful technique is based on direct perturbation theory in the bulk interactions, combined with the Renormalization Group (RG) and summed in the Random Phase Approximation (RPA). This technique has so far only been applied to the case where finite length interacting wires are attached to non-interacting Fermi liquid leads. We reformulate it in terms of the single-particle S-matrix, formally unifying treatments of junctions of different numbers of leads, and extend this method to cover the case of infinite length interacting leads obtaining results on 2-lead and 3-lead junctions in good agreement with previous bosonization and DMRG results.

I. INTRODUCTION AND CONCLUSION

A class of powerful theoretical approaches to junctions models the quantum wires as conformally invariant bulk Tomonaga-Luttinger liquids (TLL).[1–5] In the spirit of boundary conformal field theory, at low energies the junction with its boundary operators should eventually renormalize to conformally invariant boundary conditions. Possible fixed points of the renormalization group (RG) flow are then postulated, and their various properties, such as zero-temperature conductance and operator scaling dimensions, are explored. Details of the RG flow, however, are largely open to conjecture except in the vicinity of these fixed points. These approaches are often consolidated with the technique of bosonization, as the elementary excitations of TLLs are bosonic in nature, and various boundary conditions imposed by the junction are often conveniently expressed in bosonic field variables.

An alternate formalism has been independently developed in the language of fermions.[6–10] In the most primitive version of the problem, the junction system without electron-electron interactions in the quantum wires is completely non-interacting, characterized by a single-particle S-matrix. The interaction is then handled by perturbation theory; an RG equation for the S-matrix is obtained, and we gain information about the conductance by solving for the RG flow. The merit of such a formalism is that the exact crossover behavior between different fixed points can, in principle, be found to any order in interaction; indeed, it has been found that a random phase approximation (RPA) resummation of the interaction reproduces various scaling dimensions of the conductance known from bosonic methods.[8–10] (The term RPA has been used interchangeably with "ladder approximation" in Refs. 8–10.) Nevertheless, when the interaction becomes sufficiently strong in a junction of three wires (a "Y-junction"), the fixed points and the RG flow predicted by the fermionic approach and the bosonic approach begin to differ qualitatively. Also a careful analysis reveals that the β function of the S-matrix beyond one-loop order contains non-universal terms,[8,9] which depend on the precise cutoff scheme of the theory and may potentially change its predictions.

To our knowledge, many aspects of the junction problem have not been explored in the fermionic formalism. One such example is the well-known Maslov-Stone physics,[4,11,12] which recognizes the distinction between a semi-infinite TLL wire and a finite TLL wire connected to a Fermi liquid (FL) reservoir. It is predicted that the FL reservoir (or "lead") renormalizes the linear DC conductance: the conductance of a finite TLL wire attached to FL leads on both sides is e^2/h, irrespective of the interaction strength; on the other hand, the conductance of an infinite spinless TLL wire is Ke^2/h, where K is the Luttinger parameter. The Landauer formula based on a perfectly transmitting S-matrix alone cannot recover the Ke^2/h result. The case of FL leads has been studied in Refs. 8–10, but the effects of TLL leads on the conductance are not discussed.

The reasons are twofold for our interest in the effects of TLL leads on the conductance from the fermionic perspective. At the fixed points well understood in the bosonic approach, such as the perfect transmission fixed point in the two-lead junction and the chiral fixed points in the Y-junction, the agreement of these results in both approaches is a necessary validation of the fermionic approach. On the other hand, for the fixed points eluding the bosonic treatment, such as the maximally open fixed point of the Y-junction (known as the "M fixed point"),[4,9,10] these results can be directly compared to numerics[13] where available.

In this work, we adopt the RPA fermionic approach to study the conductance tensor for a generic multi-lead junction in the presence of TLL leads. Our theory makes extensive use of the scattering basis transformation of the non-interacting part of the system; as a result it is explicitly formulated on the basis of the single-particle S-matrix (much like that in Ref. 7), and is formally independent of the number of wires. This stands in contrast to previous RPA treatments of junctions attached to FL leads, whose formulation heavily depends on the parametrization of the

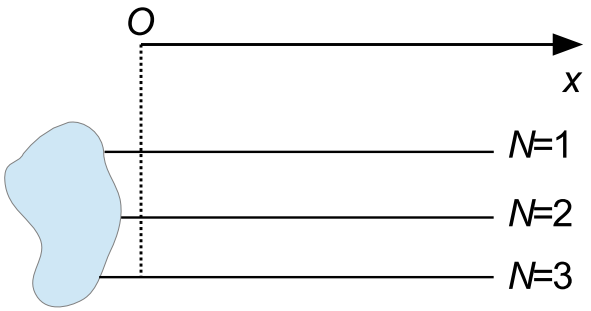

FIG. 1. Sketch of the system with the number of leads $N = 3$.

conductance tensors, different for two-lead junctions[8] and Y-junctions.[9,10] We derive a Landauer-type conductance formula, appropriate for the renormalized S-matrix, and recover the additional contribution from the TLL leads to the conductance, absent in the naive Landauer formalism. Our theory is applied to the two-lead junction and Y-junction problems, where in addition to verifying existing results on the fixed points and the phase diagrams, the conductance of the M fixed point attached to TLL leads is calculated. We summarize our findings below.

The system consists of N quantum wires of interacting spinless electrons, numbered $j = 1, 2, ..., N$, meeting at a junction which we choose as the origin $x = 0$. We align the wires so that they are parallel to the $+x$ axis; see Fig. 1.

We assume that in the absence of interactions the junction is characterized by a single-particle S-matrix, $S_{jj'}$, independent of the energy of the incident/scattered electron. This is expected to be valid at low energies in the case of a non-resonant S-matrix, which we assume. We adopt a Tomonaga-Luttinger model for the electron-electron interaction in wire j,

$$H_{\text{int}}^{j} = \int_0^\infty dx g_2^j(x)\, \psi_{jR}^\dagger(x)\, \psi_{jR}(x)\, \psi_{jL}^\dagger(x)\, \psi_{jL}(x)\,, \tag{1.1}$$

where $\psi_{jL/R}$ are low-energy left- and right-moving electrons in wire j, and $g_2^j(x \to \infty)$ is a constant. A finite $g_2^j(\infty) \neq 0$ corresponds to a TLL lead attached to wire j, while if $g_2^j(\infty) = 0$ the junction is considered to be connected to an FL lead. We define a dimensionless interaction strength

$$\alpha_j(x) = g_2^j(x) / (2\pi v_{Fj})\,, \tag{1.2}$$

where v_{Fj} is the Fermi velocity in wire j without interaction.

The linear DC conductance tensor $G_{jj'}$ of the junction is defined by $I_j = \sum_{j'} G_{jj'} V_{j'}$, where I_j is the current flowing away from the junction in wire j, and $V_{j'}$ is the bias voltage applied on lead j'. In the absence of interactions, $G_{jj'}$ is given by the Landauer formula

$$G_{jj'}^{\text{FL}} = \frac{e^2}{2\pi}\left(\delta_{jj'} - |S_{jj'}|^2\right). \tag{1.3}$$

In the first order perturbation theory in α_j, where the infrared singular corrections to physical observables are not resummed using RG methods, attaching a junction to TLL leads as compared to FL leads changes its linear DC conductance by

$$G_{jj'}^{\text{TLL}} - G_{jj'}^{\text{FL}} = -\frac{e^2}{2\pi}\sum_n \frac{1}{2}\alpha_n(\infty)\left(\delta_{jn} - |S_{jn}|^2\right)\left(\delta_{nj'} - |S_{nj'}|^2\right). \tag{1.4}$$

In the first order RG-improved perturbation theory, the bare S-matrix will be replaced by the renormalized S-matrix at temperature T in Eqs. (1.3) and (1.4).

In the RPA "bare" perturbation theory, the linear DC conductance tensors of the same junction attached to TLL leads and FL leads are related by

$$\mathbf{G}^{\mathrm{TLL}} = \left(\mathbf{1} - \mathbf{G}^{\mathrm{FL}}\mathbf{G}_c^{-1}\right)^{-1}\mathbf{G}^{\mathrm{FL}}, \tag{1.5}$$

where $\mathbf{1}$ is the $N \times N$ identity matrix, and $\mathbf{G}_c^{-1}$ is the contact resistance tensor between the wires and leads,

$$\left(G_c^{-1}\right)_{jj'} = \left(\frac{e^2}{2\pi}\right)^{-1}\frac{1}{2}\left[1 - \left(K_j^{\mathrm{L}}\right)^{-1}\right]\delta_{jj'}. \tag{1.6}$$

Here the bulk Luttinger parameter of the lead j is given by $K_j^{\mathrm{L}} = \sqrt{\left(1-\alpha_j\left(\infty\right)\right)/\left(1+\alpha_j\left(\infty\right)\right)}$. In the RPA RG-improved perturbation theory, the bare S-matrix is again replaced by the renormalized S-matrix at temperature T in Eqs. (1.3) and (1.5).

For a Z_3 symmetric Y-junction at the maximally open M fixed point, $\left|S_{jj'}\right|^2 = 4/9 - \delta_{jj'}/3$. When the junction is attached to TLL leads with dimensionless interaction strength α and Luttinger parameter K^{L}, at the first order the conductance tensor is

$$G_{jj'}^{\mathrm{TLL},M} = \left(\frac{4}{9} - \frac{8}{27}\alpha\right)\frac{e^2}{2\pi}\left(3\delta_{jj'} - 1\right). \tag{1.7}$$

In the RPA, the conductance at M becomes

$$G_{jj'}^{\mathrm{TLL},M} = \frac{4K^{\mathrm{L}}}{3K^{\mathrm{L}}+6}\frac{e^2}{2\pi}\left(3\delta_{jj'} - 1\right). \tag{1.8}$$

Eqs. (1.4)–(1.8) are the main results of this paper.

The rest of this paper is organized as follows. Section II elaborates on our model for a generic multi-lead junction, and calculates its two-point Green's function and linear DC conductance to the first order in interaction. Section III is based on perturbative RG, again to the first order in interaction. We derive the S-matrix RG equation in a Callan-Symanzik (CS) approach[8] using the two-point Green's function calculated in Section II B. In conjunction with the Kubo formula calculation, this establishes a modified Landauer formula involving the renormalized S-matrix in the case of FL leads. An additional contribution to the conductance, Eq. (1.4), is shown to arise from TLL leads. In Section IV, the conductance is found in the RPA to arbitrary order in interaction; we derive an S-matrix RG equation in the RPA, and again find the conductance [Eq. (1.5)]. Section V applies our results to the fixed points of 2-lead junctions and Y-junctions at the first order and in the RPA. In particular, we find the conductance at the M fixed point of a Z_3 symmetric Y-junction attached to TLL leads, Eqs. (1.7) and (1.8). Open questions are discussed in Section VI. In Appendix A we show details of the conductance calculations up to the first order in interaction. The Wilsonian derivation of the RG equation for the S-matrix[6] is reviewed in Appendix B. Finally, the RPA conductance calculations are explained in Appendix C.

II. FIRST ORDER PERTURBATION THEORY OF KUBO CONDUCTANCE

In this section, we establish the model Hamiltonian, and present our results for the linear DC conductance at the first order in interaction.

A. Formulation of the problem

The system is modeled by a Hamiltonian consisting of three parts:

$$H = \sum_{j=1}^{N}\left(H_{0,\mathrm{wire}}^{j} + H_{\mathrm{int}}^{j}\right) + H_{0,B}. \tag{2.1}$$

$H_{0,\mathrm{wire}}^{j}$ is the non-interacting part of the Hamiltonian for wire j, quadratic in electron operators, while the quartic H_{int}^{j} term of Eq. (1.1) describes the electron-electron interaction in wire j. The boundary term $H_{0,B}$ is quadratic, and

is responsible for electron transfer between wires across the junction. For simplicity we assume that each wire only supports one single channel, and ignore quartic interactions between wires, at the junction and between the junction and the wires.

In the continuum limit of the model, on each quantum wire we retain right- and left-movers in narrow bands of wave vectors around the Fermi points $\pm k_{Fj}$:

$$\psi_j(x) \approx e^{ik_{Fj}x}\psi_{jR}(x) + e^{-ik_{Fj}x}\psi_{jL}(x) = \int_{-D}^{D} \frac{dE}{\sqrt{2\pi v_{Fj}}} \left[\psi_{jR}(E)\, e^{i\left(\frac{E}{v_{Fj}}+k_{Fj}\right)x} + \psi_{jL}(E)\, e^{-i\left(\frac{E}{v_{Fj}}+k_{Fj}\right)x}\right], \quad (2.2)$$

where v_{Fj} is the Fermi velocity in wire j, the dispersion relation is $E = E_j(k) = v_{Fj}k$, and $D \ll v_{Fj}k_{Fj}$ is the high-energy cutoff. Left-movers ψ_{jL} are incident on the junction, scattered, and turned into right-movers $\psi_{j'R}$; ψ_{jL} and $\psi_{j'R}$ are not independent degrees of freedom, but related by the S-matrix of the junction [see also Eq. (2.5)]. The quadratic part of the wire Hamiltonian now reads

$$H^j_{0,\text{wire}} \approx iv_{Fj}\int_0^\infty dx \left[\psi^\dagger_{jL}\partial_x\psi_{jL} - \psi^\dagger_{jR}\partial_x\psi_{jR}\right](x) \approx \int_{-D}^{D} dE\, E\left(\psi^\dagger_{jR}(E)\,\psi_{jR}(E) - \psi^\dagger_{jL}(E)\,\psi_{jL}(E)\right), \quad (2.3)$$

To model the electron-electron interaction, we assume it is short-ranged and the system is away from half-filling, so that the Umklapp processes are unimportant. We further ignore processes where two chiral densities of the same chirality interact with one another, $\psi^\dagger_R\psi_R\psi^\dagger_R\psi_R$ or $\psi^\dagger_L\psi_L\psi^\dagger_L\psi_L$; these g_4 processes[14] renormalize the Fermi velocity but do not change the Luttinger parameter by themselves. For spinless fermions, this leaves us with only processes involving two chiral densities of different chiralities, or g_2 processes, $\psi^\dagger_R\psi_R\psi^\dagger_L\psi_L$. The electron-electron interaction is then represented by a spatially variant g_2 term as in Eq. (1.1).

Along the lines of Ref. 6, viewing the electron-electron interaction as a perturbation, we can diagonalize the quadratic part of the Hamiltonian. The resultant eigenstates, which form the so-called scattering basis, can be related to the S-matrix in the low-energy theory. For non-resonant scattering, which we assume throughout this paper, the S-matrix elements $S_{jj'}(E) \equiv S_{jj'}$ are independent of the electronic energy E, and the single-particle scattering state incident from wire j' with energy E' reads

$$\phi^\dagger_{j'}(E')|0\rangle = \sum_j \int_0^\infty dx \frac{1}{\sqrt{2\pi v_{Fj}}} \left(\delta_{jj'}e^{-i\frac{E'}{v_{Fj}}x}\psi^\dagger_{jL}(x) + S_{jj'}e^{i\frac{E'}{v_{Fj}}x}\psi^\dagger_{jR}(x)\right)|0\rangle + \cdots, \quad (2.4)$$

where $|0\rangle$ corresponds to the filled Dirac sea, and the omitted terms represent contributions from the junction area. Inverting Eq. (2.4) we may express the original electrons ψ in terms of the scattering basis operators ϕ,

$$\begin{aligned}\psi_{jR}(E) &= \sum_{j'=1}^N \int dE' \int_0^\infty dx \left(\frac{1}{\sqrt{2\pi v_{Fj}}}e^{i\frac{E}{v_{Fj}}x}\right)^* \left(\frac{1}{\sqrt{2\pi v_{Fj}}}S_{jj'}e^{i\frac{E'}{v_{Fj}}x}\right)\phi_{j'}(E') \\ &= \sum_{j'}\int_{-D}^{D}\frac{dE'}{2\pi}\frac{-i}{E-E'-i0}S_{jj'}\phi_{j'}(E') \end{aligned} \quad (2.5a)$$

Similarly

$$\psi_{jL}(E) = \int_{-D}^{D}\frac{dE'}{2\pi}\frac{i}{E-E'+i0}\phi_j(E') \quad (2.5b)$$

Now recast the Hamiltonian in the scattering basis. By definition, the quadratic part of the Hamiltonian is diagonal:

$$\sum_{j=1}^N H^j_{0,\text{wire}} + H_{0,B} = \sum_j \int dE\, E\phi^\dagger_j(E)\,\phi_j(E) \quad (2.6)$$

We insert the scattering basis transformation into the interaction Eq. (1.1). Allowing the energies to run freely from $-\infty$ to ∞ and calculating the energy integrals using the method of residues,[15] we find

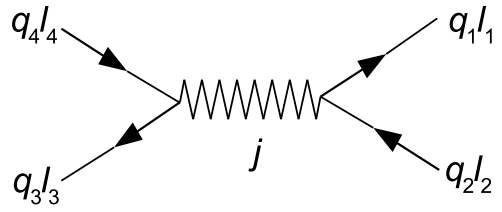

FIG. 2. Diagrammatic representation of the electron-electron interaction.

$$H_{\text{int}}^{j}=\int_{0}^{\infty}dxg_{2}^{j}\left(x\right)\sum_{l_{1}l_{2}l_{3}l_{4}}\int\frac{dE_{1}dE_{2}dE_{3}dE_{4}}{\left(2\pi\right)^{2}v_{Fj}^{2}}\phi_{l_{1}}^{\dagger}\left(E_{1}\right)\phi_{l_{2}}\left(E_{2}\right)\phi_{l_{3}}^{\dagger}\left(E_{3}\right)\phi_{l_{4}}\left(E_{4}\right)e^{i\left(-E_{1}+E_{2}+E_{3}-E_{4}\right)\frac{x}{v_{Fj}}}S_{jl_{1}}^{*}S_{jl_{2}}\delta_{jl_{3}}\delta_{jl_{4}} \tag{2.7}$$

This is a plausible manipulation, seeing that the scattering basis transformation should not introduce additional singularities at the band edge. Now

$$H_{\text{int}}^{j}=\int_{0}^{\infty}dxg_{2}^{j}\left(x\right)\sum_{l_{1}l_{2}l_{3}l_{4}}\int\frac{dE_{1}dE_{2}dE_{3}dE_{4}}{\left(2\pi\right)^{2}v_{Fj}^{2}}\varrho_{l_{1}l_{2}l_{3}l_{4}}^{j}\left(E_{1},E_{2},E_{3},E_{4};x\right)\phi_{l_{1}}^{\dagger}\left(E_{1}\right)\phi_{l_{2}}\left(E_{2}\right)\phi_{l_{3}}^{\dagger}\left(E_{3}\right)\phi_{l_{4}}\left(E_{4}\right), \tag{2.8}$$

where we introduce the function

$$\varrho_{l_{1}l_{2}l_{3}l_{4}}^{j}\left(E_{1},E_{2},E_{3},E_{4};x\right)\equiv\frac{1}{2}\left[e^{i\left(-E_{1}+E_{2}+E_{3}-E_{4}\right)\frac{x}{v_{Fj}}}S_{jl_{1}}^{*}S_{jl_{2}}\delta_{jl_{3}}\delta_{jl_{4}}+e^{i\left(-E_{3}+E_{4}+E_{1}-E_{2}\right)\frac{x}{v_{Fj}}}S_{jl_{3}}^{*}S_{jl_{4}}\delta_{jl_{1}}\delta_{jl_{2}}\right]. \tag{2.9}$$

Note that we have symmetrized the function ϱ so that $\varrho_{l_1l_2l_3l_4}^{j}\left(q_1,q_2,q_3,q_4;x\right)=\varrho_{l_3l_4l_1l_2}^{j}\left(q_3,q_4,q_1,q_2;x\right)$. This interaction is diagrammatically represented by the symmetric vertex in Fig. 2. We may well opt not to symmetrize ϱ; however, the two created electrons q_1l_1 and q_3l_3 (or the two annihilated electrons q_2l_2 and q_4l_4) would be inequivalent in that case, and the diagrammatic bookkeeping would be more difficult.

B. Two-point Green's function

The first observable we are interested in is the retarded two-point Green's function:

$$G_{jR,j'L}^{\text{R}}\left(x,x';t-t'\right)=-i\theta\left(t-t'\right)\left\langle\left[\psi_{jR}\left(x,t\right),\psi_{j'L}^{\dagger}\left(x',t'\right)\right]\right\rangle. \tag{2.10}$$

which describes the response of right-movers at coordinate x in wire j to left-movers at x' in wire j'. Intuitively, when x and x' are far away from the junction, we expect this object to be related to the probability amplitude for a left-mover in j' to become a right-mover in j, through scattering of the junction. This probability amplitude is none other than the S-matrix element, $S_{jj'}$. The usefulness of this two-point function lies in retaining both the magnitude and the phase of $S_{jj'}$, in contrast to the linear DC conductance which only involves $|S_{jj'}|^2$ and does not directly carry the phase information.

To calculate this retarded Green's function, it is convenient to consider its imaginary time counterpart which has a simple perturbation theory, $\mathcal{G}_{jR,j'L}\left(x,x';\tau-\tau'\right)=-\left\langle T_{\tau}\psi_{jR}\left(x,\tau\right)\psi_{j'L}^{\dagger}\left(x',\tau'\right)\right\rangle$, where T_τ is the imaginary time-ordering operator. Using Eq. (2.5), this becomes

$$\mathcal{G}_{jR,j'L}\left(x,x';\tau-\tau'\right)=-\sum_{j_1}\int d\epsilon_1 d\epsilon_2 e^{i\left(\frac{\epsilon_1}{v_{Fj}}x-\frac{\epsilon_2}{v_{Fj'}}x'\right)}S_{jj_1}\left\langle T_\tau\phi_{j_1}\left(\epsilon_1,\tau\right)\phi^\dagger_{j'}\left(\epsilon_2,\tau'\right)\right\rangle. \tag{2.11}$$

Switching to the interaction picture, we perform a Wick decomposition of the time-ordered product, go to the frequency space and sum over the Matsubara frequencies. The retarded correlation function is then obtained by analytic continuation $i\omega_n\to\omega^+\equiv\omega+i\eta_\omega$ where the $\eta_\omega\to 0^+$ limit is taken. The energy integrals are calculated afterwards, followed by real space integrals [which appear in Eq. (2.8)] in the end. As the perturbative calculation of the two-point function is similar to but less intricate than that of the conductance (see Section II C), we choose not to present the details for the two-point function, and only outline the results here.

To the first order in interaction, in the limit x, $x'\to\infty$, the result is

$$G^{\mathrm{R}}_{jR,j'L}\left(x,x';\omega^+\right)=\left(-2\pi i\right)e^{i\omega^+\left(\frac{x}{v_{Fj}}+\frac{x'}{v_{Fj'}}\right)}S^{\mathrm{d}(1)}_{jj'}\left(\omega^+\right), \tag{2.12}$$

where the first order "dressed S-matrix" $S^{\mathrm{d}(1)}$ is given by

$$\begin{aligned}S^{\mathrm{d}(1)}_{jj'}\left(\epsilon\right)=S_{jj'}-i\sum_n\int_0^\infty dy\,\alpha_n\left(y\right)\int\frac{d\epsilon'}{v_{Fn}}n_F\left(\epsilon'\right)\\ \times S_{jn}S^*_{nn}S_{nj'}\exp\left(2i\left(\epsilon-\epsilon'\right)\frac{y}{v_{Fn}}\right)+\delta_{jn}S_{nn}\delta_{j'n}\exp\left(-2i\left(\epsilon-\epsilon'\right)\frac{y}{v_{Fn}}\right);\end{aligned} \tag{2.13}$$

$n_F\left(\epsilon\right)=1/\left(e^{\beta\epsilon}+1\right)$ is the Fermi distribution at temperature $\beta=1/T$, and $\alpha_n\left(y\right)$ is defined in Eq. (1.2). For a non-interacting system $S^{\mathrm{d}(1)}_{jj'}\left(\epsilon\right)=S_{jj'}$; this is in agreement with our intuitive expectation.

We now perform the y integral in a simple model. Let us assume that the junction is connected through wire n to a TLL or FL lead at $x=L_n$; in other words, when $x\geq L_n$, $\alpha_n\left(x\right)=\alpha_n\left(\infty\right)$ becomes a constant independent of x and $d\alpha_n\left(x\right)/dx=0$. We further assume that the interaction inside the wire is also uniform, i.e.

$$\alpha_j\left(x\right)=\alpha_j\left(0\right)+\left[\alpha_j\left(\infty\right)-\alpha_j\left(0\right)\right]\theta\left(x-L_j\right) \tag{2.14}$$

where $\theta\left(x\right)$ is the Heaviside unit-step function. Integrating over y:

$$\begin{aligned}S^{\mathrm{d}(1)}_{jj'}\left(\epsilon\right)=S_{jj'}\left(\epsilon\right)-\sum_n\int d\epsilon'\frac{n_F\left(\epsilon'\right)}{2\left(\epsilon'-\epsilon\right)}\Big(S_{jn}S^*_{nn}S_{nj'}\left[\left(\alpha_n\left(\infty\right)-\alpha_n\left(0\right)\right)e^{2i\left(\epsilon-\epsilon'\right)\frac{L_n}{v_{Fn}}}+\alpha_n\left(0\right)\right]\\ -\delta_{jn}S_{nn}\delta_{j'n}\left[\left(\alpha_n\left(\infty\right)-\alpha_n\left(0\right)\right)e^{-2i\left(\epsilon-\epsilon'\right)\frac{L_n}{v_{Fn}}}+\alpha_n\left(0\right)\right]\Big).\end{aligned} \tag{2.15}$$

The ϵ' integral is infrared divergent, which prompts an RG resummation of leading logarithms. We will determine the renormalization of the S-matrix using Eq. (2.15) and discuss its implications in Section III.

C. Kubo conductance

We now compute the linear DC conductance in Kubo formalism. The current operator at coordinate x in wire j is first written in terms of the fermion fields:

$$\hat{I}_j\left(x\right)=ev_{Fj}\left(\psi^\dagger_{jR}\psi_{jR}-\psi^\dagger_{jL}\psi_{jL}\right)\left(x\right). \tag{2.16}$$

Note that $\hat{I}_j$ is not changed by the interaction; it is proportional to the commutator of the electron density with the Hamiltonian, but the interaction commutes with the electron density. Using Eq. (2.5) we find the imaginary time correlation function $\Omega_{jj'}\left(x,x';\tau-\tau'\right)\equiv-\left\langle T_\tau I_j\left(x,\tau\right)I_{j'}\left(x',\tau'\right)\right\rangle$ to be

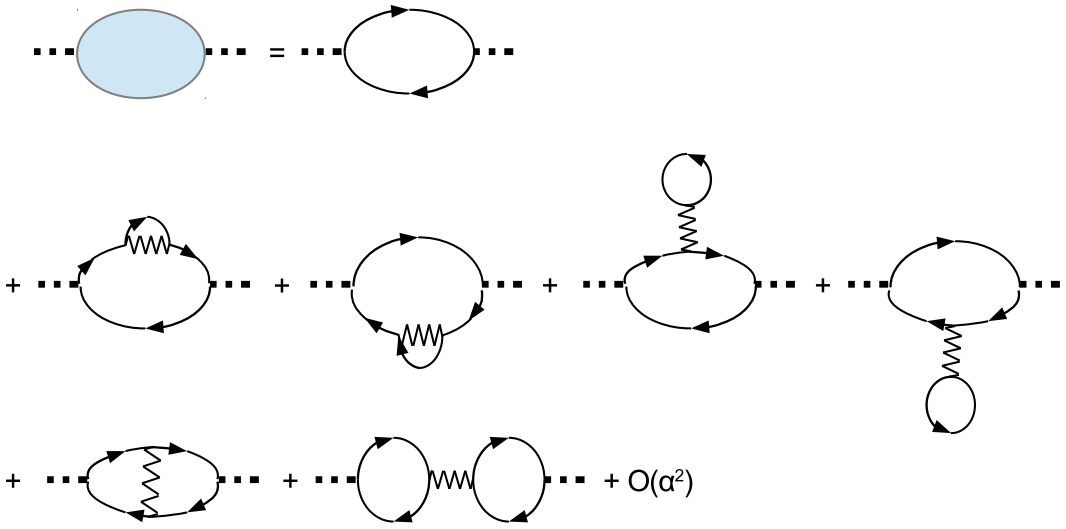

FIG. 3. Diagrams contributing to the linear DC conductance at the first order in interaction. The second line shows the self-energy dressed bubble diagrams, while vertex correction diagrams are in the third line.

$$\begin{aligned}
&\Omega_{jj'}\left(x,x';\tau-\tau'\right)\\
&=-\frac{e^2}{\left(2\pi\right)^2}\sum_{j_1j_2j_1'j_2'}\int d\epsilon_1 d\epsilon_2 d\epsilon_1' d\epsilon_2'\left[e^{i\frac{\epsilon_2-\epsilon_1}{v_{Fj}}x}S^*_{jj_1}S_{jj_2}-e^{-i\frac{\epsilon_2-\epsilon_1}{v_{Fj}}x}\delta_{jj_1}\delta_{jj_2}\right]\\
&\times\left[e^{i\frac{\epsilon_2'-\epsilon_1'}{v_{Fj'}}x'}S^*_{j'j_1'}S_{j'j_2'}-e^{-i\frac{\epsilon_2'-\epsilon_1'}{v_{Fj'}}x'}\delta_{j'j_1'}\delta_{j'j_2'}\right]\left\langle T_\tau\phi^\dagger_{j_1}\left(\epsilon_1,\tau\right)\phi_{j_2}\left(\epsilon_2,\tau\right)\phi^\dagger_{j_1'}\left(\epsilon_1',\tau'\right)\phi_{j_2'}\left(\epsilon_2',\tau'\right)\right\rangle_{\mathrm{H}}.
\end{aligned}\tag{2.17}$$

The imaginary time-ordered expectation value should be evaluated in the Heisenberg picture. The linear DC conductance $G_{jj'}$ is then given by the retarded current-current correlation function Ω,

$$G_{jj'}\left(x,x'\right)=\lim_{\omega\to0}\lim_{\eta_\omega\to0^+}\frac{i}{\omega}\left[\Omega_{jj'}\left(x,x';\omega^+\right)-\Omega_{jj'}\left(x,x';0\right)\right],\tag{2.18}$$

where again $\omega^+\equiv\omega+i\eta_\omega$. The coordinate dependence should vanish in the $\omega\to0$ limit, since where exactly we apply the bias or measure the current is inconsequential in a DC experiment.[4,12]

Eq. (2.18) is now calculated in perturbation theory as in Section II B. Some details of this mostly standard calculation are given in Appendix A; here again we only show the final results.

Feynman diagrams involved in the first order are shown in Fig. 3. In the absence of interaction, we have a single bubble diagram which leads to the usual linearized Landauer formula:[16]

$$G^{(0)}_{jj'}=\frac{e^2}{2\pi}\left(\delta_{jj'}-\left|S_{jj'}\right|^2\right)\tag{2.19}$$

Higher order diagrams can be classified into two basic types, namely self-energy diagrams and vertex corrections. At the first order, contributions from self-energy diagrams can be integrated into a Landauer-type formula:

$$G^{(0)}_{jj'}+G^{(1),\mathrm{SE}}_{jj'}=\frac{e^2}{2\pi}\left(\delta_{jj'}-\int d\epsilon\left[-n_F'\left(\epsilon\right)\right]\left|S^{\mathrm{d}(1)}_{jj'}\left(\epsilon\right)\right|^2\right)\tag{2.20}$$

where the first order "dressed S-matrix" $S^{\mathrm{d}(1)}$ has been defined in Eq. (2.13). The vertex corrections, on the other hand, contribute a completely different type of term:

$$\begin{aligned}
G_{jj'}^{(1),\mathrm{VC}}(x,x') = &-\frac{e^2}{2\pi} i \sum_n \int_0^\infty \frac{dy}{v_{Fn}} \alpha_n(y) \\
&\times \lim_{\omega\to 0} \lim_{\eta_\omega \to 0^+} \omega \left[|S_{jn}|^2 \delta_{j'n} e^{i\omega^+ \left(\frac{x}{v_{Fj}} - \frac{x'}{v_{Fj'}}\right)} e^{2i\omega^+ \frac{y}{v_{Fn}}} \theta\left(\frac{y}{v_{Fn}} - \frac{x'}{v_{Fj'}}\right) \right. \\
&-\delta_{jn}\delta_{j'n} e^{-i\omega^+ \left(\frac{x}{v_{Fj}} + \frac{x'}{v_{Fj'}}\right)} e^{2i\omega^+ \frac{y}{v_{Fn}}} \theta\left(\frac{y}{v_{Fn}} - \frac{x}{v_{Fj}}\right) \theta\left(\frac{y}{v_{Fn}} - \frac{x'}{v_{Fj'}}\right) \\
&-\delta_{jn}\delta_{j'n} e^{i\omega^+ \left(\frac{x}{v_{Fj}} + \frac{x'}{v_{Fj'}}\right)} e^{-2i\omega^+ \frac{y}{v_{Fn}}} \theta\left(\frac{x}{v_{Fj}} - \frac{y}{v_{Fn}}\right) \theta\left(\frac{x'}{v_{Fj'}} - \frac{y}{v_{Fn}}\right) \\
&- |S_{jn}|^2 |S_{nj'}|^2 e^{i\omega^+ \left(\frac{x}{v_{Fj}} + \frac{x'}{v_{Fj'}}\right)} e^{2i\omega^+ \frac{y}{v_{Fn}}} \\
&\left. +\delta_{jn} |S_{nj'}|^2 e^{-i\omega^+ \left(\frac{x}{v_{Fj}} + \frac{x'}{v_{Fj'}}\right)} e^{2i\omega^+ \frac{y}{v_{Fn}}} \theta\left(\frac{y}{v_{Fn}} - \frac{x}{v_{Fj}}\right) \right]
\end{aligned} \tag{2.21}$$

Integration by parts gives us

$$\frac{2i\omega^+}{v_{Fn}} \int_{y_l}^{y_u} dy \alpha_n(y) e^{2i\omega^+ \frac{y}{v_{Fn}}} = \alpha_n(y_u) e^{2i\omega^+ \frac{y_u}{v_{Fn}}} - \alpha_n(y_l) e^{2i\omega^+ \frac{y_l}{v_{Fn}}} - \int_{y_l}^{y_u} dy\, e^{2i\omega^+ \frac{y}{v_{Fn}}} \frac{d\alpha_n(y)}{dy}, \tag{2.22}$$

where y_u can be $v_{Fn}x/v_{Fj}$, $v_{Fn}x'/v_{Fj'}$ or ∞, and y_l can be $v_{Fn}x/v_{Fj}$, $v_{Fn}x'/v_{Fj'}$ or 0. We can let x and x' be sufficiently large so that $y_u > L_n$ is always satisfied; thus in the $d\alpha_n/dy$ term in Eq. (2.22), y_u may be replaced by L_n.

If $y_u \to \infty$, the $\alpha_n(y_u)$ term damps out due to the small imaginary part η_ω, and Eq. (2.22) becomes in the $\omega \to 0$ and $\eta_\omega \to 0$ limit

$$\frac{2i\omega^+}{v_{Fn}} \int_{y_l}^{y_u} dy \alpha_n(y) e^{2i\omega^+ \frac{y}{v_{Fn}}} = -\alpha_n(y_l) - \int_{y_l}^{L_n} dy \frac{d\alpha_n(y)}{dy} = -\alpha_n(L_n) = -\alpha_n(\infty). \tag{2.23a}$$

On the other hand, if y_u is finite, the $\alpha_n(y_u)$ term will survive the $\omega \to 0$ and $\eta_\omega \to 0$ limit:

$$\frac{2i\omega^+}{v_{Fn}} \int_{y_l}^{y_u} dy \alpha_n(y) e^{2i\omega^+ \frac{y}{v_{Fn}}} = \alpha_n(y_u) - \alpha_n(y_l) - \int_{y_l}^{L_n} dy \frac{d\alpha_n(y)}{dy} = \alpha_n(y_u) - \alpha_n(L_n) = 0. \tag{2.23b}$$

Therefore, taking the DC limit explicitly in Eq. (2.21), we find wire n contributes to the vertex correction only when it is attached to a TLL lead, and the interaction inside the wire is immaterial:

$$G_{jj'}^{(1),\mathrm{VC}}(x,x') = -\frac{e^2}{2\pi} \sum_n \frac{1}{2} \alpha_n(\infty) \left(\delta_{jn} - |S_{jn}|^2\right) \left(\delta_{nj'} - |S_{nj'}|^2\right). \tag{2.24}$$

When $\alpha_n(\infty) = 0$, as is the case for any wire n attached to an FL lead, the vertex correction due to n vanishes.

III. FIRST-ORDER CALLAN-SYMANZIK PERTURBATIVE RG

In this section, we analyze the result of Section II from the perspective of the CS formulation of RG,[8] and present a modified Landauer formula involving the renormalized S-matrix in the case of FL leads, supplemented by vertex corrections from TLL leads.

In the CS formulation of RG, we start from a field theory with a running cutoff D, and calculate low-energy physical observables (in our case the two-pointed retarded Green's function $G^{\mathrm{R}}_{jR,j'L}(\omega)$ and the linear DC conductance tensor

$G_{jj'}$) as a function of the running coupling constants of the theory (in our case the S-matrix elements $S_{jj'}(D)$). This is once again accomplished by perturbation theory in interaction, in formal analogy to Section II. However, the crucial difference is that we are now expressing certain low-energy physical quantities in terms of running coupling constants, whereas in Section II we calculate the corresponding renormalized quantities in terms of bare coupling constants. We require that when D is greater than the energy scales at which the system is probed, namely the finite temperature T and the finite frequency ω, $G^{\mathrm{R}}_{jR,j'L}(\omega)$ and $G_{jj'}$ should be independent of D. Therefore, by allowing the cutoff to run from D to $D-\delta D$, where $\delta D \ll D$, we can find the RG equation satisfied by the coupling constants $S_{jj'}(D)$.

We first apply the CS perspective to the retarded Green's function $G^{\mathrm{R}}_{jR,j'L}(\omega)$. Reducing the cutoff from D to $D-\delta D$ and demanding the right-hand side of Eq. (2.12) be a scaling invariant, we find

$$\delta S^{\mathrm{d}(1)}_{jj'}(\omega, D) \equiv S^{\mathrm{d}(1)}_{jj'}(\omega, D) - S^{\mathrm{d}(1)}_{jj'}(\omega, D-\delta D) = 0, \tag{3.1}$$

where $S^{\mathrm{d}(1)}_{jj'}(\epsilon, D)$ is Eq. (2.15) with the ϵ' integral going from $-D$ to D, and all S-matrix elements understood to be cutoff-dependent, $S_{jj'} \to S_{jj'}(D)$. To the lowest order in δD, Eq. (3.1) is equivalent to

$$\begin{aligned}
\delta S_{jj'}(\omega, D) &\equiv S_{jj'}(\omega, D) - S_{jj'}(\omega, D-\delta D) \\
&= \sum_n \int_{\delta D} d\epsilon' \frac{n_F(\epsilon')}{2(\epsilon'-\omega)} \Big(S_{jn} S^*_{nn} S_{nj'} \left[(\alpha_n(\infty) - \alpha_n(0)) e^{2i(\omega-\epsilon')\frac{L_n}{v_{Fn}}} + \alpha_n(0) \right] \\
&\quad - \delta_{jn} S_{nn} \delta_{j'n} \left[(\alpha_n(\infty) - \alpha_n(0)) e^{-2i(\omega-\epsilon')\frac{L_n}{v_{Fn}}} + \alpha_n(0) \right] \Big).
\end{aligned} \tag{3.2}$$

where $\int_{\delta D} = \int_{(D-\delta D)}^{D} + \int_{-D}^{-(D-\delta D)}$ stands for integration over fast modes.

If $D \gtrsim |\omega|$, $\epsilon' - \omega$ can be approximated as $\pm 2D$, thus giving rise to a scaling contribution $O(\delta D/D)$. If $D \gtrsim v_{Fn}/L_n$, $\exp(\pm i 2DL_n/v_{Fn})$ oscillates rapidly with D and is negligible; on the other hand, when $D \lesssim v_{Fn}/L_n$, $\exp(\pm i 2DL_n/v_{Fn}) \approx 1$. Finally, if $D \gtrsim T$, the factors $n_F(D) \approx 0$ and $n_F(-D) \approx 1$ are approximately independent of D. Therefore, to $O(\delta D/D)$, Eq. (3.2) predicts that

$$\delta S_{jj'} = -\frac{\delta D}{2D} \left(\sum_n \alpha_n(D) S_{jn} S^*_{nn} S_{nj'} - \alpha_j(D) S_{jj} \delta_{jj'} \right), \tag{3.3}$$

independent of ω, provided $D \gtrsim \max\{|\omega|, T\}$. Here we have defined a cutoff-dependent interaction strength

$$\alpha_n(D) \equiv \begin{cases} \alpha_n(0), & D \gtrsim v_{Fn}/L_n \\ \alpha_n(\infty), & D \lesssim v_{Fn}/L_n \end{cases}. \tag{3.4}$$

This means the renormalization will stop at the energy scale of the incident/scattered electron or the temperature, whichever is higher. In addition, the energy scale associated with the inverse length of wire n, v_{Fn}/L_n, determines whether the renormalization due to that wire is controlled by interaction strength in the wire $\alpha_n(0)$ or that in the lead $\alpha_n(\infty)$: the effective interaction strength crosses over from $\alpha_n(0)$ to $\alpha_n(\infty)$ as the D is reduced below v_{Fn}/L_n.

We are now in a position to write down the RG equation for the S-matrix valid to $O(\alpha)$. Restoring the explicit cutoff dependence, we have

$$-\frac{dS_{jj'}(D)}{d\ln D} = -\frac{1}{2} \sum_n \alpha_n(D) \left[S_{jn}(D) S^*_{nn}(D) S_{nj'}(D) - \delta_{j'n} S_{nn}(D) \delta_{nj} \right] \tag{3.5}$$

where the RG flow is cut off at the temperature T. This is the equation given in Refs. 6, and 7. It can be readily checked that Eq. (3.5) preserves the unitarity of the S-matrix.

We pause to remark that, as the cutoff is reduced below the inverse length of one of the wires, renormalization due to that wire is governed only by the lead to which that wire is attached. This is reasonable because a junction of finite-length TLL wires attached to FL leads should, at low energies, renormalize into a junction connected directly to FL leads.[4,8,12]

Intuitively, once the renormalization stops in Eq. (3.5), the renormalized S-matrix should represent the non-interacting part of the low-energy theory of the junction, and can be taken as an input to the Landauer formalism.

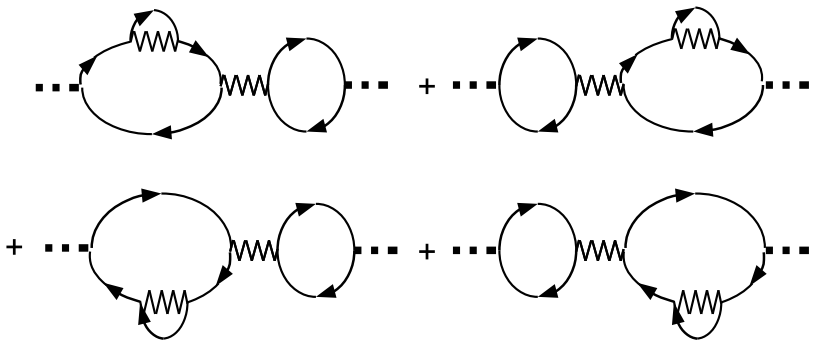

FIG. 4. Dressing of the first order vertex correction diagrams by the first order self-energy diagrams.

However, such an argument does not address the role of the low-energy residual interaction, which turns out to be especially important in the case of TLL leads. Also, in principle, the Landauer formalism is well-founded only in the absence of inelastic scattering. We therefore continue to study the conductance in the CS formulation, which fully exposes the Maslov-Stone physics, and in particular, possible deviations from the Landauer predictions.

Beginning from the simplest case where all leads are FL leads, $\alpha_n(\infty) = 0$ for all n, the vertex correction Eq. (2.24) vanishes, and the full linear DC conductance to $O(\alpha)$ is given by Eq. (2.20). Again reducing the cutoff from D to $D - \delta D$ and demanding the right-hand side of Eq. (2.20) be a scaling invariant, we have

$$\int_{\delta D} d\epsilon \left[-n_F'(\epsilon)\right] \left|S_{jj'}^{\mathrm{d}(1)}(\epsilon, D)\right|^2 + \int_{-D}^{D} d\epsilon \left[-n_F'(\epsilon)\right] \left[\left(S_{jj'}^{\mathrm{d}(1)}(\epsilon, D)\right)^* \delta S_{jj'}^{\mathrm{d}(1)}(\epsilon, D) + \text{c.c.}\right] = 0 \tag{3.6}$$

Since the derivative of the Fermi function is peaked at the Fermi energy with width T, the $\int_{\delta D}$ integral in Eq. (3.6) approximately vanishes while $D \gtrsim T$; Eq. (3.6) is thus automatically satisfied if Eq. (3.1) is enforced. The implication is that, at least in the case of FL leads, the renormalization of the conductance can be fully accounted for by the renormalization of the S-matrix. Once the cutoff D is reduced to the order of T, the perturbative correction to the S-matrix $S_{jj'}^{\mathrm{d}(1)}(\epsilon, D) - S_{jj'}(D)$ vanishes to the scaling accuracy; thus $S_{jj'}(D = T)$ may be used to approximate the dressed S-matrix in Eq. (2.20), and the conductance for a junction connected to FL leads is given by the modified Landauer formula,

$$G_{jj'}^{\mathrm{FL}} = \frac{e^2}{2\pi}\left(\delta_{jj'} - \left|S_{jj'}(T)\right|^2\right), \tag{3.7}$$

where the S-matrix is now fully renormalized according to Eq. (3.5), with the cutoff reduced to the temperature T. This is the Landauer-type formula invoked in Refs. 6, and 7.

When some of the leads are TLL leads, corrections of Eq. (2.24) must also be taken into account. It is important to note, however, that in a CS analysis of the total conductance, Eq. (3.1) remains valid to $O(\alpha)$. This is because as the cutoff is lowered, Eq. (2.24) contributes additional terms of the form of $\alpha(\infty) S^* \delta S$ to Eq. (3.6). However, by Eq. (3.1), δS is $O(\alpha)$; hence $\alpha(\infty) S^* \delta S$ is $O\left(\alpha^2\right)$, and is negligible to $O(\alpha)$.

To calculate the total conductance at $D = T$ with TLL leads, we go slightly beyond the first order and dress the $O(\alpha)$ vertex correction diagrams with $O(\alpha)$ self-energy diagrams, shown in Fig. 4. The bare S-matrix in Eq. (2.24) is then replaced by the dressed S-matrix, $S^{\mathrm{d}(1)}$:

$$\begin{aligned} G_{jj'}^{\mathrm{d}(1),\mathrm{VC}}(x, x') = -\frac{e^2}{2\pi} \sum_n \frac{1}{2} \alpha_n(\infty) \left(\delta_{jn} - \int d\epsilon_1 \left[-n_F'(\epsilon_1)\right] \left|S_{jn}^{\mathrm{d}(1)}(\epsilon_1)\right|^2\right) \\ \times \left(\delta_{nj'} - \int d\epsilon_2 \left[-n_F'(\epsilon_2)\right] \left|S_{nj'}^{\mathrm{d}(1)}(\epsilon_2)\right|^2\right). \end{aligned} \tag{3.8}$$

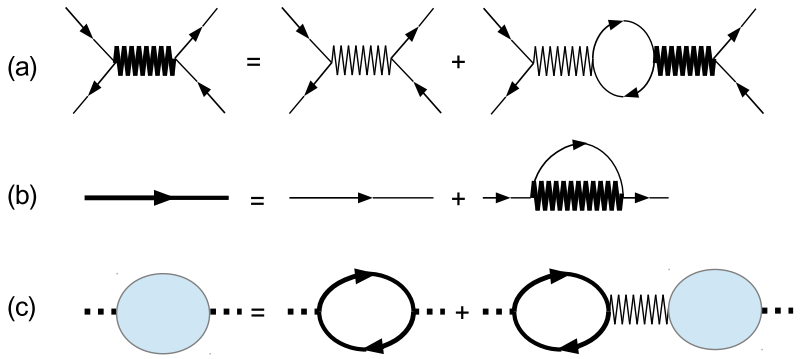

FIG. 5. The RPA diagrammatics: (a) effective interaction in the RPA represented by thick wavy lines; (b) dressed propagator in the RPA, to $O(\delta D/D)$ in RG, represented by thick straight lines; and (c) diagrams contributing to the Kubo conductance in the RPA. The dressed propagator in (b) is calculated to $O(\delta D/D)$ only, because higher order terms in $\delta D/D$ do not contribute to the renormalization of the S-matrix [Eq. (4.2)]— see Section III for an explanation in the first order context. (a) and (c) do not involve truncation at $O(\delta D/D)$ because any renormalization of the interaction [Eq. (4.3)] and the conductance [Eqs. (4.1) and (4.5)] can be attributed to the renormalization of the S-matrix. Note that (c) features a thin interaction line (rather than a thick one) to avoid double-counting.

This allows us to repeat our previous analysis for the case of FL leads, and further approximate $S^{\mathrm{d}(1)}_{jj'}$ by $S_{jj'}(D=T)$. Thus the TLL leads contribute an additional conductance of

$$G^{\mathrm{TLL}}_{jj'} - G^{\mathrm{FL}}_{jj'} = -\frac{e^2}{2\pi}\sum\nolimits_n \frac{\alpha_n(\infty)}{2}\left(\delta_{jn} - |S_{jn}(T)|^2\right)\left(\delta_{nj'} - |S_{nj'}(T)|^2\right). \tag{3.9}$$

Eqs. (3.5), (3.7) and (3.9) provide a comprehensive first-order picture for non-resonant tunneling through a junction: the interaction renormalizes the S-matrix, the renormalized S-matrix determines the conductance through a Landauer-type formula if the junction is connected to FL leads, and the residual interaction further modifies the conductance if the junction is attached to TLL leads. As will be demonstrated in Section IV, this picture is by no means limited to the first order.

IV. S-MATRIX RENORMALIZATION AND CONDUCTANCE IN THE RPA

In this section, we extends our first-order RG analysis in Section III to arbitrary order in interaction under the RPA.[8–10] The correlation function Eq. (2.17) is perturbatively evaluated for both self-energy diagrams and vertex corrections by the same procedures, except that the interaction is dressed with ring diagrams; see Fig. 5. We subsequently find the S-matrix RG equation in the CS scheme and express the conductance in terms of the renormalized S-matrix. This is once more a straightforward calculation, and we simply present the outcome, leaving the details for Appendix C.

Introduce the shorthand $W_{jj'}(D) \equiv |S_{jj'}(D)|^2$. The RPA self-energy diagrams give rise to a modified Landauer formula:

$$G^{\mathrm{FL}}_{jj'} = \frac{e^2}{2\pi}\left[\delta_{jj'} - W_{jj'}(T)\right], \tag{4.1}$$

where the renormalization of the S-matrix is governed by a generalization of Eq. (3.5),

$$-\frac{dS_{jj'}(D)}{d\ln D} = -\frac{1}{2}\sum_{n_1 n_2}\left[S_{jn_1}(D)\,\Pi_{n_1 n_2}(D)\,S^*_{n_2 n_1}(D)\,S_{n_2 j'}(D) - \delta_{j' n_1}\Pi^*_{n_1 n_2}(D)\,S_{n_2 n_1}(D)\,\delta_{n_2 j}\right]. \tag{4.2}$$

The RPA-dressed interaction is

$$\Pi(D) \equiv 2\left[\mathcal{Q}(D) - W(D)\right]^{-1}, \tag{4.3}$$

where

$$\mathcal{Q}_{jj'} = Q_j(D)\,\delta_{jj'},\ Q_j(D) = \frac{1+K_j(D)}{1-K_j(D)}, \tag{4.4}$$

with $K_j(D) = \sqrt{(1-\alpha_j(D))/(1+\alpha_j(D))}$ being the cutoff-dependent "Luttinger parameter" for wire j; $\alpha_j(D)$ is given in Eq. (3.4). To lowest order in α_j, $\Pi_{ij} = \delta_{ij}\alpha_j$. When all wires of the junction are attached to FL leads, in parallel with the $O(\alpha)$ calculation, Eq. (4.1) captures the entirety of the conductance. This is in agreement with the Kubo formula calculation in Refs. 8–10 in the language of chiral fermion densities.

When some wires are attached to TLL leads, they again provide important corrections to the DC conductance. All RPA vertex correction diagrams dressed with RPA self-energy evaluate to

$$G^{\mathrm{TLL}}_{jj'} - G^{\mathrm{FL}}_{jj'} = -\frac{e^2}{2\pi}\sum\nolimits_{n_1 n_2}\left[\delta_{jn_1} - W_{jn_1}(T)\right]\frac{1}{2}\Pi^{\mathrm{L}}_{n_1 n_2}\left[\delta_{n_2 j'} - W_{n_2 j'}(T)\right], \tag{4.5}$$

where the residual effective interaction is

$$\Pi^{\mathrm{L}} = 2\left[\mathcal{Q}^{\mathrm{L}} - W(T)\right]^{-1}, \tag{4.6}$$

and $\mathcal{Q}^{\mathrm{L}}$ is given by Eq. (4.4) with K_j replaced by $K^{\mathrm{L}}_j = \sqrt{(1-\alpha_j(\infty))/(1+\alpha_j(\infty))}$, the Luttinger parameter of the lead.

Remarkably, if we define the DC contact resistance tensor between the wires and leads,

$$\left(G_c^{-1}\right)_{jj'} = \left(\frac{e^2}{2\pi}\right)^{-1}\frac{1}{2}\left[1 - \left(K^{\mathrm{L}}_j\right)^{-1}\right]\delta_{jj'}, \tag{4.7}$$

then Eq. (4.5) can be formally recast as

$$\mathbf{G}^{\mathrm{TLL}} = \left(\mathbf{1} - \mathbf{G}^{\mathrm{FL}}\mathbf{G}_c^{-1}\right)^{-1}\mathbf{G}^{\mathrm{FL}}, \tag{4.8}$$

where $\mathbf{1}$ is the $N \times N$ identity matrix. The same relation has been derived in Refs. 4 and 5, which assume that the DC contact resistance between a finite TLL wire and an FL lead is not affected by the junction at the other end of the TLL wire. This intuitive assumption is reinforced by our calculations.

We emphasize that the inclusion of the vertex correction diagrams does not change the renormalization of the S-matrix, Eq. (4.2). The reason for this is as follows. Eq. (4.2) results from dressing the single particle propagator as shown in panel (b) of Fig. 5. The conductance is calculated in perturbation theory by replacing all bare single particle propagators (the thin lines) with the dressed ones (the thick lines) in the basic bubble diagram and the vertex correction diagrams; or equivalently, by replacing all bare S-matrix elements with the ones dressed with the RPA self-energy. As with the case at the first order, the RPA vertex correction diagrams do not introduce additional cutoff-sensitive integrals, and all cutoff-sensitive integrals originate from the dressed S-matrix. Therefore, the dressed S-matrix should be a cutoff-independent quantity when we apply the CS scheme to the conductance, regardless of whether the vertex correction diagrams contribute to the conductance. Eq. (4.2) is thus independent of vertex corrections.

Eqs. (4.1)–(4.8) are the central results of this paper. They show that at least in the RPA, in addition to the Landauer-type formula, TLL leads give rise to important corrections to the linear DC conductance which are also given in terms of the renormalized S-matrix. In the remainder of this paper, we implement these results in non-resonant tunneling through 2-lead junctions and Y-junctions.

V. FIXED POINT CONDUCTANCE

In this section we evaluate the conductance at several established fixed points of 2-lead junctions and Y-junctions attached to FL leads and TLL leads. The analysis is carried out at the first order in interaction [Eqs. (3.7) and (3.9)] and then in the RPA [Eqs. (4.1) and (4.8)]. In particular, we will examine the conductance of the maximally open M fixed point in the RPA for the Z_3 symmetric Y-junction.

For simplicity, the interactions are once more modeled by Eq. (2.14). We write $\alpha_j(0)$, the interaction strength in wire j, simply as α_j; also when the junction is connected to TLL leads, we assume the interactions in wires and leads are uniform and identical, i.e. $\alpha_j(\infty) = \alpha_j$. Of course, by definition $\alpha_j(\infty) = 0$ for FL leads.

1. 2-lead junction

In a 2-lead junction of spinless fermions away from resonance, solving the S-matrix RG equations [Eq. (3.5) at the first order and Eq. (4.2) in the RPA], we find that the only fixed points are the complete reflection fixed point [the N (Neumann) fixed point] and the perfect transmission fixed point [the D (Dirichlet) fixed point].[6,8]

At the N fixed point $W_{12} = 0$, the two wires are decoupled from each other, and we find the obvious result that the conductance $G^{N,\mathrm{FL}}_{jj'} = G^{N,\mathrm{TLL}}_{jj'} = 0$, irrespective of what leads the junction is attached to.

On the other hand, at the D fixed point $W_{12} = 1$, the backscattering between the two wires vanishes. With FL leads $G^{D,\mathrm{FL}}_{jj'} = \left(e^2/2\pi\right)(2\delta_{jj'} - 1)$, as predicted by the naive Landauer formula; with TLL leads, Eq. (3.9) predicts

$$G^{D,\mathrm{TLL}}_{jj'} = \left(1 - \frac{\alpha_1 + \alpha_2}{2}\right)\frac{e^2}{2\pi}(2\delta_{jj'} - 1) \tag{5.1}$$

at the first order, and Eq. (4.8) predicts

$$G^{D,\mathrm{TLL}}_{jj'} = \frac{2K_1K_2}{K_1 + K_2}\frac{e^2}{2\pi}(2\delta_{jj'} - 1) \tag{5.2}$$

in the RPA. Here the RPA has recovered the famous result for the conductance of two semi-infinite TLL wires.[17]

2. Y-junction

Even at the first order in interaction, the RG flow portrait for a Y-junction is more complicated than the two-lead junction.[7] Solving Eq. (3.5), we find a "non-geometrical" M fixed point whose existence and transmission probabilities generally depend on the interaction strengths, in addition to the "geometrical" fixed points N, A_j and $\chi^{\pm}$. Provided the interactions are not too strong, these are also the only fixed points allowed in the RPA.[10] N (complete reflection) and A_j (asymmetric) can be obtained by adding a third decoupled wire with label j to the N and D fixed points of the two-lead junction respectively. The conductances at N and A_j are therefore a trivial generalization of the two-lead case, and we will focus on $\chi^{\pm}$ and M alone.

At the chiral fixed points $\chi^{\pm}$, in the absence of interaction, an electron incident from wire j is perfectly transmitted to wire $j \pm 1$ (here we identify $j + 3 \equiv j$); thus the time-reversal symmetry is broken. The W matrix is given by $W_{jj'} = (1 - \delta_{jj} \mp \epsilon_{jj'})/2$, where the anti-symmetric tensor $\epsilon_{jj'}$ is defined by $\epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 1$, $\epsilon_{21} = \epsilon_{32} = \epsilon_{13} = -1$ and $\epsilon_{jj} = 0$. At the first order, inserting the W matrix into Eqs. (3.7) and (3.9), we find $G^{\chi^{\pm},\mathrm{FL}}_{jj'} = \left(e^2/2\pi\right)(3\delta_{jj'} - 1 \pm \epsilon_{jj'})/2$, and

$$G^{\chi^{\pm},\mathrm{TLL}}_{jj'} - G^{\chi^{\pm},\mathrm{FL}}_{jj'} = -\frac{e^2}{2\pi}\frac{1}{2}\left[(\alpha_j + \alpha_{j'})\left(\frac{3}{2} - \delta_{jj'}\right) + \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3)(1 - \delta_{jj'} \pm \epsilon_{jj'})\right]. \tag{5.3}$$

In the RPA, on the other hand, Eq. (4.8) gives the conductance at $\chi^{\pm}$ with TLL leads as

$$G^{\chi^{\pm},\mathrm{TLL}}_{jj'} = 2\frac{e^2}{2\pi}\frac{K_j(K_1 + K_2 + K_3)\delta_{jj'} + (\pm K_1K_2K_3\epsilon_{jj'} - K_jK_{j'})}{K_1 + K_2 + K_3 + K_1K_2K_3}, \tag{5.4}$$

which agrees with the result of bosonization analysis.[5]

The presence of the M fixed point can be inferred in a Z_3 symmetric time-reversal invariant Y-junction with attractive interactions: in this system, N is unstable, A_j is forbidden by Z_3 symmetry, and $\chi^{\pm}$ are forbidden by time-reversal symmetry, so there must be at least one stable fixed point. The W matrix has generally interaction-dependent elements at M. At the first order,

$$W_{jj'} = \begin{cases} \left(\frac{\alpha_1\alpha_2\alpha_3/\alpha_j}{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1}\right)^2, & j = j' \\ \left(1-\frac{\alpha_1\alpha_2\alpha_3/\alpha_j}{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1}\right)\left(1-\frac{\alpha_1\alpha_2\alpha_3/\alpha_{j'}}{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1}\right), & j \neq j' \end{cases}. \tag{5.5}$$

We see explicitly that M obeys time-reversal symmetry, $W_{jj'} = W_{j'j}$. Demanding $0 \leq W_{jj'} \leq 1$, we find that at the first-order M can only exist in the following situations: 1) α_1, α_2, $\alpha_3 > 0$; 2) α_1, α_2, $\alpha_3 < 0$; 3) $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 < -\alpha_1$, $\alpha_3 < -\alpha_2$; 4) $\alpha_1 < 0$, $\alpha_2 < 0$, $\alpha_3 > -\alpha_1$, $\alpha_3 > -\alpha_2$; and situations equivalent to 3) and 4) up to permuted subscripts (e.g. $(\alpha_1, \alpha_2, \alpha_3) \to (\alpha_3, \alpha_1, \alpha_2)$).

Substituting Eq. (5.5) into Eq. (3.9), we find that at the first-order the conductance at M obeys

$$G^{M,\mathrm{TLL}}_{jj'} - G^{M,\mathrm{FL}}_{jj'} = \begin{cases} -\frac{e^2}{2\pi}\frac{(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_1+\alpha_3)}{2(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)^3}\alpha_j^2\left(\alpha_1+\alpha_2+\alpha_3-\alpha_j\right)^2, & j = j' \\ \frac{e^2}{2\pi}\frac{(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_1+\alpha_3)}{2(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)^3}\left[\alpha_j\alpha_{j'}\left(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1\right)-\frac{(\alpha_1\alpha_2\alpha_3)^2}{\alpha_j\alpha_{j'}}\right], & j \neq j' \end{cases}. \tag{5.6}$$

Note that for Z_3 symmetric interactions ($\alpha_j = \alpha$), $W_{jj'} = 1/9 + \delta_{jj'}/3$ becomes independent of the interaction strength. Now $W_{jj'}$ produces the maximal transmission probability $8/9$ allowed by unitarity in a Z_3 symmetric S-matrix, and at the first order $G^{M,\mathrm{TLL}}_{jj'} - G^{M,\mathrm{FL}}_{jj'} = -(8/27)\,\alpha\left(e^2/2\pi\right)(2\delta_{jj'}-1)$. Compared to FL leads, TLL leads enhance conductance for attractive interactions and reduce conductance for repulsive interactions, as with the two-lead D fixed point.

In the RPA, the W matrix of the M fixed point is generally cumbersome, but reduces to the aforementioned maximally transmitting W matrix for Z_3 symmetric interactions. Eq. (4.8) then gives

$$G^{M,\mathrm{TLL}}_{jj'} = \frac{4K}{3K+6}\frac{e^2}{2\pi}\left(3\delta_{jj'}-1\right). \tag{5.7}$$

This result supports the findings of Ref. 13. There the M fixed point conductance of a Y-junction of infinite TLL wires is computed numerically using density matrix renormalization group (DMRG), and conjectured to be

$$G_{jj'} = \frac{2K\gamma}{2K+3\gamma-3K\gamma}\frac{e^2}{2\pi}\left(3\delta_{jj'}-1\right), \tag{5.8}$$

where it is suggested that the dimensionless parameter γ is $4/9$ based on the non-interacting limit $K = 1$.

VI. DISCUSSION AND OPEN QUESTIONS

In this paper, using the fermionic RG formalism, we calculated the linear DC conductance tensor of a junction of multiple quantum wires. We showed, both at the first order and in the RPA, that a junction attached to FL leads has a conductance tensor which obeys a linearized Landauer-type formula with a renormalized S-matrix. TLL leads modify the conductance through vertex corrections, and the conductance with FL leads may be heuristically related to the conductance with TLL leads through the contact resistance between leads. In this section, we would like to discuss some of the questions left open in our approach.

First, we have assumed that scattering by the junction is fully described by operators which are quadratic in conduction fermions and independent of other degrees of freedom. Local operators quartic in fermions are ignored, among others. This does not pose a threat to the first-order calculations, because any quartic local operator has a scaling dimension of at least $4 \times 1/2 = 2$ in the non-interacting case, and is necessarily highly irrelevant. However, it has been shown that sufficiently strong attractive bulk interactions can render quartic boundary operators relevant.[4] An example is the electron pair hopping operator at the Z_3 symmetric Y-junction, $\psi^\dagger_{1L}\psi^\dagger_{1R}\psi_3\partial_x\psi_3\,(x=0) + \mathrm{h.c.}$: it is of dimension $3/K$ at the asymmetric fixed point A_3, where K is the Luttinger parameter of all three wires, and A_3 sees

wire 3 decoupled from perfectly connected wires 1 and 2. Apparently, for very strong interactions $K > 3$, this operator becomes relevant and can potentially dominate the physical properties of the stable fixed point. Unfortunately, the present RPA analysis does not predict a scaling exponent consistent with this operator;[10] it is hence incomplete in this regard, and should not be carried too far into the regime of strongly attractive bulk interactions.

A related issue is the existence of the D fixed points in the Y-junction. Predicted by the bosonic approaches[4,5,18] but not the fermionic ones,[9,10] these fixed points are only stable for strong attractive interactions. They are most notably characterized by Andreev reflections, even when electron-electron interaction is absent in the bulk. This hints at multi-particle scattering at the junction, and rules out the possibility to represent the D fixed points by single-particle S-matrices. (Single-particle S-matrices with particle-hole channels are not feasible either since the D fixed points respect particle number conservation.)[4] The D fixed points are not predicted by the fermionic approaches, because the latter are based on the ansatz that the junction is always described by a single-particle S-matrix along the RG flow; but such an ansatz will likely be invalidated if, for instance, relevant quartic boundary operators are present. We are thus led to believe that the lack of D fixed points in the present RPA analysis does not refute their possible stability when the bulk interactions are strongly attractive.

On the other hand, even when the bulk interactions are relatively weak, it is not a priori clear to what extent the RPA is successful. In the Tomonaga-Luttinger model (which we have adopted in our bulk quantum wires), the RPA is known to be exact due to the interaction which separately conserves the numbers of right- and left-movers.[15] This is no longer the case once right- and left-movers become mixed up by the scattering at the junction. It has been pointed out that going beyond the RPA changes the renormalization of the S-matrix away from the "geometrical" fixed points, although all universal scaling exponents stay the same.[8,9] As for the "non-geometrical" M fixed point in the Y-junction, its position is generally shifted when we go beyond the RPA. Remarkably, however, if the interaction is Z_3 symmetric, not only the W matrix but also the scaling exponents at the M fixed point remain identical with the RPA results up to the third order in interaction.[9] The agreement of our RPA result with the numerics of Ref. 13 is suggestive, but more work on vertex corrections is required to verify the validity of our RPA conductance at the Z_3 symmetric M fixed point with TLL leads, Eq. (1.8).

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Appendix A: Details of zeroth and first order perturbation theory

In this appendix we present some of the crucial steps in the perturbative calculation of the conductance up to the first order in interaction, which lead to Eqs. (2.20) and (2.24). We go through the standard procedures for the conductance calculation at the zeroth order in interaction, then highlight the treatments specific to the first order.

1. Zeroth order

At the zeroth order, there is only one bubble diagram for the current-current correlation function. Wick's theorem gives

$$\begin{aligned}&\left\langle T_\tau \phi^\dagger_{j_1}(\epsilon_1,\tau)\,\phi_{j_2}(\epsilon_2,\tau)\,\phi^\dagger_{j'_1}(\epsilon'_1,\tau')\,\phi_{j'_2}(\epsilon'_2,\tau')\right\rangle\\&=-\delta_{j_2 j'_1}\delta(\epsilon_2-\epsilon'_1)\,\mathcal{G}_{j_2}(\epsilon_2,\tau-\tau')\,\delta_{j_1 j'_2}\delta(\epsilon_1-\epsilon'_2)\,\mathcal{G}_{j_1}(\epsilon_1,\tau'-\tau)\,.\end{aligned}\tag{A1}$$

Here $\mathcal{G}$ is the free scattering basis Matsubara Green's function $\mathcal{G}_j(E,i\omega_n)=1/(i\omega_n-E)$, $\omega_n=(2n+1)\pi/\beta$. We have dropped the H subscript in Eq. (2.17) when switching to the interaction picture. Going to the frequency space, doing the standard Matsubara sum

$$\frac{1}{\beta}\sum_{i\omega_n}\mathcal{G}_{j_2}(E_2,i\omega_n)\,\mathcal{G}_{j_1}(E_1,i\omega_n-ip_m)=\frac{n_F(E_2)-n_F(E_1)}{-ip_m+E_2-E_1},\tag{A2}$$

where $p_m = 2m\pi/\beta$ is a bosonic frequency, and performing analytic continuation $ip_m \to \omega^+ \equiv \omega + i\eta_\omega$ ($\eta_\omega \to 0^+$) yield the zeroth order retarded correlation function,

$$\Omega^{(0)}_{jj'}\left(x,x';\omega^+\right) = \frac{e^2}{(2\pi)^2}\sum_{j_1 j_2}\int d\epsilon_1 d\epsilon_2 \frac{n_F(\epsilon_2) - n_F(\epsilon_1)}{-\omega^+ + \epsilon_2 - \epsilon_1}\left[\delta_{jj'} e^{i(\epsilon_2-\epsilon_1)\left(\frac{x}{v_{Fj}} - \frac{x'}{v_{Fj'}}\right)}\right.$$
$$\left. + \delta_{jj'} e^{i(\epsilon_1-\epsilon_2)\left(\frac{x}{v_{Fj}} - \frac{x'}{v_{Fj'}}\right)} - |S_{j'j}|^2 e^{i(\epsilon_1-\epsilon_2)\left(\frac{x}{v_{Fj}} + \frac{x'}{v_{Fj'}}\right)} - |S_{jj'}|^2 e^{i(\epsilon_2-\epsilon_1)\left(\frac{x}{v_{Fj}} + \frac{x'}{v_{Fj'}}\right)}\right]. \tag{A3}$$

We have done the j_1 and j_2 sums using unitarity of the S-matrix. Employing contour techniques, we integrate over ϵ_1 on $(-\infty,\infty)$ for the term proportional to $n_F(\epsilon_2)$, and integrate over ϵ_2 on $(-\infty,\infty)$ for the term proportional to $n_F(\epsilon_1)$:

$$\begin{aligned}
&\Omega^{(0)}_{jj'}\left(x,x';\omega^+\right) \\
&= \frac{e^2}{(2\pi)^2}\int d\epsilon_2 (2\pi i) n_F(\epsilon_2)\left[\delta_{jj'} e^{i\omega^+\left|\frac{x}{v_{Fj}} - \frac{x'}{v_{Fj'}}\right|} - 0 - |S_{jj'}|^2 e^{i\omega^+\left(\frac{x}{v_{Fj}} + \frac{x'}{v_{Fj'}}\right)}\right] \\
&- \frac{e^2}{(2\pi)^2}\int d\epsilon_1 (2\pi i) n_F(\epsilon_1)\left[\delta_{jj'} e^{i\omega^+\left|\frac{x}{v_{Fj}} - \frac{x'}{v_{Fj'}}\right|} - 0 - |S_{jj'}|^2 e^{i\omega^+\left(\frac{x}{v_{Fj}} + \frac{x'}{v_{Fj'}}\right)}\right].
\end{aligned} \tag{A4}$$

We note that the $|S_{j'j}|^2$ term vanishes because the associated singularities are on the wrong side of the contour. Now combine the $n_F(\epsilon_2)$ and $n_F(\epsilon_1)$ terms and restore the cutoff D, recalling that $\epsilon_2 - \epsilon_1 = \omega^+$. This gives

$$\Omega^{(0)}_{jj'}\left(x,x';\omega^+\right) = i\frac{e^2}{2\pi}\int d\epsilon_2 \left[n_F(\epsilon_2) - n_F\left(\epsilon_2 - \omega^+\right)\right]\left[\delta_{jj'} e^{i\omega^+\left|\frac{x}{v_{Fj}} - \frac{x'}{v_{Fj'}}\right|} - 0 - |S_{jj'}|^2 e^{i\omega^+\left(\frac{x}{v_{Fj}} + \frac{x'}{v_{Fj'}}\right)}\right]. \tag{A5}$$

Substituting into Eq. (2.18), taking the $\eta_\omega \to 0^+$ limit and then the $\omega \to 0$ limit, we obtain Eq. (2.19).

2. First order

At the first order, as shown in Fig. 3, the bubble diagram is dressed by two types of self-energies: contraction of E_1 with E_2 or E_3 with E_4 in Eq. (2.8) (the "tadpole"), and contraction of E_1 with E_4 or E_2 with E_3. In addition, there are two types of first order vertex correction diagrams, the "cracked egg" diagram and the ring diagram.

For the self-energy diagrams and the dressed conductance bubbles we need two more types of Matsubara frequency sums. The first one is

$$\frac{1}{\beta}\sum_{i\omega_n}\mathcal{G}_j(\epsilon, i\omega_n) = n_F(\epsilon). \tag{A6}$$

The second one is

$$\begin{aligned}
&\frac{1}{\beta}\sum_{i\omega_n}\mathcal{G}_{j_1'}(\epsilon_1', i\omega_n)\mathcal{G}_{j_1}(\epsilon_1, i\omega_n)\mathcal{G}_{j_2}(\epsilon_2, i\omega_n + ip_m) \\
&= n_F(\epsilon_2)\frac{1}{(\epsilon_2 - ip_m) - \epsilon_1'}\frac{1}{(\epsilon_2 - ip_m) - \epsilon_1} - \int\frac{d\tilde{\epsilon}}{2\pi i} n_F(\tilde{\epsilon}) \\
&\times \frac{1}{\tilde{\epsilon} + ip_m - \epsilon_2}\left[\frac{1}{\tilde{\epsilon} + i0 - \epsilon_1'}\frac{1}{\tilde{\epsilon} + i0 - \epsilon_1} - \frac{1}{\tilde{\epsilon} - i0 - \epsilon_1'}\frac{1}{\tilde{\epsilon} - i0 - \epsilon_1}\right].
\end{aligned} \tag{A7}$$

To compute this sum, we consider the following contour integral,

$$\oint\frac{dz}{2\pi i} n_F(z)\frac{1}{z - \epsilon_1'}\frac{1}{z - \epsilon_1}\frac{1}{z + ip_m - \epsilon_2}, \tag{A8}$$

where the integration contour is wrapped around the branch cut on the real axis,[19] so that poles inside the contour are $z = i\omega_n$ (n running over all integers) and also $z = \epsilon_2 - ip_m$. The $n_F(\epsilon_2)$ term in Eq. (A7) comes from $z = \epsilon_2 - ip_m$, and the $n_F(\tilde{\epsilon})$ term comes from the branch cut $z = 0$.

We ignore the tadpole-type self-energy diagrams, again on the grounds that they only modify the chemical potential. The other type of self-energy diagrams turn out to dress the S-matrix as in Eq. (2.15). One instance of these diagrams reads

$$\begin{aligned}
&\Omega_{jj'}^{(1),\mathrm{SE,non\text{-}tadpole},1}(x,x';\tau-\tau') = -\frac{e^2}{(2\pi)^2}\sum_{j_1j_2j_1'j_2'}\int d\epsilon_1 d\epsilon_2 d\epsilon_1' d\epsilon_2' \\
&\times\left[e^{i\frac{(\epsilon_2-\epsilon_1)}{v_{Fj}}x}S_{jj_1}^* S_{jj_2} - e^{-i\frac{(\epsilon_2-\epsilon_1)}{v_{Fj}}x}\delta_{jj_1}\delta_{jj_2}\right]\left[e^{i\frac{(\epsilon_2'-\epsilon_1')}{v_{Fj'}}x'}S_{j'j_1'}^* S_{j'j_2'} - e^{-i\frac{(\epsilon_2'-\epsilon_1')}{v_{Fj'}}x'}\delta_{j'j_1'}\delta_{j'j_2'}\right] \\
&\times(-)\int_0^\beta d\tau_1\sum_n\int dy\, g_2^n(y)\sum_{l_1l_2l_3l_4}\int\frac{dE_1dE_2dE_3dE_4}{(2\pi)^2 v_{Fn}^2}\varrho_{l_1l_2l_3l_4}^n(E_1,E_2,E_3,E_4;y) \\
&\times\delta_{j_2j_1'}\delta(\epsilon_2-\epsilon_1')\mathcal{G}_{j_2}(\epsilon_2,\tau-\tau')\delta_{j_2'l_1}\delta(\epsilon_2'-E_1)\mathcal{G}_{j_2'}(\epsilon_2',\tau'-\tau_1) \\
&\times\delta_{l_2l_3}\delta(E_2-E_3)\mathcal{G}_{l_2}(E_2,0)\delta_{l_4j_1}\delta(E_4-\epsilon_1)\mathcal{G}_{j_1}(\epsilon_1,\tau_1-\tau).
\end{aligned} \tag{A9}$$

Going to the frequency space, performing Matsubara sums and analytic continuation, we find

$$\begin{aligned}
&\Omega_{jj'}^{(1),\mathrm{SE,non\text{-}tadpole},1}\left(x,x';\omega^+\right) = -\frac{e^2}{(2\pi)^2}\sum_{j_1j_2j_2'}\int d\epsilon_1 d\epsilon_2 d\epsilon_2' \\
&\times\left[e^{i\frac{(\epsilon_2-\epsilon_1)}{v_{Fj}}x}S_{jj_1}^* S_{jj_2} - e^{-i\frac{(\epsilon_2-\epsilon_1)}{v_{Fj}}x}\delta_{jj_1}\delta_{jj_2}\right]\left[e^{i\frac{(\epsilon_2'-\epsilon_2)}{v_{Fj'}}x'}S_{j'j_2}^* S_{j'j_2'} - e^{-i\frac{(\epsilon_2'-\epsilon_2)}{v_{Fj'}}x'}\delta_{j'j_2}\delta_{j'j_2'}\right] \\
&\times(-)\sum_n\int dy\, g_2^n(y)\int\frac{dE_2}{(2\pi)^2 v_{Fn}^2}n_F(E_2)\varrho_{j_2'nnj_1}^n(\epsilon_2',E_2,E_2,\epsilon_1;y) \\
&\times\left[n_F(\epsilon_2)\frac{1}{(\epsilon_2-\omega^+)-\epsilon_2'}\frac{1}{(\epsilon_2-\omega^+)-\epsilon_1} - \int\frac{d\tilde{\epsilon}}{2\pi i}n_F(\tilde{\epsilon})\frac{1}{\tilde{\epsilon}+\omega^+-\epsilon_2}\right. \\
&\left.\times\left(\frac{1}{\tilde{\epsilon}+i0-\epsilon_2'}\frac{1}{\tilde{\epsilon}+i0-\epsilon_1} - \frac{1}{\tilde{\epsilon}-i0-\epsilon_2'}\frac{1}{\tilde{\epsilon}-i0-\epsilon_1}\right)\right].
\end{aligned} \tag{A10}$$

Carrying out the ϵ_1 and ϵ_2' integrations, and also the ϵ_2 integration in the $n_F(\tilde{\epsilon})$ term, this becomes in the x, $x' \to \infty$ limit

$$\begin{aligned}
&\Omega_{jj'}^{(1),\mathrm{SE,non\text{-}tadpole},1}\left(x,x';\omega^+\right) = -\frac{e^2}{2\pi}\int d\epsilon_2\left[n_F(\epsilon_2) - n_F\left(\epsilon_2-\omega^+\right)\right] \\
&\times(-)\sum_n\int dy\,\alpha_n(y)\int\frac{dE_2}{v_{Fn}}n_F(E_2)\frac{1}{2}e^{i\omega^+\left(\frac{x}{v_{Fj}}+\frac{x'}{v_{Fj'}}\right)} \\
&\times S_{jj'}\left[S_{jn}^* S_{nn} S_{nj'}^* e^{2i\left(E_2-\left(\epsilon_2-\omega^+\right)\right)\frac{y}{v_{Fn}}} + \delta_{jn}S_{nn}^*\delta_{j'n}e^{-2i\left(E_2-\left(\epsilon_2-\omega^+\right)\right)\frac{y}{v_{Fn}}}\right].
\end{aligned} \tag{A11}$$

This is just one of the four terms which reproduce Eqs. (2.20) and (2.13) when inserted in Eq. (2.18). Another identical term comes from contracting E_1 with E_4 (completely equivalent to contracting E_2 with E_3 which we have done). The remaining two terms have all their electron propagators reverted, so that their contributions to the conductance are the complex conjugate of the first two terms. This concludes the derivation of Eqs. (2.20) and (2.13).

Neither type of vertex corrections to the conductance requires Matsubara sums other than Eq. (A2). An example of the ring diagram is

$$\Omega_{jj'}^{(1),\mathrm{VC,ring},1}(x,x';\tau-\tau') = -\frac{e^2}{(2\pi)^2}\sum_{j_1j_2j_1'j_2'}\int d\epsilon_1 d\epsilon_2 d\epsilon_1' d\epsilon_2'$$
$$\times\left[e^{i\frac{(\epsilon_2-\epsilon_1)}{v_{Fj}}x}S_{jj_1}^* S_{jj_2} - e^{-i\frac{(\epsilon_2-\epsilon_1)}{v_{Fj}}x}\delta_{jj_1}\delta_{jj_2}\right]\left[e^{i\frac{(\epsilon_2'-\epsilon_1')}{v_{Fj'}}x'}S_{j'j_1'}^* S_{j'j_2'} - e^{-i\frac{(\epsilon_2'-\epsilon_1')}{v_{Fj'}}x'}\delta_{j'j_1'}\delta_{j'j_2'}\right]$$
$$\times(-)\int_0^\beta d\tau_1\sum_n\int dy\, g_2^n(y)\sum_{l_1l_2l_3l_4}\int\frac{dE_1dE_2dE_3dE_4}{(2\pi)^2 v_{Fn}^2}\varrho_{l_1l_2l_3l_4}^n(E_1,E_2,E_3,E_4;y)$$
$$\times\delta_{j_2l_1}\delta(\epsilon_2-E_1)\mathcal{G}_{j_2}(\epsilon_2,\tau-\tau_1)\delta_{l_4j_1'}\delta(\epsilon_1'-E_4)\mathcal{G}_{j_1'}(\epsilon_1',\tau_1-\tau')$$
$$\times\delta_{j_2'l_3}\delta(\epsilon_2'-E_3)\mathcal{G}_{j_2'}(\epsilon_2',\tau'-\tau_1)\delta_{j_1l_2}\delta(E_2-\epsilon_1)\mathcal{G}_{j_1}(\epsilon_1,\tau_1-\tau). \tag{A12}$$

Going to the frequency space, performing Matsubara sums and analytic continuation:

$$\Omega_{jj'}^{(1),\mathrm{VC,ring},1}(x,x';\omega^+) = -\frac{e^2}{(2\pi)^2}\sum_{j_1j_2j_1'j_2'}\int d\epsilon_1 d\epsilon_2 d\epsilon_1' d\epsilon_2'$$
$$\times\left[e^{i\frac{(\epsilon_2-\epsilon_1)}{v_{Fj}}x}S_{jj_1}^* S_{jj_2} - e^{-i\frac{(\epsilon_2-\epsilon_1)}{v_{Fj}}x}\delta_{jj_1}\delta_{jj_2}\right]\left[e^{i\frac{(\epsilon_2'-\epsilon_1')}{v_{Fj'}}x'}S_{j'j_1'}^* S_{j'j_2'} - e^{-i\frac{(\epsilon_2'-\epsilon_1')}{v_{Fj'}}x'}\delta_{j'j_1'}\delta_{j'j_2'}\right]$$
$$\times\sum_n\int dy\, g_2^n(y)\frac{1}{(2\pi)^2 v_{Fn}^2}\varrho_{j_2j_1j_2'j_1'}^n(\epsilon_2,\epsilon_1,\epsilon_2',\epsilon_1';y)\frac{n_F(\epsilon_2)-n_F(\epsilon_1)}{\epsilon_2-\epsilon_1-\omega^+}\frac{n_F(\epsilon_2')-n_F(\epsilon_1')}{\epsilon_2'-\epsilon_1'+\omega^+}. \tag{A13}$$

Integrating over the energies as before, we find

$$\Omega_{jj'}^{(1),\mathrm{VC,ring},1}(x,x';\omega^+) = -\frac{e^2}{2\pi}\int d\epsilon_1 d\epsilon_1'(-)\left[n_F(\epsilon_1+\omega^+)-n_F(\epsilon_1)\right]\left[n_F(\epsilon_1'-\omega^+)-n_F(\epsilon_1')\right]$$
$$\times\sum_n\int dy\,\alpha_n(y_1)\frac{1}{v_{Fn}}\frac{1}{2}\left[|S_{jn}|^2\delta_{j'n}e^{i\omega^+\left(\frac{x}{v_{Fj}}-\frac{x'}{v_{Fj'}}\right)}e^{2i\omega^+\frac{y}{v_{Fn}}}\theta\left(\frac{y}{v_{Fn}}-\frac{x'}{v_{Fj'}}\right)\right.$$
$$-\delta_{jn}\delta_{j'n}e^{-i\omega^+\left(\frac{x}{v_{Fj}}+\frac{x'}{v_{Fj'}}\right)}e^{2i\omega^+\frac{y}{v_{Fn}}}\theta\left(\frac{y}{v_{Fn}}-\frac{x}{v_{Fj}}\right)\theta\left(\frac{y}{v_{Fn}}-\frac{x'}{v_{Fj'}}\right)$$
$$-\delta_{jn}\delta_{j'n}e^{i\omega^+\left(\frac{x}{v_{Fj}}+\frac{x'}{v_{Fj'}}\right)}e^{-2i\omega^+\frac{y}{v_{Fn}}}\theta\left(\frac{x}{v_{Fj}}-\frac{y}{v_{Fn}}\right)\theta\left(\frac{x'}{v_{Fj'}}-\frac{y}{v_{Fn}}\right)$$
$$\left.-|S_{jn}|^2|S_{nj'}|^2 e^{i\omega^+\left(\frac{x}{v_{Fj}}+\frac{x'}{v_{Fj'}}\right)}e^{2i\omega^+\frac{y}{v_{Fn}}}+\delta_{jn}|S_{nj'}|^2 e^{-i\omega^+\left(\frac{x}{v_{Fj}}-\frac{x'}{v_{Fj'}}\right)}e^{2i\omega^+\frac{y}{v_{Fn}}}\theta\left(\frac{y}{v_{Fn}}-\frac{x}{v_{Fj}}\right)\right]. \tag{A14}$$

There exists an analogous term with all electron lines reverted. Upon substitution into Eq. (2.18) these two terms reproduce Eq. (2.21).

Finally, by summing over all dummy wire indices, we can show that the "cracked egg" contribution to the DC conductance is proportional to $\delta_{jj'}$. On the other hand, due to current conservation and the absence of equilibrium currents, the full DC conductance $G_{jj'}$ obeys

$$\sum_j G_{jj'} = \sum_{j'} G_{jj'} = 0; \tag{A15}$$

this must also be true at $O(\alpha)$. Since Eq. (A15) is already satisfied by Eqs. (2.20) and (2.24), it must also be separately satisfied by the "cracked egg" diagrams. But $\sum_j \delta_{jj'} = \sum_{j'}\delta_{jj'} = 1$, and we infer that the "cracked egg" diagrams must be identically zero.

Appendix B: Details of the Wilsonian approach to S-matrix renormalization

In this appendix, we review the derivation of the S-matrix RG equation using the Wilsonian scaling approach in Ref. 6.

Starting from Eq. (2.8), we reduce the energy cutoff D to $D-\delta D$ ($\delta D \ll D$), and integrate out the so-called "fast modes" with energies in one of the two slices $(-D+\delta D,-D)$ and $(D-\delta D,D)$. This procedure generates corrections of $O(\alpha\delta D/D)$ to the quadratic part of the Hamiltonian [Eq. (2.6)] as well as the quartic part [Eq. (2.8)]. We assume that the corrections to the quartic part are unimportant; the rationale is that the quartic part originates entirely from the bulk, so it should renormalize independently of the junction. In fact, since the quartic part is free of Umklapp processes, it should be exactly marginal in the RG sense.[20] Meanwhile, the renormalized quadratic part becomes off-diagonal and must be diagonalized with a new scattering basis, which is in turn associated with a running (i.e. cutoff-dependent) S-matrix.

The quadratic correction generated by Eq. (2.8) reads

$$\delta H_0^j = -2\int_0^\infty dx g_2^j(x)\sum_{l_1l_2l_3}\int_{\delta D}dE_1\int_{-D}^{D}\frac{dE_2dE_3}{(2\pi)^2 v_{Fj}^2}\varrho^j_{l_1l_2l_3l_1}(E_1,E_2,E_3,E_1;x)\,n_F(E_1)\,\phi^\dagger_{l_3}(E_3)\,\phi_{l_2}(E_2). \tag{B1}$$

The E_2E_3 contraction is equivalent to the E_1E_4 contraction; hence the factor of 2. The E_1E_2 and E_3E_4 contractions are discarded because, once we sum over l taking into account the S-matrix unitarity $\sum_{l_1}|S_{jl_1}|^2=1$, we find they only harmlessly shift the chemical potential.[20]

We let ϕ' be the renormalized scattering basis after integrating out fast modes. ϕ' is related to ϕ by another S-matrix, $S^\delta_{jj'}$, which only weakly deviates from the $N\times N$ identity matrix:

$$\phi_j(E)=\int\frac{dE'}{2\pi}\left[\frac{i}{E-E'+i0}\phi'_j(E')+\frac{-i}{E-E'-i0}\sum_{j'}S^\delta_{jj'}(E';E)\,\phi'_{j'}(E')\right] \tag{B2}$$

The inverse transformation is obtained by calculating anti-commutators:

$$\left(\phi'_{j'}(E')\right)^\dagger=\sum_j\int\frac{dE}{2\pi}\left[\frac{i}{E-E'+i0}\delta_{jj'}+\frac{-i}{E-E'-i0}S^\delta_{jj'}(E';E)\right]\phi^\dagger_j(E) \tag{B3}$$

By definition ϕ' diagonalizes the renormalized quadratic Hamiltonian,

$$\left[\sum_j H^j_{0,\text{wire}}+H_{0,B}+\sum_j\delta H^j_0,\left(\phi'_{j'}(E')\right)^\dagger\right]=E'\left(\phi'_{j'}(E')\right)^\dagger \tag{B4}$$

Substituting Eq. (B3) into the above, we find to $O(\delta D/D)$

$$\begin{aligned}i\left[\delta_{jj'}-S^\delta_{jj'}(E';E)\right]=\delta D\int_0^\infty dx\Bigg\{&n_F(D)\left[\frac{\alpha_j(x)}{v_{Fj}}S^*_{jj}S_{jj'}e^{-i\frac{2D-E-E'}{v_{Fj}}x}+\frac{\alpha_{j'}(x)}{v_{Fj'}}S_{j'j'}S^*_{j'j}e^{i\frac{2D-E-E'}{v_{Fj'}}x}\right]\\&+n_F(-D)\left[\frac{\alpha_j(x)}{v_{Fj}}S^*_{jj}S_{jj'}e^{i\frac{2D+E+E'}{v_{Fj}}x}+\frac{\alpha_{j'}(x)}{v_{Fj'}}S_{j'j'}S^*_{j'j}e^{-i\frac{2D+E+E'}{v_{Fj'}}x}\right]\Bigg\}\end{aligned} \tag{B5}$$

For the simple model Eq. (2.14), integrating over x, we find

$$\begin{aligned}\delta_{jj'}-S^\delta_{jj'}(E';E)=\delta D\Bigg\{&n_F(D)\left[S^*_{jj}S_{jj'}\frac{(\alpha_j(0)-\alpha_j(\infty))\,e^{-i\frac{2D-E-E'}{v_{Fj}}L_j}-\alpha_j(0)}{2D-E-E'}\right.\\&\left.+S_{j'j'}S^*_{j'j}\frac{(\alpha_{j'}(0)-\alpha_{j'}(\infty))\,e^{i\frac{2D-E-E'}{v_{Fj'}}L_{j'}}-\alpha_{j'}(0)}{2D-E-E'}\right]+n_F(-D)\left[S^*_{jj}S_{jj'}\frac{(\alpha_j(0)-\alpha_j(\infty))\,e^{i\frac{2D+E+E'}{v_{Fj}}L_j}-\alpha_j(0)}{-2D-E-E'}\right.\\&\left.+S_{j'j'}S^*_{j'j}\frac{(\alpha_{j'}(0)-\alpha_{j'}(\infty))\,e^{-i\frac{2D+E+E'}{v_{Fj'}}L_{j'}}-\alpha_{j'}(0)}{-2D-E-E'}\right]\Bigg\}\end{aligned} \tag{B6}$$

When we assume $D \gtrsim \max\{|E|, |E'|, T\}$, and apply the same considerations below Eq. (3.2), Eq. (B6) becomes

$$S^{\delta}_{jj'}(E;E') = \delta_{jj'} - \frac{\delta D}{2D}\left(\alpha_j(D) S^*_{jj} S_{jj'} - \alpha_{j'}(D) S_{j'j'} S^*_{j'j}\right). \tag{B7}$$

The renormalized S-matrix $S + \delta S$ relates ϕ' to the original fermions ψ. Inserting Eq. (B2) into Eq. (2.5) we find that δS and S^{δ} obey the simple matrix relation $\delta S = SS^{\delta} - S$, and according to Eq. (B7), $\delta S_{jj'}$ is given by none other than Eq. (3.3). Thus to the first order in interaction the CS approach and the Wilsonian approach predict the same S-matrix renormalization, Eq. (3.5).

Appendix C: Details of the RPA

In this appendix we expound the RPA calculations that lead to Eqs. (4.2) and (4.5).

1. RPA conductance

The RPA self-energy beyond the first order involves a new type of Matsubara sum. For instance, at the third order in interaction, we need

$$\begin{aligned}
&\frac{1}{\beta}\sum_{ip_m} \frac{n_F(E_4) - n_F(E_3)}{ip_m + E_4 - E_3} \frac{n_F(E_8) - n_F(E_7)}{ip_m + E_8 - E_7} \frac{1}{i(p_m + \omega_n) - E_2} \\
&= n_F(E_2) \frac{n_F(E_4) - n_F(E_3)}{E_2 - i\omega_n + E_4 - E_3} \frac{n_F(E_8) - n_F(E_7)}{E_2 - i\omega_n + E_8 - E_7} \\
&+ \int \frac{d\tilde{\epsilon}}{2\pi i} n_B(\tilde{\epsilon}) \frac{1}{\tilde{\epsilon} + i\omega_n - E_2} \left(\frac{n_F(E_4) - n_F(E_3)}{(\tilde{\epsilon} + i0) + E_4 - E_3} \frac{n_F(E_8) - n_F(E_7)}{(\tilde{\epsilon} + i0) + E_8 - E_7} \right. \\
&\left. - \frac{n_F(E_4) - n_F(E_3)}{(\tilde{\epsilon} - i0) + E_4 - E_3} \frac{n_F(E_8) - n_F(E_7)}{(\tilde{\epsilon} - i0) + E_8 - E_7} \right)
\end{aligned} \tag{C1}$$

where $n_B(\epsilon) = 1/\left(e^{\beta\epsilon} - 1\right)$ is the Bose distribution. To derive Eq. (C1) we again wrap the integration contour around the branch cut at the real axis. The fraction with numerator $n_F(E_3) - n_F(E_4)$ originates from the fermion loop with loop energy E_3 and E_4; at the lth order there will be $l-1$ loops present. ip_m is the bosonic frequency carried by the interaction lines; after ip_m, $i\omega_n$ is also summed over following Eq. (A7).

After we perform analytic continuation and integrate over the loop momenta, as x, $x' \to \infty$, the three most important terms in the correlation function at the third order are

$$\begin{aligned}
&\Omega^{(3),\mathrm{SE,RPA},E_2}_{jj'}\left(x, x'; \omega^+\right) = \frac{e^2}{2\pi} e^{i\omega^+\left(\frac{x}{v_{Fj}} + \frac{x'}{v_{Fj'}}\right)} \int d\epsilon'_1 \left[n_F(\epsilon'_1) - n_F\left(\epsilon'_1 - \omega^+\right)\right] S_{jj'} \\
&\times \sum_{n_1 n_2 n_3} \int_0^{\infty} d\tilde{y}_1 d\tilde{y}_2 d\tilde{y}_3 \alpha_{n_1}(v_{Fn_1}\tilde{y}_1) \alpha_{n_2}(v_{Fn_2}\tilde{y}_2) \alpha_{n_3}(v_{Fn_3}\tilde{y}_3) \int dE_2 n_F(E_2) \tilde{E}_2^2 \\
&\times \Big[S^*_{jn_3} S_{n_1 n_3} S^*_{n_1 j'} |S_{n_2 n_1}|^2 |S_{n_3 n_2}|^2 e^{2i\tilde{E}_2(\tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3)} \\
&+ S^*_{jn_3} S_{n_1 n_3} S^*_{n_1 j'} \delta_{n_3 n_2} \delta_{n_2 n_1} \theta(\tilde{y}_1 - \tilde{y}_2) \theta(\tilde{y}_3 - \tilde{y}_2) e^{2i\tilde{E}_2(\tilde{y}_1 - \tilde{y}_2 + \tilde{y}_3)} \\
&+ \delta_{jn_3} \delta_{n_3 n_1} S^*_{n_1 j'} \delta_{n_3 n_2} |S_{n_2 n_1}|^2 \theta(\tilde{y}_2 - \tilde{y}_3) e^{2i\tilde{E}_2(\tilde{y}_1 + \tilde{y}_2 - \tilde{y}_3)} \\
&+ S^*_{jn_3} \delta_{n_3 n_1} \delta_{n_1 j'} |S_{n_3 n_2}|^2 \delta_{n_2 n_1} \theta(\tilde{y}_2 - \tilde{y}_1) e^{2i\tilde{E}_2(-\tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3)} \\
&+ \delta_{jn_3} S^*_{n_3 n_1} \delta_{n_1 j'} \delta_{n_3 n_2} \delta_{n_2 n_1} \theta(\tilde{y}_2 - \tilde{y}_1) \theta(\tilde{y}_2 - \tilde{y}_3) e^{2i\tilde{E}_2(-\tilde{y}_1 + \tilde{y}_2 - \tilde{y}_3)} \Big]
\end{aligned} \tag{C2a}$$

where we have substituted $\tilde{y}_n = y_n / v_{Fn}$ and $\tilde{E}_2 = E_2 - \epsilon'_1 + \omega^+$,

$$\Omega_{jj'}^{(3),\mathrm{SE,RPA},\tilde{\epsilon}+}\left(x,x';\omega^{+}\right)=\frac{e^{2}}{2\pi}e^{i\omega^{+}\left(\frac{x}{v_{Fj}}+\frac{x'}{v_{Fj'}}\right)}\int d\epsilon_{1}'\left[n_{F}\left(\epsilon_{1}'\right)-n_{F}\left(\epsilon_{1}'-\omega^{+}\right)\right]S_{jj'}$$
$$\times\sum_{n_1n_2n_3}\int_{0}^{\infty}d\tilde{y}_{1}d\tilde{y}_{2}d\tilde{y}_{3}\alpha_{n_1}\left(v_{Fn_1}\tilde{y}_{1}\right)\alpha_{n_2}\left(v_{Fn_2}\tilde{y}_{2}\right)\alpha_{n_3}\left(v_{Fn_3}\tilde{y}_{3}\right)\int d\tilde{\epsilon}n_{B}\left(\tilde{\epsilon}\right)\tilde{\epsilon}^{2}$$
$$\times\Big[\delta_{jn_3}\delta_{n_3n_1}S_{n_1j'}^{*}\delta_{n_3n_2}\left|S_{n_2n_1}\right|^{2}\theta\left(\tilde{y}_{2}-\tilde{y}_{3}\right)\theta\left(\tilde{y}_{3}-\tilde{y}_{1}\right)e^{2i(\tilde{\epsilon}+i0)(\tilde{y}_1+\tilde{y}_2-\tilde{y}_3)}$$
$$+S_{jn_3}^{*}\delta_{n_3n_1}\delta_{n_1j'}\left|S_{n_3n_2}\right|^{2}\delta_{n_2n_1}\theta\left(\tilde{y}_{2}-\tilde{y}_{1}\right)\theta\left(\tilde{y}_{1}-\tilde{y}_{3}\right)e^{2i(\tilde{\epsilon}+i0)(-\tilde{y}_1+\tilde{y}_2+\tilde{y}_3)}$$
$$+\delta_{jn_3}S_{n_3n_1}^{*}\delta_{n_1j'}\delta_{n_3n_2}\delta_{n_2n_1}\theta\left(\tilde{y}_{2}-\tilde{y}_{1}\right)\theta\left(\tilde{y}_{2}-\tilde{y}_{3}\right)e^{2i(\tilde{\epsilon}+i0)(-\tilde{y}_1+\tilde{y}_2-\tilde{y}_3)}\Big] \tag{C2b}$$

and finally

$$\Omega_{jj'}^{(3),\mathrm{SE,RPA},\tilde{\epsilon}-}\left(x,x';\omega^{+}\right)=\frac{e^{2}}{2\pi}e^{i\omega^{+}\left(\frac{x}{v_{Fj}}+\frac{x'}{v_{Fj'}}\right)}\int d\epsilon_{1}'\left[n_{F}\left(\epsilon_{1}'\right)-n_{F}\left(\epsilon_{1}'-\omega^{+}\right)\right]S_{jj'}$$
$$\times\sum_{n_1n_2n_3}\int_{0}^{\infty}d\tilde{y}_{1}d\tilde{y}_{2}d\tilde{y}_{3}\alpha_{n_1}\left(v_{Fn_1}\tilde{y}_{1}\right)\alpha_{n_2}\left(v_{Fn_2}\tilde{y}_{2}\right)\alpha_{n_3}\left(v_{Fn_3}\tilde{y}_{3}\right)\int d\tilde{\epsilon}n_{B}\left(\tilde{\epsilon}\right)\tilde{\epsilon}^{2}$$
$$\times\Big[-\delta_{jn_3}\delta_{n_3n_1}S_{n_1j'}^{*}\delta_{n_1n_2}\left|S_{n_2n_3}\right|^{2}\theta\left(\tilde{y}_{2}-\tilde{y}_{1}\right)\theta\left(\tilde{y}_{3}-\tilde{y}_{1}\right)e^{2i(\tilde{\epsilon}-i0)(\tilde{y}_1-\tilde{y}_2-\tilde{y}_3)}$$
$$-\delta_{jn_3}S_{n_3n_1}^{*}\delta_{n_1j'}\delta_{n_1n_2}\delta_{n_2n_3}\theta\left(\tilde{y}_{1}-\tilde{y}_{2}\right)\theta\left(\tilde{y}_{3}-\tilde{y}_{2}\right)e^{2i(\tilde{\epsilon}-i0)(-\tilde{y}_1+\tilde{y}_2-\tilde{y}_3)}$$
$$-S_{jn_3}^{*}\delta_{n_3n_1}\delta_{n_1j'}\left|S_{n_1n_2}\right|^{2}\delta_{n_2n_3}\theta\left(\tilde{y}_{2}-\tilde{y}_{3}\right)\theta\left(\tilde{y}_{1}-\tilde{y}_{3}\right)e^{2i(\tilde{\epsilon}-i0)(-\tilde{y}_1-\tilde{y}_2+\tilde{y}_3)}$$
$$-\delta_{jn_3}S_{n_3n_1}^{*}\delta_{n_1j'}\left|S_{n_1n_2}\right|^{2}\left|S_{n_2n_3}\right|^{2}e^{2i(\tilde{\epsilon}-i0)(-\tilde{y}_1-\tilde{y}_2-\tilde{y}_3)}\Big] \tag{C2c}$$

plus similar terms with all electron lines reverted. $\tilde{y}_j \equiv y_j/v_{Fj}$ runs between 0 and ∞, $j = 1, 2, 3$. These three terms come from lines 2, 3 and 4 of Eq. (C1) respectively.

In the DC limit, the zeroth order contribution and the self-energy corrections to the conductance again constitute a Landauer-type formula with a dressed S-matrix, similar to Eq. (2.20). Now we reduce the cutoff and demand the conductance be cutoff-independent. Once the $\tilde{y}$ integrals are performed, it is obvious that the cutoff-sensitive integrals are the E_2 integral and the $\tilde{\epsilon}$ integral.

We are in a position to discuss the real space integrals. We first focus on the simplest case where the interactions in wires and leads are uniform and identical, $\alpha_{n_1}(y) = \alpha_{n_1}$ for any n_1, so that all α's factor out. At the third order, we find the following integrals:

$$I_{1}\left(E^{+}\right)\equiv\int_{0}^{\infty}d\tilde{y}_{1}d\tilde{y}_{2}d\tilde{y}_{3}e^{iE^{+}(\tilde{y}_1+\tilde{y}_3)}e^{iE^{+}(\tilde{y}_1-2\tilde{y}_2+\tilde{y}_3)}\theta\left(\tilde{y}_{1}-\tilde{y}_{2}\right)\theta\left(\tilde{y}_{3}-\tilde{y}_{2}\right) \tag{C3}$$

which appears alongside the factors $\delta_{n_1n_2}\delta_{n_2n_3}$, and

$$\int_{0}^{\infty}d\tilde{y}_{1}e^{2iE^{+}\tilde{y}_1}=\frac{i}{2E^{+}} \tag{C4}$$

which appears alongside, for example, $W_{n_1n_2}W_{n_2n_3}$. (More accurately, Eq. (C4) comes with each "node" n_2 as long as n_2 is not sandwiched between two Kronecker δ factors.) Here $E^+ \equiv E + i0$ may be replaced by $\tilde{E}_2$ or $(\pm\tilde{\epsilon} + i0)$. At higher orders, we need to evaluate the integral

$$I_{M}\left(E^{+}\right)\equiv\prod_{l=1}^{2M+1}\left(\int_{0}^{\infty}d\tilde{y}_{l}\right)e^{iE^{+}(\tilde{y}_1+\tilde{y}_{2M+1})}\prod_{j=1}^{M}\left[e^{iE^{+}(\tilde{y}_{2j-1}+\tilde{y}_{2j+1}-2\tilde{y}_{2j})}\theta\left(\tilde{y}_{2j-1}-\tilde{y}_{2j}\right)\theta\left(\tilde{y}_{2j+1}-\tilde{y}_{2j}\right)\right] \tag{C5}$$

This is accompanied by a string of $2M$ consecutive δ factors uninterrupted by W factors, $\delta_{n_1n_2}\delta_{n_2n_3}\cdots\delta_{n_{2M}n_{2M+1}}$. We will prove in Section C 2 that

$$I_M\left(E^+\right)=\left(\frac{i}{2E^+}\right)^{2M+1}C_M \tag{C6}$$

where $C_M=(2M)!/\left[M!\,(M+1)!\right]$ is the Mth Catalan number.[21] The first few Catalan numbers $0\le M\le 5$ are 1, 1, 2, 5, 14, 42.

At this stage we can combine the $\left(E^+\right)^{-1}$ factors in Eqs. (C4) and (C6) with the $\tilde{E}_2$ or $\tilde{\epsilon}$ factors. At each order there will be a single $\left(E^+\right)^{-1}$ factor left unpaired, which gives the leading-log renormalization $\delta D/D$ as the cutoff is reduced from D to $D-\delta D$. Collecting terms of all orders we see the S-matrix RG equation is of the form of Eq. (4.2), but the interaction $\Pi\left(D\right)$ is given by

$$\begin{aligned}\frac{\Pi_{jj'}}{2}=&\frac{\alpha_j}{2}\delta_{jj'}+\frac{\alpha_j}{2}\frac{\alpha_{j'}}{2}\left\{W_{jj'}+\sum_{n_1}\frac{\alpha_{n_1}}{2}\left[\delta_{jn_1}\delta_{n_1j'}+W_{jn_1}W_{n_1j'}\right]\right.\\&+\sum_{n_1n_2}\frac{\alpha_{n_1}}{2}\frac{\alpha_{n_2}}{2}\left[\delta_{jn_1}\delta_{n_1n_2}W_{n_2j'}+W_{jn_1}\delta_{n_1n_2}\delta_{n_2j'}+W_{jn_1}W_{n_1n_2}W_{n_2j'}\right]\\&+\sum_{n_1n_2n_3}\frac{\alpha_{n_1}}{2}\frac{\alpha_{n_2}}{2}\frac{\alpha_{n_3}}{2}\left[2\delta_{jn_1}\delta_{n_1n_2}\delta_{n_2n_3}\delta_{n_3j'}+\delta_{jn_1}\delta_{n_1n_2}W_{n_2n_3}W_{n_3j'}\right.\\&\left.\left.+W_{jn_1}\delta_{n_1n_2}\delta_{n_2n_3}W_{n_3j'}+W_{jn_1}W_{n_1n_2}\delta_{n_2n_3}\delta_{n_3j'}+W_{jn_1}W_{n_1n_2}W_{n_2n_3}W_{n_3j'}\right]+\cdots\right\}\end{aligned} \tag{C7}$$

The rules to write down terms in Eq. (C7) are as follows. At $O\left(\alpha^m\right)$, there is a total number of $(m-1)$ factors of δ and W. The δ factors always appear in even-length strings separated by the W factors. Each string of δ of length $2M$ is associated with a multiplicative coefficient of the Mth Catalan number C_M. For instance, at $O\left(\alpha^{17}\right)$ there is a term $W\delta\delta\delta\delta\delta\delta W W W\delta\delta\delta\delta\delta W W$, whose prefactor will be $C_3C_2=5\times 2=10$.

We can resum Eq. (C7) by observing that we can uniquely construct every term containing a least one factor of W, by adding to an existing term a (possibly empty) even-length string of δ followed by one factor of W; e.g. the term $\delta\delta\delta\delta W W\delta\delta W$ is uniquely constructed as $\delta\delta\delta\delta/W/W\delta\delta W$. In other words, Π satisfies the relation

$$\frac{\Pi_{jj'}}{2}=\frac{\bar{\Pi}_{jj'}}{2}+\sum_{l_1l_2}\frac{\bar{\Pi}_{jl_1}}{2}W_{l_1l_2}\frac{\Pi_{l_2j'}}{2}. \tag{C8}$$

Here $\bar{\Pi}$ is the part of Π which does not contain any factors of W:

$$\begin{aligned}\frac{\bar{\Pi}_{jj'}}{2}&=\frac{\alpha_j}{2}\delta_{jj'}+\frac{\alpha_j}{2}\frac{\alpha_{j'}}{2}\left[\sum_{n_1}\frac{\alpha_{n_1}}{2}\delta_{jn_1}\delta_{n_1j'}+\sum_{n_1n_2n_3}\frac{\alpha_{n_1}}{2}\frac{\alpha_{n_2}}{2}\frac{\alpha_{n_3}}{2}2\delta_{jn_1}\delta_{n_1n_2}\delta_{n_2n_3}\delta_{n_3j'}+\cdots\right]\\&=\frac{\alpha_j}{2}\delta_{jj'}\sum_{M=0}^{\infty}C_M\left(\frac{\alpha_j}{2}\right)^{2M}=\frac{\alpha_j}{1+\sqrt{1-\alpha_j^2}}\delta_{jj'}.\end{aligned} \tag{C9}$$

In the last line we have used the generating function of Catalan numbers,[21]

$$\sum_{M=0}^{\infty}C_Mx^M=\frac{2}{1+\sqrt{1-4x}}. \tag{C10}$$

Inserting Eq. (C9) into Eq. (C8) and solving for Π, we obtain Eq. (4.3) in the case of spatially uniform interactions, $\alpha_n\left(y\right)=\alpha_n$.

We now argue that the cutoff-dependence of the Luttinger parameter is through Eq. (3.4) as is the case with the first order calculation. To this end, notice that it is values of $\tilde{y}_n$ between 0 and $O\left(1/E^+\right)$ that dominate the integral in Eq. (C5). Therefore, when $D=\mathrm{Re}\,E^+\gtrsim v_{Fn}/L_n$, the integral is governed by $v_{Fn}\tilde{y}_n\lesssim L_n$; in this range of $\tilde{y}_n$, $\alpha_n\left(v_{Fn}\tilde{y}_n\right)=\alpha_n\left(0\right)$. On the other hand, when $D\ll v_{Fn}/L_n$, the integral is controlled mainly by $v_{Fn}\tilde{y}_n\gg L_n$,

where $\alpha_n(v_{Fn}\tilde{y}_n) = \alpha_n(\infty)$. This justifies the crossover behavior given by Eqs. (3.4) and (4.4), and concludes the calculation of the self-energy terms in the RPA conductance.

Calculations of the RPA vertex corrections, or the ring diagrams, are completely in parallel with the first-order vertex corrections except Eq. (C5) appears in the real space integrals. Here E^+ in Eq. (C5) should be substituted for ω^+. At the mth order, all m factors of $1/\omega^+$ in Eqs. (C4) and (C6) can be paired with the $m+1$ factors of ω^+ from loop energy integrals; the single unpaired ω^+ will be combined with the $1/\omega$ factor in Eq. (2.18) so that the conductance is finite in the DC limit. Also, all interaction strengths appearing here are those in the leads $\alpha_n(\infty)$; this is because in the DC limit $\omega \lesssim v_{Fn}/L_n$ for any lead n, and we may refer to our argument in the previous paragraph for $D \ll v_{Fn}/L_n$. Eventually, taking into account the dressing of the electron lines, we recover Eq. (4.5).

2. Real space integral Eq. (C6)

To prove Eq. (C6), we adopt the following change of variables in Eq. (C5): $z_0 = \tilde{y}_1$, $z_{2j-1} = \tilde{y}_{2j-1} - \tilde{y}_{2j}$, $z_{2j} = \tilde{y}_{2j+1} - \tilde{y}_{2j}$, $1 \le j \le M$. The absolute value of the Jacobian of this change of variables is simply $\left|(-1)^M\right| = 1$. We also introduce the shorthand $s_j = \sum_{l=0}^{j}(-1)^l z_l$. Eq. (C5) then becomes

$$I_M\left(E^+\right) = \int_0^\infty dz_0 \prod_{l=1}^{M}\left(\int_0^{s_{2l-2}} dz_{2l-1}\int_0^\infty dz_{2l}\right)\prod_{j=0}^{M} e^{2iE^+ z_{2j}} \tag{C11}$$

Now consider the auxiliary object,

$$\tilde{I}_M\left(E^+, z_0\right) \equiv \prod_{l=1}^{M}\left(\int_0^{s_{2l-2}} dz_{2l-1}\int_0^\infty dz_{2l}\right)\prod_{j=0}^{M} e^{2iE^+ z_{2j}} \equiv \left(2iE^+\right)^{-2M} e^{2iE^+ z_0}\sum_{l=0}^{M}\frac{T_{M,l}}{l!}\left(-2iE^+ z_0\right)^l \tag{C12}$$

where $T_{M,l}$ are dimensionless coefficients; obviously $\tilde{I}_0\left(E^+, z_0\right) = e^{2iE^+ z_0}$ and $T_{0,0} = 1$. $\tilde{I}_M$ obeys the recurrence relation

$$\tilde{I}_{M+1}\left(E^+, z_0\right) = \int_0^{z_0} dz_1 e^{2iE^+ z_1}\int_0^\infty dz_2 \tilde{I}_M\left(E^+, z_0 - z_1 + z_2\right). \tag{C13}$$

Inserting Eq. (C12) into Eq. (C13), we find that $T_{M,l}$ satisfies the simple recurrence relation $T_{M+1,l} = \sum_{j=l-1}^{M} T_{M,j}$, and that $T_{M+1,0} = 0$ ($M \ge 0$). Such a recurrence relation leads to the Catalan's triangle,[21]

$$T_{M,l} = \frac{(2M-l-1)!\,l}{M!\,(M-l)!}\ \ (M \ge 1). \tag{C14}$$

Therefore,

$$I_M\left(E^+\right) = \int_0^\infty dz_0 \tilde{I}_M\left(E^+, z_0\right) = -\left(2iE^+\right)^{-2M-1}\sum_{l=0}^{M} T_{M,l} \tag{C15}$$

Noting that $\sum_{l=0}^{M} T_{M,l} = C_M$, which is a property of Catalan's triangle, we immediately recover Eq. (C6).

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