A central limit theorem for Lebesgue integrals of random fields

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Abstract

In this paper we show a central limit theorem for Lebesgue integrals of stationary $BL(\theta)$-dependent random fields as the integration domain grows in Van Hove-sense. Our method is to use the (known) analogue result for discrete sums. As applications we obtain various multivariate versions of this central limit theorem.

1 Introduction

Random fields are collections of random variables indexed by the Euclidean space $\mathbb{R}^d$. They have applications in various branches of science, e.g. in medicine [1, 13], in geostatistics [5, 15] or in material science [10, 14].

The aim of the present paper is to establish a central limit theorem for integrals $\int_{W_n} X(t)\,dt$, where $(W_n)_{n\in\mathbb{N}}$ is a sequence of compact subsets of $\mathbb{R}^d$ and $(X(t))_{t\in\mathbb{R}^d}$ is a random field. The sequence $(W_n)_{n\in\mathbb{N}}$ of integration domains is assumed to *grow in Van Hove-sense (VH-sense)*, i.e.

$$\lim_{n\to\infty} \lambda_d((\operatorname{bd} W_n) + B^d)/\lambda_d(W_n) = 0,$$

where λ_d denotes the Lebesgue measure, $\operatorname{bd} W$ is the boundary of $W \subseteq \mathbb{R}^d$, $A+B := \{a+b \mid a \in A,\ b \in B\}$ for two subsets $A, B \subseteq \mathbb{R}^d$ and $B^d := \{x \in \mathbb{R}^d \mid \|x\| \leq 1\}$ is the closed Euclidean unit ball.

The main result of the present paper is the following (the notion of $BL(\theta)$-dependence will be defined in Subsection 2.1).

Theorem 1. *Let $\theta = (\theta_r)_{r\in\mathbb{N}}$ be a monotonically decreasing zero sequence. Let $(X(t))_{t\in\mathbb{R}^d}$ be a measurable, stationary, $BL(\theta)$-dependent $\mathbb{R}$-valued random field such that*

$$\int_{\mathbb{R}^d} |\operatorname{Cov}\big(X(0), X(t)\big)|\,dt < \infty.$$

Let $(W_n)_{n\in\mathbb{N}}$ be a VH-growing sequence of subsets of $\mathbb{R}^d$. Then

$$\frac{\int_{W_n} X(t)\,dt - \mathbb{E}\,X(0)\lambda_d(W_n)}{\sqrt{\lambda_d(W_n)}} \to \mathcal{N}(0,\sigma^2), \quad n\to\infty,$$

in distribution, where

$$\sigma^2 := \int_{\mathbb{R}^d} \operatorname{Cov}\big(X(0), X(t)\big)\,dt.$$

There is a wide literature on similar results, where mixing conditions are assumed instead of $BL(\theta)$-dependence, see e.g. [4, 7, 8, 9]. For $BL(\theta)$-dependent random fields there are no central limit theorems for Lebesgue integrals up to now. However, there are such results for discrete sums [2] and for Lebesgue measures of excursion sets [3]. In the latter paper the random field is in fact assumed to be quasi-associated, which is a slightly stronger assumption than $BL(\theta)$- dependence.

This paper is organized as follows: In Section 2 we collect preliminaries about associated random variables, random fields and functions of bounded variation. Section 3 is devoted to the proof of the main theorem. In Section 4 we present several examples how the main result can be extended to a multivariate central limit theorem. The case that the random field is of the form $(f(X(t)))_{t\in\mathbb{R}^d}$ for some deterministic function $f : \mathbb{R} \to \mathbb{R}^s$ and some random $\mathbb{R}$-valued field $(X(t))_{t\in\mathbb{R}^d}$ will be of particular interest.

2 Preliminaries

2.1 Association concepts

In this subsection we introduce different association concepts and discuss their relations.

We start with the broadest appearing in this paper, namely $BL(\theta)$-dependence.

For finite subsets $I, J \subseteq \mathbb{R}^d$ we put $\operatorname{dist}(I,J) := \min\{\|x-y\|_1 : x \in I,\, y \in J\}$, where $\|\cdot\|_1$ is the ℓ_1-norm. For two Lipschitz functions $f : \mathbb{R}^{n_1} \to \mathbb{R}$ and $g : \mathbb{R}^{n_2} \to \mathbb{R}$ we put

$$\Psi(n_1, n_2, f, g) = \min\{n_1, n_2\} \operatorname{Lip}(f) \operatorname{Lip}(g),$$

where

$$\operatorname{Lip}(f) := \sup\Big\{ \frac{|f(x) - f(y)|}{\|x-y\|_1} \mid x, y \in \mathbb{R}^n,\ x \neq y \Big\}$$

denotes the (optimal) Lipschitz constant of a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$.

For a random field $(X(t))_{t\in\mathbb{R}^d}$, a finite subset $I = \{t_1, \ldots, t_n\} \subseteq \mathbb{R}^d$ with n elements and a function f on $\mathbb{R}^n$ we abbreviate $f(X_I) := f(X(t_1), \ldots, X(t_n))$. If such an abbreviation X_I appears more than once within one formula, then always the same enumeration of the elements of I has to be used.

For a set M let $\#M$ denote the number of elements of M.

Furthermore, for $\Delta > 0$ we put

$$T(\Delta) := \{(j_1/\Delta, \ldots, j_d/\Delta) \mid (j_1, \ldots, j_d) \in \mathbb{Z}^d\}.$$

Def. 2. Let $\theta = (\theta_r)_{r\in\mathbb{N}}$ be a monotonically decreasing sequence with $\lim_{r\to\infty} \theta_r = 0$.

(i) An $\mathbb{R}^s$-valued random field $(X(t))_{t\in\mathbb{R}^d}$ is called *$BL(\theta)$-dependent* if for any $\Delta > 1$ and any disjoint, finite sets $I, J \subseteq T(\Delta)$ with $\operatorname{dist}(I,J) \geq r$ and all bounded Lipschitz functions $f : \mathbb{R}^{s\cdot\#I} \to \mathbb{R}$ and $g : \mathbb{R}^{s\cdot\#J} \to \mathbb{R}$ we have
$$\operatorname{Cov}(f(X_I), g(X_J)) \leq \Psi(\#I, \#J, f, g)\Delta^d \theta_r.$$

(ii) An $\mathbb{R}^s$-valued random field $(X(t))_{t\in\mathbb{Z}^d}$ is called *$BL(\theta)$-dependent* if for any disjoint, finite sets $I, J \subseteq \mathbb{Z}^d$ with $\operatorname{dist}(I,J) \geq r$ and all bounded Lipschitz functions $f : \mathbb{R}^{s\cdot\#I} \to \mathbb{R}$ and $g : \mathbb{R}^{s\cdot\#J} \to \mathbb{R}$ we have
$$\operatorname{Cov}(f(X_I), g(X_J)) \leq \Psi(\#I, \#J, f, g)\theta_r.$$

Lemma 3. *Let $\theta = (\theta_r)_{r\in\mathbb{N}}$ be a monotonically decreasing sequence with $\lim_{r\to\infty} \theta_r = 0$. For $T = \mathbb{Z}^d$ or $T = \mathbb{R}^d$, let $(X^{(n)}(t))_{t\in T}, n \in \mathbb{N}$, be a sequence of $BL(\theta)$-dependent random fields such that the finite-dimensional distributions converge to those of a field $(X(t))_{t\in T}$. Then $(X(t))_{t\in T}$ is also $BL(\theta)$-dependent.*

Proof: By the definition of convergence in probability we get $\lim_{n\to\infty} \mathbb{E}\, f(X_I^{(n)}) g(X_J^{(n)}) = \mathbb{E}\, f(X_I) g(X_J)$, $\lim_{n\to\infty} \mathbb{E}\, f(X_I^{(n)}) = \mathbb{E}\, f(X_I)$ and $\lim_{n\to\infty} \mathbb{E}\, g(X_J^{(n)}) = \mathbb{E}\, g(X_J)$ for any finite sets $I, J \subseteq T$ and bounded Lipschitz continuous functions $f : \mathbb{R}^{\#I} \to \mathbb{R}$ and $g : \mathbb{R}^{\#J} \to \mathbb{R}$, which yields the assertion. □

Lemma 4. *Let $\theta = (\theta_r)_{r\in\mathbb{N}}$ be a monotonically decreasing sequence with $\lim_{r\to\infty} \theta_r = 0$. Let $(X(t))_{t\in T}$ be a $BL(\theta)$-dependent random field and let $f : \mathbb{R}^s \to \mathbb{R}^{s'}$ be a Lipschitz function. Then there is a monotonically decreasing sequence $\theta' = (\theta'_r)_{r\in\mathbb{N}}$ with $\lim_{r\to\infty} \theta'_r = 0$ such that $(f(X(t)))_{t\in T}$ is $BL(\theta')$-dependent.*

Proof: For a function $f : V \to W$ and a finite set I let f_I denote the function $V^{\#I} \to W^{\#I}$, $(x_1, \ldots, x_{\#I}) \mapsto (f(x_1), \ldots, f(x_{\#I}))$. We put $\theta'_r := \operatorname{Lip}(f)^2 \cdot \theta_r$. Let I, J be two disjoint finite sets with $\operatorname{dist}(I,J) \geq r$ and let $\tilde{f} : \mathbb{R}^{s'\cdot\#I} \to \mathbb{R}$ and $g : \mathbb{R}^{s'\cdot\#J} \to \mathbb{R}$ be two Lipschitz functions. Then

$$\begin{aligned}\operatorname{Cov}(\tilde{f}(f_I(X_I)), g(f_J(X_J))) &\leq \min\{\#I, \#J\} \cdot \operatorname{Lip}(\tilde{f} \circ f_I) \cdot \operatorname{Lip}(g \circ f_J) \cdot \Delta^d \cdot \theta_r \\ &\leq \min\{\#I, \#J\} \cdot \operatorname{Lip}(\tilde{f}) \cdot \operatorname{Lip}(g) \cdot \Delta^d \cdot \theta'_r. \qquad \square\end{aligned}$$

An $\mathbb{R}^s$-valued random field $(X(t))_{t\in T}$ is called *positively associated* (PA) if

$$\operatorname{Cov}(f(X_I), g(X_J)) \geq 0$$

for any finite sets $I, J \subseteq T$ and functions $f : \mathbb{R}^{s\cdot \#I} \to \mathbb{R}$ and $g : \mathbb{R}^{s\cdot \#J} \to \mathbb{R}$ which are bounded and monotonically increasing in every coordinate.

For a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ we define coordinate-wise Lipschitz constants by

$$\mathrm{Lip}_k(f) = \sup\Big\{ \frac{|f(x_1,\dots,x_{k-1},y_k,x_{k+1},\dots,x_n) - f(x_1,\dots,x_{k-1},z_k,x_{k+1},\dots,x_n)|}{|y_k - z_k|} \mid x_1,\dots,x_{k-1},x_{k+1},\dots,x_n,y_k,z_k \in \mathbb{R},\ y_k \neq z_k \Big\},\ \ k \in \{1,\dots,n\}.$$

An $\mathbb{R}^s$-valued random field $(X(t))_{t\in T}$ with $\mathbb{E}[X_k(t)^2] < \infty, t \in T, k = 1,\dots,s$, is called *quasi-associated* (QA) if

$$|\operatorname{Cov}(f(X_I), g(X_J))| \le \sum_{t\in I}\sum_{k=1}^{s}\sum_{u\in J}\sum_{l=1}^{s} \mathrm{Lip}_{t,k}(f)\cdot \mathrm{Lip}_{u,l}(g) |\operatorname{Cov}(X_k(t), X_l(u))|$$

for any finite sets $I, J \subseteq T$ and Lipschitz continuous functions $f : \mathbb{R}^{s\cdot\#I} \to \mathbb{R}$ and $g : \mathbb{R}^{s\cdot\#J} \to \mathbb{R}$.

It is well known that every PA random field is also QA, see e.g. Theorem 5.3 in [2, p. 89] (this theorem is only formulated in the special case $s = 1$ and $T = \mathbb{Z}^d$, but the proof holds in the present setting).

Lemma 5. *Let $(X(t))_{t\in\mathbb{R}^d}$ be an $\mathbb{R}^s$-valued QA random field. Assume that there are $c > 0$ and $\epsilon > 0$ with*

$$\operatorname{Cov}(X_i(t_1), X_j(t_2)) \le c \cdot \|t_1 - t_2\|_\infty^{-d-\epsilon}$$

for $t_1, t_2 \in \mathbb{R}^d$ and $i, j = 1,\dots,s$. Then $(X(t))_{t\in\mathbb{R}^d}$ is $BL(\theta)$-dependent for some monotonically decreasing zero sequence θ.

Proof: Let $r > 0$, $\Delta > 1$ and let $I, J \subseteq T(\Delta)$ be finite with $\operatorname{dist}(I, J) \ge r$, w.l.o.g. $\#I \le \#J$. Moreover, let $f : \mathbb{R}^{s\cdot\#I} \to \mathbb{R}$ and $g : \mathbb{R}^{s\cdot\#J} \to \mathbb{R}$ be bounded and Lipschitz continuous. Then

$$\begin{aligned}
\operatorname{Cov}(f(X_I), g(X_J)) &\le \sum_{t\in I}\sum_{k=1}^{s}\sum_{u\in J}\sum_{l=1}^{s} \mathrm{Lip}_{t,k}(f)\cdot \mathrm{Lip}_{u,l}(g) \operatorname{Cov}(X_k(t), X_l(u)) \\
&\le s^2 \cdot \#I \cdot \mathrm{Lip}(f) \cdot \mathrm{Lip}(g) \cdot \max_{t,k,l} \sum_{u\in J} \operatorname{Cov}(X_k(t), X_l(u)) \\
&\le s^2 \cdot \#I \cdot \mathrm{Lip}(f) \cdot \mathrm{Lip}(g) \cdot \max_{t\in I} \sum_{u\in J} c \cdot \|t-u\|_\infty^{-d-\epsilon}.
\end{aligned}$$

We have for fixed $t \in I$, if $r > 1$,

$$\begin{aligned}
\sum_{u\in J} \|t-u\|_\infty^{-d-\epsilon} &\le \sum_{s=\lceil r\Delta\rceil}^{\infty} \big(\frac{s}{\Delta}\big)^{-d-\epsilon} \cdot \#\{v \in T(\Delta) \mid \|v\|_\infty = \frac{s}{\Delta}\} \\
&= \sum_{s=\lceil r\Delta\rceil}^{\infty} \big(\frac{s}{\Delta}\big)^{-d-\epsilon} \cdot \big((2s+1)^d - (2s-1)^d\big) \\
&= \Delta^{d+\epsilon} \sum_{s=\lceil r\Delta\rceil}^{\infty} \sum_{\iota=0}^{d-1} s^{-d-\epsilon} \binom{d}{\iota} (1 + (-1)^{d-\iota-1})(2s)^\iota \\
&\le \Delta^{d+\epsilon} \int_{\lceil r\Delta\rceil-1}^{\infty} \sum_{\iota=0}^{d-1} \binom{d}{\iota} (1 + (-1)^{d-\iota-1}) 2^\iota s^{-d-\epsilon+\iota}\, ds \\
&\le \Delta^{d+\epsilon} \sum_{\iota=0}^{d-1} \binom{d}{\iota} (1 + (-1)^{d-\iota-1}) 2^\iota \frac{(r\Delta - \Delta)^{-d-\epsilon+\iota+1}}{d+\epsilon-\iota-1} \\
&\le \Delta^{d} \sum_{\iota=0}^{d-1} \binom{d}{\iota} (1 + (-1)^{d-\iota-1}) 2^\iota \frac{(r-1)^{-d-\epsilon+\iota+1}}{d+\epsilon-\iota-1}.
\end{aligned}$$

Putting

$$\theta_r := \begin{cases} c \cdot s^2 \sum_{\iota=0}^{d-1} \binom{d}{\iota}(1+(-1)^{d-\iota-1})2^\iota \frac{(r-1)^{-d-\epsilon+\iota+1}}{d+\epsilon-\iota-1} & \text{for } r>1, \\ \frac{3^d}{\Delta^d} \cdot \max_{i=1,\dots,s} \operatorname{Var}(X_i(0)) + \theta_2 & \text{for } r=1, \end{cases}$$

we obtain

$$\operatorname{Cov}(f(X_I), g(X_J)) \le \min\{\#I, \#J\} \cdot \operatorname{Lip}(f) \cdot \operatorname{Lip}(g) \cdot \Delta^d \theta_r. \quad \square$$

2.2 Random fields

After having introduced the association concepts in subsection 2.1, we will now collect various other preliminaries concerning random fields.

The following theorem (see [6, Ch. III, § 3] and [12, Prop. 3.1]) says that for stationary random fields stochastic continuity and measurability are essentially equivalent.

Theorem 6. *(i) Let $(X(t))_{t\in\mathbb{R}^d}$ be a stochastically continuous random field. Then there is a measurable modification of $(X(t))_{t\in\mathbb{R}^d}$.*

(ii) Let $(X(t))_{t\in\mathbb{R}^d}$ be a stationary and measurable random field. Then $(X(t))_{t\in\mathbb{R}^d}$ is stochastically continuous.

Lemma 7. *Let $(X_t)_{t\in\mathbb{R}^d}$ be a stationary, stochastically continuous random field with $\mathbb{E}\, X(0)^j < \infty$ for $j > 0$. Then $(X_t)_{t\in\mathbb{R}^d}$ is continuous in j-mean.*

Proof: Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d$ converging to a point $t \in \mathbb{R}^d$. Then

$$\lim_{n\to\infty} \mathbb{E}\,|X(t_n)-X(t)|^j = \lim_{n\to\infty} \int_0^\infty \mathbb{P}(|X(t_n)-X(t)|^j \ge x)\, dx = \int_0^\infty \lim_{n\to\infty} \mathbb{P}(|X(t_n)-X(t)| \ge \sqrt[j]{x})\, dx = 0.$$

We have been allowed to interchange limit and integral, since

$$\mathbb{P}(|X(t_n)-X(t)| \ge \sqrt[j]{x}) \le \mathbb{P}(|X(t_n)| \ge \tfrac{\sqrt[j]{x}}{2}) + \mathbb{P}(|X(t)| \ge \tfrac{\sqrt[j]{x}}{2}) = 2\mathbb{P}(|X(t)| \ge \tfrac{\sqrt[j]{x}}{2})$$

due to the stationarity and

$$\int_0^\infty 2\mathbb{P}(|X(t)| \ge \tfrac{\sqrt[j]{x}}{2})\, dx = \int_0^\infty 2\mathbb{P}(|2\cdot X(t)|^j \ge x)\, dx = 2^{j+1}\mathbb{E}\,|X(0)|^j < \infty. \quad \square$$

Lemma 8. *Let $(X(t))_{t\in\mathbb{R}^d}$ and $(Y(t))_{t\in\mathbb{R}^d}$ be two stochastically continuous and measurable random fields having the same distribution. Let $A_1, \dots, A_m \subseteq \mathbb{R}^d$ be bounded Borel sets. Assume that $\int_{A_i} X(t)\, dt$ is defined a.s. for $i = 1, \dots, m$, i.e. not both the positive part and the negative part of these integrals are infinite. Then $\int_{A_1} Y(t)\, dt, \dots, \int_{A_m} Y(t)\, dt$ are defined a.s. as well and*

$$\Big(\int_{A_1} X(t)\, dt, \dots, \int_{A_m} X(t)\, dt\Big) \overset{d}{=} \Big(\int_{A_1} Y(t)\, dt, \dots, \int_{A_m} Y(t)\, dt\Big).$$

Proof: By the Monotone Convergence Theorem, we may assume w.l.o.g. that there is some $N \in \mathbb{N}$ such that $X(t) \in [-N, N]$ and $Y(t) \in [-N, N]$ for all $t \in \mathbb{R}^d$.

We define processes $(X^n(t))_{t\in\mathbb{R}^d}$ and $(Y^n(t))_{t\in\mathbb{R}^d}$ by putting

$$X^n(t_1, \dots, t_d) := X\Big(\frac{z_1}{n}, \dots, \frac{z_d}{n}\Big), \quad \text{for all } t_1 \in \Big[\frac{z_1}{n}, \frac{z_1+1}{n}\Big), \dots, t_d \in \Big[\frac{z_d}{n}, \frac{z_d+1}{n}\Big),\ z_1, \dots, z_d \in \mathbb{Z}.$$

We get

$$\Big\{ \int_{A_i} X^n(t)\, dt \Big| i = 1, \dots, m \Big\} \tag{1}$$

$$= \Big\{ \sum_{z_1,\dots,z_d\in\mathbb{Z}} \lambda_d\Big(A_i \cap [\frac{z_1}{n}, \frac{z_1+1}{n}) \times \cdots \times [\frac{z_d}{n}, \frac{z_d+1}{n})\Big) X\big(\frac{z_1}{n}, \dots, \frac{z_d}{n}\big) \Big| i = 1, \dots, m \Big\}$$

$$\overset{d}{=} \Big\{ \sum_{z_1,\dots,z_d\in\mathbb{Z}} \lambda_d\Big(A_i \cap [\frac{z_1}{n}, \frac{z_1+1}{n}) \times \cdots \times [\frac{z_d}{n}, \frac{z_d+1}{n})\Big) Y\big(\frac{z_1}{n}, \dots, \frac{z_d}{n}\big) \Big| i = 1, \dots, m \Big\}$$

$$= \Big\{ \int_{A_i} Y^n(t)\, dt \Big| i = 1, \dots, m \Big\}. \tag{2}$$

For $\epsilon, \delta > 0$ we get

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^m \Big|\int_{A_i} X^n(t)\,dt - \int_{A_i} X(t)\,dt\Big| > \epsilon\Big) &\le \mathbb{P}\Big(\sum_{i=1}^m \int_{A_i} |X^n(t) - X(t)|\,dt > \epsilon\Big)\\ &\le \frac{\mathbb{E}\sum_{i=1}^m \int_{A_i} |X^n(t) - X(t)|\,dt}{\epsilon}\\ &= \frac{\sum_{i=1}^m \int_{A_i} \mathbb{E}\,|X^n(t) - X(t)|\,dt}{\epsilon}\\ &\le \frac{\sum_{i=1}^m \int_{A_i} \big(\delta + \mathbb{P}(|X^n(t) - X(t)| > \delta)\cdot 2N\big)\,dt}{\epsilon}\\ &\overset{n\to\infty}{\longrightarrow} \frac{\sum_{i=1}^m \int_{A_i} \delta\,dt}{\epsilon}\\ &= \frac{\delta \cdot \sum_{i=1}^m \lambda_d(A_i)}{\epsilon}.\end{aligned}$$

The limit relation holds by the Majorized Convergence Theorem, since the assumption that $(X(t))_{t\in\mathbb{R}}$ is stochastically continuous implies that $X^n(t)$ converges to $X(t)$. Since $\delta > 0$ was arbitrary, we get

$$\mathbb{P}\Big(\sum_{i=1}^m \Big|\int_{A_i} X^n(t)\,dt - \int_{A_i} X(t)\,dt\Big| > \epsilon\Big) \overset{n\to\infty}{\longrightarrow} 0$$

and the same way

$$\mathbb{P}\Big(\sum_{i=1}^m \Big|\int_{A_i} Y^n(t)\,dt - \int_{A_i} Y(t)\,dt\Big| > \epsilon\Big) \overset{n\to\infty}{\longrightarrow} 0.$$

Now (2) yields the assertion. □

2.3 Functions of bounded variation

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be of *locally bounded variation* if there is a monotonically increasing function $\alpha : \mathbb{R} \to \mathbb{R}$ and a monotonically decreasing function $\beta : \mathbb{R} \to \mathbb{R}$ such that $f = \alpha + \beta$. We denote the set of such functions α and β by A resp. B. We put

$$f^+(x) := \begin{cases} \inf\{\alpha(x) \mid \alpha \in A,\, \alpha(0) = f(0)\} & \text{if } x > 0\\ f(0) & \text{if } x = 0\\ \sup\{\alpha(x) \mid \alpha \in A,\, \alpha(0) = f(0)\} & \text{if } x < 0.\end{cases}$$

It is easy to see that $f^+ \in A$ and $f^- := f - f^+ \in B$. We put $h_f := f^+ - f^-$.

Lemma 9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of locally bounded variation. Then $f = g \circ h_f$ for a Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$ of Lipschitz constant* 1.

Proof: For each $x \in \mathbb{R}$, for which there is $t \in \mathbb{R}$ with $h_f(t) = x$, define $g(x) := f(t)$. Now g is well-defined, since for $t_1, t_2 \in \mathbb{R}$ with $h_f(t_1) = h_f(t_2)$, f is constant on $[t_1, t_2]$. Clearly, $f = g \circ h_f$. Moreover, g -defined on a subset of $\mathbb{R}$ so far- is Lipschitz continuous with Lipschitz constant 1. Indeed, let $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, be two points for which there are $t_1, t_2 \in \mathbb{R}$ with $h_f(t_1) = x_1$ and $h_f(t_2) = x_2$. Then

$$h_f(t_2) - h_f(t_1) = f^+(t_2) - f^+(t_1) - (f^-(t_2) - f^-(t_1)) \ge |f^+(t_2) - f^+(t_1) + (f^-(t_2) - f^-(t_1))| = |f(t_2) - f(t_1)|.$$

Hence $x_2 - x_1 \ge |g(x_2) - g(x_1)|$.

It remains to show that g has a Lipschitz continuous extension to the whole of $\mathbb{R}$. The domain of g is $\mathbb{R}$ minus the union of countable many, disjoint intervals. For a point x lying on the boundary of the domain of g but not in the domain of g, choose a sequence $(x_n)_{n\in\mathbb{N}}$ such that $g(x_n)$ is defined for all $n \in \mathbb{N}$. Then $(g(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, since g is Lipschitz continuous, and hence convergent. Since $(g(x_n))_{n\in\mathbb{N}}$ is convergent for *every* such sequence $(x_n)_{n\in\mathbb{N}}$, the limit is independent of the choice of the sequence. So we can put $g(x) := \lim_{n\to\infty} g(x_n)$. It is easy to see that this extension still has Lipschitz constant 1. Now all gaps in the domain of g are *open* intervals. So they can be filled by affine functions. Clearly, the Lipschitz constant is preserved again. □

3 The univariate CLT

In this section we will prove Theorem 1.
Proof: For $j = (j_1, \dots, j_d) \in \mathbb{Z}^d$ we put $Q_j = \times_{i=1}^d [j_i, j_i + 1)$ and $Z(j) := \int_{Q_j} X(t)\,dt - \mathbb{E}\,X(0)$. We will show that this random field $(Z(j))_{j\in\mathbb{Z}^d}$ fulfills the assumptions of Theorem 1.12 of [2, p. 178]. The collection

$$Z_n(j) := \frac{1}{n^d} \sum_{k_1,\dots,k_d=1}^{n} X(j_1 + \tfrac{k_1}{n}, \dots, j_d + \tfrac{k_d}{n}) - \mathbb{E}\,X(0),\ j \in \mathbb{Z}^d,$$

is $BL(\theta')$-dependent for any $n \in \mathbb{N}$, where $\theta'_r := \theta_{r-d}$. Indeed, let $I, J \subseteq \mathbb{Z}^d$ and let $f : \mathbb{R}^{\#I} \to \mathbb{R}$ and $g : \mathbb{R}^{\#J} \to \mathbb{R}$ be bounded Lipschitz functions. Put $\tilde{I} = I + \{1/n, 2/n, \dots, 1\}^d$, $\tilde{J} = J + \{1/n, 2/n, \dots, 1\}^d$,

$$\tilde{f} : \mathbb{R}^{\#I\cdot n^d} \to \mathbb{R},\ (x_{1,1}, \dots, x_{\#I,n^d}) \to f\Big(\frac{1}{n^d}\sum_{\ell=1}^{n^d} x_{1,\ell} - \mathbb{E}X(0), \dots, \frac{1}{n^d}\sum_{\ell=1}^{n^d} x_{\#I,\ell} - \mathbb{E}X(0)\Big)$$

and

$$\tilde{g} : \mathbb{R}^{\#J\cdot n^d} \to \mathbb{R},\ (x_{1,1}, \dots, x_{\#J,n^d}) \to g\Big(\frac{1}{n^d}\sum_{\ell=1}^{n^d} x_{1,\ell} - \mathbb{E}X(0), \dots, \frac{1}{n^d}\sum_{\ell=1}^{n^d} x_{\#J,\ell} - \mathbb{E}X(0)\Big).$$

Then we have $f(Z_{n,I}) = \tilde{f}(X_{\tilde{I}})$, $g(Z_{n,J}) = \tilde{g}(X_{\tilde{J}})$, $\operatorname{Lip}(\tilde{f}) = \operatorname{Lip}(f)/n^d$, $\operatorname{Lip}(\tilde{g}) = \operatorname{Lip}(g)/n^d$ and $\operatorname{dist}(\tilde{I}, \tilde{J}) \geq \operatorname{dist}(I, J) - d$. So

$$\begin{aligned}
\operatorname{Cov}(f(Z_{n,I}), g(Z_{n,J})) &= \operatorname{Cov}(\tilde{f}(X_{\tilde{I}}), \tilde{g}(X_{\tilde{J}})) \\
&\leq \min\{\#I \cdot n^d, \#J \cdot n^d\} \operatorname{Lip}(\tilde{f}) \operatorname{Lip}(\tilde{g}) n^d \theta_{r-d} \\
&= \min\{\#I, \#J\} \operatorname{Lip}(f) \operatorname{Lip}(g) \theta'_r.
\end{aligned}$$

By Lemma 3, the field $(Z(j))_{j\in\mathbb{Z}^d}$ is $BL(\theta')$-dependent if we can show that the finite-dimensional distributions of $(Z_n(j))_{j\in\mathbb{Z}^d}$ converge to those of $(Z(j))_{j\in\mathbb{Z}^d}$. First we will show

$$\lim_{n\to\infty} \mathbb{E}\,|Z_n(j) - Z(j)| = 0, \quad j \in \mathbb{Z}^d. \tag{3}$$

Let $\epsilon > 0$. Due to Lemma 7, the field $(X(t))_{t\in\mathbb{R}^d}$ is continuous in 1-mean and hence there is n such that

$$\mathbb{E}\,|X(0) - X(t)| < \epsilon \text{ for all } t \in [0, \tfrac{1}{n}]^d.$$

Since $(X_t)_{t\in\mathbb{R}^d}$ is stationary, this implies

$$\mathbb{E}\,|X(j_1 + \tfrac{k_1}{n}, \dots, j_d + \tfrac{k_d}{n}) - X(t)| < \epsilon \text{ for all } t \in [j_1 + \tfrac{k_1-1}{n}, j_1 + \tfrac{k_1}{n}] \times \dots \times [j_d + \tfrac{k_d-1}{n}, j_d + \tfrac{k_d}{n}].$$

Hence $\mathbb{E}\,|Z_n(j) - Z(j)| < \epsilon$, which finishes the proof of (3).

Now let $j^{(1)}, \dots, j^{(r)} \in \mathbb{Z}^d$ and let $\delta > 0$. From Markov's inequality we get

$$\mathbb{P}\Big(\sum_{l=1}^{r} |Z_n(j^{(l)}) - Z(j^{(l)})| > \delta\Big) \leq \frac{\sum_{l=1}^r \mathbb{E}\,|Z_n(j^{(l)}) - Z(j^{(l)})|}{\delta} \to 0, \quad n \to \infty.$$

So the finite-dimensional distributions of $(Z_n(j))_{j\in\mathbb{Z}^d}$ converge to those of $(Z(j))_{j\in\mathbb{Z}^d}$ and hence $(Z(j))_{j\in\mathbb{Z}^d}$ is $BL(\theta)$-dependent.

By Lemma 8, the assumption that $(X(t))_{t\in\mathbb{R}^d}$ is stationary implies that $(Z(j))_{j\in\mathbb{Z}^d}$ is stationary. Moreover, $(Z(j))_{j\in\mathbb{Z}^d}$ is centered, since

$$\mathbb{E}\,Z(0) = \mathbb{E}\int_{[0,1)^d} X(t)\,dt - \mathbb{E}\,X(0) = \int_{[0,1)^d} \mathbb{E}\,X(t)\,dt - \mathbb{E}\,X(0) = 0.$$

Further,

$$\begin{aligned}\sum_{j\in\mathbb{Z}^d} \operatorname{Cov}\big(Z(0),Z(j)\big) &= \sum_{j\in\mathbb{Z}^d}\int_{[0,1)^d}\int_{j+[0,1)^d} \operatorname{Cov}\big(X(s),X(t)\big)\,dt\,ds\\ &= \int_{[0,1)^d}\int_{\mathbb{R}^d} \operatorname{Cov}\big(X(0),X(t-s)\big)\,dt\,ds\\ &= \int_{[0,1)^d}\int_{\mathbb{R}^d} \operatorname{Cov}\big(X(0),X(t)\big)\,dt\,ds\\ &= \int_{\mathbb{R}^d} \operatorname{Cov}\big(X(0),X(t)\big)\,dt.\end{aligned}$$

We put $Q_n := \{j \in \mathbb{Z}^d \mid j+[0,1)^d \subseteq W_n\}$ and $W_n^- := \bigcup_{j\in Q_n}\big(j+[0,1)^d\big)$. As explained in the proof of [3, Theorem 1.2], the assumption that $(W_n)_{n\in\mathbb{N}}$ is VH-growing implies that $(Q_n)_{n\in\mathbb{N}}$ is regular growing. Now Theorem 1.12 of [2, p. 178] implies that

$$\frac{\int_{W_n^-} X(t)\,dt - \lambda_d(W_n^-)\mathbb{E}\,X(0)}{\sqrt{\lambda_d(W_n^-)}} = \frac{\sum_{j\in Q_n} Z(j)}{\sqrt{\#Q_n}} \to \mathcal{N}(0,\sigma^2), \quad n\to\infty.$$

If we can show that

$$\frac{\int_{W_n\setminus W_n^-} X(t)\,dt - \lambda_d(W_n\setminus W_n^-)\mathbb{E}\,X(0)}{\sqrt{\lambda_d(W_n)}} \xrightarrow{P} 0, \quad n\to\infty, \tag{4}$$

then Slutzki's theorem will imply the assertion, since, clearly, $\sqrt{\lambda_d(W_n^-)}/\sqrt{\lambda_d(W_n)} \to 1$. We get

$$\begin{aligned}\operatorname{Var}\Big(\int_{W_n\setminus W_n^-} X(t)\,dt\Big) &= \int_{W_n\setminus W_n^-}\int_{W_n\setminus W_n^-} \operatorname{Cov}(X(s),X(t))\,dt\,ds\\ &\le \int_{W_n\setminus W_n^-}\int_{\mathbb{R}^d} |\operatorname{Cov}(X(s),X(t))|\,dt\,ds\\ &= \lambda_d(W_n\setminus W_n^-)\int_{\mathbb{R}^d} |\operatorname{Cov}(X(0),X(t))|\,dt.\end{aligned}$$

Since $(W_n)_{n\in\mathbb{N}}$ is VH-growing, we get

$$\operatorname{Var}\Big(\frac{\int_{W_n\setminus W_n^-} X(t)\,dt}{\sqrt{\lambda_d(W_n)}}\Big) = \frac{\operatorname{Var}\big(\int_{W_n\setminus W_n^-} X(t)\,dt\big)}{\lambda_d(W_n)} \to 0, \quad n\to\infty.$$

By the Chebyshev inequality this implies (4). □

4 The multivariate CLT

In this section we extend Theorem 1 in various ways to multivariate central limit theorems.

Theorem 10. *Let $\theta = (\theta_r)_{r\in\mathbb{N}}$ be a monotonically decreasing zero sequence. Let $(X(t))_{t\in\mathbb{R}^d}$ be an $\mathbb{R}^s$-valued random field. Assume that $(X(t))_{t\in\mathbb{R}^d}$ is stationary, measurable, BL(θ)-dependent and fulfills*

$$\int_{\mathbb{R}^d} |\operatorname{Cov}(X_i(0),X_j(t))|\,dt < \infty, \quad i,j=1,\dots,s.$$

Let $(W_n)_{n\in\mathbb{N}}$ be a VH-growing sequence of subsets of $\mathbb{R}^d$. Then

$$\Big(\frac{\int_{W_n} X_1(t)\,dt - \mathbb{E}\,X_1(0)\lambda_d(W_n)}{\sqrt{\lambda_d(W_n)}},\dots,\frac{\int_{W_n} X_s(t)\,dt - \mathbb{E}\,X_s(0)\lambda_d(W_n)}{\sqrt{\lambda_d(W_n)}}\Big) \to \mathcal{N}(0,\Sigma), \quad n\to\infty,$$

in distribution, where Σ is the matrix with entries

$$\int_{\mathbb{R}^d} \operatorname{Cov}(X_i(0),X_j(t))\,dt, \quad i,j=1,\dots,s.$$

Proof: Let $u = (u_1, \dots, u_s) \in \mathbb{R}^s$. Then $(\langle X(t), u\rangle)_{t\in\mathbb{R}^d}$ is $BL(\theta')$-dependent for a monotonically decreasing sequence $\theta' = (\theta'_r)_{r\in\mathbb{N}}$ with $\lim_{r\to\infty}\theta'_r = 0$ due to Lemma 4. Obviously, $(\langle X(t), u\rangle)_{t\in\mathbb{R}^d}$ is stationary and measurable. We have

$$\int_{\mathbb{R}^d} \mathrm{Cov}(\langle X(0), u\rangle, \langle X(t), u\rangle)\, dt = \sum_{i=1}^{s}\sum_{j=1}^{s} u_i u_j \int_{\mathbb{R}^d} \mathrm{Cov}(X_i(0), X_j(t))\, dt = u^T \Sigma u.$$

In particular, the integral is defined. So Theorem 1 implies

$$\Big\langle \Big(\frac{\int_{W_n} X_1(t)\, dt - \mathbb{E}\, X_1(0)\lambda_d(W_n)}{\sqrt{\lambda_d(W_n)}}, \dots, \frac{\int_{W_n} X_s(t)\, dt - \mathbb{E}\, X_s(0)\lambda_d(W_n)}{\sqrt{\lambda_d(W_n)}} \Big), u \Big\rangle$$
$$= \frac{\int_{W_n} \langle X(t), u\rangle\, dt - \mathbb{E}\,\langle X(0), u\rangle \lambda_d(W_n)}{\sqrt{\lambda_d(W_n)}} \to \mathcal{N}(0, u^T\Sigma u), \quad n\to\infty.$$

Since $\langle Y, u\rangle \sim \mathcal{N}(0, u^T\Sigma u)$ for a random vector $Y \sim \mathcal{N}(0,\Sigma)$, the Theorem of Cramér and Wold implies the assertion. □

Corollary 11. *Let $(X(t))_{t\in\mathbb{R}^d}$ be a stationary, measurable $\mathbb{R}$-valued random field and let $f_1, \dots, f_s : \mathbb{R}\to\mathbb{R}$ be functions. Let $(W_n)_{n\in\mathbb{N}}$ be a VH-growing sequence of subsets of $\mathbb{R}^d$. Assume that one of the following conditions holds:*

(i) The field $(X(t))_{t\in\mathbb{R}^d}$ is $BL(\theta)$-dependent for a monotonically decreasing zero sequence $\theta = (\theta_r)_{r\in\mathbb{N}}$, the maps $f_1, \dots, f_s$ are Lipschitz continuous and

$$\int_{\mathbb{R}^d} \big| \mathrm{Cov}\left(f_i(X(0)), f_j(X(t))\right)\big|\, dt < \infty, \quad i,j = 1,\dots,s.$$

(ii) The field $(X(t))_{t\in\mathbb{R}^d}$ is QA and there are $c>0$ and $\epsilon>0$ with

$$\mathrm{Cov}(X(0), X(t)) \le c\cdot \|t\|_\infty^{-d-\epsilon}, \quad t\in\mathbb{R}^d. \tag{5}$$

The maps $f_1, \dots, f_s$ are Lipschitz continuous.

(iii) The field $(X(t))_{t\in\mathbb{R}^d}$ is PA with $\mathbb{E}\, X(0)^2 < \infty$. The maps $f_1,\dots,f_s$ are of locally bounded variation with $\mathbb{E}[h_{f_i}(X(0))^2] < \infty$, $i = 1,\dots,s$, and there are $c>0$ and $\epsilon>0$ with

$$\mathrm{Cov}\left(h_{f_i}(X(0)), h_{f_j}(X(t))\right) \le c\cdot\|t\|_\infty^{-d-\epsilon}, \qquad t\in\mathbb{R}^d,\ i,j=1,\dots,s. \tag{6}$$

Then

$$\Big(\frac{\int_{W_n} f_1(X(t))\, dt - \mathbb{E}\, f_1(X(0))\lambda_d(W_n)}{\sqrt{\lambda_d(W_n)}}, \dots, \frac{\int_{W_n} f_s(X(t))\, dt - \mathbb{E}\, f_s(X(0))\lambda_d(W_n)}{\sqrt{\lambda_d(W_n)}}\Big) \to \mathcal{N}(0,\Sigma),$$

as $n\to\infty$ in distribution, where Σ is the matrix with entries

$$\int_{\mathbb{R}^d} \mathrm{Cov}\left(f_i(X(0)), f_j(X(t))\right) dt, \quad i,j=1,\dots,s.$$

Part (i) of this corollary is an immediate consequence of Lemma 4 and Theorem 10.

Proof of Corollary 11(ii): The field $(X(t))_{t\in\mathbb{R}^d}$ is $BL(\theta)$-dependent by Lemma 5 and thus Lemma 4 implies that the field $(f_1(X(t)), \dots, f_s(X(t)))_{t\in\mathbb{R}^d}$ is also $BL(\theta)$-dependent.

In order to check the integrability assumptions from part (i), we put

$$f_j^{(N)} : x \mapsto \begin{cases} -N & \text{if } f_j(x) < -N \\ f_j(x) & \text{if } f_j(x)\in[-N,N] \\ N & \text{if } f_j(x) > N. \end{cases}$$

Since $(X(t))_{t\in\mathbb{R}^d}$ is QA, we get

$$\begin{aligned}|\operatorname{Cov}\left(f_i^{(N)}(X(0)), f_j^{(N)}(X(t))\right)| &\le \operatorname{Lip}(f_i^{(N)}) \cdot \operatorname{Lip}(f_j^{(N)}) \cdot |\operatorname{Cov}(X(0), X(t))| \\ &\le \operatorname{Lip}(f_i) \cdot \operatorname{Lip}(f_j) \cdot |\operatorname{Cov}(X(0), X(t))|.\end{aligned}$$

By the Monotone Convergence Theorem, applied to both summands of $\mathbb{E}[f_i^{(N)}(X(0))f_j^{(N)}(X(t))] - \mathbb{E}[f_i^{(N)}(X(0))] \cdot \mathbb{E}[f_j^{(N)}(X(t))]$, this yields

$$|\operatorname{Cov}\left(f_i(X(0)), f_j(X(t))\right)| \le \operatorname{Lip}(f_i) \cdot \operatorname{Lip}(f_j) \cdot |\operatorname{Cov}(X(0), X(t))|.$$

Moreover, (5) implies

$$\int_{\mathbb{R}^d} \left|\operatorname{Cov}\left(X(0), X(t)\right)\right| dt < \infty$$

and hence

$$\int_{\mathbb{R}^d} \left|\operatorname{Cov}\left(f_i(X(0)), f_j(X(t))\right)\right| dt < \infty, \quad i, j = 1, \ldots, s.$$

So part (i) yields the assertion.

□

Proof of Corollary 11(iii): Since $(X(t))_{t\in\mathbb{R}^d}$ is PA, the random field $(h_{f_1}(X(t)), \ldots, h_{f_s}(X(t)))_{t\in\mathbb{R}^d}$ is also PA, see Theorem 1.8(d) of [2, p. 7], and therefore QA. By Lemma 5 it is $BL(\theta)$-dependent for some monotonically decreasing zero sequence θ. Hence $(f_1(X(t)), \ldots, f_s(X(t)))_{t\in\mathbb{R}^d}$ is $BL(\theta')$-dependent for some monotonically decreasing zero sequence θ' by Lemma 9 and Lemma 4.

Clearly, the field $(f_1(X(t)), \ldots, f_s(X(t)))_{t\in\mathbb{R}^d}$ is also stationary and measurable.

Moreover, (6) implies

$$\int_{\mathbb{R}^d} \left|\operatorname{Cov}\left(h_{f_i}(X(0)), h_{f_j}(X(t))\right)\right| dt < \infty, \quad i, j = 1, \ldots, s.$$

Now Lemma 9 and the QA property of $(h_{f_1}(X(t)), \ldots, h_{f_s}(X(t)))_{t\in\mathbb{R}^d}$ give

$$\int_{\mathbb{R}^d} \left|\operatorname{Cov}\left(f_i(X(0)), f_j(X(t))\right)\right| dt \le \int_{\mathbb{R}^d} \left|\operatorname{Cov}\left(h_{f_i}(X(0)), h_{f_j}(X(t))\right)\right| dt < \infty, \quad i, j = 1, \ldots, s.$$

So Theorem 10 yields the assertion. □

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