A comparative study of $2d$ Ising model at different boundary conditions using Cellular Automata

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Abstract

Using Cellular Automata, we simulate spin systems corresponding to $2d$ Ising model with various kinds of boundary conditions (bcs). The appearance of spontaneous magnetization in the absence of magnetic field is studied with a 64×64 square lattice with five different bcs, i.e., periodic, adiabatic, reflexive, fixed ($+1$ or -1) bcs with three initial conditions (all spins up, all spins down and random orientation of spins). In the context of $2d$ Ising model, we have calculated the magnetisation, energy, specific heat, susceptibility and entropy with each of the bcs and observed that the phase transition occurs around $T_c = 2.269$ as obtained by Onsager. We compare the behaviour of magnetisation vs temperature for different types of bcs by calculating the number of points close to the line of zero magnetisation after $T > T_c$ at various lattice sizes. We observe that the periodic, adiabatic and reflexive bcs give closer approximation to the value of T_c than fixed $+1$ and fixed -1 bcs with all three initial conditions for lattice size less than 70×70. However, for lattice size between 70×70 and 100×100, fixed $+1$ bc and fixed -1 bc give closer approximation to the T_c with initial conditions all spin up configuration and all spin down configuration respectively.

1 Introduction

The phenomenon of magnetism belongs to one of the oldest observations in nature which is yet to be understood at a fundamental level. One remarkable effect is the appearance of spontaneous magnetization giving rise to ferromagnetism when certain materials are cooled down below a critical temperature called Curie temperature in the absence of any external applied magnetic field. The $2d$ Ising model is represented by a square lattice of particles, each carrying one of the two spins states with magnetic moments ± 1. Each particle at a node is assigned a definite orientation. Spins of these particles cause a magnetic field whose strength decreases with increase in distance in the lattice. For simplification, we consider only the nearest neighbour interaction i.e., no other particle is located closer to one of them. In $2d$ Ising model, an ordinary particle has four nearest neighbours at east, west, north and south direction of the particle. These spin interactions contribute to the energy of the whole system. The energy of a spin configuration $s = \{s_{ij}, s_{ij} \in \{+1, -1\}, i, j = 1, \ldots, N\}$, with N as the order of sqaure lattice is given by the Hamiltonian

$$H(s) = -\sum_{i,j=1}^{N} J_{ij} s_{ij}(s_{ij-1} + s_{ij+1} + s_{i-1j} + s_{i+1j}) - \mu \sum_{kl} H_{kl} s_{kl} \qquad (1.1)$$

Where J_{ij} is the exchange interaction among s_{ij} with their four neigbours, μ is the magnetic moment and H_{kl} is the external magnetic field at $(kl)^{th}$ spin. For simulation purpose, we have to define a finite system with $N^2 < \infty$. We study different bcs under which the interaction energy will be maximum. In a periodic bc, the matrix J_{ij} defines a nearest neighbourhood topology of a loop and for other bcs nearest neighborhood topology is a square of a square lattice. A $2d$ Ising model with N^2 particles has 2^{N^2} spin configurations.

The partition function in Boltzmann statistics is given by

$$Z_\beta = \sum_s e^{-\beta H(s)} \qquad (1.2)$$

where $\beta = \frac{1}{k_B T}$. If we consider a 8×8 lattice, the space of states s has 2^{64} elements and it is a daunting task to compute Z_β. To find a concise formula for Z_β, the thermodynamic limit $N \to \infty$ is considered in analytical calculation. Basing on the tranfer matrix method with pbc, Onsager has solved the $2d$ Ising model [1]. Kotecky et. al [2] have studied magnetization of the Ising model under minus fixed bc. Still the $2d$ Ising model with other bcs are yet to be solved. So, here we consider five bcs to simulate $2d$ Ising model.

Explicit formulation of the spontaneous magnetization of the $2d$ Ising model with pbc for $N \to \infty$ was carried out in reference [3] and the magnetization m_β in terms β is found to be,

$$m_\beta = \begin{cases} (1 - (\sinh(2J\beta))^{-4})^{1/8}, & \beta > \beta_c \\ 0, & \beta \leq \beta_c \end{cases}$$

where

$$\begin{aligned} \beta_c &= \frac{\log(1 + \sqrt{2})}{2J} \\ \Rightarrow T_c &= 2.269. \end{aligned}$$

where $k_B = 1$ and $J = 1$ for a ferromagnetic substance.

Magnetisation in terms of T and T_c is given by,

$$m = \begin{cases} (1 - (\sinh(\log(1 + \sqrt{2})\frac{T_c}{T})^{-4})^{1/8}, & T < T_c \\ 0, & T \geq T_c \end{cases} \tag{1.3}$$

From *equation 3*, it is seen that magnetisation has at least two different possible directions and the average magnetization is zero in the absence of external magnetic field at $T > T_c$. We consider the above theory to compare among different bcs.

Cellular Automaton is a mathematical model in which the state of a cell interact with neighbours and then updates the state according to a specific rule in $2d$ CA [4]. This transition rule depends on the problem on which one is interested. While dealing with different dimensions, CA models are categorised as $1d$, $2d$, $3d$ CA etc. In $2d$ CA, cells may be square, trianglular, hexagonal, polygon type. State of the cell is given in terms of any finite number. The number of neighbours depend on the dimesion and the specific approach to the problem. To simulate the Ising model, we can create a 2 state CA, for spin up state $(+1)$ and spin down state (-1). For $1d$ model, we can consider two or four neighborhoods, for $2d$ we can consider four, six or eight neighborhoods and for $3d$ we can consider six or twenty six neighborhoods [4]. Both the CA model and the Ising model have similar characteristics. However, in Ising model case, before T_c states of the cells are either all in up state or all in down state and after T_c, the net magnetisation becomes zero and the pattern become random (half of $+1$ and half of -1 spins). So, it is a big challenge to find a specific rule in CA that satisfies the above behaviour of the Ising model.

Numerical methods like Markov chain, Metropolis [5], Spin-Flipping, Wolff algorithm [6] take a lot of time to simulate the Ising model. Monte Carlo is one of the simulation methods which has been widely used for studying Ising models [7]. Lot of work has been done for mapping Ising models using CA. A deterministic CA (DCA) is mostly used for this purpose. Domany and Kinzel [8] modelled a DCA in triangular lattice with conditional probabilities as the transition rule to map Ising problems in

two dimensions with representation of directed percolation. The so called Q2R CA [9] is a deterministic, reversible, nonergodic and fast method that is used for the microcanonical Ising model. Many authors have produced results based on this model [10, 11, 12]. The Creutz CA [13] has simulated the $2d$ Ising model successfully near the critical region under periodic bc and using this Creutz CA, the Ising model simulations in higher dimensions e.g., in $3d$ [14], $4d$ [15], $8d$ [16] have been done. Although the Q2R and Creutz CA models are deterministic and fast, it has been demonstrated that the probabilistic model of the CA like Metropolis algorithm is more realistic for description of the Ising model even though the random number generation makes it slower. Probabilistic CA model under periodic bc is applied to anisotropic-layer Ising and Potts models to find the critical point and shift exponent in ref. [17]. Implementation of the Ising model using two dimensional CA under different bcs other than period has not yet been studied.

The paper organised as follows: in *section 2*, we discuss the basic theory to treat a $2d$ CA and in *section 3*, we discuss how to implement it in the Ising model. The simulation result and discussions are given in *section 4*. The comparision of the five bcs with three different initial conditions are discussed in *section 5*. Our conclusion and future perspective are discussed in *section 6*.

2 Two dimensional CA

Two dimensional CA is described by finite states of cells (s), neighborhood cells (n) and its distance among neighbourhood (r), boundary conditions and transition functions or rules (f). In our 2D CA model, $s = \{s_{ij}, s_{ij} \in -1/+1\}$, number of neighbours $n = 9$ (the central cell and its eight neigbours), $r = 1$ and we consider all five bcs. We can construct 2^{2^9} total number of updating rules. All these rules can be derived from nine basic rules $Rule_1$, $Rule_2$, $Rule_4$, $Rule_8$, $Rule_{16}$, $Rule_{32}$, $Rule_{64}$, $Rule_{128}$ and $Rule_{256}$ [18] as shown in the following table.

Table 1: Nine basic rules and their positions in the lattice.

2^6	2^7	2^8
$(i-1, j-1)$	$(i-1, j)$	$(i-1, j+1)$
2^5	2^0	2^1
$(i, j-1)$	(i, j)	$(i, j+1)$
2^4	2^3	2^2
$(i+1, j-1)$	$(i+1, j)$	$(i+1, j+1)$

Neighbourhoods of extreme cells are taken care of by bc. In fixed bc, the extreme

cells are connected to -1 or $+1$ state. If it is connected to $+1$ state, it is called fixed $+1$ bc (f1bc) and if it is connected to -1 state, then it is called fixed -1 bc (f-1bc). If the extreme cells are adjacent to each other then it is called periodic bc (pbc). In adiabatic bc (abc), the extreme cells replicate its state and in reflexive bc (rbc), mirror states replace the extreme cells.

Below we show how a composite rule like $Rule_{170}$ can be calculated by using the basic rules (*Example 1*) and in next example, we show how $Rule_2$ is applied to a matrix of dimension 3×3 with rbc *Example 2* [18].

Example 1 *$Rule_{170}$ can be uniquely expressed in terms of the basic rule matrices as follows:*

170 can be expressed as

$$170 = 2^1 + 2^3 + 2^5 + 2^7.$$
$$Rule_{170} = Rule_2 + Rule_8 + Rule_{32} + Rule_{128}.$$

Example 2 *Transformation of a particular 3×3 matrix by applying $Rule_2$ with rbc is given as:*

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{pmatrix}$$

If the same rule is applied to all the elements of the matrix, then it is called *uniform CA* and if different rules are applied to all the elements of the matrix or block of elements then it is called *nonuniform CA*. At different time intervals, if different rules are applied to the matrix then it is called varying CA e.g., *probabilistic CA*. With the application of these rules, elements (states) of the matrix changes at successive intervals as shown in the following equation.

$$s_{N \times N}^{t+1} = f(s_{N \times N}^t, t) \tag{2.1}$$
$$s_{N \times N}^{t+1} = f_{N \times N}^t \diamondsuit s_{N \times N}^t$$

where f is a time varying rule and $\diamondsuit$ represents a binary operation.

3   Implemetation of isotropic $2d$ Ising model

Consider a square lattice (s) with N rows and N columns. Lattice has then N^2 sites. Each of the site s_{ij}, $i, j = 1, \ldots, N$ has one of the ± 1 spin, which are two states in CA. So, there are 2^{N^2} spin configurations. We consider the nearest neighbor interactions, so the number of neighbor is 4. We include the five different bcs as

1. pbc : $s_{ij+N} = s_{ij}$ and $s_{i+Nj} = s_{ij}$.

2. abc : $s_{ij+N} = s_{ij+N-1}$ and $s_{i+Nj} = s_{i+N-1j}$.

3. rbc : $s_{ij+N} = s_{ij+N-2}$ and $s_{i+Nj} = s_{i+N-2j}$.

4. f1bc : $s_{ij+N} = +1$ and $s_{i+Nj} = +1$.

5. f-1bc : $s_{ij+N} = -1$ and $s_{i+Nj} = -1$.

Average magnetization for the configuration is defined as,

$$\langle M \rangle = \sum_{i,j=1}^{N} s_{ij} \tag{3.1}$$

and the average magnetization per spin is given by,

$$\langle m \rangle = \frac{\langle M \rangle}{N^2} \tag{3.2}$$

Energy for the configuration s is defined as,

$$E(s) = -\frac{J}{2} \sum_{i,j=1}^{N} s_{ij} Rule_{170}(s_{ij}) \tag{3.3}$$

Here, $Rule_{170}(s_{ij})$ is a four neighbourhood matrix that interacts with s_{ij}. The factor of 1/2 has been put to remove the double counting of energy otherwise the interacting energy will be computed twice. $J_{ij} = J$ (isotropic) for 4 neighbours, or else, $J_{ij} = 0$.

The configuration energy per spin is

$$\langle e \rangle = \frac{E(s)}{N^2} \tag{3.4}$$

For updating the lattice in next iterartion, we use the probabilistic approach by constructing a probalistic CA. We use following procedure.

First we calculate the change in energy, $\Delta E(s^t) = E(s^t) - E(s^{t-1})$ i.e., the energy difference at successive time intervals. We do not consider the case when $\Delta E < 0$, because it is obvious that after a finite time, system falls to ground state, and there can not be a state with lower energy. In our approach, we consider always $E(s^t) \geq E(s^{t-1})$. Next we calculate the probability of each spin of the spin configuration s at time t (which is the number of iteration) of each of the site by using the Boltzmann factor

$$p_t = \frac{p(E(s^t))}{p(E(s^{t-1}))} = e^{-\frac{\Delta E(s^t)}{k_B T}} \tag{3.5}$$

With the above probability for each site, we construct a probability weighted matrix. This matrix leads to our probabilistic CA matrix (PCA^t) by comparing with a random matrix and multiplying by a probability factor 0.1 to normalise the rule.

Successive spin configurations are obtained from

$$[s_{ij}^{t+1}]_{N \times N} = [PCA_{ij}^{t}]_{N \times N}[s_{ij}^{t}]_{N \times N} \tag{3.6}$$

After a finite iterartion we calculate the enegry of the system (e), magnetisation, susceptibility (χ), specific heat (C_v) and entropy (S), where,

$$\chi = \frac{N^2}{k_B T}(\langle M^2 \rangle - \langle M \rangle^2) \tag{3.7}$$

$$C_v = \frac{N^2}{k_B T^2}(\langle E^2 \rangle - \langle E \rangle^2) \tag{3.8}$$

$$S = -k_B(r_1 P_1 \log_2 P_1 - r_2 P_2 \log_2 P_2) \tag{3.9}$$

Where $r1$ is the total number of spin up states, $r2$ is the total number of spin down states, P_1 is the probability of spin up states and P_2 is the probability of spin down states in the lattice s. Our probabilistic CA matrix updates in successive time and every spin that is updated in the direction of higher energy will be unflipped in the next iteration. This algorithm checks the time complexity better than the Metropolis algorithm [5] that transits one spin at a time.

4 Simulation results and discussions

In this work, we have considered square lattice of different sizes with $J = 1$ and $k_B = 1$. We do not consider external magnetic field H. Here, temperature T ranges from 0.1 to 5.0. We have carried the simulation with all the three initial conditions and with all five bcs. The optimal lattice size and maximum iteration are decided by the simulation result, which is relevant to study the phase transition.

4.1 Simulation to find maximum iteration

In this simulation, we have found the maximum iteration time (t_{max}) by applying our transition rule to compare between different bcs and for claculation of magnetisation per site m, enegry per site e, χ, C_v and S. For our initial guess of $t_{max} = 2^{14}$, we have considered lattice sizes 4×4, 8×8, 16×16, 32×32, 64×64 and 128×128 with all three initial conditions and all the five bcs. In figure 1, with abc, 64×64 and 128×128 lattice sizes show $m = 0$ after $T > T_c$ i.e., magnetisation curves fluctuate around zero line. After several runs on different bcs with all three initial conditions, we have found t_{max}, by taking lattice sizes 64×64 and 128×128. Here, we have chosen 64×64 lattice size for less computation.
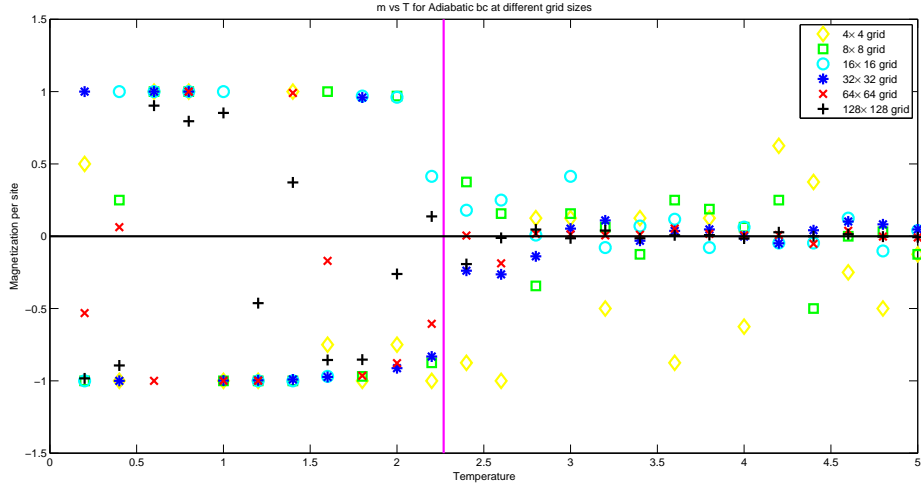
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Figure 1: Comparison among lattice sizes with randomly oriented spin configuration as initial condition. Magenta line represents a parallel line to magnetisation per site at $T = T_c$.

Similar simulation procedure has been applied to find the optimal t_{max} with lattice size is 64×64 at different iteration t_{max} i.e., 2^{11}, 2^{12}, 2^{13}, 2^{14}, 2^{15} and 2^{16}. One of the simulation result given in figure 2 shows that $t_{max} = 2^{15}$ is the best.
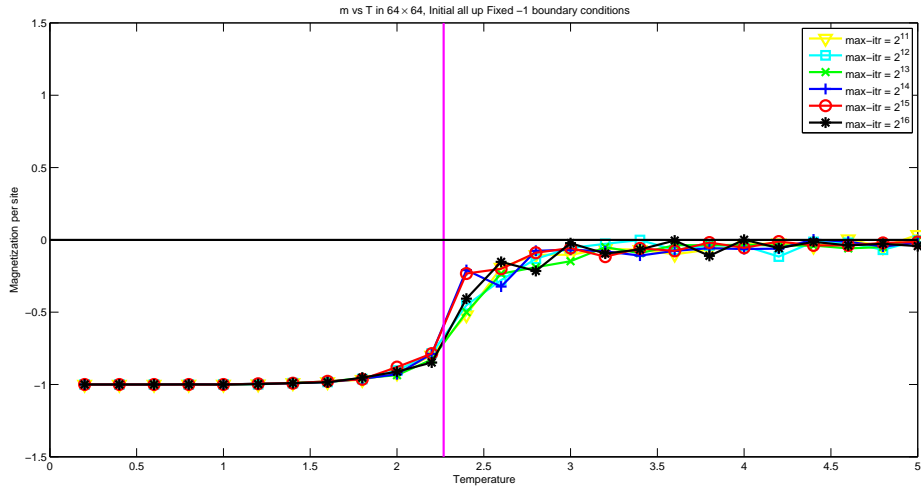
Figure 2: Comparison among different t_{max} with all down spin configuration as initial condition.

4.2 Phase transition with pbc, abc, rbc, f1bc and f-1bc

In this simulation, we have considered 64×64 lattice size and $t_{max} = 2^{15}$ with all bcs and the temperature ranging from 0.1 to 5.0 with increment of 0.1 unit. In figure 3, we have plotted e vs T, m vs T, m vs e, χ vs T, C_v vs T and S vs T with initial condition all up with all bcs. Figures 4 and 5 show similar plots with initial condition all down and random spin configuration respectively. We compare, the simulation result of magnetisation with all five bcs with all three initial conditions with the exact solution given by Onsager which are shown in m vs T graphs in figures 3(a), 4(a) and 5(a). Between temperature $T = 2$ and $T = 2.5$, one finds that the enegry gradually increases, magnetisation gradually decreases to zero. The susceptibility and specific heat also change, initially they increase upto to T_c and then start decreasing as shown in figures 3(d), 4(d), 5(d) and 3(e), 4(e), 5(e) respectively. Entropy gradually increases between tempearture $T = 2$ and $T = 2.5$ and the stays at maximum which are shown in figures 3(f), 4(f) and 5(f). So, a phase transition is clearly visible in between $T = 2$ and $T = 2.5$ with all three initial conditions with all five bcs. In m vs e graphs shown in figures 3(c), 4(c) and 5(c), the higher density states indicate four states. We find two low temperature ground states around ($M = \pm 1$, $E = 4$) with all three initial conditions and all five bcs. The high temperature phase is centered at ($M = 0$, $E = 1$) with all three initial conditions with all five bcs. Then the other state is around ($M = 0$, $E = 3.5$), which is a low-temperature metastable states with all five bcs and this happens only in case of random initial condition. With initial condition all up spins and f-1bc, one can find ground state at ($M = -1$, $E = 4$) and for other bcs at ($M = +1$, $E = 4$) and with initial condition all down spins and f1bc, one can find ground state at ($M = +1$, $E = 4$) and for other bcs at ($M = -1$, $E = 4$).

5 Comparison among boundary conditions

Starting with three different initial conditions, from figure 2, one finds that the magnetisation meet the zero line after $T > T_c$ differently in all bcs. In each bc, the simulation with different initial conditions, meet the zero line after $T > T_c$ differently which is close to exact solution T_c.

With one simulation for all bcs, it is not possible to predict which bc is closer to T_c. So, we analyse the points for magnetisation in the range $-0.1 \leq m \leq 0.1$ and $-0.2 \leq m \leq 0.2$ which are close to the zero line of magnetisation (where magnetisation is zero) after $T > T_c$. We call such points as converging points.
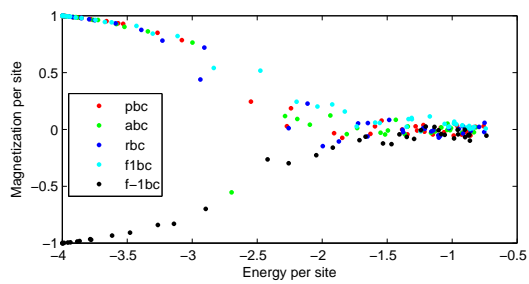
For the above purpose, we have taken different lattice sizes ranging from 5×5 to 60×60 with increment of 5; from 60×60 to 100×100 with increment of 10 and temperature ranging from 0.1 to 5.0 with small increment of 0.05 units. Figure 6 shows
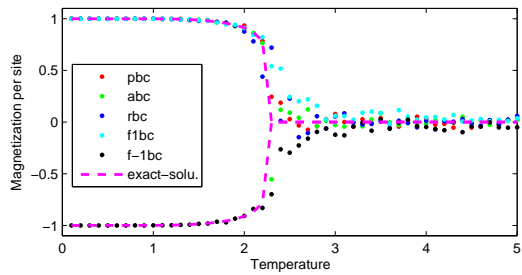
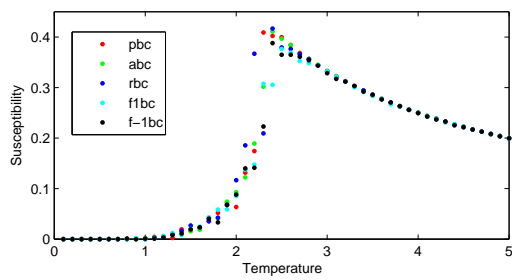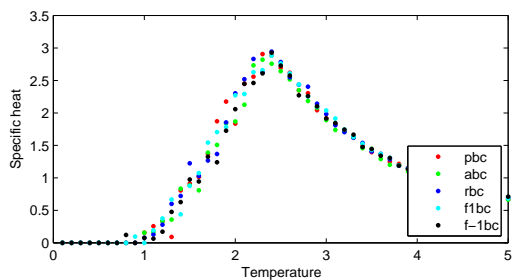(a) m versus T for all bcs.
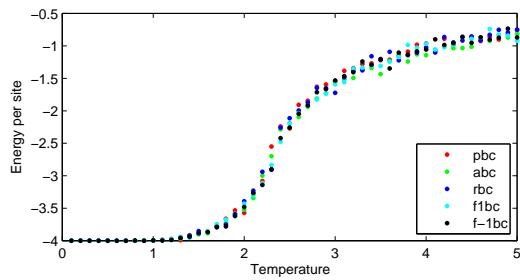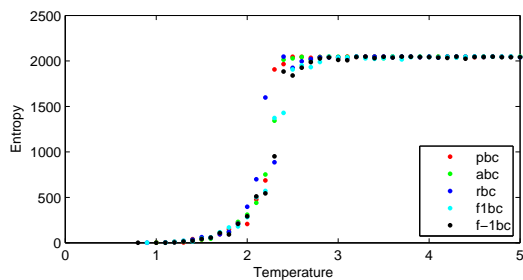

(b) e versus T for all bcs.

(c) m versus e for all bcs.

(d) χ versus T for all bcs.

(e) C_v versus T for all bcs.

(f) S versus T for all bcs.

Figure 3: Initial condition with all up spin configuration for all five bcs.

(a) m versus T for all bcs.

(b) e versus T for all bcs.

(c) m versus e for all bcs.

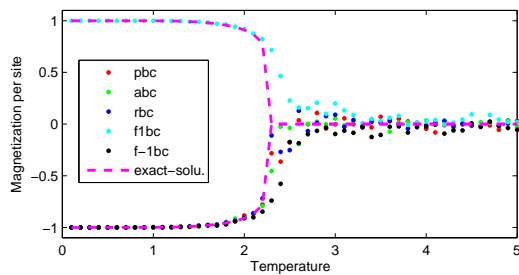(d) χ versus T for all bcs.
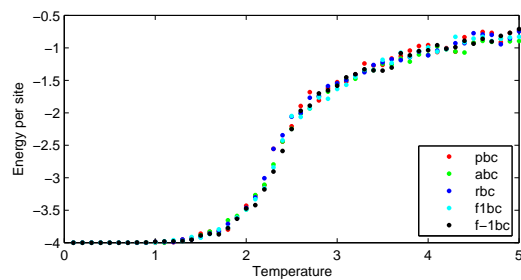

(e) C_v versus T for all bcs.

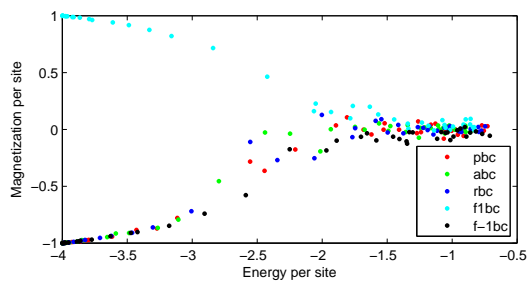(f) S versus T for all bcs.

Figure 4: Initial condition with all down spin configuration for all five bcs.

(a) m versus T for all bcs.

(b) e versus T for all bcs.

(c) m versus e for all bcs.

(d) χ versus T for all bcs.

(e) C_v versus T for all bcs.
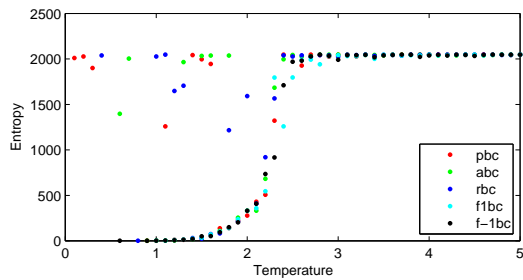
(f) S versus T for all bcs.

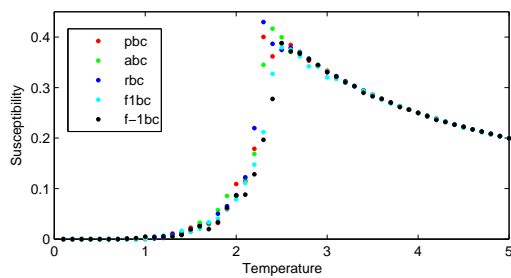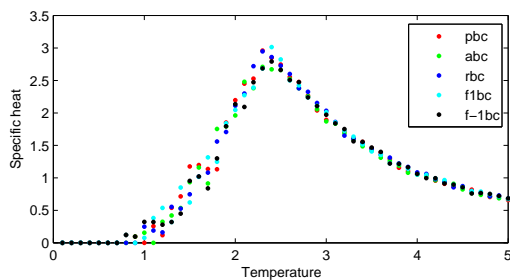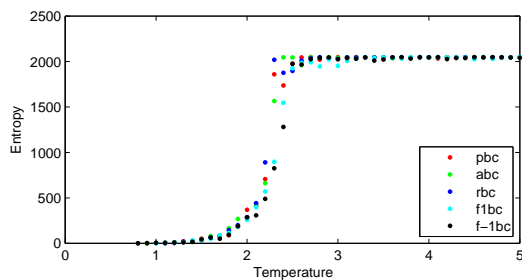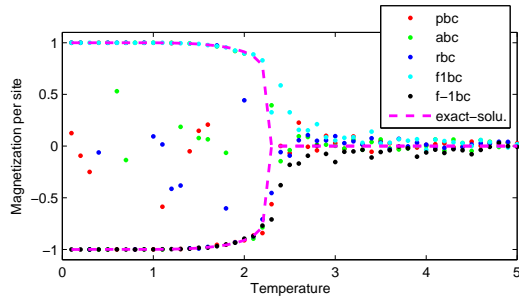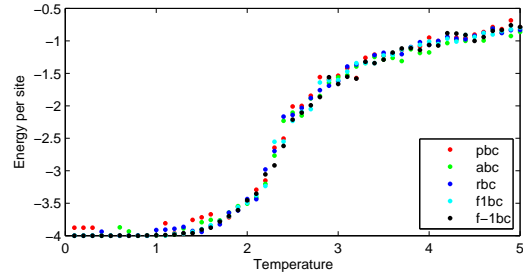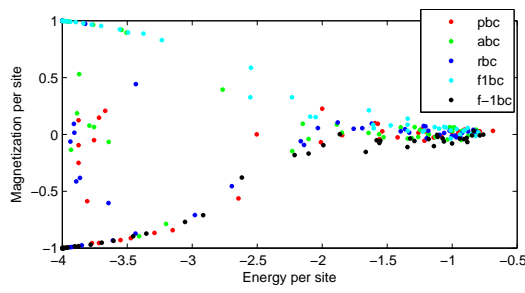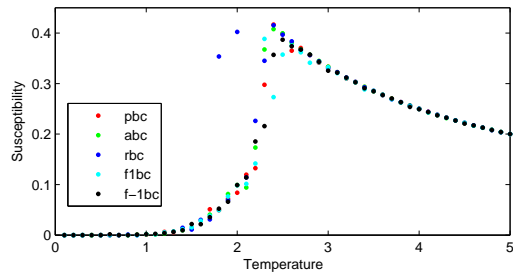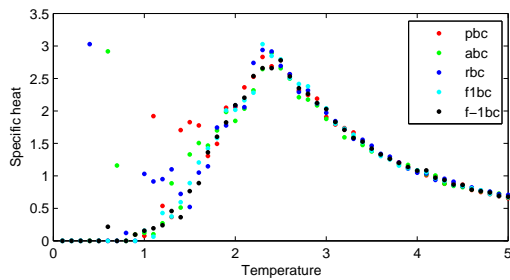Figure 5: Initial condition with random spin configuration for all five bcs.

converging points in the above mentioned range of m. We have counted the number of coverging points as defined above. Their average percentage are shown in table 2, table 3 and table 4 by taking average of ten simulation result of each bc with three different initial conditions.
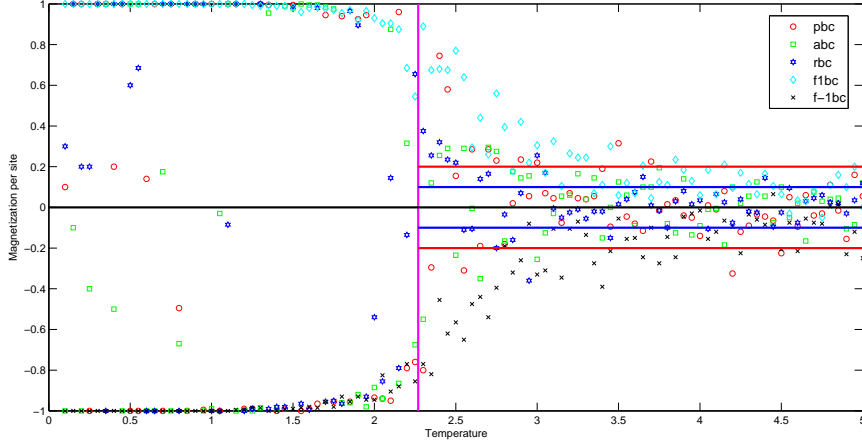
Figure 6: Converging points of bcs for 20×20 lattice size after $T > T_c$. Red lines are for magnetisation $m = +0.2$ and $m = -0.2$ and blue lines are for magnetisation $m = +0.1$ and $m = -0.1$ with the initial condition of random spin configuration.

Lattice $size \le 30 \times 30$ with pbc, abc and rbc, one finds more convergent points in both cases $-0.1 \le m \le 0.1$ and $-0.2 \le m \le 0.2$ than f1bc and f-1bc but among the three with rbc shows more converging points in all three initial conditions. With lattice size between 30×30 and 70×70 with pbc, abc, rbc, one finds more convergent points in both the cases $-0.1 \le m \le 0.1$ and $-0.2 \le m \le 0.2$ in comparison to f1bc and f-1bc with random initial case. For all initial cases with $-0.1 \le m \le 0.1$, the result is better for pbc, abc and rbc. For $-0.2 \le m \le 0.2$, the results are better for f1bc with initial condition all down spins and f-1bc with initial condition all up spins. For lattice size greater than 70×70 and less than equal to 100×100 with initial conditions of all up spins and f-1bc and initial condition of all down spins and f1bc, one observe more converging points in both cases $-0.1 \le m \le 0.1$ and $-0.2 \le m \le 0.2$. For random initial condition with pbc, rbc and abc, one finds more converging points in both the range of magnetisation per site.

6   Conclusion

We have observed a phase transition with all the bcs with all initial conditions around the critical temperature T_c. We have thus observed that with different initial conditions

on different lattice sizes ($\leq 30 \times 30$), one can take care of boundary spins by not only pbc but also by abc and rbc. Further in our analysis, rbc shows best result among them in the average case of lattice size $\leq 30 \times 30$. And for lattice size greater than 70×70 with f1bc and f-1bc are best to use than other bcs in case of initial spin configuration with all up or all down spins. In case of random initial spin configuration, it is better to use either pbc, abc or rbc when the lattice size less than equal to 100×100. Further one can study the behaviour of all five bcs for lattice $size > 100^2$ with all initial conditions. From the simulation point of view, our method takes lesser time than the Metropolis algorithm [5]. This observation is expected to find the approximate values of critical exponents more accurately which we plan to study next. We can check the simulation time by generating random numbers through CA and also we can find a deterministic CA for this purpose.

Acknowledgement

Table 2: Percentage of converging points among bcs for initial condition all spins up

Lattice size	m	pbc	abc	rbc	f1bc	f-1bc
5×5	±0.1	22.2	26.4	23.8	13	14.6
	±0.2	21	22	28.4	14	14.2
10×10	±0.1	25.8	25	26.2	11	11.8
	±0.2	21.6	25	24.8	14.4	12.8
15×15	±0.1	26.4	27.6	25.2	11	10
	±0.2	23.2	23.2	23.8	14	15.2
20×20	±0.1	23.6	24.6	26.4	13.2	11.8
	±0.2	22.4	22.4	23	16	16.2
25×25	±0.1	24	23.8	22.8	14.6	14.6
	±0.2	21.8	21.4	21.4	17	18.2
30×30	±0.1	22.2	24	22.6	14.6	16.8
	±0.2	20.8	21	21	16.8	20.4
35×35	±0.1	22.4	22.6	22.2	14.8	17.8
	±0.2	19.8	20.2	20.8	18	21.6
40×40	±0.1	22.4	21.8	21.2	16.2	19.2
	±0.2	20.4	20	20.2	17.8	22.2
45×45	±0.1	21.8	21.6	21.8	16.8	18.6
	±0.2	20.4	20	20	18	22
50×50	±0.1	20.8	21.2	20.6	16.8	20
	±0.2	20	20	20	18.4	22
55×55	±0.1	20.4	21.2	21.4	17.4	19.4
	±0.2	19.8	19.8	19.8	18.8	21.6
60×60	±0.1	20.8	20.8	20.8	17.8	19.6
	±0.2	20	19.8	19.8	18.8	21.6
70×70	±0.1	20.4	20.2	20.2	18.4	20.4
	±0.2	20	20	19.8	19	21.6
80×80	±0.1	20.4	19.8	20.2	18.4	21.2
	±0.2	19.8	19.8	19.8	19.2	21
90×90	±0.1	20	20	20.2	18.8	21.2
	±0.2	20.2	19.8	20	19	21.2
100×100	±0.1	20.4	20	19.8	18.4	21.2
	±0.2	20	19.8	20.4	19	21

Table 3: Percentage of converging points among bcs for initial condition all spins down

Lattice size	m	pbc	abc	rbc	f1bc	f-1bc
5×5	± 0.1	30.8	26.5	16.4	12.6	14
	± 0.2	25.2	24.4	20.6	14.4	15.4
10×10	± 0.1	21	25	29.8	11.6	12.6
	± 0.2	22	25.6	25.6	13.6	13.4
15×15	± 0.1	22	26	25.6	15.6	11
	± 0.2	22.8	23	24	15.6	14
20×20	± 0.1	24.4	25.2	26.2	12.4	11.6
	± 0.2	22	23	23.2	16.8	15
25×25	± 0.1	23.6	23.4	25.8	15.6	12
	± 0.2	21.6	20.8	22.2	18.6	16.6
30×30	± 0.1	23.6	22.2	22.8	16	15.6
	± 0.2	20.8	20.4	21.2	20.6	17.4
35×35	± 0.1	22.6	21.4	22.6	17	16
	± 0.2	20.6	20	20.6	21.6	17.6
40×40	± 0.1	21.2	21.6	22.6	19	15.8
	± 0.2	19.8	20	20.4	21.6	18.2
45×45	± 0.1	22.2	21.6	21.4	18.8	15.6
	± 0.2	20	20	20.4	21.8	18
50×50	± 0.1	21.2	21.2	21.4	19.8	16.4
	± 0.2	19.6	20.2	20.2	21.8	18.6
55×55	± 0.1	21.2	21.2	21	20	16.6
	± 0.2	19.8	19.8	19.8	22	18.8
60×60	± 0.1	21	20.8	20.4	20.6	17.2
	± 0.2	19.6	20	20	21.6	18.8
70×70	± 0.1	20.4	20.2	20.4	21	18.4
	± 0.2	20	20	19.8	21.4	19
80×80	± 0.1	19.4	20	20.2	21.6	18.8
	± 0.2	20	20	19.6	21	19.2
90×90	± 0.1	20.2	19.8	20	21.4	18.6
	± 0.2	20	20	19.8	21	19
100×100	± 0.1	20.2	20	20.2	21.4	18.2
	± 0.2	20	19.8	20	21	19.2

Table 4: Percentage of converging points among bcs for initial condition all spins random

Lattice size	m	pbc	abc	rbc	f1bc	f-1bc
5×5	± 0.1	20.2	24.2	34	8.8	12.8
	± 0.2	21.6	20	31.6	12.4	14.4
10×10	± 0.1	26.4	25.4	27	12.6	8.8
	± 0.2	23.6	24.6	25.6	13.6	12.8
15×15	± 0.1	25.6	24	26	12.4	12.4
	± 0.2	24.4	23.6	24.2	12.8	14.8
20×20	± 0.1	25.2	24	25.6	12.6	12.8
	± 0.2	23.6	22.6	23.2	15.8	14.6
25×25	± 0.1	24	25.4	23.6	13	14.6
	± 0.2	21.2	22.8	22.4	15.8	18
30×30	± 0.1	24.2	24.2	23	14.6	14.4
	± 0.2	21	21.6	21.6	18	17.8
35×35	± 0.1	23.8	22.2	24.6	14.6	14.2
	± 0.2	21.2	21.6	22	17.4	17.6
40×40	± 0.1	23	22.4	23.8	14.8	15.4
	± 0.2	21.4	20.6	21.6	18.2	18.2
45×45	± 0.1	23.2	22.4	23.2	16.6	15.2
	± 0.2	21.4	21	21.8	18.2	18
50×50	± 0.1	22.2	23	23	16	16
	± 0.2	20.6	20.8	20.8	18.6	18.8
55×55	± 0.1	22.8	22.6	23	16.2	15.8
	± 0.2	20.8	20.8	21	18.8	18.4
60×60	± 0.1	22	22	21.4	17.6	16.8
	± 0.2	20.8	20.8	20.8	19.2	18.4
70×70	± 0.1	21.2	21.4	22	17.6	18
	± 0.2	20.6	20.6	20.8	19.4	19.2
80×80	± 0.1	21.4	21.2	21.4	18.4	17.6
	± 0.2	20.2	20	20.2	19.6	19.6
90×90	± 0.1	21.2	21.2	21.2	18.2	18
	± 0.2	20.2	20.2	20	19.6	19.2
100×100	± 0.1	20.8	20.8	20.8	18.8	18.8
	± 0.2	20	20	20	19.8	19.6

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