AVERAGING ALGEBRAS, REWRITING SYSTEMS AND GRÖBNER-SHIRSHOV BASES

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Abstract. In this paper, we study the averaging operator by assigning a rewriting system to it. We obtain some basic results on the kind of rewriting system we used. In particular, we obtain a sufficient and necessary condition for the confluence. We supply the relationship between rewriting systems and Gröbner-Shirshov bases based on bracketed polynomials. As an application, we give a basis of the free unitary averaging algebra on a non-empsty set.

Contents

1. Introduction

There is an extensive literature on averaging operators under various contexts, motivated largely by the fact that they are generalizations of conditional expectation in probability theory. Let us recall briefly. As early as 1895, O. Reynolds in a famous paper on turbulence theory [33] already studied the averaging operator because of the closed relationship between averaging operators and Reynolds operators—an idempotent operator is an averaging operator if and only if it is a Reynolds operator. Kolmogoroff and Kampé de Fériet defined explicitly the averaging operator in 1930s [25, 29] and began their study on it in a series of papers. G. Birkhoff continued the study of averaging operators using the method of functional analysis [6]. S. T. C. Moy discussed the relationship of averaging operators with conditional expectation in probability theory and studied the connection between averaging operators and integration theory in probability [30]. J. L. Kelley [24] characterized the idempotent averaging operators on the Banach algebra of all real valued continuous functions vanishing at the infinity on a locally compact Hausdorff space. G. C. Rota [35] in 1964 showed that a continuous Reynolds operator on the algebra $L^\infty(S,\Sigma,m)$ of bounded measurable functions on a measure space (S,Σ,m) is an averaging operator if and only if it has closed range.

In the above literatures, most studies on averaging operators are for various special algebras and the topics are largely analytic. The algebraic study on averaging operators has also been

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deepened and generalized. W. Cao [12] studied averaging operators in the general context from an algebraic point of view. He gave the explicit construction of free commutative averaging algebras and investigated the Lie algebra structures induced naturally from averaging operators. By way of analogy with the associative algebra as the enveloping algebra of the Lie algebra, J. L. Loday defined the diassociative algebra as the enveloping algebra of the Leibniz algebra [28]. M. Aguiar showed that a diassociative algebra can be derived from an averaging associative algebra [1]. A further algebraic and combinatorial study of averaging operators was carried out in [21].

The averaging operators are interested also because they are closely related to Reynolds operators, symmetric operators and Rota-Baxter operators [15, 38, 10]. It is worth mentioning that the Rota-Baxter operator has broad connections with many areas in mathematics and mathematical physics [3, 4, 20]. In particular, L. Guo and J. Pei in a recent paper [21] obtained a connection between averaging operators and Rota-Baxter operators: the algebraic structures resulted from the actions of the two operators are Koszul dual to each other. In that paper [21], L. Guo and J. Pei also gave a basis of the free nonunitary (noncommutative) averaging algebra on a non-empty set. It is natural to consider the case of the free unitary (noncommutative) averaging algebra on a non-empty set, our main object of study in the present paper.

Gröbner and Gröbner-Shirshov bases theory was initiated independently by Shirshov [37], Hironaka [23] and Buchberger [11]. It has been proved to be very useful in different branches of mathematics, including commutative algebras and combinatorial algebras, see [7, 8, 9]. Abstract rewriting system is a branch of theoretical computer science, combining elements of logic, universal algebra, automated theorem proving and functional programming [2, 31]. The theories of Gröbner-Shirshov bases and rewriting systems are successfully applied to study operators and operator polynomial identities [17, 22].

In the present paper along this line, using the theories of Gröbner-Shirshov bases and rewriting systems, we construct a basis of the free unitary (noncommutative) averaging algebra on a non-empty set. Terminating and confluence are essential and desirable properties of a rewriting system. To use the tools of Gröbner-Shirshov bases and rewriting systems, we obtain a sufficient and necessary condition for the confluence of the kind of rewriting system we used. We supply the relationship between Gröbner-Shirshov bases and rewriting systems based on bracketed polynomials. Applying the method we obtained for checking confluence, we successfully prove that the rewriting system associated to the averaging operator is confluent and then convergent with a suitable order. Let us emphasize that there are a lot of forks in the process of checking confluence. We handle technically most of them in a unified way. These techniques can also be used to study other operators. It is well known that in the category of any given algebraic structure, the free objects play a central role in study other objects. Thus as an application, we give a basis of the free unitary (noncommutative) averaging algebra on a non-empty set.

Our characterization of averaging operators in terms of Gröbner-Shirshov bases and rewriting systems reveals the power of this approach. It would be interesting to further study operators and operator polynomial identities by making use of the two related theories: Gröbner-Shirshov bases and rewriting systems.

The *layout of the paper* is as follows. In Section 2, we first recall the concepts of averaging algebras and free operated algebras. We next recall some necessary backgrounds of Gröbner-Shirshov bases and rewriting systems. We obtain some basic results on the kind of rewriting system we used. In particular, we obtain a sufficient and necessary condition to characterize the confluence (Theorem 2.36). We end this section by supplying the relationship between the two powerful tools—Gröbner-Shirshov bases and rewriting systems (Theorem 2.41). Section 3 is devoted to a basis of the free unitary averaging algebra on a non-empty set. In order to achieve this purpose, we assign a rewriting system to the averaging operator (Eq. (19)). We show this rewriting system is convergent (Theorem 3.10). We end this section by giving a basis of the free unitary (noncommutative) averaging algebra on a non-empty set (Theorem 3.11).

Some remark on *notation*. We fix a domain $\mathbf{k}$ and a non-empty set X. Denote by $\mathbf{k}^\times := \mathbf{k} \setminus \{0\}$ the subset of nonzero elements. We denote the $\mathbf{k}$-span of a set Y by $\mathbf{k}Y$. For an algebra, we mean a unitary associative noncommutative $\mathbf{k}$-algebra, unless specified otherwise. For any set Y, let $M(Y)$ be the free monoid on Y with identity 1. We use $\sqcup$ for disjoint union.

2. Gröbner-Shirshov bases and rewriting systems

In this section, we first recall the definition of averaging algebras and characterize free averaging algebras as quotients of free operated algebras. We then recall some backgrounds on Gröbner-Shirshov bases and rewriting systems.

2.1. **Free averaging algebras.** An averaging algebra in the noncommutative context is given as follows.

Definition 2.1. A linear operator A on a $\mathbf{k}$-algebra R is called an *averaging operator* if

$$A(u_1)A(u_2) = A(A(u_1)u_2) = A(u_1A(u_2)) \text{ for all } u_1, u_2 \in R.$$

A $\mathbf{k}$-algebra R together with an averaging operator A on R is called an *averaging algebra*.

To characterize the free averaging algebra, let us recall the free operated algebra [9, 19, 27].

Definition 2.2. An **operated monoid** (resp. **operated k-algebra**, resp. **operated k-module**) is a monoid (resp. $\mathbf{k}$-algebra, resp. $\mathbf{k}$-module) U together with a map (resp. $\mathbf{k}$-linear map, resp. $\mathbf{k}$-linear map) $P_U : U \to U$. A morphism from an operated monoid (resp. $\mathbf{k}$-algebra, resp. $\mathbf{k}$-module) (U, P_U) to an operated monoid (resp. $\mathbf{k}$-algebra, resp. $\mathbf{k}$-module) (V, P_V) is a monoid (resp. $\mathbf{k}$-algebra, resp. $\mathbf{k}$-module) homomorphism $f : U \to V$ such that $f \circ P_U = P_V \circ f$.

For any set Y, define

$$\lfloor Y \rfloor := \{\lfloor y \rfloor \mid y \in Y\},$$

which is a disjoint copy of Y. The following is the construction of the free operated monoid on the set X, proceeding via the finite stage $\mathfrak{M}_n(X)$ recursively defined as follows. Define

$$\mathfrak{M}_0(X) := M(X) \text{ and } \mathfrak{M}_1(X) := M(X \sqcup \lfloor \mathfrak{M}_0(X) \rfloor).$$

Then the inclusion $X \hookrightarrow X \sqcup \lfloor \mathfrak{M}_0 \rfloor$ induces a monomorphism

$$i_0 : \mathfrak{M}_0(X) = M(X) \hookrightarrow \mathfrak{M}_1(X) = M(X \sqcup \lfloor \mathfrak{M}_0 \rfloor)$$

of monoids through which we identify $\mathfrak{M}_0(X)$ with its image in $\mathfrak{M}_1(X)$. Suppose that $\mathfrak{M}_{n-1}(X)$ has been defined and the embedding

$$i_{n-2,n-1} : \mathfrak{M}_{n-2}(X) \hookrightarrow \mathfrak{M}_{n-1}(X)$$

has been obtained for $n \geqslant 2$ and consider the case of n. Define

$$\mathfrak{M}_n(X) := M(X \sqcup \lfloor \mathfrak{M}_{n-1}(X) \rfloor).$$

Since $\mathfrak{M}_{n-1}(X) = M(X \sqcup \lfloor \mathfrak{M}_{n-2}(X) \rfloor)$ is the free monoid on the set $X \sqcup \lfloor \mathfrak{M}_{n-2}(X) \rfloor$, the injection

$$X \sqcup \lfloor \mathfrak{M}_{n-2}(X) \rfloor \hookrightarrow X \sqcup \lfloor \mathfrak{M}_{n-1}(X) \rfloor$$

induces a monoid embedding

$$\mathfrak{M}_{n-1}(X) = M(X \sqcup \lfloor \mathfrak{M}_{n-2}(X) \rfloor) \hookrightarrow \mathfrak{M}_n(X) = M(X \sqcup \lfloor \mathfrak{M}_{n-1}(X) \rfloor).$$

Finally we define the monoid

$$\mathfrak{M}(X) := \varinjlim \mathfrak{M}_n(X) = \bigcup_{n \geqslant 0} \mathfrak{M}_n(X).$$

The elements in $\mathfrak{M}(X)$ are called *bracketed words* or *bracketed monomials on* X. When X is finite, we may also just list its elements, as in $\mathfrak{M}(x_1, x_2)$ if $X = \{x_1, x_2\}$. For any $u \in \mathfrak{M}(X) \setminus \{1\}$, u can be written uniquely as a product:

$$u = u_1 \cdots u_n, \text{ for some } n \geqslant 1,\ u_i \in X \sqcup \lfloor \mathfrak{M}(X) \rfloor,\ 1 \leqslant i \leqslant n. \tag{1}$$

The *breadth* of u, denoted by $|u|$, is defined to be n. If $u = 1$, define $|u| = 0$.

Let $\mathbf{k}\mathfrak{M}(X)$ be the free module with the basis $\mathfrak{M}(X)$. Using $\mathbf{k}$-linearity, the concatenation product on $\mathfrak{M}(X)$ can be extended to a multiplication on $\mathbf{k}\mathfrak{M}(X)$, turning $\mathbf{k}\mathfrak{M}(X)$ into a $\mathbf{k}$-algebra. Define an operator $\lfloor\ \rfloor : \mathfrak{M}(X) \to \mathfrak{M}(X)$ by assigning

$$u \mapsto \lfloor u \rfloor,\ u \in \mathfrak{M}(X).$$

By $\mathbf{k}$-linearly, the operator $\lfloor\ \rfloor : \mathfrak{M}(X) \to \mathfrak{M}(X)$ can be extended to a linear operator $\lfloor\ \rfloor : \mathbf{k}\mathfrak{M}(X) \to \mathbf{k}\mathfrak{M}(X)$, turning $(\mathbf{k}\mathfrak{M}(X), \lfloor\ \rfloor)$ into an operated $\mathbf{k}$-algebra. The elements in $\mathbf{k}\mathfrak{M}(X)$ are called *bracketed polynomials* or *operated polynomials* on X.

Lemma 2.3. [19, Coro. 3.6, 3.7] *With structures as above,*

(*a*) *the* $(\mathfrak{M}(X), \lfloor\ \rfloor)$ *together with the natural embedding* $i : X \to \mathfrak{M}(X)$ *is the free operated monoid on* X*; and*

(*b*) *the* $(\mathbf{k}\mathfrak{M}(X), \lfloor\ \rfloor)$ *together with the natural embedding* $i : X \to \mathbf{k}\mathfrak{M}(X)$ *is the free operated* $\mathbf{k}$*-algebra on* X.

Definition 2.4. Let (R, P) be an operated $\mathbf{k}$-algebra.

(*a*) An element $\phi(x_1, \ldots, x_k) \in \mathbf{k}\mathfrak{M}(X)$ (or $\phi(x_1, \ldots, x_k) = 0$) is called an *operated polynomial identity* (OPI), where $k \geqslant 1$ and $x_1, \ldots, x_k \in X$.

(*b*) Let $\phi = \phi(x_1, \ldots, x_k) \in \mathbf{k}\mathfrak{M}(X)$ be an OPI. Given any $u_1, \ldots, u_k \in R$, there is a set map $f : x_i \mapsto u_i, 1 \leqslant i \leqslant k$ and we define

$$\phi(u_1, \ldots, u_k) := \widetilde{f}(\phi(x_1, \ldots, x_k)),$$

where $\widetilde{f} : \mathbf{k}\mathfrak{M}(x_1, \ldots, x_k) \to R$ is the unique morphism of operated algebras that extends the set map f, using the universal property of $\mathbf{k}\mathfrak{M}(x_1, \ldots, x_k)$ as the free operated $\mathbf{k}$-algebra on $\{x_1, \ldots, x_k\}$. Informally, $\phi(u_1, \ldots, u_k)$ is the element of R obtained from $\phi(x_1, \ldots, x_k)$ by replacing each x_i by u_i, $1 \leqslant i \leqslant k$.

(*c*) Let $\Phi \subseteq \mathbf{k}\mathfrak{M}(X)$ be a set of OPIs. We call Φ *is satisfied* by R if

$$\phi(u_1, \ldots, u_k) = 0,\ \forall \phi(x_1, \ldots, x_k) \in \Phi,\ \forall u_1, \ldots, u_k \in R.$$

In this case, we speak that R is a Φ*-algebra* and P is a Φ*-operator*.

(*d*) Let $S \subseteq \mathbf{k}\mathfrak{M}(X)$ be a set. The *operated ideal* $\mathrm{Id}(S)$ of $\mathbf{k}\mathfrak{M}(X)$ generated by S is the smallest operated ideal containing S.

Let us proceed some examples.

Example 2.5. The differential operator as an algebraic abstraction of derivation in analysis leads to the differential algebra, which is an algebraic study of differential equations and has been largely successful in many important areas [26, 32, 34]. The differential operator $d = \lfloor\ \rfloor$ fulfils the following OPI

$$\phi(x_1, x_2) = \lfloor x_1x_2\rfloor - \lfloor x_1\rfloor x_2 - x_1\lfloor x_2\rfloor.$$

Example 2.6. The Rota-Baxter operator $P = \lfloor\ \rfloor$ of weight λ has played important role in mathematics and physics[4, 20, 36], satisfying the OPI

$$\phi(x_1, x_2) = \lfloor x_1\rfloor\lfloor x_2\rfloor - \lfloor x_1\lfloor x_2\rfloor\rfloor - \lfloor\lfloor x_1\rfloor x_2\rfloor - \lambda\lfloor x_1x_2\rfloor,$$

where $\lambda \in \mathbf{k}$ is a fixed constant.

Example 2.7. From Definition 2.1, the averaging operator $A = \lfloor\ \rfloor$ (noncommutative) is defined by the OPIs

$$\begin{aligned}\phi(x_1, x_2) &= \lfloor x_1\rfloor\lfloor x_2\rfloor - \lfloor\lfloor x_1\rfloor x_2\rfloor,\\ \psi(x_1, x_2) &= \lfloor x_1\lfloor x_2\rfloor\rfloor - \lfloor\lfloor x_1\rfloor x_2\rfloor.\end{aligned} \tag{2}$$

Example 2.8. O. Reynolds [33] introduced the concept of Reynolds operators into fluid dynamics, and Kampé de Fériet [14] named it in his study on the various spaces of functions. The Reynolds operator is defined by the OPI

$$\phi(x_1, x_2) = \lfloor\lfloor x_1\rfloor\lfloor x_2\rfloor\rfloor + \lfloor x_1\rfloor\lfloor x_2\rfloor - \lfloor x_1\lfloor x_2\rfloor\rfloor - \lfloor\lfloor x_1\rfloor x_2\rfloor.$$

Definition 2.9. (a) Let $\phi = \phi(x_1, \dots, x_k) \in \mathbf{k}\mathfrak{M}(X)$ be an OPI with $k \geqslant 1$. Define

$$S_\phi(X) := \{\, \phi(u_1, \dots, u_k) \mid u_1, \dots, u_k \in \mathfrak{M}(X)\,\}. \tag{3}$$

(b) Let Φ be a set of OPIs. Define

$$S_\Phi(X) := \bigcup_{\phi\in\Phi} S_\phi(X). \tag{4}$$

It is well-known that

Proposition 2.10. [13, Prop. 1.3.6] *Let $\Phi \subseteq \mathbf{k}\mathfrak{M}(X)$ a set of OPIs. Then the quotient operated algebra $\mathbf{k}\mathfrak{M}(X)/\mathrm{Id}(S_\Phi(X))$ is the free Φ-algebra on X.*

In particular, we have

Proposition 2.11. *Let $\phi(x_1, x_2)$, $\psi(x_1, x_2)$ defined in Eq. (2). Then the quotient operated algebra $\mathbf{k}\mathfrak{M}(X)/\mathrm{Id}(S_\phi(X) \cup S_\psi(X))$ is the free averaging algebra on X.*

2.2. **Gröbner Shirshov bases.** In this subsection, we provide some backgrounds on Gröbner-Shirshov bases [9, 18, 22].

Definition 2.12. Let $\star$ be a symbol not in X and $X^\star = X \sqcup \{\star\}$.

(a) By a *$\star$-bracketed word* on X, we mean any bracketed word in $\mathfrak{M}(X^\star)$ with exactly one occurrence of $\star$, counting multiplicities. The set of all $\star$-bracketed words on X is denoted by $\mathfrak{M}^\star(X)$.

(b) For $q \in \mathfrak{M}^\star(X)$ and $u \in \mathfrak{M}(X)$, we define $q|_u := q|_{\star\mapsto u}$ to be the bracketed word on X obtained by replacing the symbol $\star$ in q by u.

(c) For $q \in \mathfrak{M}^\star(X)$ and $s = \sum_i c_i u_i \in \mathbf{k}\mathfrak{M}(X)$, where $c_i \in \mathbf{k}$ and $u_i \in \mathfrak{M}(X)$, we define

$$q|_s := \sum_i c_i q|_{u_i}.$$

(d) A bracketed word $u \in \mathfrak{M}(X)$ is a *subword* of another bracketed word $w \in \mathfrak{M}(X)$ if $w = q|_u$ for some $q \in \mathfrak{M}^\star(X)$.

Generally, with $\star_1, \star_2$ distinct symbols not in X, set $X^{\star 2} := X \sqcup \{\star_1, \star_2\}$.

(e) We define an $(\star_1, \star_2)$-*bracketed word on* X to be a bracketed word in $\mathfrak{M}(X^{\star 2})$ with exactly one occurrence of each of $\star_i$, $i = 1, 2$. The set of all $(\star_1, \star_2)$-bracketed words on X is denoted by $\mathfrak{M}^{\star_1,\star_2}(X)$.

(f) For $q \in \mathfrak{M}^{\star_1,\star_2}(X)$ and $u_1, u_2 \in \mathbf{k}\mathfrak{M}^{\star_1,\star_2}(X)$, we define

$$q|_{u_1,u_2} := q|_{\star_1 \mapsto u_1, \star_2 \mapsto u_2}$$

to be obtained by replacing the letters $\star_i$ in q by u_i for $i = 1, 2$.

Remark 2.13. Recall [22] that $q|_{u_1,u_2} = (q^{\star_1}|_{u_1})|_{u_2} = (q^{\star_2}|_{u_2})|_{u_1}$, where $q^{\star_1}$ is viewed as a $\star_1$-bracketed word on $X \sqcup \{\star_2\}$ and $q^{\star_2}$ as a $\star_2$-bracketed word on $X \sqcup \{\star_1\}$.

We record the following obvious properties of subwords, which will be used later.

Lemma 2.14. *Let $u, v, w \in \mathfrak{M}(X)$.*

(a) *If u is a subword of $\lfloor v \rfloor$, then either $u = \lfloor v \rfloor$ or u is a subword of v.*
(b) *If $\lfloor u \rfloor$ is a subword of vw, then either $\lfloor u \rfloor$ is a subword of v or $\lfloor u \rfloor$ is a subword of w.*

Proof. (a) Suppose $u \neq \lfloor v \rfloor$. Since u is a subword of $\lfloor v \rfloor$, then $\lfloor v \rfloor = q|_u$ for some $q \in \mathfrak{M}^\star(X)$ by Definition 2.12 (d). Since $u \neq \lfloor v \rfloor$, it follows that $q \neq \star$. Thus $q = \lfloor p \rfloor$ for some $p \in \mathfrak{M}^\star(X)$ by $\lfloor v \rfloor = q|_u$. Therefore $\lfloor v \rfloor = q|_u = \lfloor p|_u \rfloor$ and so $v = p|_u$, as required.

(b). This is followed by the breadth of $\lfloor u \rfloor$ is 1. □

The operated ideals in $\mathbf{k}\mathfrak{M}(X)$ can be characterized by $\star$-bracketed words [9, 22].

Lemma 2.15. ([22, Lem. 3.2]) *Let $S \subseteq \mathbf{k}\mathfrak{M}(X)$. Then*

$$\mathrm{Id}(S) = \left\{ \sum_{i=1}^{n} c_i q_i|_{s_i} \,\middle|\, n \geqslant 1 \text{ and } c_i \in \mathbf{k}^\times, q_i \in \mathfrak{M}^\star(X), s_i \in S \text{ for } 1 \leqslant i \leqslant n \right\}. \tag{5}$$

Definition 2.16. A *monomial order on* $\mathfrak{M}(X)$ is a well-order $\leqslant$ on $\mathfrak{M}(X)$ such that

$$u < v \Longrightarrow q|_u < q|_v, \quad \forall u, v \in \mathfrak{M}(X), \forall q \in \mathfrak{M}^\star(X).$$

Definition 2.17. Let $s \in \mathbf{k}\mathfrak{M}(X)$ and $\leqslant$ a linear order on $\mathfrak{M}(X)$.

(a) Let $s \notin \mathbf{k}$. The *leading monomial* of s, denoted by $\overline{s}$, is the largest monomial appearing in s. The *leading coefficient of* s, denoted by c_s, is the coefficient of $\overline{s}$ in s.
(b) If $s \in \mathbf{k}$, we define the *leading monomial of* s to be 1 and the *leading coefficient of* s to be $c_s = s$.
(c) s is called *monic with respect to* $\leqslant$ if $s \notin \mathbf{k}$ and $c_s = 1$. A subset $S \subseteq \mathbf{k}\mathfrak{M}(X)$ is called *monic with respect to* $\leqslant$ if every $s \in S$ is monic with respect to $\leqslant$.
(d) Define $R(s) := c_s\overline{s} - s$. So $s = c_s\overline{s} - R(s)$.

We will not need the precise definition of Gröbner-Shirshov bases for our construction. So we will not recall it for now and the authors are refereed to [7] and references therein. Suffices it to say that we need the Composition-Diamond Lemma, the corner stone of Gröbner-Shirshov basis theories.

Lemma 2.18. (Composition-Diamond Lemma [9, 22]) *Let $\leqslant$ a monomial order on $\mathfrak{M}(X)$ and $S \subseteq \mathbf{k}\mathfrak{M}(X)$ monic with respect to $\leqslant$. Then the following conditions are equivalent.*

(*a*) *S is a Gröbner-Shirshov basis in $\mathbf{k}\mathfrak{M}(X)$.*

(*b*) *$\eta(\mathrm{Irr}(S))$ is a $\mathbf{k}$-basis of $\mathbf{k}\mathfrak{M}(X)/\mathrm{Id}(S)$, where $\eta\colon \mathbf{k}\mathfrak{M}(X) \to \mathbf{k}\mathfrak{M}(X)/\mathrm{Id}(S)$ is the canonical homomorphism of $\mathbf{k}$-modules and*

$$\mathrm{Irr}(S) := \mathfrak{M}(X) \setminus \{q|_{\overline{s}} \mid s \in S\}. \tag{6}$$

More precisely as $\mathbf{k}$-modules,

$$\mathbf{k}\mathfrak{M}(X) = \mathbf{k}\mathrm{Irr}(S) \oplus \mathrm{Id}(S).$$

2.3. **Term-rewriting systems.** In this subsection, we give a method for checking confluence of term-rewriting systems. Let us recall some basic notations and results [17].

Definition 2.19. Let V be a free $\mathbf{k}$-module with a given $\mathbf{k}$-basis W and $f, g \in V$.

(*a*) The *support* $\mathrm{Supp}(f)$ of f is the set of monomials (with non-zero coefficients) of f. Here we use the convention that $\mathrm{Supp}(0) = \emptyset$.

(*b*) We write $f \dotplus g$ to indicate that $\mathrm{Supp}(f) \cap \mathrm{Supp}(g) = \emptyset$ and say $f + g$ is a *direct sum* of f and g. If this is the case, we also use $f \dotplus g$ for the sum $f + g$.

(*c*) For $w \in \mathrm{Supp}(f)$ with the coefficient c_w, we define $R_w(f) := c_w w - f \in V$ and so $f = c_w w \dotplus (-R_w(f))$.

Lemma 2.20. [17, Lem. 2.12] *Let V be a free $\mathbf{k}$-module with a $\mathbf{k}$-basis W and $f, g \in V$. If $f \dotplus g$, then $cf \dotplus dg$ for any $c, d \in \mathbf{k}$.*

Remark 2.21. Using the notation $\dotplus$, the equation $s = c_s\overline{s} - R(s)$ in Definition 2.17 (*d*) can be written in more detail as $s = c_s\overline{s} \dotplus (-R(s))$.

The following is the concept of term-rewriting systems.

Definition 2.22. Let V be a free $\mathbf{k}$-module with a $\mathbf{k}$-basis W. A *term-rewriting system* Π *on V with respect to W* is a binary relation $\Pi \subseteq W \times V$. An element $(t, v) \in \Pi$ is called a *(term) rewriting rule* of Π, denoted by $t \to v$. The term-rewriting system Π is called *simple* if $t \dotplus v$ for all $t \to v \in \Pi$.

Remark 2.23. Now we explain the requirement that the term-rewriting system Π is simple. Suppose Π is not simple. Then by Definition 2.22, there is a rewriting rule $t \to v$ such that $t \in \mathrm{Supp}(v)$. Assume $v = ct \dotplus (-R_t(v))$ for some $c \in \mathbf{k}^{\times}$. Then

$$t \to_{\Pi} v = ct \dotplus (-R_t(f)) \to_{\Pi} cv - R_t(v) = c^2 t \dotplus (-c-1)R_t(v) \to_{\Pi} \cdots.$$

So as long as c is not a nilpotent element, Π is not terminating. In the remainder of this paper, we always assume that the term-rewriting system is simple, unless specified otherwise.

Definition 2.24. Let V be a free $\mathbf{k}$-module with a $\mathbf{k}$-basis W, Π a simple term-rewriting system on V with respect to W and $f, g \in V$.

(*a*) We speak that f *rewrites* to g *in one-step*, denoted by $f \to_\Pi g$ or $f \xrightarrow{(t,v)}_\Pi g$, if

$$f = c_t t \dotplus (-R_t(f)) \ \text{and} \ g = c_t v - R_t(f) \ \text{for some } c_t \in \mathbf{k}^\times \ \text{and} \ t \to v \in \Pi.$$

(*b*) The reflexive-transitive closure of the binary relation $\to_\Pi$ on V is denoted by $\overset{*}{\to}_\Pi$. If $f \overset{*}{\to}_\Pi g$ (resp. $f \overset{*}{\not\to}_\Pi g$), we speak that f *rewrites (resp. doesn't rewrite) to* g *with respect to* Π.
(*c*) We call f and g are *joinable*, denoted by $f \downarrow_\Pi g$, if there exists $h \in V$ such that $f \overset{*}{\to}_\Pi h$ and $g \overset{*}{\to}_\Pi h$.
(*d*) We say f *a normal form* if no more rewriting rules can apply.

Remark 2.25. Let $f, g \in V$.

(*a*) By Definition 2.24 (*b*), $f \overset{*}{\to}_\Pi f$ and

$$f \overset{*}{\to} g \Longleftrightarrow f =: f_0 \to_\Pi f_1 \to_\Pi \cdots \to_\Pi f_n := g \ \text{for some } n \geqslant 0, f_i \in V, 0 \leqslant i \leqslant n.$$

(*b*) If $f \overset{*}{\to}_\Pi g$, then $f \downarrow_\Pi g$ by $g \overset{*}{\to}_\Pi g$. In particular, $f \downarrow_\Pi f$ by $f \overset{*}{\to}_\Pi f$.

The following definitions are adapted from abstract rewriting systems [2, 5].

Definition 2.26. Let V be a free $\mathbf{k}$-module with a $\mathbf{k}$-basis W, Π a simple term-rewriting system on V with respect to W.

(*a*) Π is *terminating* if there is no infinite chain of one-step rewriting
$$f_0 \to_\Pi f_1 \to_\Pi f_2 \cdots \quad .$$
(*b*) $f \in V$ is *locally confluent* if for every local fork $(h \,{}_\Pi\!\leftarrow f \to_\Pi g)$, we have $g \downarrow_\Pi h$.
(*c*) $f \in V$ is *confluent* if for every fork $(h \,{}_\Pi\!\overset{*}{\leftarrow} f \overset{*}{\to}_\Pi g)$, we have $g \downarrow_\Pi h$.
(*d*) Π is *locally confluent (resp. confluent)* if every $f \in V$ is locally confluent (resp. confluent).
(*e*) Π is *convergent* if it is both terminating and confluent.

A well-known result on rewriting systems is Newman's Lemma.

Lemma 2.27. ([2, Lem. 2.7.2]) *A terminating rewriting system is confluent if and only if it is locally confluent.*

The following result will be used later.

Lemma 2.28. ([17, Thm. 2.20]) *Let V be a free* $\mathbf{k}$*-module with a* $\mathbf{k}$*-basis W and Π a simple term-rewriting system on V with respect to W. If Π is confluent, then, for all $m \geqslant 1$ and $f_1, \ldots, f_m, g_1, \ldots, g_m \in V$,*

$$f_i \downarrow_\Pi g_i \quad (1 \leqslant i \leqslant m), \ \text{and} \ \sum_{i=1}^m g_i = 0 \implies \left(\sum_{i=1}^m f_i\right) \overset{*}{\to}_\Pi 0.$$

Remark 2.29. If Π is confluent and $f \downarrow_\Pi g$, then $f - g \overset{*}{\to}_\Pi 0$ by $-g \downarrow_\Pi -g$ and Lemma 2.28.

The following is a concept strong than locally confluence and similar to Buchberger's S-polynomials.

Definition 2.30. Let V be a free $\mathbf{k}$-module with a $\mathbf{k}$-basis W, Π a simple term-rewriting system on V with respect to W.

(a) *A local base-fork* is a fork $(cv_1\ {}_{\Pi}\!\leftarrow ct \to_{\Pi} cv_2)$, where $c \in \mathbf{k}^{\times}$ and $t \to v_1, t \to v_2 \in \Pi$ are rewriting rules.

(b) The term-rewriting system Π is called *locally base-confluent* if for every local base-fork $(cv_1\ {}_{\Pi}\!\leftarrow ct \to_{\Pi} cv_2)$, we have $c(v_1 - v_2) \overset{*}{\to}_{\Pi} 0$.

(c) Π is *compatible* with a linear order $\leqslant$ on W if $\overline{v} < t$ for each $t \to v \in \Pi$.

Lemma 2.31. ([17, Lem. 2.22]) *Let V be a free $\mathbf{k}$-module with a $\mathbf{k}$-basis W and let Π be a simple term-rewriting system on V which is compatible with a well order $\leqslant$ on W. If Π is locally base-confluent, then it is locally confluent.*

The following concept is followed from general abstract rewriting systems [5, Def. 1.1.6].

Definition 2.32. Let V be a free $\mathbf{k}$-modules with a $\mathbf{k}$-basis W and let Π be a simple term-rewriting system on V with respect to W. Let $Y \subseteq W$ and $\Pi_{\mathbf{k}Y} := \Pi \cap (Y \times \mathbf{k}Y)$. We call $\Pi_{\mathbf{k}Y}$ a *sub-term-rewriting system* of Π on $\mathbf{k}Y$ with respect to Y, denoted by $\Pi_{\mathbf{k}Y} \leqslant \Pi$, if $\mathbf{k}Y$ is closed under Π, i.e., for any $f \in \mathbf{k}Y$ and any $g \in V$, $f \to_{\Pi} g$ implies $g \in \mathbf{k}Y$.

Remark 2.33. Since Π is simple, $\Pi_{\mathbf{k}Y}$ is also simple. Indeed, let $t \to v \in \Pi_{\mathbf{k}Y}$ be a rewriting rule with $t \in Y$ and $v \in \mathbf{k}Y$. Then $t \to v \in \Pi$ by $\Pi_{\mathbf{k}Y} \subseteq \Pi$. Since Π is simple, we have $t \notin \mathrm{Supp}(v)$ by Definition 2.22 and so $\Pi_{\mathbf{k}Y}$ is simple.

We record the following properties.

Lemma 2.34. *Let V be a free $\mathbf{k}$-module with a $\mathbf{k}$-basis W, and let Π be a simple term-rewriting system on V with respect to W.*

(a) *If $t \in \mathrm{Supp}(cf)$ with $t \in W$, $c \in \mathbf{k}^{\times}$ and $f \in V$, then $t \in \mathrm{Supp}(f)$.*

(b) *If $cf \to_{\Pi} g$ with $c \in \mathbf{k}^{\times}$ and $f, g \in V$, then $g = cg'$ for some $g' \in V$.*

(c) *If $cf = 0$ with $c \in \mathbf{k}^{\times}$ and $f \in V$, then $f = 0$.*

(d) *For $c \in \mathbf{k}^{\times}$ and $f, g \in V$ with $f \neq g$, $f \to_{\Pi} g$ if and only if $cf \to_{\Pi} cg$.*

Proof. (a) Suppose for a contrary that $t \notin \mathrm{Supp}(f)$. Since W is a $\mathbf{k}$-basis of V, by Definition 2.19 (a), we may write $f = \sum_i c_i w_i$, where each $c_i \in \mathbf{k}^{\times}$ and $w_i \in W \setminus \{t\}$. Then $cf = \sum_i cc_i w_i$. Since $w_i \neq t$ for each i, we have $t \notin \mathrm{Supp}(cf)$, a contradiction.

(b) Suppose $cf \xrightarrow{(t,v)}_{\Pi} g$ for some $t \to v \in \Pi$. Then $t \in \mathrm{Supp}(cf)$ and so $t \in \mathrm{Supp}(f)$ by Item (a). Write $f = c_t t \dotplus (-R_t(f))$ with $c_t \in \mathbf{k}^{\times}$. Then by Lemma 2.20,

$$cf = cc_t t \dotplus (-cR_t(f)) \xrightarrow{(t,v)}_{\Pi} cc_t v - cR_t(f) = c(c_t v - R_t(f)) = g,$$

as required.

(c) Since W is a $\mathbf{k}$-basis of V, we may write $f = \sum_i c_i w_i$ with $c_i \in \mathbf{k}$ and $w_i \in W$ for each i. Then $cf = \sum_i cc_i w_i = 0$ and so $cc_i = 0$ for each i. Since $\mathbf{k}$ is a domain by our hypothesis and $c \neq 0$, we have $c_i = 0$ for each i, that is, $f = 0$.

(d) Suppose $f \xrightarrow{(t,v)}_{\Pi} g$ for some $t \to v \in \Pi$. By Definition 2.24 (a), we may write

$$f = dt \dotplus (-R_t(f)) \ \text{ and } \ g = dv - R_t(f) \ \text{ for some } d \in \mathbf{k}^{\times}.$$

Then by Lemma 2.20,

$$cf = cdt \dotplus (-cR_t(f)) \ \text{ and } \ cg = cdv - cR_t(f)$$

and so $cf \xrightarrow{(t,v)}_\Pi cg$. Conversely, suppose $cf \xrightarrow{(t,v)}_\Pi cg$ for some $t \to v \in \Pi$. Then $t \in \mathrm{Supp}(cf)$ and so $t \in \mathrm{Supp}(f)$ by Item (*a*). Write $f = c_t t \dotplus (-R_t(f))$ with $c_t \in \mathbf{k}^\times$. Then from Lemma 2.20,

$$cf = cc_t t \dotplus (-cR_t(f)) \xrightarrow{(t,v)}_\Pi cc_t v - cR_t(f) = cg.$$

Since $c \in \mathbf{k}^\times$, we get $c_t v - R_t(f) = g$ by Item (*c*) and so $f \to_\Pi g$. □

Lemma 2.35. *Let V be a free $\mathbf{k}$-module with a $\mathbf{k}$-basis W, and let Π be a simple term-rewriting system on V with respect to W. Let $f, g \in V$ and $c \in \mathbf{k}^\times$. Then $f \overset{*}{\to}_\Pi g$ if and only if $cf \overset{*}{\to}_\Pi cg$.*

Proof. ($\Rightarrow$) If $f = g$, then $cf = cg$ and $cf \overset{*}{\to}_\Pi cg$ by Remark 2.25 (*a*). Suppose $f \neq g$. Let $n \geqslant 1$ be the least number such that f rewrites to g in n steps. Then

$$f = f_0 \to_\Pi f_1 \to_\Pi \cdots \to_\Pi f_n = g \tag{7}$$

for some distinct $f_i \in V$, $0 \leqslant i \leqslant n$ and so by Lemma 2.34 (*d*),

$$cf = cf_0 \to_\Pi cf_1 \to_\Pi \cdots \to_\Pi cf_n = cg. \tag{8}$$

Hence $cf \overset{*}{\to}_\Pi cg$.

($\Leftarrow$) If $cf = cg$, then $f = g$ by Lemma 2.34 (*c*) and so $f \overset{*}{\to}_\Pi g$ by Remark 2.25 (*a*). Suppose $cf \neq cg$. Let $n \geqslant 1$ be the least number such that cf rewrites to cg in n steps. Then by Lemma 2.34 (*b*), Eq (8) holds for some distinct $cf_i \in V$, $0 \leqslant i \leqslant n$. Using Lemma 2.34 (*c*), $f_i \in V$ are distinct for $0 \leqslant i \leqslant n$. From Lemma 2.34 (*d*), Eq. (7) is valid and so $f \overset{*}{\to}_\Pi g$. □

Theorem 2.36. *Let V be a free $\mathbf{k}$-module with a $\mathbf{k}$-basis W and let Π be a simple terminating term-rewriting system on V with respect to W. Suppose $\leqslant$ is a well-order on W compatible with Π. Then Π is confluent if and only if w is locally confluent for any $w \in W$.*

Proof. ($\Rightarrow$) Since Π is confluent, Π is locally confluent by Definition 2.26, that is, every element in V is locally confluent. From $W \subseteq V$, w is locally confluent for any $w \in W$.

($\Leftarrow$) To show Π is confluent, it is enough to show Π is locally confluent by Lemma 2.27. In view of Lemma 2.31, we are left to prove that Π is locally base-confluent, that is, for any local base-fork $(cv_1 \ {}_\Pi\!\leftarrow cw \to_\Pi cv_2)$, we have $cv_1 - cv_2 \overset{*}{\to}_\Pi 0$. Suppose for a contrary that Π is not locally base-confluent. Then the set

$$\mathfrak{C} = \{w \in W \,\Big|\, \text{there is a local fork base-fork } (cv_1 \ {}_\Pi\!\leftarrow cw \to_\Pi cv_2) \\ \text{for some } c \in \mathbf{k}^\times, v_1, v_2 \in V \text{ such that } cv_1 - cv_2 \overset{*}{\not\to}_\Pi 0\}$$

is non-empty. Since $\leqslant$ is a well-order, $\mathfrak{C}$ has the least element w with respect to $\leqslant$. Thus there is a local base-fork

$$(cv_1 \ {}_\Pi\!\leftarrow cw \to_\Pi cv_2) \ \text{ with } w \to v_1, w \to v_2 \in \Pi \tag{9}$$

such that

$$cv_1 - cv_2 \overset{*}{\not\to}_\Pi 0 \ \text{ for some } \ c \in \mathbf{k}^\times, v_1, v_2 \in V. \tag{10}$$

Let

$$Y := \{y \in W \mid y < w\} \ \text{ and } \ \Pi_{\mathbf{k}Y} = \Pi \cap (Y \times \mathbf{k}Y). \tag{11}$$

Since $\leqslant$ is compatible with Π, we have $\mathrm{Supp}(v_1), \mathrm{Supp}(v_2) \subseteq Y$ and so $Y \neq \emptyset$. Furthermore, $\Pi_{\mathbf{k}Y} \leqslant \Pi$ is a sub-term-rewriting system of Π. Indeed, let $f \overset{*}{\to}_\Pi g$ with $f \in \mathbf{k}Y$, since $\leqslant$ is compatible with Π, we get $\overline{g} \leqslant \overline{f} < w$ and so $g \in \mathbf{k}Y$. Thus $\Pi_{\mathbf{k}Y}$ is closed under Π and so $\Pi_{\mathbf{k}Y} \leqslant \Pi$ by Definition 2.32.

For any local base-fork $(du_1\ {}_{\Pi_{\mathbf{k}Y}}\!\leftarrow dy \rightarrow_{\Pi_{\mathbf{k}Y}} du_2)$ of $\Pi_{\mathbf{k}Y}$ with $d \in \mathbf{k}^{\times}$, $y \in Y$ and $u_1, u_2 \in \mathbf{k}Y$, it induces a local base-fork $(du_1\ {}_{\Pi}\!\leftarrow dy \rightarrow_{\Pi} du_2)$ by $\Pi_{\mathbf{k}Y} \subseteq \Pi$. Since $y \in Y$, we have $y < w$ and $y \notin \mathfrak{C}$ by the minimality of w. So $du_1 - du_2 \overset{*}{\rightarrow}_{\Pi} 0$ by the definition of $\mathfrak{C}$. Claim

$$f \overset{*}{\rightarrow}_{\Pi} g \Longrightarrow f \overset{*}{\rightarrow}_{\Pi_{\mathbf{k}Y}} g \ \text{ for } \ f, g \in \mathbf{k}Y. \tag{12}$$

Since $du_1 - du_2 \in \mathbf{k}Y$ by $u_1, u_2 \in \mathbf{k}Y$, we have $du_1 - du_2 \overset{*}{\rightarrow}_{\Pi_{\mathbf{k}Y}} 0$ by the Claim. Thus $\Pi_{\mathbf{k}Y}$ is locally base-confluent and so is locally confluent by Lemma 2.31. Since Π is terminating and $\Pi_{\mathbf{k}Y} \leqslant \Pi$, $\Pi_{\mathbf{k}Y}$ is terminating. Therefore $\Pi_{\mathbf{k}Y}$ is confluent by Lemma 2.27.

For the local fork in Eq. (9), it induces a local fork $(v_1\ {}_{\Pi}\!\leftarrow w \rightarrow_{\Pi} v_2)$ by Lemma 2.34 (*d*). Since $w \in W$ is confluent by our hypothesis, it follows that $v_1 \downarrow_{\Pi} v_2$. So there is $u \in V$ such that $v_1 \overset{*}{\rightarrow}_{\Pi} u$ and $v_2 \overset{*}{\rightarrow}_{\Pi} u$ by Definition 2.24 (*c*). From Lemma 2.35,

$$cv_1 \overset{*}{\rightarrow}_{\Pi} cu \ \text{ and } \ cv_2 \overset{*}{\rightarrow}_{\Pi} cu.$$

From $cv_1 \in \mathbf{k}Y$ and $\Pi_{\mathbf{k}Y} \leqslant \Pi$ is closed under Π, we have $cu \in \mathbf{k}Y$. So by the Claim of Eq. (12),

$$cv_1 \overset{*}{\rightarrow}_{\Pi_{\mathbf{k}Y}} cu \ \text{ and } \ cv_2 \overset{*}{\rightarrow}_{\Pi_{\mathbf{k}Y}} cu.$$

This means that $cv_1 \downarrow_{\Pi_{\mathbf{k}Y}} cv_2$. Since $\Pi_{\mathbf{k}Y}$ is confluent, $cv_1 - cv_2 \overset{*}{\rightarrow}_{\Pi_{\mathbf{k}Y}} 0$ by Remark 2.29. Hence $cv_1 - cv_2 \overset{*}{\rightarrow}_{\Pi} 0$ by $\Pi_{\mathbf{k}Y} \subseteq \Pi$, contradicting Eq. (10). We are left to prove the Claim. □

proof of Claim. We want to show Eq. (12). Suppose $f \overset{*}{\rightarrow}_{\Pi} g$ with $f, g \in \mathbf{k}Y$. If $f = g$, then $f \overset{*}{\rightarrow}_{\Pi_{\mathbf{k}Y}} g$ by Remark 2.25 (*a*). Assume $f \neq g$ and let $n \geqslant 1$ be least number such that

$$f =: f_0 \rightarrow_{\Pi} f_1 \rightarrow_{\Pi} \cdots \rightarrow_{\Pi} f_n := g \ \text{ with } f_i \in V \text{ are distinct}, 0 \leqslant i \leqslant n.$$

Since $f_0 = f \in \mathbf{k}Y$ and Π is compatible with $\leqslant$, we have $f_i \in \mathbf{k}Y$ for $0 \leqslant i \leqslant n$. We prove the Claim by induction on $n \geqslant 1$. For the initial step of $n = 1$, suppose $f = f_0 \overset{(t,v)}{\rightarrow}_{\Pi} f_1 = g$ for some $t \rightarrow v \in \Pi$. Then $t \in \mathrm{Supp}(f) \subseteq Y$. This follows that $t < w$ by Eq. (11). Since Π is compatible with $\leqslant$, we have $\overline{v} < t < w$ and so $v \in \mathbf{k}Y$. Thus $t \rightarrow v \in Y \times \mathbf{k}Y$ and so $t \rightarrow v \in \Pi \cap (Y \times \mathbf{k}Y) = \Pi_{\mathbf{k}Y}$. This implies that $f = f_0 \overset{(t,v)}{\longrightarrow}_{\Pi_{\mathbf{k}Y}} f_1 = g$ by $f_0, f_1 \in \mathbf{k}Y$ and $f_0 \overset{(t,v)}{\longrightarrow}_{\Pi} f_1$. For the induction step, we have $f = f_0 \overset{*}{\rightarrow}_{\Pi_{\mathbf{k}Y}} f_1$ and $f_1 \overset{*}{\rightarrow}_{\Pi_{\mathbf{k}Y}} f_n = g$ by induction hypothesis and so $f \overset{*}{\rightarrow}_{\Pi_{\mathbf{k}Y}} g$, as required. □

2.4. **Term-rewriting systems and Gröbner-Shirshov bases.** In this subsection, we supply the relationship between Gröbner-Shirshov bases and term-rewriting systems based on bracketed polynomials. A term-rewriting system can be assigned to a given set S of OPIs [17].

Definition 2.37. Let $\leqslant$ be a linear order on $\mathfrak{M}(X)$ and $S \subseteq \mathbf{k}\mathfrak{M}(X)$ monic with respect to $\leqslant$. Define a term-rewriting system associated to S as

$$\Pi_S := \{\, q|_{\overline{s}} \rightarrow q|_{R(s)} \mid s = \overline{s} \dotplus (-R(s)) \in S,\ q \in \mathfrak{M}^{\star}(X) \,\} \subseteq \mathfrak{M}(X) \times \mathbf{k}\mathfrak{M}(X). \tag{13}$$

For notation clarify, we denote $\rightarrow_{\Pi_S}$ (resp. $\overset{*}{\rightarrow}_{\Pi_S}$, resp. $\downarrow_{\Pi_S}$) by $\rightarrow_S$ (resp. $\overset{*}{\rightarrow}_S$, resp. $\downarrow_S$). In more detail when a specific $s \in S$ is used in one step rewriting, we replace $\rightarrow_S$ by $\rightarrow_s$. If $\leqslant$ is a monomial order on $\mathfrak{M}(X)$, we have $\overline{q|_{R(s)}} = q|_{\overline{R(s)}} < q|_{\overline{s}}$ by $\overline{R(s)} < \overline{s}$. So Π_S is compatible with $\leqslant$ in the sense in Definition 2.30 (*c*).

Remark 2.38. Let $f, g \in \mathbf{k}\mathfrak{M}(X)$.

(a) If $f \to_S g$, then we can write $f = cq|_{\overline{s}} \dotplus f'$ and $g = cq|_{R(s)} + f'$ for some $c \in \mathbf{k}^{\times}$, $q \in \mathfrak{M}^{\star}(X)$, $s \in S$ and $f' \in \mathbf{k}\mathfrak{M}(X)$ by Definition 2.24 (a). So $f - g = cq|_{\overline{s}-R(s)} = cq|_s \in \mathrm{Id}(S)$ by Lemma 2.15.

(b) If $f \xrightarrow{*}_S g$, then $f =: f_0 \to_S f_1 \to_S \cdots \to_S f_n := g$ for some $n \geqslant 0$, $f_i \in \mathbf{k}\mathfrak{M}(X), 0 \leqslant i \leqslant n$. If $n = 0$, then $f = g$ and $f - g \in \mathrm{Id}(S)$. If $n \geqslant 1$, then by Item (a),

$$f - g = (f_0 - f_1) + (f_1 - f_2) + \cdots + (f_{n-1} - f_n) \in \mathrm{Id}(S).$$

Lemma 2.39. *If $u \dotplus v$, then $q|_u \dotplus q|_v$ for any $q \in \mathfrak{M}^{\star}(X)$ and $u, v \in \mathbf{k}\mathfrak{M}(X)$.*

Proof. Write $u = \sum_i c_i u_i$ and $v = \sum_j d_j v_j$, where each $c_i, d_j \in \mathbf{k}^{\times}$ and $u_i, v_j \in \mathfrak{M}(X)$. Then

$$q|_u = \sum_i c_i q|_{u_i} \text{ and } q|_v = \sum_j d_j q|_{v_j}.$$

Suppose for a contrary that $q|_u \dotplus q|_v$ fails. Then $q|_{u_i} = q|_{v_j}$ by Definition 2.19 for some i, j. This implies that $u_i = v_j \in \mathrm{Supp}(u) \cap \mathrm{Supp}(v)$, contradicting that $u \dotplus v$. $\square$

The following results are characterized in [16]. For completeness, we record the proof here.

Lemma 2.40. *Let $\leqslant$ be a linear order on $\mathfrak{M}(X)$ and $S \subseteq \mathbf{k}\mathfrak{M}(X)$ monic with respect to $\leqslant$.*

(a) *If Π_S is confluent, then, $u \in \mathrm{Id}(S)$ if and only if $u \xrightarrow{*}_{\Pi_S} 0$.*
(b) *If Π_S is confluent, then $\mathrm{Id}(S) \cap \mathbf{k}\mathrm{Irr}(S) = 0$.*
(c) *If Π_S is terminating and $\mathrm{Id}(S) \cap \mathbf{k}\mathrm{Irr}(S) = 0$, then Π_S is confluent.*
(d) *If Π_S is terminating, then $\mathbf{k}\mathfrak{M}(X) = \mathrm{Id}(S) + \mathbf{k}\mathrm{Irr}(S)$,*

where $\mathrm{Irr}(S) = \mathfrak{M}(X) \setminus \{q|_{\overline{s}} \mid s \in S\}$.

Proof. Note that $\mathbf{k}\mathrm{Irr}(S)$ is precisely the set of normal forms of Π_S.

(a) If $u \xrightarrow{*}_{\Pi_S} 0$, then $u \in \mathrm{Id}(S)$ by Remark 2.38 (b). Conversely, let $u \in \mathrm{Id}(S)$. By Eq. (5), we have

$$u = \sum_{i=1}^{n} c_i q_i|_{s_i}, \text{ where } c_i \in \mathbf{k}^{\times}, s_i \in S, q_i \in \mathfrak{M}^{\star}(X), 1 \leqslant i \leqslant n.$$

For each $s_i = \overline{s_i} \dotplus (-R(s_i))$ with $1 \leqslant i \leqslant n$, it follows from Lemmas 2.20 and 2.39 that

$$c_i q_i|_{s_i} = c_i q_i|_{\overline{s_i}} \dotplus (-c_i q_i|_{R(s_i)}) \to_{\Pi_S} c_i q_i|_{R(s_i)} - c_i q_i|_{R(s_i)} = 0 \text{ and so } c_i q_i|_{s_i} \downarrow_{\Pi_S} 0$$

by Remark 2.25 (b). Since Π_S is confluent, $u = \sum_{i=1}^{n} c_i q_i|_{s_i} \xrightarrow{*}_{\Pi_S} 0$ by Lemma 2.28.

(b) If $\mathrm{Id}(S) \cap \mathbf{k}\mathrm{Irr}(S) \neq 0$, let $0 \neq w \in \mathrm{Id}(S) \cap \mathbf{k}\mathrm{Irr}(S)$. Since $w \in \mathbf{k}\mathrm{Irr}(S)$, w is of normal form. On the other hand, from $w \in \mathrm{Id}(S)$ and Item (a), we have $w \xrightarrow{*}_{\Pi_S} 0$. So w has two normal forms w and 0, contradicting that Π_S is confluent.

(c) Suppose for a contrary that Π_S is not confluent. Since Π_S is terminating, there is $w \in \mathbf{k}\mathfrak{M}(X)$ such that w has two distinct normal forms, say u and v. Thus $u, v \in \mathbf{k}\mathrm{Irr}(S)$ and so $u - v \in \mathbf{k}\mathrm{Irr}(S)$. Since $w \xrightarrow{*}_{\Pi} u$ and $w \xrightarrow{*} v_{\Pi}$, we have $w - u, w - v \in \mathrm{Id}(S)$ by Remark 2.38 (b). Hence $0 \neq u - v \in \mathrm{Id}(S) \cap \mathbf{k}\mathrm{Irr}(S)$, a contradiction.

(d) Let $w \in \mathbf{k}\mathfrak{M}(X)$, since Π_S is terminating, w has a normal form $u \in \mathbf{k}\mathrm{Irr}(S)$ and $w \xrightarrow{*}_{\Pi} u$. From Remark 2.38 ($b$), we have $w - u \in \mathrm{Id}(S)$ and so $w \in \mathrm{Id}(S) + \mathbf{k}\mathrm{Irr}(S)$. $\square$

Theorem 2.41. *Let $\leqslant$ be a monomial order on $\mathfrak{M}(X)$ and $S \subseteq \mathbf{k}\mathfrak{M}(X)$ monic with respect to $\leqslant$. Then the followings are equivalent.*

(*a*) Π_S *is convergent.*
(*b*) Π_S *is confluent.*
(*c*) $\mathrm{Id}(S) \cap \mathbf{k}\mathrm{Irr}(S) = 0$.
(*d*) $\mathrm{Id}(S) \oplus \mathbf{k}\mathrm{Irr}(S) = \mathbf{k}\mathfrak{M}(X)$.
(*e*) S *is a Gr obner-Shirshov basis in* $\mathbf{k}\mathfrak{M}(X)$,

where $\mathrm{Irr}(S) = \mathfrak{M}(X) \setminus \{q|_{\overline{s}} \mid s \in S\}$.

Proof. Since $\leqslant$ is a monomial order on $\mathfrak{M}(X)$, Π_S is terminating [17]. So Item (*a*) and Item (*b*) are equivalent. The equivalence of Item (*b*) and Item (*c*) is followed from Items (*b*) and (*c*) in Lemma 2.40.

Clearly, Item (*d*) implies Item (*c*). The converse is employed Item (*d*) in Lemma 2.40. At last, the equivalence of Item (*d*) and Item (*e*) is obtained from Lemma 2.18. □

3. A basis of the free averaging algebra

In this section, we give a basis of the free averaging algebra. We begin with a lemma.

Lemma 3.1. *Let* $S \subseteq \mathbf{k}\mathfrak{M}(X)$, $q \in \mathfrak{M}^\star(X)$ *and* $\leqslant$ *a linear order on* $\mathfrak{M}(X)$. *Then*

(*a*) *If* $u \overset{*}{\to}_S v$ *with* $u, v \in \mathbf{k}\mathfrak{M}(X)$, *then* $q|_u \overset{*}{\to}_S q|_v$.
(*b*) *If* $u \downarrow_S v$, *then* $q|_u \downarrow_S q|_v$.

Proof. (*a*) If $u = v$, then $q|_u = q|_v$ and $q|_u \overset{*}{\to}_S q|_v$ by Remark 2.25 (*a*). Suppose $u \neq v$. Let $m \geqslant 1$ be the least number such that u rewrites to v in m steps. We prove the result by induction on m. For the initial step $m = 1$, since $u \to_S v$, we may write

$$u = cp|_{\overline{s}} \dotplus u' \;\text{ and }\; v = cp|_{R(s)} + u' \;\text{ for some }\; c \in \mathbf{k}^\times, s \in S, p \in \mathfrak{M}^\star(X), u' \in \mathbf{k}\mathfrak{M}(X).$$

Then from Lemma 2.39,

$$q|_u = c(q|_p)|_{\overline{s}} \dotplus q|_{u'} \to_S c(q|_p)|_{R(s)} + q|_{u'} = q|_{cp|_{R(s)}+u'} = q|_v.$$

Assume the result is true for $m \leqslant n$ and consider the case of $m = n + 1 \geqslant 2$. Then we can write $u \to_S w \overset{*}{\to}_S v$ for some $u \neq w \in \mathbf{k}\mathfrak{M}(X)$. By the minimality of m, we have $w \neq v$. Using induction hypothesis, we get $q|_u \overset{*}{\to}_S q|_w$ and $q|_w \overset{*}{\to}_S q|_v$. This implies that $q|_u \overset{*}{\to}_S q|_v$, as required.

(*b*) Since $u \downarrow_S v$, we may suppose by Definition 2.24 (*c*) that $u \overset{*}{\to}_S w$ and $v \overset{*}{\to}_S w$ for some $w \in \mathbf{k}\mathfrak{M}(X)$. Then by Item (*a*), we have $q|_u \overset{*}{\to}_S q|_w$ and $q|_v \overset{*}{\to}_S q|_w$. So $q|_u \downarrow_S q|_v$. This completes the proof. □

The following is a concept finer than subwords, including the information of placements [39].

Definition 3.2. Let $w \in \mathfrak{M}(X)$ such that

$$q_1|_{u_1} = w = q_2|_{u_2} \;\text{ for some }\; u_1, u_2 \in \mathfrak{M}(X), q_1, q_2 \in \mathfrak{M}^\star(X). \tag{14}$$

The two placements (u_1, q_1) and (u_2, q_2) are called

(*a*) *separated* if there exist $p \in \mathfrak{M}^{\star_1,\star_2}(X)$ and $a, b \in \mathfrak{M}(X)$ such that $q_1|_{\star_1} = p|_{\star_1, b}$, $q_2|_{\star_2} = p|_{a, \star_2}$, and $w = p|_{a, b}$;
(*b*) *nested* if there exists $q \in \mathfrak{M}^\star(X)$ such that either $q_2 = q_1|_q$ or $q_1 = q_2|_q$;
(*c*) *intersecting* if there exist $q \in \mathfrak{M}^\star(X)$ and $a, b, c \in \mathfrak{M}(X)\setminus\{1\}$ such that $w = q|_{abc}$ and either
  (i) $q_1 = q|_{\star c}$ and $q_2 = q|_{a\star}$; or
  (ii) $q_1 = q|_{a\star}$ and $q_2 = q|_{\star c}$.

Lemma 3.3. [39, Thm. 4.11] *Let $w \in \mathfrak{M}(X)$. For any two placements (u_1, q_1) and (u_2, q_2) in w, exactly one of the following is true:*

(*a*) (u_1, q_1) *and* (u_2, q_2) *are separated;*
(*b*) (u_1, q_1) *and* (u_2, q_2) *are nested;*
(*c*) (u_1, q_1) *and* (u_2, q_2) *are intersecting.*

Now we fix some notations which will be used through out the remainder of the paper. For any $u \in \mathfrak{M}(X)$, define recursively $\lfloor u\rfloor^{(1)} := \lfloor u\rfloor$ and $\lfloor u\rfloor^{(k+1)} := \lfloor\lfloor u\rfloor^{(k)}\rfloor$ for $k \geqslant 1$. Recall from Example 2.7 that

$$\phi(x_1, x_2) := \lfloor x_1\rfloor\lfloor x_2\rfloor - \lfloor\lfloor x_1\rfloor x_2\rfloor \text{ and } \psi(x_1, x_2) := \lfloor x_1\lfloor x_2\rfloor\rfloor - \lfloor\lfloor x_1\rfloor x_2\rfloor$$

are the OPIs defining the averaging operator. Let $\leqslant$ be a well-order on X such that $x_1 < x_2$. Then $\leqslant$ can be extended to the monomial order $\leqslant_{\mathrm{db}}$ on $\mathfrak{M}(X)$ [17], which will be used through out in the remainder of the paper. With respect to $\leqslant_{\mathrm{db}}$, we have

$$\begin{aligned} \overline{\phi(x_1, x_2)} &= \lfloor x_1\rfloor\lfloor x_2\rfloor, \quad R(\phi(x_1, x_2)) = \lfloor\lfloor x_1\rfloor x_2\rfloor, \\ \overline{\psi(x_1, x_2)} &= \lfloor x_1\lfloor x_2\rfloor\rfloor, \quad R(\psi(x_1, x_2)) = \lfloor\lfloor x_1\rfloor x_2\rfloor. \end{aligned} \tag{15}$$

The term-rewriting system associated to $\phi(x_1, x_2), \psi(x_1, x_2)$ is not confluent. For example, for the element $\lfloor\lfloor x_1\rfloor\lfloor x_2\rfloor\rfloor \in \mathfrak{M}(X)$, on the one hand,

$$\lfloor\lfloor x_1\rfloor\lfloor x_2\rfloor\rfloor \to_{\phi(x_1,x_2)} \lfloor\lfloor\lfloor x_1\rfloor x_2\rfloor\rfloor = \lfloor\lfloor x_1\rfloor x_2\rfloor^{(2)},$$

which is in normal form. On the other hand,

$$\lfloor\lfloor x_1\rfloor\lfloor x_2\rfloor\rfloor \to_{\psi(x_1,x_2)} \lfloor\lfloor\lfloor x_1\rfloor\rfloor x_2\rfloor = \lfloor\lfloor x_1\rfloor^{(2)} x_2\rfloor,$$

which is in normal form. So the element $\lfloor\lfloor x_1\rfloor\lfloor x_2\rfloor\rfloor$ is not confluent. For the desired confluence, we need more rewriting rules. Let

$$\varphi(x_1, x_2) := \lfloor\lfloor x_1\rfloor x_2\rfloor^{(2)} - \lfloor\lfloor x_1\rfloor^{(2)} x_2\rfloor \text{ and } \Phi := \{\phi(x_1, x_2),\ \psi(x_1, x_2),\ \varphi(x_1, x_2)\}. \tag{16}$$

With respect to $\leqslant_{\mathrm{db}}$, we have

$$\overline{\varphi(x_1, x_2)} = \lfloor\lfloor x_1\rfloor x_2\rfloor^{(2)} \text{ and } R(\varphi(x_1, x_2)) = \lfloor\lfloor x_1\rfloor^{(2)} x_2\rfloor. \tag{17}$$

Let $u_1, u_2 \in \mathfrak{M}(X)$. Then by Eq. (3),

$$\phi(u_1, u_2) = \lfloor u_1\rfloor\lfloor u_2\rfloor - \lfloor\lfloor u_1\rfloor u_2\rfloor \in S_\phi(X),$$

and by Lemma 2.15,

$$\lfloor\lfloor u_1\rfloor\lfloor u_2\rfloor\rfloor - \lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)} = \lfloor\star\rfloor|_{\phi(u_1,u_2)} \in \mathrm{Id}(S_\phi(X)) \subseteq \mathrm{Id}(S_\phi(X) \cup S_\psi(X)).$$

With the same argument,

$$\lfloor\lfloor u_1\rfloor\lfloor u_2\rfloor\rfloor - \lfloor\lfloor u_1\rfloor^{(2)} u_2\rfloor = \psi(\lfloor u_1\rfloor, u_2) \in S_\psi(X) \subseteq \mathrm{Id}(S_\psi(X)) \subseteq \mathrm{Id}(S_\phi(X) \cup S_\psi(X)).$$

This implies that

$$\begin{aligned} &\varphi(u_1, u_2) = \lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)} - \lfloor\lfloor u_1\rfloor^{(2)} u_2\rfloor \\ =&\lfloor\lfloor u_1\rfloor\lfloor u_2\rfloor\rfloor - \lfloor\lfloor u_1\rfloor^{(2)} u_2\rfloor - (\lfloor\lfloor u_1\rfloor\lfloor u_2\rfloor\rfloor - \lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)}) \in \mathrm{Id}(S_\phi(X) \cup S_\psi(X)) \end{aligned}$$

and so $\mathrm{Id}(S_\varphi(X)) \subseteq \mathrm{Id}(S_\phi(X) \cup S_\psi(X))$. Hence by Eqs. (4) and (16),

$$\mathrm{Id}(S_\Phi(X)) = \mathrm{Id}(S_\phi(X) \cup S_\psi(X)). \tag{18}$$

Remark 3.4. If $u_2 = 1$, then $\varphi(u_1, u_2)$ degenerates to

$$\varphi(u_1, u_2) = \lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)} - \lfloor\lfloor u_1\rfloor^{(2)} u_2\rfloor = \lfloor u_1\rfloor^{(3)} - \lfloor u_1\rfloor^{(3)} = 0.$$

So we always assume $u_2 \neq 1$ in $\varphi(u_1, u_2)$. This is our running hypothesis in the remainder of the paper.

Remark 3.5. From Eqs. (15) and (17), we have

(*a*) for any $\alpha(x_1, x_2) \in \Phi$ and $u_1, u_2 \in \mathfrak{M}(X)$, $\underline{R(\alpha(u_1, u_2))} \in \mathfrak{M}(X)$ is a monomial.
(*b*) for any $u_1, u_2 \in \mathfrak{M}(X)$, the breadth $|\overline{\phi(u_1, u_2)}| = 2$ and $|\overline{\psi(u_1, u_2)}| = |\overline{\varphi(u_1, u_2)}| = 1$.

Recall Φ is fixed in Eq. (16). In Eq. (13), taking $S = S_\Phi(X)$ defined in Eq. (4), we get a term-rewriting system associated to Φ (with respect to $\leqslant_{\rm db}$)

$$\Pi_\Phi := \Pi_{S_\Phi(X)} = \{\, q|_{\overline{\alpha(u_1,u_2)}} \to q|_{R(\alpha(u_1,u_2))} \mid \alpha(x_1, x_2) \in \Phi, q \in \mathfrak{M}^\star(X), u_1, u_2 \in \mathfrak{M}(X)\}. \tag{19}$$

For notation clarity, we abbreviate $\to_{\alpha(u_1,u_2)}$ as $\to_\alpha$. Now we are in the position to consider the confluence of the term-rewriting system Π_Φ. By Theorem 2.36, we only need to consider the confluence of basis elements. Take a local fork of a basis element $w \in \mathfrak{M}(X)$:

$$(q_1|_{R(\alpha(u_1,u_2))}\ {}_\alpha\!\leftarrow q_1|_{\overline{\alpha(u_1,u_2)}} = w = q_2|_{\overline{\beta(v_1,v_2)}} \to_\beta q_2|_{R(\beta(v_1,v_2))}),$$

where

$$\alpha(x_1, x_2), \beta(x_1, x_2) \in \Phi,\ u_i, v_i \in \mathfrak{M}(X),\ i = 1, 2.$$

According to Lemma 3.3, the two placements $(\overline{\alpha(u_1, u_2)}, q_1)$ and $(\overline{\beta(v_1, v_2)}, q_2)$ are separated, or intersecting, or nested. We consider firstly the former two cases.

Lemma 3.6. *Let $\alpha(x_1, x_2), \beta(x_1, x_2) \in \Phi$ and $q_1|_{\overline{\alpha(u_1,u_2)}} = q_2|_{\overline{\beta(v_1,v_2)}}$ for some $q_1, q_2 \in \mathfrak{M}^\star(X)$ and $u_i, v_i \in \mathfrak{M}(X)$, $i = 1, 2$. If the placements $(\overline{\alpha(u_1, u_2)}, q_1)$ and $(\overline{\beta(v_1, v_2)}, q_2)$ are separated, then $q_1|_{R(\alpha(u_1,u_2))} \downarrow_\Phi q_2|_{R(\beta(v_1,v_2))}$.*

Proof. In view of Definition 3.2 (*a*), there exists $p \in \mathfrak{M}^{\star_1,\star_2}(X)$ such that

$$q_1|_{\star_1} = p|_{\star_1, \overline{\beta(v_1,v_2)}} \text{ and } q_2|_{\star_2} = p|_{\overline{\alpha(u_1,u_2)},\, \star_2}.$$

On the one hand,

$$q_1|_{R(\alpha(u_1,u_2))} = p|_{R(\alpha(u_1,u_2)), \overline{\beta(v_1,v_2)}} \to_\beta p|_{R(\alpha(u_1,u_2)), R(\beta(v_1,v_2))}, \tag{20}$$

where the last step employs the facts that $R(\alpha(u_1, u_2))$ is a monomial by Remark 3.5 (*a*) and so is $p|_{R(\alpha(u_1,u_2)), \overline{\beta(v_1,v_2)}}$. On the other hand,

$$q_2|_{R(\beta(v_1,v_2))} = p|_{\overline{\alpha(u_1,u_2)},\, R(\beta(v_1,v_2))} \to_\alpha p|_{R(\alpha(u_1,u_2)), R(\beta(v_1,v_2))}. \tag{21}$$

Comparing Eqs (20) and (21), we conclude that $q_1|_{R(\alpha(u_1,u_2))} \downarrow_\Phi q_2|_{R(\beta(v_1,v_2))}$. ◻

Lemma 3.7. *Let $\alpha(x_1, x_2), \beta(x_1, x_2) \in \Phi$ and $q_1|_{\overline{\alpha(u_1,u_2)}} = q_2|_{\overline{\beta(v_1,v_2)}}$ for some $q_1, q_2 \in \mathfrak{M}^\star(X)$ and $u_i, v_i \in \mathfrak{M}(X)$, $i = 1, 2$. If the placements $(\overline{\alpha(u_1, u_2)}, q_1)$ and $(\overline{\beta(v_1, v_2)}, q_2)$ are intersecting, then $q_1|_{R(\alpha(u_1,u_2))} \downarrow_\Phi q_2|_{R(\beta(v_1,v_2))}$.*

Proof. If the two placements $(\overline{\alpha(u_1, u_2)}, q_1)$ and $(\overline{\beta(v_1, v_2)}, q_2)$ are intersecting, by symmetry, we may assume that Item (*c*) (i) in Definition 3.2 holds. Then $q_1 \neq q_2$, because if $q_1 = q_2$, then $\star c = a\star$, a contradiction. So

$$q|_{\overline{\alpha(u_1,u_2)}\,c} = q_1|_{\overline{\alpha(u_1,u_2)}} = q_2|_{\overline{\beta(v_1,v_2)}} = q|_{a\,\overline{\beta(v_1,v_2)}} = q|_{abc} \tag{22}$$

and

$$\overline{\alpha(u_1,u_2)}\,c = a\,\overline{\beta(v_1,v_2)} = abc.$$

This implies that

$$\overline{\alpha(u_1,u_2)} = ab \text{ and } \overline{\beta(v_1,v_2)} = bc. \tag{23}$$

If the breadth $|\overline{\alpha(u_1,u_2)}| = 1$, then $a = 1$ or $b = 1$, both contradicting that $a, b \neq 1$ in Definition 3.2 (c). Similarly, if the breadth $|\overline{\beta(v_1,v_2)}| = 1$, then $b = 1$ or $c = 1$, again a contradiction. So $|\overline{\alpha(u_1,u_2)}| \neq 1$ and $|\overline{\beta(v_1,v_2)}| \neq 1$. Hence by Remark 3.5 ($b$),

$$\alpha(x_1,x_2) = \beta(x_1,x_2) = \phi(x_1,x_2) = \lfloor x_1\rfloor\lfloor x_2\rfloor - \lfloor\lfloor x_1\rfloor x_2\rfloor.$$

From Eq. (23), we have

$$\overline{\alpha(u_1,u_2)} = \lfloor u_1\rfloor\lfloor u_2\rfloor = ab \text{ and } \overline{\beta(v_1,v_2)} = \lfloor v_1\rfloor\lfloor v_2\rfloor = bc$$

and so $\lfloor u_1\rfloor = a$, $\lfloor u_2\rfloor = b = \lfloor v_1\rfloor$, $u_2 = v_1$ and $\lfloor v_2\rfloor = c$. From Eqs. (15) and (17),

$$R(\alpha(u_1,u_2))c = R(\phi(u_1,u_2))c = \lfloor\lfloor u_1\rfloor u_2\rfloor\lfloor v_2\rfloor \to_\phi \lfloor\lfloor\lfloor u_1\rfloor u_2\rfloor v_2\rfloor$$

and

$$aR(\beta(v_1,v_2)) = aR(\phi(v_1,v_2)) = aR(\phi(u_2,v_2)) = \lfloor u_1\rfloor\lfloor\lfloor u_2\rfloor v_2\rfloor \to_\phi \lfloor\lfloor u_1\rfloor\lfloor u_2\rfloor v_2\rfloor \to_\phi \lfloor\lfloor\lfloor u_1\rfloor u_2\rfloor v_2\rfloor.$$

So $R(\alpha(u_1,u_2))c \downarrow_\Phi aR(\beta(v_1,v_2))$. This follows from Eq. (22) and Lemma 3.1 (b) that

$$q_1|_{R(\alpha(u_1,u_2))} = q|_{R(\alpha(u_1,u_2))\,c} \downarrow_\Phi q|_{a\,R(\beta(v_1,v_2))} = q_2|_{R(\beta(v_1,v_2))},$$

as required. □

Next, let us turn to consider the nested case. We need the following lemmas. The first is on the leading monomials of OPIs in Φ.

Lemma 3.8. *Let $\alpha(x_1,x_2), \beta(x_1,x_2) \in \Phi$ and $\overline{\alpha(u_1,u_2)} = \overline{\beta(v_1,v_2)}$ for some $u_i, v_i \in \mathfrak{M}(X)$, $i = 1, 2$. Then exactly one of the following is true:*

- (a) $\alpha(x_1,x_2) = \beta(x_1,x_2)$, $u_1 = v_1$, $u_2 = v_2$;
- (b) $\alpha(x_1,x_2) = \psi(x_1,x_2)$, $\beta(x_1,x_2) = \varphi(x_1,x_2)$, $u_1 = 1$, $u_2 = \lfloor v_1\rfloor v_2$;
- (c) $\alpha(x_1,x_2) = \varphi(x_1,x_2)$, $\beta(x_1,x_2) = \psi(x_1,x_2)$, $v_1 = 1$, $v_2 = \lfloor u_1\rfloor u_2$.

Proof. According to whether α and β are equal, we have the following cases to consider.

Case 1. $\alpha(x_1,x_2) = \beta(x_1,x_2)$. Then Items ($b$) and ($c$) fail. We show Item ($a$) is valid. Consider firstly that $\alpha(x_1,x_2) = \phi(x_1,x_2)$. Then

$$\lfloor u_1\rfloor\lfloor u_2\rfloor = \overline{\alpha(u_1,u_2)} = \overline{\beta(v_1,v_2)} = \lfloor v_1\rfloor\lfloor v_2\rfloor.$$

By the unique decomposition of bracketed words in Eq. (1), we have $\lfloor u_1\rfloor = \lfloor v_1\rfloor$ and $\lfloor u_2\rfloor = \lfloor v_2\rfloor$. This implies $u_1 = v_1$ and $u_2 = v_2$. Consider secondly that $\alpha(x_1,x_2) = \psi(x_1,x_2)$. Then $\lfloor u_1\lfloor u_2\rfloor\rfloor = \overline{\alpha(u_1,u_2)} = \overline{\beta(v_1,v_2)} = \lfloor v_1\lfloor v_2\rfloor\rfloor$ and so $u_1\lfloor u_2\rfloor = v_1\lfloor v_2\rfloor$. This also implies $u_1 = v_1$, $\lfloor u_2\rfloor = \lfloor v_2\rfloor$ and $u_2 = v_2$. At last, consider $\alpha(x_1,x_2) = \varphi(x_1,x_2)$. Then $\lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)} = \overline{\alpha(u_1,u_2)} = \overline{\beta(v_1,v_2)} = \lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}$ and so $\lfloor\lfloor u_1\rfloor u_2\rfloor = \lfloor\lfloor v_1\rfloor v_2\rfloor$. Thus $\lfloor u_1\rfloor u_2 = \lfloor v_1\rfloor v_2$ and so $u_1 = v_1$ and $u_2 = v_2$.

Case 2. $\alpha(x_1,x_2) \neq \beta(x_1,x_2)$. Then Item ($a$) fails.

Suppose firstly that one of $\alpha(x_1,x_2)$ and $\beta(x_1,x_2)$ is $\phi(x_1,x_2)$. By symmetry, we may let $\alpha(x_1,x_2) = \phi(x_1,x_2)$. Then $\beta(x_1,x_2) \neq \phi(x_1,x_2)$. From Remark 3.5 ($b$), $|\overline{\alpha(u_1,u_2)}| = |\overline{\phi(u_1,u_2)}| = 2$ and $|\overline{\beta(v_1,v_2)}| = 1$. This implies that $\overline{\alpha(u_1,u_2)} \neq \overline{\beta(v_1,v_2)}$, contradicting our hypothesis. Suppose $\alpha(x_1,x_2), \beta(x_1,x_2) \neq \phi(x_1,x_2)$. Then we have the following two subcases.

Case 2.1. $\alpha(x_1, x_2) = \psi(x_1, x_2)$ and $\beta(x_1, x_2) = \varphi(x_1, x_2)$. Then Item ($c$) fails and

$$\lfloor u_1 \lfloor u_2 \rfloor \rfloor = \overline{\psi(u_1, u_2)} = \overline{\alpha(u_1, u_2)} = \overline{\beta(v_1, v_2)} = \overline{\varphi(v_1, v_2)} = \lfloor \lfloor v_1 \rfloor v_2 \rfloor^{(2)}.$$

So $u_1 \lfloor u_2 \rfloor = \lfloor \lfloor v_1 \rfloor v_2 \rfloor$. This implies that $u_1 = 1$, $\lfloor u_2 \rfloor = \lfloor \lfloor v_1 \rfloor v_2 \rfloor$ and $u_2 = \lfloor v_1 \rfloor v_2$ and so Item (b) is valid.

Case 2.2. $\alpha(x_1, x_2) = \varphi(x_1, x_2)$ and $\beta(x_1, x_2) = \psi(x_1, x_2)$. Then Item ($b$) fails and

$$\lfloor \lfloor u_1 \rfloor u_2 \rfloor^{(2)} = \overline{\varphi(u_1, u_2)} = \overline{\alpha(u_1, u_2)} = \overline{\beta(v_1, v_2)} = \overline{\psi(v_1, v_2)} = \lfloor v_1 \lfloor v_2 \rfloor \rfloor.$$

This follows that $\lfloor \lfloor u_1 \rfloor u_2 \rfloor = v_1 \lfloor v_2 \rfloor$. So $v_1 = 1$, $v_2 = \lfloor u_1 \rfloor u_2$ and Item (c) is valid. □

Lemma 3.9. *Let $\alpha(x_1, x_2), \beta(x_1, x_2) \in \Phi$ and $q_1|_{\overline{\alpha(u_1,u_2)}} = q_2|_{\overline{\beta(v_1,v_2)}}$ for some $q_1, q_2 \in \mathfrak{M}^\star(X)$ and $u_i, v_i \in \mathfrak{M}(X)$, $i = 1, 2$. If $q_2 = q_1|_q$ for some $q \in \mathfrak{M}^\star(X)$ and $\overline{\beta(v_1, v_2)}$ is a subword of u_1 or u_2, then $q_1|_{R(\alpha(u_1,u_2))} \downarrow_\Phi q_2|_{R(\beta(v_1,v_2))}$.*

Proof. For clarity, write

$$\alpha := \alpha(u_1, u_2) \text{ and } \beta := \beta(v_1, v_2).$$

By symmetry we may assume that $\overline{\beta}$ is a subword of u_1 and so $u_1 = q'|_{\overline{\beta}}$ for some $q' \in \mathfrak{M}^\star(X)$. As $\alpha(x_1, x_2)$ is linear on each variable and $R(\alpha(u_1, u_2))$ is a monomial by Remark 3.5 (a), we may write

$$\overline{\alpha} = \overline{\alpha(u_1, u_2)} = p|_{u_1,u_2} \text{ and } R(\alpha) = R(\alpha(u_1, u_2)) = p'|_{u_1,u_2} \text{ for some } p, p' \in \mathfrak{M}^\star(X). \tag{24}$$

Since $q_2 = q_1|_q$ by our hypothesis, we have

$$q_1|_{\overline{\alpha}} = q_2|_{\overline{\beta}} = q_1|_{q|_{\overline{\beta}}},$$

and so

$$q|_{\overline{\beta}} = \overline{\alpha} = p|_{u_1,u_2} = p|_{q'|_{\overline{\beta}},u_2} = (p|_{q',u_2})|_{\overline{\beta}}.$$

Hence

$$q = p|_{q',u_2} = \overline{\alpha(q', u_2)}, \tag{25}$$

where the second equation employs Eq. (24). So on the one hand, we have

$$q_1|_{R(\alpha)} = q_1|_{p'|_{u_1,u_2}} = q_1|_{p'|_{q'|_{\overline{\beta}},u_2}} \to_\beta q_1|_{p'|_{q'|_{R(\beta)},u_2}}, \tag{26}$$

where the first step is followed from Eq. (24). On the other hand, we have

$$q_2|_{R(\beta)} = q_1|_{q|_{R(\beta)}} = q_1|_{\overline{\alpha(q'|_{R(\beta)}, u_2)}} \to_\alpha q_1|_{R(\alpha(q'|_{R(\beta)}, u_2))} = q_1|_{p'|_{q'|_{R(\beta)},u_2}}, \tag{27}$$

where the first step is followed from the hypothesis $q_2 = q_1|_q$, the second from Eq. (25) and the last from Eq. (24). Comparing Eqs (26) and (27), we obtain $q_1|_{R(\alpha)} \downarrow_\Phi q_2|_{R(\beta)}$. This completes the proof.

□

As an application of Theorem 2.36, we have

Theorem 3.10. *The term-rewriting system Π_Φ defined in Eq. (19) is convergent.*

Proof. Since $\leqslant_{\rm db}$ we used is a monomial order on $\mathfrak{M}(X)$, Π_Φ is terminating [17]. By Definition 2.26 (e), we are left to show that Π_Φ is confluent. From Theorem 2.36, it is sufficient to prove that Π_Φ is locally confluent for any basis element. Let

$$(q_1|_{R(\alpha(u_1,u_2))}\ {}_\Phi\!\leftarrow q_1|_{\overline{\alpha(u_1,u_2)}} = w = q_2|_{\overline{\beta(v_1,v_2)}} \to_\Phi q_2|_{R(\beta(v_1,v_2))})$$

be an arbitrary local fork of a basis element w, where

$$\alpha(x_1,x_2),\beta(x_1,x_2)\in\Phi,\ u_i,v_i\in\mathfrak{M}(X),\ i=1,2.$$

We only need to show that

$$q_1|_{R(\alpha(u_1,u_2))}\downarrow_\Phi q_2|_{R(\beta(v_1,v_2))}. \tag{28}$$

According to Lemma 3.3, the two placements $(\overline{\alpha(u_1,u_2)},q_1)$ and $(\overline{\beta(v_1,v_2)},q_2)$ are separated, or nested, or intersecting. If they are separated or intersecting, then by Lemmas 3.6 and 3.7, Eq. (28) holds. If the two placements $(\overline{\alpha(u_1,u_2)},q_1)$ and $(\overline{\beta(v_1,v_2)},q_2)$ are nested, by symmetry in Definition 3.2 (*b*), we may assume that $q_2=q_1|_q$. If $\overline{\beta(v_1,v_2)}$ is a subword of u_1 or u_2, then by Lemma 3.9, Eq. (28) holds.

Suppose $\overline{\beta(v_1,v_2)}$ is not a subword of u_1 and u_2. Note

$$q_1|_{\overline{\alpha(u_1,u_2)}}=q_2|_{\overline{\beta(v_1,v_2)}}=q_1|_{q|_{\overline{\beta(v_1,v_2)}}}\ \text{ and so }\overline{\alpha(u_1,u_2)}=q|_{\overline{\beta(v_1,v_2)}}. \tag{29}$$

Since $q_2=q_1|_q$, Eq. (28) is equivalent to

$$q_1|_{R(\alpha(u_1,u_2))}\downarrow_\Phi q_1|_{q|_{R(\beta(v_1,v_2))}}.$$

So to prove Eq. (28), by Lemma 3.1 (*b*), it is enough to show that

$$R(\alpha(u_1,u_2))\downarrow_\Phi q|_{R(\beta(v_1,v_2))}. \tag{30}$$

If $q=\star$, then $\overline{\alpha(u_1,u_2)}=\overline{\beta(v_1,v_2)}$. By Lemma 3.8, exactly one of the three items there holds. If Item (*a*) holds, then $R(\alpha(u_1,u_2))=R(\beta(v_1,v_2))$ and Eq. (30) is valid by $q=\star$. Since Item (*b*) and Item (*c*) are symmetric, we consider that Item (*b*) holds. Then

$$\alpha(x_1,x_2)=\psi(x_1,x_2),\ \beta(x_1,x_2)=\varphi(x_1,x_2),\ u_1=1,u_2=\lfloor v_1\rfloor v_2.$$

This follows from Eqs. (15) and (17) that

$$R(\alpha(u_1,u_2))=\lfloor\lfloor u_1\rfloor u_2\rfloor=\lfloor\lfloor 1\rfloor\lfloor v_1\rfloor v_2\rfloor\to_\phi\lfloor\lfloor\lfloor 1\rfloor v_1\rfloor v_2\rfloor$$

and

$$q|_{R(\beta(v_1,v_2))}=\star|_{\lfloor\lfloor v_1\rfloor^{(2)}v_2\rfloor}=\lfloor\lfloor v_1\rfloor^{(2)}v_2\rfloor=\lfloor\lfloor 1\lfloor v_1\rfloor\rfloor v_2\rfloor\to_\psi\lfloor\lfloor\lfloor 1\rfloor v_1\rfloor v_2\rfloor.$$

Hence Eq. (30) is valid.

Summing up, we are left to consider the case of that

$$q_2=q_1|_q,\ \overline{\alpha(u_1,u_2)}=q|_{\overline{\beta(v_1,v_2)}},\ q\neq\star\ \text{ and }\overline{\beta(v_1,v_2)}\text{ is not a subword of }u_1\text{ and }u_2. \tag{31}$$

Then

$$q_1\neq q_2\ \text{ and }\overline{\alpha(u_1,u_2)}\neq\overline{\beta(v_1,v_2)}. \tag{32}$$

We have the following cases to consider.

Case 1. $\alpha(x_1,x_2)=\phi(x_1,x_2)$. Then $\overline{\alpha(u_1,u_2)}=\lfloor u_1\rfloor\lfloor u_2\rfloor$ by Eq. (15).

If $\beta(x_1,x_2)=\phi(x_1,x_2)$, then

$$\lfloor u_1\rfloor\lfloor u_2\rfloor=\overline{\alpha(u_1,u_2)}=q|_{\overline{\beta(v_1,v_2)}}=q|_{\lfloor v_1\rfloor\lfloor v_2\rfloor},$$

that is, $\lfloor v_1\rfloor\lfloor v_2\rfloor$ is a subword of $\lfloor u_1\rfloor\lfloor u_2\rfloor$. By Eq. (32), $\lfloor v_1\rfloor\lfloor v_2\rfloor\neq\lfloor u_1\rfloor\lfloor u_2\rfloor$. So $\lfloor v_1\rfloor\lfloor v_2\rfloor$ is a subword of $\lfloor u_1\rfloor$ or $\lfloor u_2\rfloor$. Since $\lfloor v_1\rfloor\lfloor v_2\rfloor\neq\lfloor u_1\rfloor,\lfloor u_2\rfloor$ by comparing the breadth, $\lfloor v_1\rfloor\lfloor v_2\rfloor$ is a subword of u_1 or u_2 by Lemma 2.14 (*a*), contradicting Eq. (31). So $\beta(x_1,x_2)\neq\phi(x_1,x_2)$.

Subcase 1.1. $\beta(x_1,x_2)=\psi(x_1,x_2)$. In this subcase, we have

$$\lfloor u_1\rfloor\lfloor u_2\rfloor=\overline{\alpha(u_1,u_2)}=q|_{\overline{\beta(v_1,v_2)}}=q|_{\lfloor v_1\lfloor v_2\rfloor\rfloor}, \tag{33}$$

that is, $\lfloor v_1\lfloor v_2\rfloor\rfloor$ is a subword of $\lfloor u_1\rfloor\lfloor u_2\rfloor$. By Lemma 2.14 (*b*), either $\lfloor v_1\lfloor v_2\rfloor\rfloor$ is a subword of $\lfloor u_1\rfloor$ or $\lfloor v_1\lfloor v_2\rfloor\rfloor$ is a subword of $\lfloor u_2\rfloor$. Note that $\overline{\beta(v_1,v_2)} = \lfloor v_1\lfloor v_2\rfloor\rfloor$ is not a subword of u_1 and u_2 by Eq. (31). From Lemma 2.14 (*a*) and Eq. (33), either

$$\lfloor v_1\lfloor v_2\rfloor\rfloor = \lfloor u_1\rfloor \text{ and } q = \star\lfloor u_2\rfloor, \tag{34}$$

or

$$\lfloor v_1\lfloor v_2\rfloor\rfloor = \lfloor u_2\rfloor \text{ and } q = \lfloor u_1\rfloor\star. \tag{35}$$

For the former case of Eq. (34), we have

$$R(\phi(u_1,u_2)) = \lfloor\lfloor u_1\rfloor u_2\rfloor = \lfloor\lfloor v_1\lfloor v_2\rfloor\rfloor u_2\rfloor \to_\psi \lfloor\lfloor\lfloor v_1\rfloor v_2\rfloor u_2\rfloor$$

and

$$q|_{R(\psi(v_1,v_2))} = (\star\lfloor u_2\rfloor)|_{\lfloor\lfloor v_1\rfloor v_2\rfloor} = \lfloor\lfloor v_1\rfloor v_2\rfloor\lfloor u_2\rfloor \to_\phi \lfloor\lfloor\lfloor v_1\rfloor v_2\rfloor u_2\rfloor.$$

Hence $R(\phi(u_1,u_2)) \downarrow_\Phi q|_{R(\psi(v_1,v_2))}$ and Eq. (30) holds, as needed. For the later case of Eq. (35), we have $u_2 = v_1\lfloor v_2\rfloor$. So

$$R(\phi(u_1,u_2)) = \lfloor\lfloor u_1\rfloor u_2\rfloor = \lfloor\lfloor u_1\rfloor v_1\lfloor v_2\rfloor\rfloor \to_\psi \lfloor\lfloor\lfloor u_1\rfloor v_1\rfloor v_2\rfloor$$

and

$$q|_{R(\psi(v_1,v_2))} = (\lfloor u_1\rfloor\star)|_{\lfloor\lfloor v_1\rfloor v_2\rfloor} = \lfloor u_1\rfloor\lfloor\lfloor v_1\rfloor v_2\rfloor \to_\phi \lfloor\lfloor u_1\rfloor\lfloor v_1\rfloor v_2\rfloor \to_\phi \lfloor\lfloor\lfloor u_1\rfloor v_1\rfloor v_2\rfloor.$$

Hence $R(\phi(u_1,u_2)) \downarrow_\Phi q|_{R(\psi(v_1,v_2))}$ and Eq. (30) holds, as needed.

Subcase 1.2. $\beta(x_1,x_2) = \varphi(x_1,x_2)$. In this subcase, we have

$$\lfloor u_1\rfloor\lfloor u_2\rfloor = \overline{\alpha(u_1,u_2)} = q|_{\overline{\beta(v_1,v_2)}} = q|_{\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}}, \tag{36}$$

that is, $\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}$ is a subword of $\lfloor u_1\rfloor\lfloor u_2\rfloor$. By Lemma 2.14 (*b*), either $\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}$ is a subword of $\lfloor u_1\rfloor$ or $\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}$ is a subword of or $\lfloor u_2\rfloor$. Since $\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}$ is not a subword or u_1 and u_2 by Eq. (31), from Lemma 2.14 (*a*) and Eq (36), either

$$\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)} = \lfloor u_1\rfloor \text{ and } q = \star\lfloor u_2\rfloor \tag{37}$$

or

$$\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)} = \lfloor u_2\rfloor \text{ and } q = \lfloor u_1\rfloor\star. \tag{38}$$

Consider firstly the former case of Eq. (37). We have

$$R(\phi(u_1,u_2)) = \lfloor\lfloor u_1\rfloor u_2\rfloor = \lfloor\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)} u_2\rfloor \to_\varphi \lfloor\lfloor\lfloor v_1\rfloor^{(2)} v_2\rfloor u_2\rfloor$$

and

$$q|_{R(\varphi(v_1,v_2))} = (\star\lfloor u_2\rfloor)|_{\lfloor\lfloor v_1\rfloor^{(2)} v_2\rfloor} = \lfloor\lfloor v_1\rfloor^{(2)} v_2\rfloor\lfloor u_2\rfloor \to_\phi \lfloor\lfloor\lfloor v_1\rfloor^{(2)} v_2\rfloor u_2\rfloor.$$

Hence $R(\phi(u_1,u_2)) \downarrow_\Phi q|_{R(\varphi(v_1,v_2))}$ and Eq. (30) holds. For the later case of Eq. (38), we have $u_2 = \lfloor\lfloor v_1\rfloor v_2\rfloor$. Then

$$\begin{aligned} R(\phi(u_1,u_2)) &= \lfloor\lfloor u_1\rfloor u_2\rfloor = \lfloor\lfloor u_1\rfloor\lfloor\lfloor v_1\rfloor v_2\rfloor\rfloor \to_\phi \lfloor\lfloor\lfloor u_1\rfloor\lfloor v_1\rfloor v_2\rfloor\rfloor = \lfloor\lfloor u_1\rfloor\lfloor v_1\rfloor v_2\rfloor^{(2)} \to_\phi \lfloor\lfloor\lfloor u_1\rfloor v_1\rfloor v_2\rfloor^{(2)} \\ &\to_\varphi \lfloor\lfloor\lfloor u_1\rfloor v_1\rfloor^{(2)} v_2\rfloor \to_\varphi \lfloor\lfloor\lfloor u_1\rfloor^{(2)} v_1\rfloor v_2\rfloor \end{aligned}$$

and

$$\begin{aligned} q|_{R(\varphi(v_1,v_2))} &= (\lfloor u_1\rfloor\star)|_{\lfloor\lfloor v_1\rfloor^{(2)} v_2\rfloor} = \lfloor u_1\rfloor\lfloor\lfloor v_1\rfloor^{(2)} v_2\rfloor \to_\phi \lfloor\lfloor u_1\rfloor\lfloor v_1\rfloor^{(2)} v_2\rfloor \to_\phi \lfloor\lfloor\lfloor u_1\rfloor\lfloor v_1\rfloor\rfloor v_2\rfloor \\ &\to_\phi \lfloor\lfloor\lfloor\lfloor u_1\rfloor v_1\rfloor\rfloor v_2\rfloor = \lfloor\lfloor\lfloor u_1\rfloor v_1\rfloor^{(2)} v_2\rfloor \to_\varphi \lfloor\lfloor\lfloor u_1\rfloor^{(2)} v_1\rfloor v_2\rfloor. \end{aligned}$$

Hence $R(\phi(u_1,u_2)) \downarrow_\Phi q|_{R(\varphi(v_1,v_2))}$ and Eq. (30) holds, as needed.

Case 2. $\alpha(x_1,x_2) = \psi(x_1,x_2)$. Then $\overline{\alpha(u_1,u_2)} = \lfloor u_1\lfloor u_2\rfloor\rfloor$ by Eq. (15).

Case 2.1. $\beta(x_1, x_2) = \phi(x_1, x_2)$. In this subcase, we have

$$\lfloor u_1 \lfloor u_2 \rfloor \rfloor = \overline{\alpha(u_1, u_2)} = q|_{\overline{\beta(v_1,v_2)}} = q|_{\lfloor v_1 \rfloor \lfloor v_2 \rfloor}, \tag{39}$$

that is, $\lfloor v_1 \rfloor \lfloor v_2 \rfloor$ is a subword of $\lfloor u_1 \lfloor u_2 \rfloor \rfloor$. Since $\lfloor v_1 \rfloor \lfloor v_2 \rfloor \neq \lfloor u_1 \lfloor u_2 \rfloor \rfloor$ by Eq. (32), it follows from Lemma 2.14 (*a*) that $\lfloor v_1 \rfloor \lfloor v_2 \rfloor$ is a subword of $u_1 \lfloor u_2 \rfloor$. Note $\lfloor v_1 \rfloor \lfloor v_2 \rfloor$ is not a subword of u_1 or u_2 by Eq. (31). So $a \lfloor v_1 \rfloor \lfloor v_2 \rfloor = u_1 \lfloor u_2 \rfloor$ for some $a \in \mathfrak{M}(X)$ and $q = \lfloor a \star \rfloor$ by Eq. (39). Then $a \lfloor v_1 \rfloor = u_1$, $\lfloor v_2 \rfloor = \lfloor u_2 \rfloor$, $v_2 = u_2$. This follows that

$$R(\psi(u_1, u_2)) = \lfloor \lfloor u_1 \rfloor u_2 \rfloor = \lfloor \lfloor a \lfloor v_1 \rfloor \rfloor u_2 \rfloor \to_\psi \lfloor \lfloor \lfloor a \rfloor v_1 \rfloor u_2 \rfloor$$

and

$$q|_{R(\phi(v_1,v_2))} = (\lfloor a \star \rfloor)|_{\lfloor \lfloor v_1 \rfloor v_2 \rfloor} = \lfloor a \lfloor \lfloor v_1 \rfloor v_2 \rfloor \rfloor = \lfloor a \lfloor \lfloor v_1 \rfloor u_2 \rfloor \rfloor \to_\psi \lfloor \lfloor a \rfloor \lfloor v_1 \rfloor u_2 \rfloor \to_\phi \lfloor \lfloor \lfloor a \rfloor v_1 \rfloor u_2 \rfloor.$$

Hence $R(\psi(u_1, u_2)) \downarrow_\Phi q|_{R(\phi(v_1,v_2))}$ and Eq. (30) holds, as needed.

Case 2.2 $\beta(x_1, x_2) = \psi(x_1, x_2)$. In this subcase, we have

$$\lfloor u_1 \lfloor u_2 \rfloor \rfloor = \overline{\alpha(u_1, u_2)} = q|_{\overline{\beta(v_1,v_2)}} = q|_{\lfloor v_1 \lfloor v_2 \rfloor \rfloor}, \tag{40}$$

that is, $\lfloor v_1 \lfloor v_2 \rfloor \rfloor$ is a subword of $\lfloor u_1 \lfloor u_2 \rfloor \rfloor$. By Lemma 2.14 (*a*) and $\lfloor v_1 \lfloor v_2 \rfloor \rfloor \neq \lfloor u_1 \lfloor u_2 \rfloor \rfloor$ from Eq. (32), $\lfloor v_1 \lfloor v_2 \rfloor \rfloor$ is a subword of $u_1 \lfloor u_2 \rfloor$. Note $\lfloor v_1 \lfloor v_2 \rfloor \rfloor$ is not a subword of u_1 and u_2 by Eq. (31). So by Lemma 2.14 (*b*), $\lfloor v_1 \lfloor v_2 \rfloor \rfloor$ is a subword of $\lfloor u_2 \rfloor$. From Lemma 2.14 (*a*), we have $\lfloor v_1 \lfloor v_2 \rfloor \rfloor = \lfloor u_2 \rfloor$, $v_1 \lfloor v_2 \rfloor = u_2$ and $q = \lfloor u_1 \star \rfloor$ by Eq. (40). Thus

$$R(\psi(u_1, u_2)) = \lfloor \lfloor u_1 \rfloor u_2 \rfloor = \lfloor \lfloor u_1 \rfloor v_1 \lfloor v_2 \rfloor \rfloor \to_\psi \lfloor \lfloor \lfloor u_1 \rfloor v_1 \rfloor v_2 \rfloor$$

and

$$q|_{R(\psi(v_1,v_2))} = (\lfloor u_1 \star \rfloor)|_{\lfloor \lfloor v_1 \rfloor v_2 \rfloor} = \lfloor u_1 \lfloor \lfloor v_1 \rfloor v_2 \rfloor \rfloor \to_\psi \lfloor \lfloor u_1 \rfloor \lfloor v_1 \rfloor v_2 \rfloor \to_\phi \lfloor \lfloor \lfloor u_1 \rfloor v_1 \rfloor v_2 \rfloor.$$

Hence $R(\psi(u_1, u_2)) \downarrow_\Phi q|_{R(\psi(v_1,v_2))}$ and Eq. (30) holds, as needed.

Case 2.3. $\beta(x_1, x_2) = \varphi(x_1, x_2)$. In this subcase, we have

$$\lfloor u_1 \lfloor u_2 \rfloor \rfloor = \overline{\alpha(u_1, u_2)} = q|_{\overline{\beta(v_1,v_2)}} = q|_{\lfloor \lfloor v_1 \rfloor v_2 \rfloor^{(2)}}, \tag{41}$$

that is, $\lfloor \lfloor v_1 \rfloor v_2 \rfloor^{(2)}$ is a subword of $\lfloor u_1 \lfloor u_2 \rfloor \rfloor$. By Lemma 2.14 (*a*) and $\lfloor \lfloor v_1 \rfloor v_2 \rfloor^{(2)} \neq \lfloor u_1 \lfloor u_2 \rfloor \rfloor$ from Eq. (32), $\lfloor \lfloor v_1 \rfloor v_2 \rfloor^{(2)}$ is a subword of $u_1 \lfloor u_2 \rfloor$. Note from Eq. (31), $\lfloor \lfloor v_1 \rfloor v_2 \rfloor^{(2)}$ is not a subword of u_1 and u_2. So by Lemma 2.14 (*b*), $\lfloor \lfloor v_1 \rfloor v_2 \rfloor^{(2)}$ is a subword of $\lfloor u_2 \rfloor$. By Lemma 2.14 (*a*), $\lfloor \lfloor v_1 \rfloor v_2 \rfloor^{(2)} = \lfloor u_2 \rfloor$ and then $q = \lfloor u_1 \star \rfloor$ by Eq. (41). This implies $\lfloor \lfloor v_1 \rfloor v_2 \rfloor = u_2$. Thus

$$\begin{aligned} R(\psi(u_1, u_2)) &= \lfloor \lfloor u_1 \rfloor u_2 \rfloor = \lfloor \lfloor u_1 \rfloor \lfloor \lfloor v_1 \rfloor v_2 \rfloor \rfloor \to_\phi \lfloor \lfloor \lfloor u_1 \rfloor \lfloor v_1 \rfloor v_2 \rfloor \rfloor = \lfloor \lfloor u_1 \rfloor \lfloor v_1 \rfloor v_2 \rfloor^{(2)} \\ &\to_\phi \lfloor \lfloor \lfloor u_1 \rfloor v_1 \rfloor v_2 \rfloor^{(2)} \to_\varphi \lfloor \lfloor \lfloor u_1 \rfloor v_1 \rfloor^{(2)} v_2 \rfloor \to_\varphi \lfloor \lfloor \lfloor u_1 \rfloor^{(2)} v_1 \rfloor v_2 \rfloor \end{aligned}$$

and

$$\begin{aligned} q|_{R(\varphi(v_1,v_2))} &= (\lfloor u_1 \star \rfloor)|_{\lfloor \lfloor v_1 \rfloor^{(2)} v_2 \rfloor} = \lfloor u_1 \lfloor \lfloor v_1 \rfloor^{(2)} v_2 \rfloor \rfloor \to_\psi \lfloor \lfloor u_1 \rfloor \lfloor v_1 \rfloor^{(2)} v_2 \rfloor \to_\phi \lfloor \lfloor \lfloor u_1 \rfloor \lfloor v_1 \rfloor \rfloor v_2 \rfloor \\ &\to_\phi \lfloor \lfloor \lfloor \lfloor u_1 \rfloor v_1 \rfloor \rfloor v_2 \rfloor = \lfloor \lfloor \lfloor u_1 \rfloor v_1 \rfloor^{(2)} v_2 \rfloor \to_\varphi \lfloor \lfloor \lfloor u_1 \rfloor^{(2)} v_1 \rfloor v_2 \rfloor. \end{aligned}$$

Hence $R(\psi(u_1, u_2)) \downarrow_\Phi q|_{R(\varphi(v_1,v_2))}$ and Eq. (30) holds, as needed.

Case 3. $\alpha(x_1, x_2) = \varphi(x_1, x_2)$. Then $\overline{\alpha(u_1, u_2)} = \lfloor \lfloor u_1 \rfloor u_2 \rfloor^{(2)}$ by Eq. (17).

Subcase 3.1. $\beta(x_1, x_2) = \phi(x_1, x_2)$. In this subcase,

$$\lfloor \lfloor u_1 \rfloor u_2 \rfloor^{(2)} = \overline{\alpha(u_1, u_2)} = q|_{\overline{\beta(v_1,v_2)}} = q|_{\lfloor v_1 \rfloor \lfloor v_2 \rfloor}, \tag{42}$$

that is, $\lfloor v_1 \rfloor \lfloor v_2 \rfloor$ is a subword of $\lfloor \lfloor u_1 \rfloor u_2 \rfloor^{(2)}$. As $\lfloor v_1 \rfloor \lfloor v_2 \rfloor \neq \lfloor \lfloor u_1 \rfloor u_2 \rfloor^{(2)}$ by Eq. (31), $\lfloor v_1 \rfloor \lfloor v_2 \rfloor$ is a subword of $\lfloor \lfloor u_1 \rfloor u_2 \rfloor$ by Lemma 2.14 (*a*). Again using Lemma 2.14 (*a*), $\lfloor v_1 \rfloor \lfloor v_2 \rfloor$ is a subword of

$\lfloor u_1\rfloor u_2$ by $\lfloor v_1\rfloor\lfloor v_2\rfloor \neq \lfloor\lfloor u_1\rfloor u_2\rfloor$. From Eq. (31), $\overline{\beta(v_1,v_2)} = \lfloor v_1\rfloor\lfloor v_2\rfloor$ is not a subword of u_1 and u_2. Hence $\lfloor v_1\rfloor\lfloor v_2\rfloor a = \lfloor u_1\rfloor u_2$ for some $a\in\mathfrak{M}(X)$ and so $q = \lfloor\star a\rfloor^{(2)}$ by Eq. (42). This implies that $\lfloor v_1\rfloor = \lfloor u_1\rfloor$, $v_1 = u_1$ and $\lfloor v_2\rfloor a = u_2$. Hence

$$R(\varphi(u_1,u_2)) = \lfloor\lfloor u_1\rfloor^{(2)}u_2\rfloor = \lfloor\lfloor u_1\rfloor^{(2)}\lfloor v_2\rfloor a\rfloor \to_\phi \lfloor\lfloor\lfloor u_1\rfloor^{(2)}v_2\rfloor a\rfloor$$

and

$$\begin{aligned} q|_{R(\phi(v_1,v_2))} &= (\lfloor\star a\rfloor^{(2)})|_{\lfloor\lfloor v_1\rfloor v_2\rfloor} = \lfloor\lfloor\lfloor v_1\rfloor v_2\rfloor a\rfloor^{(2)} = \lfloor\lfloor\lfloor u_1\rfloor v_2\rfloor a\rfloor^{(2)} \\ &\to_\varphi \lfloor\lfloor\lfloor u_1\rfloor v_2\rfloor^{(2)}a\rfloor \to_\varphi \lfloor\lfloor\lfloor u_1\rfloor^{(2)}v_2\rfloor a\rfloor. \end{aligned}$$

Hence $R(\varphi(u_1,u_2)) \downarrow_\Phi q|_{R(\phi(v_1,v_2))}$ and Eq. (30) holds, as needed.

Subcase 3.2. $\beta(x_1,x_2) = \psi(x_1,x_2)$. In this subcase,

$$\lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)} = \overline{\alpha(u_1,u_2)} = q|_{\overline{\beta(v_1,v_2)}} = q|_{\lfloor v_1\lfloor v_2\rfloor\rfloor}, \tag{43}$$

that is, $\lfloor v_1\lfloor v_2\rfloor\rfloor$ is a subword of $\lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)}$. Since $\lfloor v_1\lfloor v_2\rfloor\rfloor \neq \lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)}$ by Eq. (32), $\lfloor v_1\lfloor v_2\rfloor\rfloor$ is a subword of $\lfloor\lfloor u_1\rfloor u_2\rfloor$ by Lemma 2.14 (a). Again using Lemma 2.14 (a), either $\lfloor v_1\lfloor v_2\rfloor\rfloor = \lfloor\lfloor u_1\rfloor u_2\rfloor$ or $\lfloor v_1\lfloor v_2\rfloor\rfloor$ is a subword of $\lfloor u_1\rfloor u_2$.

For the former case of $\lfloor v_1\lfloor v_2\rfloor\rfloor = \lfloor\lfloor u_1\rfloor u_2\rfloor$, we have $q = \lfloor\star\rfloor$ by Eq. (43) and $v_1\lfloor v_2\rfloor = \lfloor u_1\rfloor u_2$. This implies that $v_1 = \lfloor u_1\rfloor v_1'$ and $u_2 = u_2'\lfloor v_2\rfloor$ for some $v_1', u_2' \in \mathfrak{M}(X)$. Then

$$\lfloor u_1\rfloor v_1'\lfloor v_2\rfloor = v_1\lfloor v_2\rfloor = \lfloor u_1\rfloor u_2 = \lfloor u_1\rfloor u_2'\lfloor v_2\rfloor \text{ and so } v_1' = u_2' =: a.$$

Then $v_1 = \lfloor u_1\rfloor a$ and $u_2 = a\lfloor v_2\rfloor$. This follows that

$$R(\varphi(u_1,u_2)) = \lfloor\lfloor u_1\rfloor^{(2)}u_2\rfloor = \lfloor\lfloor u_1\rfloor^{(2)}a\lfloor v_2\rfloor\rfloor \to_\psi \lfloor\lfloor\lfloor u_1\rfloor^{(2)}a\rfloor v_2\rfloor$$

and

$$\begin{aligned} q|_{R(\psi(v_1,v_2))} &= \lfloor\star\rfloor|_{\lfloor\lfloor v_1\rfloor v_2\rfloor} = \lfloor\lfloor\lfloor v_1\rfloor v_2\rfloor\rfloor = \lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)} = \lfloor\lfloor\lfloor u_1\rfloor a\rfloor v_2\rfloor^{(2)} \\ &\to_\varphi \lfloor\lfloor\lfloor u_1\rfloor a\rfloor^{(2)}v_2\rfloor \to_\varphi \lfloor\lfloor\lfloor u_1\rfloor^{(2)}a\rfloor v_2\rfloor. \end{aligned}$$

Hence $R(\varphi(u_1,u_2)) \downarrow_\Phi q|_{R(\psi(v_1,v_2))}$ and Eq. (30) holds, as needed.

Consider the latter case that $\lfloor v_1\lfloor v_2\rfloor\rfloor$ is a subword of $\lfloor u_1\rfloor u_2$. By Eq. (31), $\overline{\beta(v_1,v_2)} = \lfloor\lfloor v_1\rfloor v_2\rfloor$ is not a subword of u_1 and u_2. So from Lemma 2.14 (b), $\lfloor v_1\lfloor v_2\rfloor\rfloor$ is a subword of $\lfloor u_1\rfloor$. Using Lemma 2.14 (a), we have $\lfloor v_1\lfloor v_2\rfloor\rfloor = \lfloor u_1\rfloor$ and so $q = \lfloor\star u_2\rfloor^{(2)}$ by Eq. (43). Then $v_1\lfloor v_2\rfloor = u_1$. So we have

$$R(\varphi(u_1,u_2)) = \lfloor\lfloor u_1\rfloor^{(2)}u_2\rfloor = \lfloor\lfloor v_1\lfloor v_2\rfloor\rfloor^{(2)}u_2\rfloor \to_\psi \lfloor\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}u_2\rfloor \to_\varphi \lfloor\lfloor\lfloor v_1\rfloor^{(2)}v_2\rfloor u_2\rfloor$$

and

$$q|_{R(\psi(v_1,v_2))} = (\lfloor\star u_2\rfloor^{(2)})|_{\lfloor\lfloor v_1\rfloor v_2\rfloor} = \lfloor\lfloor\lfloor v_1\rfloor v_2\rfloor u_2\rfloor^{(2)} \to_\varphi \lfloor\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}u_2\rfloor \to_\varphi \lfloor\lfloor\lfloor v_1\rfloor^{(2)}v_2\rfloor u_2\rfloor.$$

Hence $R(\varphi(u_1,u_2)) \downarrow_\Phi q|_{R(\psi(v_1,v_2))}$ and Eq. (30) holds, as needed.

Subcase 3.3. $\beta(x_1,x_2) = \varphi(x_1,x_2)$. In this subsection, we have

$$\lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)} = \overline{\alpha(u_1,u_2)} = q|_{\overline{\beta(v_1,v_2)}} = q|_{\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}}, \tag{44}$$

that is, $\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}$ is a subword of $\lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)}$. By Eq. (32), $\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)} \neq \lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)}$. So from Lemma 2.14 ($a$), $\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}$ is a subword of $\lfloor\lfloor u_1\rfloor u_2\rfloor$. Again using Lemma 2.14 (a), either $\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)} = \lfloor\lfloor u_1\rfloor u_2\rfloor$ or $\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}$ is a subword of $\lfloor u_1\rfloor u_2$.

For the former case, we have $q = \lfloor\star\rfloor$ by Eq. (44) and $\lfloor\lfloor v_1\rfloor v_2\rfloor = \lfloor u_1\rfloor u_2$. This implies that $u_2 = 1$, $\lfloor\lfloor v_1\rfloor v_2\rfloor = \lfloor u_1\rfloor$ and $\lfloor v_1\rfloor v_2 = u_1$. Then

$$R(\varphi(u_1,u_2)) = \lfloor\lfloor u_1\rfloor^{(2)}u_2\rfloor = \lfloor u_1\rfloor^{(3)} = \lfloor\lfloor v_1\rfloor v_2\rfloor^{(3)} \to_\varphi \lfloor\lfloor v_1\rfloor^{(2)}v_2\rfloor^{(2)} \to_\varphi \lfloor\lfloor v_1\rfloor^{(3)}v_2\rfloor$$

and

$$q|_{R(\varphi(v_1,v_2))} = \lfloor\star\rfloor|_{\lfloor\lfloor v_1\rfloor^{(2)}v_2\rfloor} = \lfloor\lfloor\lfloor v_1\rfloor^{(2)}v_2\rfloor\rfloor = \lfloor\lfloor v_1\rfloor^{(2)}v_2\rfloor^{(2)} \to_\varphi \lfloor\lfloor v_1\rfloor^{(3)}v_2\rfloor.$$

Hence $R(\varphi(u_1,u_2)) \downarrow_\Phi q|_{R(\varphi(v_1,v_2))}$ and Eq. (30) holds, as needed.

Consider the later case of that $\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}$ is a subword of $\lfloor u_1\rfloor u_2$. By Eq. (31), $\overline{\beta(v_1,v_2)} = \lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}$ is not a subword of u_1 and u_2. So from Lemma 2.14 (b), $\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)}$ is a subword of $\lfloor u_1\rfloor$. Using Lemma 2.14 (a), we have $\lfloor\lfloor v_1\rfloor v_2\rfloor^{(2)} = \lfloor u_1\rfloor$ and so $q = \lfloor\star u_2\rfloor^{(2)}$ by Eq. (44). Thus we have

$$R(\varphi(u_1,u_2)) = \lfloor\lfloor u_1\rfloor^{(2)}u_2\rfloor = \lfloor\lfloor\lfloor v_1\rfloor v_2\rfloor^{(3)}u_2\rfloor \to_\varphi \lfloor\lfloor\lfloor v_1\rfloor^{(2)}v_2\rfloor^{(2)}u_2\rfloor \to_\varphi \lfloor\lfloor\lfloor v_1\rfloor^{(3)}v_2\rfloor u_2\rfloor$$

and

$$q|_{R(\varphi(v_1,v_2))} = (\lfloor\star u_2\rfloor^{(2)})|_{\lfloor\lfloor v_1\rfloor^{(2)}v_2\rfloor} = \lfloor\lfloor\lfloor v_1\rfloor^{(2)}v_2\rfloor u_2\rfloor^{(2)} \to_\varphi \lfloor\lfloor\lfloor v_1\rfloor^{(2)}v_2\rfloor^{(2)}u_2\rfloor \to_\varphi \lfloor\lfloor\lfloor v_1\rfloor^{(3)}v_2\rfloor u_2\rfloor$$

Hence $R(\varphi(u_1,u_2)) \downarrow_\Phi q|_{R(\varphi(v_1,v_2))}$ and Eq. (30) holds, as needed. This completes the proof. □

Recall from Remark 3.4 that $u_2 \neq 1$ in $\varphi(u_1,u_2)$. So we define

$$\begin{aligned}
M &:= \{q|_{\overline{\phi(u_1,u_2)}},\ q|_{\overline{\psi(u_1,u_2)}} \mid q \in \mathfrak{M}^\star(X), u_1,u_2 \in \mathfrak{M}(X)\},\\
N &:= \{q|_{\overline{\varphi(u_1,u_2)}} \mid q \in \mathfrak{M}^\star(X), u_1 \in \mathfrak{M}(X), u_2 \in \mathfrak{M}(X)\setminus\{1\}\},\\
N_1 &:= \{q|_{\overline{\varphi(u_1,u_2)}} \mid q \in \mathfrak{M}^\star(X), u_1,u_2 \in \mathfrak{M}(X)\},\\
N_2 &:= \{q|_{\overline{\varphi(u_1,1)}} \mid q \in \mathfrak{M}^\star(X), u_1 \in \mathfrak{M}(X)\}.
\end{aligned} \tag{45}$$

Then $N = N_1 \setminus N_2$. From Eqs. (15) and (17),

$$q|_{\overline{\varphi(u_1,1)}} = q|_{\lfloor u_1\rfloor^{(3)}} = q|_{\lfloor 1\lfloor u_1\rfloor^{(2)}\rfloor} = q|_{\overline{\psi(1,\lfloor u_1\rfloor)}} \in M$$

and so $N_2 \subseteq M$. Thus

$$M \cup N = M \cup (N_1 \setminus N_2) = M \cup N_1.$$

Hence

$$\begin{aligned}
&\{q|_{\overline{s}} \mid q \in \mathfrak{M}^\star(X), s \in S_\Phi(X)\}\\
=&\{q|_{\overline{s}} \mid q \in \mathfrak{M}^\star(X), s \in S_\phi(X) \cup S_\psi(X)\} \cup \{q|_{\overline{s}} \mid q \in \mathfrak{M}^\star(X), s \in S_\varphi(X)\}\\
=&M \cup N = M \cup N_1\\
=&\{q|_{\lfloor u_1\rfloor\lfloor u_2\rfloor},\ q|_{\lfloor u_1\lfloor u_2\rfloor\rfloor},\ q|_{\lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)}} \mid q \in \mathfrak{M}^\star(X), u_1,u_2 \in \mathfrak{M}(X)\},
\end{aligned} \tag{46}$$

where the second step employs Remark 3.4. Now we are ready to give our main result. From Proposition 2.11 and Eq. (18), $\mathbf{k}\mathfrak{M}(X)/\mathrm{Id}(S_\Phi(X))$ is the free averaging algebra on X.

Theorem 3.11. *The* $\mathrm{Irr}(S_\Phi(X))$ *is a* $\mathbf{k}$*-basis of the free (unitary) averaging algebra* $\mathbf{k}\mathfrak{M}(X)/\mathrm{Id}(S_\Phi(X))$ *on* X. *More precisely,*

$$\mathbf{k}\mathfrak{M}(X) = \mathrm{Id}(S_\Phi(X)) \oplus \mathbf{k}\mathrm{Irr}(S_\Phi(X)),$$

where

$$\mathrm{Irr}(S_\Phi(X)) = \mathfrak{M}(X) \setminus \{q|_{\lfloor u_1\rfloor\lfloor u_2\rfloor},\ q|_{\lfloor u_1\lfloor u_2\rfloor\rfloor},\ q|_{\lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)}} \mid q \in \mathfrak{M}^\star(X), u_1,u_2 \in \mathfrak{M}(X)\}.$$

Proof. By Theorem 3.10, $\Pi_\Phi = \Pi_{S_\Phi(X)}$ is convergent. Using Theorems 2.41 to $S = S_\Phi(X)$, we have

$$\mathbf{k}\mathfrak{M}(X) = \mathrm{Id}(S_\Phi(X)) \oplus \mathbf{k}\mathrm{Irr}(S_\Phi(X)),$$

where

$$\begin{aligned}\mathrm{Irr}(S_\Phi(X)) =& \mathfrak{M}(X) \setminus \{q|_{\bar{s}} \mid q \in \mathfrak{M}^\star(X), s \in S_\Phi(X)\}\\ =& \mathfrak{M}(X) \setminus \{q|_{\lfloor u_1\rfloor\lfloor u_2\rfloor},\ q|_{\lfloor u_1\lfloor u_2\rfloor\rfloor},\ q|_{\lfloor\lfloor u_1\rfloor u_2\rfloor^{(2)}} \mid q \in \mathfrak{M}^\star(X), u_1, u_2 \in \mathfrak{M}(X)\}\end{aligned}$$

by Eq. (46). □

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