HÖLDER CONTINUITY OF THE LIOUVILLE QUANTUM GRAVITY MEASURE

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ABSTRACT. We show that for certain Hölder continuously parameterized families of measures ν_t on a regular plane domain D, the total 'Liouville quantum' masses of the random measures $\widetilde{\nu}_t$ obtained as limits of the circle averages of the Gaussian free field on the ν_t vary Hölder continuously with the parameter t. In particular, this implies that if the Liouville quantum gravity measure μ on D has Hausdorff dimension larger than 1 then almost surely the orthogonal projections of μ in all directions are simultaneously absolutely continuous with respect to Lebesgue measure. As a consequence, almost surely μ has positive Fourier dimension. We give further applications to the Hölder continuity of the Liouville quantum masses of self-similar measures, and the Liouville quantum lengths of planar curves.

1. INTRODUCTION

1.1. **Overview.** Random multiplicative cascades were introduced by Mandelbrot [12] as a model to explain energy dissipation and intermittency in Kolmogorov's model of fully developed turbulance. These random cascade measures and the consequences of their martingale structure were studied in detail by Kahane and Peyrière [10, 17]. Whilst random cascade measures, with their underlying self-similarity, have been widely used, they are to some extent artificial in that their construction depends on a preferred range of scales and they are intrinsically non-isotropic and not translation invariant.

In 1985, Kahane [9] constructed what he termed 'Gaussian multiplicative chaos' with analogous properties but which overcame these drawbacks, with the lognormal hierarchy of multiplicative cascades replaced by correlated exponentials of a Gaussian process. The construction has two stages. First a log-correlated Gaussian field is defined on a domain D, that is a random distribution Γ with a logarithmic covariance structure. The Gaussian multiplicative chaos (GMC) measure is then defined as a normalized exponential of Γ. There are technical difficulties in the

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construction of the GMC measure since Γ is a random Schwartz distribution rather than a random function, and this is generally addressed using continuous approximations to Γ. Kahane used the partial sums of a sequence of independent Gaussian processes to approximate Γ and showed the uniqueness of the measure, i.e., the law of the GMC does not depend on the choice of the approximating sequence. More recently, Duplantier and Sheffield constructed the measure [3] by using a circle average approximation of Γ where Γ is the Gaussian Free Field (GFF) on a plane domain D with a regular boundary. They also pointed out that two-dimensional Gaussian multiplicative chaos, which depends on a parameter γ $(0<\gamma<2)$, may be regarded as giving a rigorous interpretation of the Liouville measure that occurs in Liouville quantum gravity (LQG) and this name has become attached to the two-dimensional case. Full surveys of this area may be found in [2, 3, 18].

Recently, there has been renewed interest in geometrical properties of classes of deterministic or random fractal sets and measures. This includes investigation the Hausdorff dimensions of sections or projections of sets and measures, the absolute continuity of projections and Hölder continuity of their intersection with families of lines or curves.

A version of Marstrand's projection theorem [13] states that if a fractal measure μ in the plane has Hausdorff dimension $\dim_H \mu$ larger than 1, then its orthogonal projection $\pi_\theta\mu$ in direction θ is absolutely continuous with respect to Lebesgue measure except for a set of θ of Lebesgue measure 0. There has been considerable interest of late in identifying classes of measure for which there are no exceptional directions, or at least for which the set of exceptional directions is very small.

Peres and Shmerkin [16] and Hochman and Shmerkin [8], showed that for self-similar measures with $\dim_H \mu > 1$ such that the rotations underlying the defining similarities are dense in the rotation group, the projected measures have dimension 1 in all directions, and Shmerkin and Solomyak [20] showed that they are absolutely continuous except for a set of directions of Hausdorff dimension 0. Falconer and Jin [6, 7] obtained similar results for random self-similar measures and in particular their analysis included Mandelbrot's random cascade measures [12]. Shmerkin and Suomala [21] have studied such problems for other classes of random sets.

It is natural to ask similar questions concerning the geometry of the LQG measure μ which has Hausdorff dimension $\dim_H \mu = 2-\gamma^2/2$. For any measure ν on $\overline{D}$ we denote by $\widetilde{\nu}$ the limit of the circle averages of the GFF on ν and write $\|\widetilde{\nu}\|$ for its total mass, which might be thought of as the 'quantum mass' of ν. Our main result, Theorem 1.6, shows that for a family of measures $\{\nu_t : t \in \mathcal{T}\}$ on $\overline{D}$ with a Hölder continuous parameterization by a metric space $\mathcal{T}$, almost surely $\|\widetilde{\nu}_t\|$ varies Hölder continuously with the parameter t. This has many applications. Firstly Theorem 2.1 asserts that if $\dim_H \mu > 1$ $(0 < \gamma < \sqrt{2})$ then almost surely the orthogonal projections of μ in all directions are simultaneously absolutely continuous with respect to Lebesgue measure. A consequence, Corollary 2.5, is that, almost surely, the measure μ has positive Fourier dimension. Another application, Theorem 2.6, is that the quantum mass of self-similar measures is Hölder continuous in the underlying similarities. Furthermore, Theorem 2.8 concludes that if we define the LQG measure simultaneously on certain parametreized families of Jordan curves in D, their mass, which in a sense is the Liouville quantum length of the curves, is Hölder continuous. We also show in Theorem 1.7 that if ν is a measure that satisfies a density bound with exponent α, so in particular is at least α-dimensional, then $\widetilde{\nu}$ has Hausdorff dimension at least $\alpha - \gamma^2/2$.

The proof of Hölder continuity of $\{\|\widetilde{\nu}_t\| : t \in \mathcal{T}\}$ is inspired by the paper [21] of Shmerkin and Suomala on Hölder properties of 'compound Poisson cascades' types of random measures. The difference here is that the circle average of the GFF dose not have the spatial independence and the uniform bounded density properties needed in [21]. Hence we have to use a different approach (Lemma 1.2 and Lemma 1.3) to estimate the convergence speed, and the Kolmogorov continuity type argument in our case is more complicated (Proposition 1.4). Nevertheless it seems possible to relax some of the conditions in [21] using our approach.

1.2. **Gaussian Free Fields.** Let D be a regular planar domain, namely a simply-connected open subset of $\mathbb{R}^2$ with a regular boundary, that is, for every point $x \in \partial D$ there exists a continuous path $u(t)$, $0 \le t \le 1$, such that $u(0) = x$ and $u(t) \in D^c$ for $0 < t \le 1$. The Green function G_D on $D \times D$ is given by

$$G_D(x, y) = \log \frac{1}{|x-y|} - \mathbb{E}^x \left(\log \frac{1}{|W_\tau - y|} \right),$$

where the expectation $\mathbb{E}^x$ is taken with respect to the probability measure P^x under which W is a planar Brownian motion started from x, and τ is the first exit time of W in D, i.e., $\tau = \inf\{t \ge 0 : W_t \notin D\}$. The Green function is conformally invariant in the sense that if $f : D \mapsto D'$ is a conformal mapping, then

$$G_D(x, y) = G_{f(D)}(f(x), f(y)).$$

Let $\mathcal{M}^+$ be the set of finite measures supported in D such that

$$\int_D \int_D G_D(x, y)\, d\mu(x) d\mu(y) < \infty.$$

Let $\mathscr{M}$ be the vector space of signed measures $\mu^+ - \mu^-$, where $\mu^+, \mu^- \in \mathscr{M}^+$. Let $(\Gamma(\mu), \mu \in \mathscr{M})$ be a centered Gaussian process on $\mathscr{M}$ with covariance function

$$\mathbb{E}(\Gamma(\mu)\Gamma(\nu)) = \int_D G_D(x,y)\,d\mu(x)\,d\nu(y).$$

Then Γ is called a *Gaussian free field* (GFF) on D.

Let O be a regular subdomain of D. Then Γ may be decomposed into a sum:

$$\Gamma = \Gamma^O + \Gamma_O, \tag{1.1}$$

where Γ^O and Γ_O are two independent Gaussian processes on $\mathscr{M}$ with covariance functions G_O and $G_D - G_O$ respectively. Moreover, there is a version of the process such that Γ^O vanishes on all measures supported in $D \setminus O$, and Γ_O restricted to O is harmonic, that is there exists a harmonic function h_O on O such that for any measure μ supported in O,

$$\Gamma_O(\mu) = \int_O h_O(x)\,\mu(dx).$$

In fact $h_O(x) = \Gamma(\mu_{O,x})$ for $x \in O$, where $\mu_{O,x}$ is the exit distribution of O for a Brownian motion started from x. Furthermore, if we denote by $\mathscr{F}_{D\setminus O}$ the σ-algebra generated by all $\Gamma(\mu)$ for which $\mu \in \mathscr{M}$ is supported by $D \setminus O$, then Γ^O is independent of $\mathscr{F}_{D\setminus O}$.

For more details on Gaussian free fields, see, for example, [2, 18, 19, 23].

1.3. **Liouville quantum gravity.** For $x \in D$ and $\epsilon > 0$ let $\rho_{x,\epsilon}$ be normalized Lebesgue measure on $\{y \in D : |x-y| = \epsilon\}$, the circle centered at x with radius ϵ in D. Fix $\gamma \in [0,2)$. For $\epsilon > 0$ let

$$\mu_\epsilon(dx) = \epsilon^{\gamma^2/2} e^{\gamma\Gamma(\rho_{x,\epsilon})}\,dx, \quad x \in D. \tag{1.2}$$

Then almost surely the weak limit $\mu = \text{w-lim}_{\epsilon\to 0}\mu_\epsilon$ exists and the measure μ is called *Liouville quantum gravity* (LQG) on D, see [3]. Since $\Gamma(\rho_{x,\epsilon})$ is centered Gaussian,

$$\mathbb{E}\left(e^{\gamma\Gamma(\rho_{x,\epsilon})}\right) = e^{\frac{\gamma^2}{2}\mathrm{Var}(\Gamma(\rho_{x,\epsilon}))}.$$

Using the conformal invariance of GFF it may be shown that, provided that $B(x,\epsilon) \subset D$, where $B(x,\epsilon)$ is the open ball of centre x and radius ϵ,

$$\mathrm{Var}(\Gamma(\rho_{x,\epsilon})) = -\log\epsilon + \log R(x,D), \tag{1.3}$$

where $R(x,D)$ is the conformal radius of x in D, given by $R(x,D) = |f'(0)|$ where $f : \mathbb{D} \mapsto D$ is a conformal mapping from the unit disc $\mathbb{D}$ onto D with $f(0) = x$. (It is well-known, for example using the Schwarz lemma and the Koebe 1/4 theorem, that

$$\mathrm{dist}(x,\partial D) \le R(x,D) \le 4\,\mathrm{dist}(x,\partial D), \tag{1.4}$$

where $\operatorname{dist}(x, \partial D)$ is the Euclidean distance from x to the boundary of D.) This gives

$$\mathbb{E}\left(e^{\gamma\Gamma(\rho_{x,\epsilon})}\right) = \epsilon^{-\gamma^2/2} R(x,D)^{\gamma^2/2} dx, \tag{1.5}$$

and so

$$\mathbb{E}(\mu(dx)) = R(x,D)^{\gamma^2/2} dx, \quad x \in D.$$

For more details on LQG, see for example [2, 3].

1.4. **Assumptions and main results.** We assume that the domain D is bounded. Let $(\mathcal{T}, d)$ be a compact metric space. Let ν be a positive finite measure on a measurable space $(E, \mathcal{E})$. For each $t \in \mathcal{T}$ we assign a measurable set I_t and a measurable function f_t,

$$I_t \in \mathcal{E}, \quad f_t : I_t \mapsto f_t(I_t) \subset \overline{D},$$

and define the push-forward measure on D by

$$\nu_t := \nu \circ f_t^{-1},$$

with the convention that ν_t is the null measure if $\nu(I_t) = 0$.

To help fix ideas, I_t may typically be a real interval with f_t a continuous injection, so that $f_t(I_t)$ is a curve in $\overline{D}$ that supports the measure ν_t.

For $t \in \mathcal{T}$ and $r > 0$ denote open balls in $(\mathcal{T}, d)$ by $B_d(t,r) = \{s \in \mathcal{T} : d(s,t) < r\}$. Throughout the paper we make the following three assumptions

(A1) There exist constants $C_1, \alpha_1 > 0$ such that for all $x \in \mathbb{R}^2$ and $r > 0$,

$$\sup_{t \in \mathcal{T}} \nu_t(B(x,r)) \le C_1 r^{\alpha_1};$$

(A2) There exist constants $C_2, r_2, \alpha_2, \alpha_2' > 0$ such that for all $s, t \in \mathcal{T}$ with $d(s,t) \le r_2$ and $I_s \cap I_t \neq \emptyset$,

$$\sup_{u \in I_s \cap I_t} |f_s(u) - f_t(u)| \le C_2 d(s,t)^{\alpha_2}$$

and

$$\nu(I_s \Delta I_t) \le C_2 d(s,t)^{\alpha_2'}$$

(A3) There exist an increasing sequence of sets of points $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \cdots$ in $\mathcal{T}$ and constants $C_3, \alpha_3 > 0$ such that for each $n \ge 1$, $\#\mathcal{T}_n \le C_3 2^{n\alpha_3}$ and $\{B_d(t, 2^{-n}) : t \in \mathcal{T}_n\}$ forms a covering of $\mathcal{T}$ such that each point in $\mathcal{T}$ is covered by at most C_3 balls. In particular $\mathcal{T}_* := \bigcup_{n=1}^{\infty} \mathcal{T}_n$ forms a countable dense subset of $\mathcal{T}$. Furthermore, for all $s, t \in \mathcal{T}$ with $d(s,t) \le C_3 2^{-n}$, and all $m \ge n+1$ there exist $s_m, t_m \in \mathcal{T}_m$ such that $d(s, s_m) \le 2^{-m}$, $d(t, t_m) \le 2^{-m}$ and $d(s_m, t_m) \le C_3 2^{-n}$. By increasing α_3 if necessary, we may further assume that $(\frac{\alpha_2}{2} \wedge \alpha_2') \le \alpha_3$.

Remark 1.1. *Condition* (A3) *may seem a little cumbersome, but it holds, for example, for metric spaces that can be bi-Lipschitz embedded into a finite dimensional Euclidean space, and in particular for spaces parameterizing translations and rotations in a natural way.*

For $t \in \mathscr{T}$ and $n \geq 1$ define circle averages of Γ on ν_t by

$$\widetilde{\nu}_{t,n}(dx) = 2^{-n\gamma^2/2} \mathrm{e}^{\gamma\Gamma(\rho_{x,2^{-n}})} \nu_t(dx), \quad x \in D, \tag{1.6}$$

and let

$$Y_{t,n} := \|\widetilde{\nu}_{t,n}\| \tag{1.7}$$

be the total mass of $\widetilde{\nu}_{t,n}$. Let $\widetilde{\nu}_t = \text{w-lim}_{n\to\infty} \widetilde{\nu}_{t,n}$ be γ-LQG on ν_t and $Y_t = \|\widetilde{\nu}_t\|$ be its total mass if it exists.

The following two lemmas, which will be proved in Sections 3.1 and 3.2, concern the expected convergence speed of $Y_{t,n}$ and the Hölder exponents of $Y_{t,n}$, respectively.

Lemma 1.2. *For* $1 \leq p \leq 2$ *there exists a constant* C_p *such that for all* $t \in \mathscr{T}$ *and* $n \geq 1$,

$$\mathbb{E}(|Y_{t,n+1} - Y_{t,n}|^p) \leq C_p 2^{-n(\alpha_1 - \frac{\gamma^2}{2}p)(p-1)}. \tag{1.8}$$

Lemma 1.3. *For* $q > 1$ *and* $0 < \eta < 1/2$ *there exists a constant* $C_{q,\eta}$ *such that for all* $0 < r < r_2$ *and* $s, t \in \mathscr{T}$ *with* $d(s,t) \leq r$ *and all* $n \geq 1$,

$$\mathbb{E}\left(\max_{1 \leq k \leq n} |Y_{s,k} - Y_{t,k}|^q \right) \leq C_{q,\eta} r^{q((\eta\alpha_2)\wedge\alpha_2')} 2^{nq(\frac{1}{2} + \frac{\gamma^2}{2}(q-1))}. \tag{1.9}$$

Using Lemma 1.2 and Lemma 1.3 we will show the uniform Hölder regularity of $t \mapsto Y_{t,n}$, obtaining a bound for the Hölder exponent. The proof, given in Section 1.4, is reminiscent of that of the Kolmogorov-Chentsov theorem.

Proposition 1.4. *If* $\alpha_1 - \frac{\gamma^2}{2} > 0$ *then there are numbers* $C, \beta > 0$ *such that almost surely there exists a (random) integer* N *such that for all* $s, t \in \mathscr{T}$ *with* $d(s,t) \leq 2^{-N}$,

$$\sup_{n \geq 1} |Y_{s,n} - Y_{t,n}| \leq C d(s,t)^{\beta}. \tag{1.10}$$

In particular, the exponent β *can be any number satisfying* $0 < \beta < \beta_0$ *where*

$$\beta_0 := \left(\frac{\frac{\alpha_2}{2} \wedge \alpha_2'}{\sqrt{\alpha_3} + \sqrt{\alpha_3 + (\frac{\alpha_2}{2} \wedge \alpha_2')\left(\frac{(2\alpha_1+1)}{\gamma^2} - \frac{2\sqrt{2\alpha_1}}{\gamma}\right)}} \right)^2 \left(\sqrt{\frac{2\alpha_1}{\gamma^2}} - 1 \right)^2. \tag{1.11}$$

Remark 1.5. *If $\gamma \to 0$, corresponding to* $\dim_H \mu \to 2$, *then* $\beta_0 \to \frac{2(\frac{\alpha_2}{2} \wedge \alpha_2')\alpha_1}{2\alpha_1+1}$.

We can now quickly deduce our main result from Proposition 1.4.

Theorem 1.6. *If $\alpha_1 - \frac{\gamma^2}{2} > 0$, then almost surely the sequence of mappings $\{t \mapsto Y_{t,n}\}_{n=1}^{\infty}$ converges uniformly on $(\mathscr{T}, d)$ to a limit $t \mapsto Y_t$. Moreover, Y_t is β-Hölder continuous in d, for all $0 < \beta < \beta_0$, where β_0 is given by* (1.11).

Proof. Proposition 1.4 implies that for each $\beta < \beta_0$, almost surely the sequence of β-Hölder continuous functions $\{t \mapsto Y_{t,n}\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. Taking $p > 1$ such that $\alpha_1 - \frac{\gamma^2}{2}p > 0$ in Lemma 1.2 and using the Borel-Cantelli lemma, almost surely for all $t \in \mathscr{T}_*$ the sequence $\{Y_{t,n}\}_{n=1}^{\infty}$ is Cauchy and so convergent. Since $\mathscr{T}_*$ is dense in $\mathscr{T}$, this pointwise convergence together with the equicontinuity implies that $\{t \mapsto Y_{t,n}\}_{n=1}^{\infty}$ converges uniformly to some function $t \mapsto Y_t$ which must be β-Hölder continuous since the $\{t \mapsto Y_{t,n}\}_{n=1}^{\infty}$ are uniformly β-Hölder, as required. □

We will give a range of applications of Theorem 1.6 in Section 2.

The next theorem gives a lower bound for the Hausdorff dimension of a single LQG measure. Recall that the Hausdorff dimension of a measure τ is defined by $\dim_H \tau = \inf\{\dim_H E : \tau(E) > 0\}$.

Theorem 1.7. *Let ν be a positive Borel measure on D satisfying* (A1) *in the sense that for some $C_1 > 0$ and $\alpha_1 > \gamma^2/2$, $\nu(B(x,r)) \le C_1 r^{\alpha_1}$ when $B(x,r) \cap D \ne \emptyset$. Define the γ-LQG measure of ν via the circle averages*

$$\widetilde{\nu}_n(dx) = 2^{-n\gamma^2/2} \mathrm{e}^{\gamma \Gamma(\rho_{x,2^{-n}})}\, d\nu(x), \ x \in D, \tag{1.12}$$

and $\widetilde{\nu} = \text{w-lim}_{n\to\infty} \widetilde{\nu}_n$. Then, almost surely,

$$\dim_H \widetilde{\nu} \ge \alpha_1 - \frac{\gamma^2}{2}.$$

Remark 1.8. *It is tempting to hope that for suitable $\mathscr{T}$, almost surely,*

$$\dim_H \widetilde{\nu}_t \ge \alpha_1 - \frac{\gamma^2}{2} \text{ for all } t \in \mathscr{T} \tag{1.13}$$

under the assumptions (A1)-(A3), *but we are unable to show this. The difficulty comes from the fact that the random integer N in Proposition 1.4 for $Y_{t,n} = \widetilde{\nu}_{t,n}(D)$ may become larger even if we replace D by a subset $S \subset D$. If one could estimate the Hölder regularity of $\widetilde{\nu}_{t,n}(S)$ as in Proposition 1.4 simultaneously for, say, all dyadic squares $S \subset D$, then it is not hard to show that* (1.13) *follows. But this may require further assumptions.*

2. APPLICATIONS

2.1. **Absolute continuity of projections.** We first apply Theorem 1.6 to show that, almost surely, the orthogonal projections of the Liouville quantum gravity measure μ in all directions are absolutely continuous provided that $\dim_H \mu > 1$, and then use this to show this implies that the Fourier transform of the LQG measure decays polynomialy. We write π_θ for the orthogonal projection onto the line through the origin in direction perpendicular to the unit vector θ, and $\pi_\theta \tau = \tau \circ \pi_\theta^{-1}$ for the projection of a measure τ on $\mathbb{R}^2$ in the obvious way.

We prove the theorem for a rotund convex domain $D \subset \mathbb{R}^2$. We call a bounded open convex D *rotund* if its boundary ∂D has continuously varying radius of curvature that is bounded away from 0 and ∞.

Theorem 2.1. *Let $0 < \gamma < \sqrt{2}$ and let μ be γ-LQG on a rotund convex domain D, so that almost surely, $\dim_H \mu = 2 - \frac{\gamma^2}{2} > 1$. Then, almost surely, for all $\theta \in [0,\pi)$ the projected measure $\pi_\theta \mu$ is absolutely continuous with respect to Lebesgue measure with a β-Hölder continuous Radon-Nikodym derivative for all $0 < \beta < \beta_0$, where β_0 is given by (1.11) with $\alpha_1 = \alpha_2 = 1$, $\alpha_2' = \frac{1}{2}$ and $\alpha_3 = 2$.*

Remark 2.2. *Note that, with these values, $\beta_0 \to \frac{1}{3}$ as $\gamma \searrow 0$.*

For $(\theta, u) \in [0,\pi) \times \mathbb{R}$ let $l_{(\theta,u)}$ be the straight line in $\mathbb{R}^2$ in direction θ and perpendicular distance u from the origin. We identify these lines $l_{(\theta,u)}$ with the parameters (θ, u) and define a metric d by

$$d(l_{(\theta,u)}, l_{(\theta',u')}) \equiv d\big((\theta,u),(\theta',u')\big) = |u-u'| + \min\{|\theta-\theta'|, \pi-|\theta-\theta'|\}. \tag{2.1}$$

We write $L(l)$ for the length of the chord $l \cap \overline{D}$ provided the line l intersects $\overline{D}$. We require a geometrical lemma on the Hölder continuity of chord lengths of plane convex sets.

Lemma 2.3. *Let $D \subset \mathbb{R}^2$ be a rotund convex domain. There is a constant c_0 depending only on D such that for all l, l' that intersect $\overline{D}$*

$$\big|L(l) - L(l')\big| \leq c_0 d(l,l')^{1/2}. \tag{2.2}$$

Proof. It is convenient to work with an alternative geometrical interpretation of the metric d. Given a line l and $\epsilon > 0$ let $S_\infty(l,\epsilon)$ be the infinite strip $\{x \in \mathbb{R}^2 : |x-y| \leq \epsilon$ for some $y \in l\}$. For $M > 0$ let $R_M(l,\epsilon)$ be the rectangle $\{x \in S_\infty(l,\epsilon) : |x \cdot \theta| \leq M\}$ where here we regard θ as a unit vector in the direction of l and '$\cdot$' denotes the scalar product. Fix M sufficiently large so that for all lines l and $\epsilon > 0$,

$$S_\infty(l,\epsilon) \cap \overline{D} = R_M(l,\epsilon) \cap \overline{D}.$$

Write

$$E_M(l,\epsilon) = \{l' : l' \cap \partial R_M(l,\epsilon) = \{x_-, x_+\} \text{ where } x_\pm \cdot \theta = \pm M\},$$

for the set of lines that enter and exit the rectangle $R_M(l,\epsilon)$ across its two 'narrow' sides.

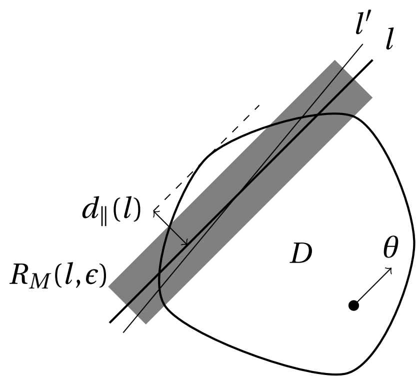

FIGURE 1

It is easy to see that there are constants $\epsilon_0, \lambda > 0$ depending only on D (taking into account M and the position of D relative to the origin) such that if $d(l,l') \le \lambda\epsilon \le \lambda\epsilon_0$ then $l' \in E_M(l,\epsilon)$. Thus (2.2) will follow if there is a constant c_1 such that for all l that intersect $\overline{D}$ and all sufficiently small ϵ,

$$\text{if } l' \in E_M(l,\epsilon) \text{ then } |L(l) - L(l')| \le c_1\epsilon^{1/2}. \tag{2.3}$$

Write $0 < \rho_{\min} \le \rho_{\max} < \infty$ for the minimum and maximum radii of curvature of ∂D. For a line l that intersects $\overline{D}$ let $d_\parallel(l)$ denote the perpendicular distance between l and the closest parallel tangent to ∂D, see Figure 1. We consider two cases.

(a) $\epsilon \le \frac{1}{4}\rho_{\min}$, $\frac{1}{2}d_\parallel(l) \le \epsilon$. Here both of the 'long' sides of the rectangle $R_M(l,\epsilon)$ are within distance $d_\parallel + \epsilon \le 3\epsilon < \rho_{\min}$ of the tangent to ∂D parallel to l, so that if $l' \in E_M(l,\epsilon)$ then $d_\parallel(l') \le 3\epsilon$. By simple geometry, $L(l), L(l') \le (2\rho_{\max})^{1/2}(3\epsilon)^{1/2}$, so (2.3) holds with $c_1 = (2\rho_{\max})^{1/2}3^{1/2}$.

(b) $\epsilon \le \frac{1}{4}\rho_{\min}$, $\frac{1}{2}d_\parallel(l) \ge \epsilon$. In this case, all $l' \in E_M(l,\epsilon)$ are distance at least $d_\parallel(l) - \epsilon \ge \frac{1}{2}\epsilon$ from their parallel tangents to ∂D. In particular, the angles between every $l' \in E_M(l,\epsilon)$ and the tangents to ∂D at either end of l' are at least ϕ where $\cos\phi = \left(\rho_{\max} - \frac{1}{2}\epsilon\right)/\rho_{\max}$. Both $l, l' \in E_M(l,\epsilon)$ intersect ∂D at points on each of its arcs of intersection with $R_M(l,\epsilon)$, so that l and l' intersect each of these arcs at points within distance

$$\frac{2\epsilon}{\sin\phi} \le \frac{2\epsilon}{\left(1-\left(1-\frac{1}{2}\frac{\epsilon}{\rho_{\max}}\right)^2\right)^{1/2}} \le 2(2\rho_{\max})^{1/2}\epsilon^{1/2}$$

of each other, where we have used $\epsilon/\rho_{\max} \le \frac{1}{2}$ in the second estimate. Applying the triangle inequality (twice) to the points of $l \cap \partial D$ and $l' \cap \partial D$ inequality (2.3) follows with $c_1 = 4(2\rho_{\max})^{1/2}$. □

Remark 2.4. *Note that* (2.2) *remains true taking d to be any reasonable metric on the lines. Moreover, it is easy to obtain a Hölder exponent of 1 if we restrict to lines that intersect $\overline{D} \setminus (\partial D)_\delta$ for given $\delta > 0$, where $(\partial D)_\delta$ is the δ-neighbourhood of the boundary of D.*

Proof of Theorem 2.1. Fix some point of D as origin and choose R such that $D \subset B(0,R)$. Let ν be Lebesgue measure on the interval $E = [-R,R]$ Let $\mathscr{T} = \left\{(\theta,u) \in [0,\pi) \times \mathbb{R} : l_{(\theta,u)} \cap \overline{D} \neq \emptyset\right\}$. For $(\theta,u) \in \mathscr{T}$ let

$$I_{(\theta,u)} = \pi_{\theta+\pi/2}(l_{(\theta,u)} \cap \overline{D})$$

where $\pi_{\theta+\pi/2}$ denotes orthogonal projection onto the line through 0 in direction θ, which we identify with $\mathbb{R}$ in the natural way. Let

$$f_{(\theta,u)}(v) = u\mathrm{e}^{i(\theta+\pi/2)} + v\mathrm{e}^{i\theta}, \quad v \in I_{(\theta,u)},$$

where we identify $\mathbb{R}^2$ with $\mathbb{C}$. Then

$$\nu_{(\theta,u)} := \nu \circ f^{-1}_{(\theta,u)}$$

is just 1-dimensional Lebesgue measure on the chord $l_{(\theta,u)} \cap \overline{D}$ of D, with the convention that $\nu_{(\theta,u)}$ is the null measure when $l_{(\theta,u)}$ is tangent to D. It is easy to see that $(\mathscr{T}, d)$ is compact and satisfies (A3) for $\alpha_3 = 2$, and also that $\{\nu_{(\theta,u)} : (\theta,u) \in \mathscr{T}\}$ satisfies (A1) for $C_1 = 1$ and $\alpha_1 = 1$.

For condition (A2), for $(\theta,u), (\theta',u') \in \mathscr{T}$ and $v \in E$,

$$\begin{aligned} \left|f_{(\theta,u)}(v) - f_{(\theta',u')}(v)\right| &\le \left(|v|+|u|\right)\left|1-\mathrm{e}^{i(\theta-\theta')}\right| + |u-u'| \\ &\le 2\sqrt{2}R\left|1-\cos(\theta-\theta')\right|^{1/2} + |u-u'| \\ &\le 2\sqrt{2}R\left(\min\left\{|\theta-\theta'|, \pi-|\theta-\theta'|\right\} + |u-u'|\right), \end{aligned}$$

where R is such that $D \subset B(0,R)$. Also, by Lemma 2.3,

$$\nu\left(I_{(\theta,u)} \Delta I_{(\theta',u')}\right) = \left|L\left(l_{(\theta,u)}\right) - L\left(l_{(\theta',u')}\right)\right| \le c_0 d\left(l_{(\theta,u)}, l_{(\theta',u')}\right)^{1/2}.$$

This gives (A2) for $C_2 = \max\{2\sqrt{2}R, c_0\}$, $\alpha_2 = 1$ and $\alpha_2' = \frac{1}{2}$.

For $(\theta,u) \in \mathscr{T}$ and $n \ge 1$ let $\widetilde{\nu}_{(\theta,u),n}$ and $Y_{(\theta,u),n}$ be given as in (1.6) and (1.7). The conclusions of Theorem 1.6 hold in this setting, with β_0 given by (1.11) with these $\alpha_1, \alpha_2, \alpha_2', \alpha_3$. In particular, for $\beta < \beta_0$, we may assume that, as happens almost surely, $Y_{(\theta,u),n}$ converges uniformly on $\mathscr{T}$

to a β-Hölder continuous $Y_{(\theta,u)}$, and for all n the measure $\mu_{2^{-n}}$, given by (1.2) is absolutely continuous and coverges weakly to μ.

Now fix θ and let $(u,v) \in \mathbb{R}^2$ be coordinates in directions $\theta + \frac{\pi}{2}$ and θ. Let $\phi(u,v) \equiv \phi(u)$ be continuous on $\mathbb{R}^2$ and independent of the second variable. Since $\widetilde{\nu}_{(\theta,u),n}$ are absolutely continuous measures, using (1.2), (1.6) and Fubini's theorem,

$$\begin{aligned}
\int_{(u,v)\in D} \phi(u)\, d\mu_{2^{-n}}(u,v) &= \int_{(u,v)\in D} \phi(u) 2^{-n\gamma^2/2} \mathrm{e}^{\gamma\Gamma(\rho_{(u,v),2^{-n}})}\, dv\, du \\
&= \int_{(u,v)\in D} \phi(u) 2^{-n\gamma^2/2} \mathrm{e}^{\gamma\Gamma(\rho_{(u,v),2^{-n}})}\, d\nu_{(\theta,u)}(v)\, du \\
&= \int_{u_-(\theta)}^{u_+(\theta)} \phi(u) \|\widetilde{\nu}_{(\theta,u),n}\|\, du \\
&= \int_{u_-(\theta)}^{u_+(\theta)} \phi(u) Y_{(\theta,u),n}\, du,
\end{aligned}$$

where $u_-(\theta)$ and $u_+(\theta)$ are the values of u corresponding to the tangents to D in direction θ. Letting $n \to \infty$ and using the weak convergence of $\mu_{2^{-n}}$ and the uniform convergence of $Y_{(\theta,u),n}$,

$$\int_{u_-(\theta)}^{u_+(\theta)} \phi(u)\, d(\pi_\theta \mu)(u) = \int_{(u,v)\in D} \phi(u)\, d\mu(u,v) = \int_{u_-(\theta)}^{u_+(\theta)} \phi(u) Y_{(\theta,u)}\, du. \tag{2.4}$$

Thus $d(\pi_\theta\mu)(u) = Y_{(\theta,u)}\, du$ on $[u_-(\theta), u_+(\theta)]$, so as $Y_{(\theta,u)}$ is β-Hölder continuous on $[u_-(\theta), u_+(\theta)]$ we conclude that $\pi_\theta\mu$ is absolutely continuous with a β-Hölder Radon-Nikodym derivative. □

Theorem 2.1 leads to a bound on the rate of decay of the Fourier transform $\widehat{\mu}$ of μ, or equivalently on the Fourier dimension of the measure (see [4, 14] for recent discussions on Fourier dimensions).

Corollary 2.5. *Let $\gamma < \sqrt{2}$, let μ be γ-LQG on a rotund convex domain D and let $0 < \beta < \beta_0$ where β_0 is as in Theorem 2.1. Then, almost surely, there is a random constant C such that*

$$|\widehat{\mu}(\xi)| \le C|\xi|^{-\beta}, \qquad \xi \in \mathbb{R}^2, \tag{2.5}$$

so in particular μ has Fourier dimension at least $2\beta_0$.

Proof. We use the same notation as in the proof of Theorem 2.1. Almost surely, $\mathscr{T} \ni (\theta,u) \mapsto Y_{(\theta,u)}$ is β-Hölder continuous, that is, for some $C_\beta > 0$,

$$|Y_{(\theta,u)} - Y_{(\theta',u')}| \le C_\beta d\big((\theta,u),(\theta',u')\big)^\beta.$$

For $\theta \in [0,\pi)$ and $j \in \{u_-(\theta), u_+(\theta)\}$,

$$\mathbb{E}\big(\lim_{u\to j} Y_{(\theta,u)}\big) \le \lim_{u\to j} \mathbb{E}\big(Y_{(\theta,u)}\big) = \lim_{u\to j} \mathbb{E}\big(\|\widetilde{\nu}_{(\theta,u)}\|\big) = 0,$$

since $\lim_{u\to j}\|v_{(\theta,u)}\|=0$. As $\lim_{u\to j}Y_{(\theta,u)}\geq 0$, this implies that almost surely $\lim_{u\to j}Y_{(\theta,u)}=0$. By taking a countable dense subset of $[0,\pi)$ and applying Hölder continuity, we have that almost surely $Y_{(\theta,j)}=0$ for all $\theta\in[0,\pi)$ and $j\in\{u_-(\theta),u_+(\theta)\}$. This means that we can extend $Y_{(\theta,u)}$ to all $u\in\mathbb{R}$ by letting $Y_{(\theta,u)}=0$ for $u\notin[u_-(\theta),u_+(\theta)]$, with the extended function is still β-Hölder continuous with the same constant C_β.

Write the transform variable $\xi=\tilde{\xi}\theta$ where here we regard $\theta\in[0,\pi)$ as a unit vector and $\tilde{\xi}\in\mathbb{R}$. From (2.4)

(2.6)
$$\widehat{\mu}(\tilde{\xi}\theta)=\int_D \mathrm{e}^{\mathrm{i}(\tilde{\xi}\theta)\cdot x}\mu(dx)=\int_{u_-(\theta)}^{u_+(\theta)}\mathrm{e}^{\mathrm{i}\tilde{\xi}u}d(\pi_\theta\mu)(u)=\int_{u_-(\theta)}^{u_+(\theta)}\mathrm{e}^{\mathrm{i}\tilde{\xi}u}Y_{(\theta,u)}du.$$

Let $M>\max\{|u_-(\theta)|,|u_+(\theta)|\}+1$. Then $Y_{(\theta,u)}$ is supported in $[u_-(\theta),u_+(\theta)]\subset[-M,M]$. Using an argument attributed to Zygmund, for $|\tilde{\xi}|>\pi$,

$$\int_{-M}^{M}\mathrm{e}^{\mathrm{i}\tilde{\xi}u}Y_{(\theta,u)}du=\int_{-M}^{M}\mathrm{e}^{\mathrm{i}\tilde{\xi}(u+\pi/\tilde{\xi})}Y_{(\theta,u+\pi/\tilde{\xi})}du=-\int_{-M}^{M}\mathrm{e}^{\mathrm{i}\tilde{\xi}u}Y_{(\theta,u+\pi/\tilde{\xi})}du.$$

The first and third integrals both equal the transform, so

$$\left|\widehat{\mu}(\tilde{\xi}\theta)\right|=\frac{1}{2}\left|\int_{-M}^{M}\mathrm{e}^{\mathrm{i}\tilde{\xi}u}\left[Y_{(\theta,u)}-Y_{(\theta,u+\pi/\tilde{\xi})}\right]du\right|\leq MC_\beta\left(\frac{\pi}{\tilde{\xi}}\right)^\beta$$

by the Hölder condition, giving (2.5). □

2.2. **Liouville quantum gravity on families of self-similar sets.** Let $m\geq 2$ be an integer. Let $\mathscr{S}=(0,1)^m\times SO(\mathbb{R},2)^m\times(\mathbb{R}^2)^m$ be endowed with the product metric d. For each $t=(\vec{r},\vec{O},\vec{x})\in\mathscr{S}$ the set of m mappings

$$\mathscr{I}_t=\left\{g_i^t(\cdot)=r_iO_i(\cdot)+x_i:1\leq i\leq m\right\}$$

forms an iterated function system (IFS) of contracting similarity mappings. Such an IFS defines a unique non-empty compact set $F_t\subset\mathbb{R}^2$ that satisfies $F_t=\bigcup_{i=1}^m g_i^t(F_t)$, known as a self-similar set, see, for example, [5] for details of IFSs and self-similar sets and measures. Let $E=\{1,\dots,m\}^{\mathbb{N}}$ be the symbolic space endowed with the standard product topology and Borel σ-algebra $\mathscr{E}$. In the usual way, the points of F_t are coded by the canonical projection $f_t:E\to F_t$ given by

$$f_t(\underline{i})=f_t(i_1i_2\cdots)=\lim_{n\to\infty}g_{i_1}^t\circ\cdots\circ g_{i_n}^t(x_0),$$

which is independent of the choice of $x_0\in\mathbb{R}^2$.

Let ν be a Bernoulli measure on E with respect to a probability vector $p=(p_1,\dots,p_m)$. For $t\in\mathscr{S}$ let $\nu_t=\nu\circ f_t^{-1}$; then ν_t is a self-similar probability measure on $\mathbb{R}^2$ in the sense that $\nu_t=\sum_{i=1}^m p_i\,\nu_t\circ(g_i^t)^{-1}$.

Let $D\subset\mathbb{R}^2$ be a bounded regular domain. Let $\mathscr{T}$ be a compact subset of $\mathscr{S}$ such that for all $t\in\mathscr{T}$, $F_t\equiv f_t(E)\subset D$ and the open set condition

(OSC) is satisfied, that is there exists a non-empty open set U_t such that $U_t \supset \bigcup_{i=1}^m g_i^t(U_t)$ with this union disjoint. Take $I_t = E$ for all $t \in \mathscr{T}$. We claim that $\{(g_i^t, I_t) : t \in \mathscr{T}\}$ satisfies assumptions (A1)-(A3).

A standard estimate using OSC shows that

$$\nu_t(B(x,r)) \le C_1 r^{\alpha_1}, \quad x \in \mathbb{R}^2, r > 0, \tag{2.7}$$

where $\alpha_1 = \min_{t\in\mathscr{T},1\le i\le m} \log p_i / \log r_{t,i}$ and $C_1 > 0$ for (A1). Moreover,

$$\begin{aligned}
|f_s(\underline{i}) - f_t(\underline{i})| &\le \lim_{n\to\infty} |g^s_{i_1} \circ \cdots \circ g^s_{i_n}(x_0) - g^t_{i_1} \circ \cdots \circ g^t_{i_n}(x_0)| \\
&\le \lim_{n\to\infty} \Big\{ \big|(g^s_{i_1} - g^t_{i_1}) \circ g^s_{i_2} \circ \cdots \circ g^s_{i_n}(x_0)\big| + \big|g^t_{i_1} \circ (g^s_{i_2} - g^t_{i_2}) \circ g^s_{i_3} \circ \cdots \circ g^s_{i_n}(x_0)\big| \\
&\qquad + \cdots + \big|g^t_{i_1} \circ \cdots \circ (g^s_{i_n} - g^t_{i_n})(x_0)\big| \Big\} \\
&\le \sum_{n=0}^{\infty} r_+^n c_0 d(s,t) = C_2 d(s,t),
\end{aligned}$$

using that the g_i^t are uniformly Lipschitz on $\mathscr{T}$ and their contraction ratios are bounded by $r_+ = \max_{t\in\mathscr{T},1\le i\le m}\{r_i\} < 1$. Trivially $\nu(I_s \Delta I_t) = \nu(\emptyset) = 0$, so (A2) is satisfied. Condition (A3) holds as $\mathscr{T}$ is a compact subset of the locally Euclidean $4m$-dimensional manifold $\mathscr{S}$.

Hence the assumptions (A1)-(A3) are satisfied. Thus, in this context, Theorem 1.6 yields the following theorem.

Theorem 2.6. *Let $\gamma < \sqrt{2\alpha_1}$ where $\alpha_1 = \min_{t\in\mathscr{T},1\le i\le m} \log p_i / \log r_i$ and let $\{\widetilde{\nu}_t : t \in \mathscr{T}\}$ be γ-LQG on D for the family of self-similar measures $\{\nu_t : t \in \mathscr{T}\}$. Then almost surely the function*

$$L : \mathscr{T} \ni t \mapsto \|\widetilde{\nu}_t\|$$

is β-Hölder continuous for all $0 < \beta < \beta_0$, where β_0 is given by (1.11) with $\alpha_2 = 1$, α_2' arbitrary and $\alpha_3 = 4m$.

Remark 2.7. *One might hope that Theorem 2.6 would hold with $\alpha_1 = \min_{t\in\mathscr{T}} \dim_H \nu_t$, that is where $\alpha_1 = \min_{t\in\mathscr{T}} \sum_{i=1}^m p_i^q \log p_i \big/ \sum_{i=1}^m p_i^q \log r_i$, see [5], where $\gamma < \sqrt{2\alpha_1}$ corresponds to $\dim_H \nu_t > 2 - \dim_H \mu$. This is the case if for all $t \in \mathscr{T}$, $r_i = p_i^{1/s}$ for all i, when $\dim_H \nu_t = \dim_H F_t = s$ and ν_t is a constant multiple of the restriction of s-dimensional Hausdorff measure to F_t.*

However, in general, inequality (2.7) holds only for ν_t-almost all x when $\alpha_1 = \dim_H \nu_t$, with C_1 dependent on x, and this is not enough to apply Theorem 1.6. With some effort it can be shown that Lemma 1.2 still holds under the weaker assumption (A1′): There exist constants $C_1, \alpha_1, \epsilon_1 > 0$ such that for all $n \ge 1$ and $q \in (1, 1+\epsilon_1)$,

$$\sup_{t\in\mathscr{T}} \sum_{S\in\mathscr{S}_n} \nu_t(S)^q \le C_1 2^{-n\alpha_1(q-1)},$$

where $\mathscr{S}_n$ is any dyadic square partition of D of side length 2^{-n}. This would give the best value $\alpha_1 = \min_{t\in\mathscr{T}} \dim_H \nu_t$.

2.3. **Liouville quantum length on families of curves.** Let D be a bounded regular domain. Let $\mathscr{T} = [0,T]$ and let d be Euclidean distance on $\mathscr{T}$. Let ν be a positive Borel measure on a bounded interval $E = [L,R]$ and let $\phi : E \mapsto \overline{D}$ be a measurable function. Let $\{g_t : t \in \mathscr{T}\}$ be a family of mappings from $\overline{D}$ to $\overline{D}$. For $t \in \mathscr{T}$ let $I_t = [L_t, R_t]$ be a subinterval of E; let $f_t = g_t \circ \phi$ and $\nu_t = \nu \circ f_t^{-1}$. Trivially (A3) is satisfied in this setting, and we also assume (A1) and (A2), that is, there exist constants $C_1, \alpha_1 > 0$ such that

$$\nu_t(B(x,r)) \le C_1 r^{\alpha_1} \tag{2.8}$$

for all $B(x,r) \subset D$, and there exist $C_2, r_2, \alpha_2, \alpha_2' > 0$ such that for all $|s-t| \le r_2$ with $s,t \in \mathscr{T}$ and $I_s \cap I_t \ne \emptyset$,

$$\sup_{u\in I_s\cap I_t} |f_s(u) - f_t(u)| \le C_2 |s-t|^{\alpha_2} \tag{2.9}$$

and

$$|L_s - L_t| + |R_s - R_t| \le C_2 |s-t|^{\alpha_2'}. \tag{2.10}$$

We may apply Theorem 1.6 in this context to obtain the following theorem.

Theorem 2.8. *Let $\gamma < \sqrt{2\alpha_1}$ and let $\{\widetilde{\nu}_t : t \in [0,T]\}$ be γ-LQG in D for the family of measures $\{\nu_t : t \in [0,T]\}$. Then almost surely, for some $\beta > 0$ the function*

$$L : [0,T] \ni t \mapsto \|\widetilde{\nu}_t\|$$

is β-Hölder continuous.

If $\{f_t(I_t) : t \in [0,T]\}$ is a curve in D parameterized by t, and ν is the Lebesgue measure on $[0,T]$, then $L(t) = \|\widetilde{\nu}_t\|$ becomes the 'Liouville quantum length' of $f_t(I_t)$. By chosing a countable dense subset $\mathscr{T}_*$ of $\mathscr{T}$ and applying Lemma 1.2, Theorem 2.8 implies that almost surely L is strictly increasing and Hölder continuous in t.

Remark 2.9. *It is not difficult to verify* (2.8), (2.9) *and* (2.10) *in various situations, such as when ϕ is an algebraic curve or a Hölder continuous function, and f_t is smooth. It may also be possible to verify these inequalities for certain Loewner chains, and in particular for SLE curves, as long as the curve is independent of the GFF.*

Remark 2.10. *A challenge would be to extend Theorem 2.8 to families of random curves depending on GFF but with a coupling property, such as the flow lines of the GFF introduced in* [15]. *These flow lines are locally*

SLE curves with parameter $\kappa = \gamma^2$*, hence the Hausdorff dimension is* $\alpha_1 = 1 + \frac{\kappa}{8} = 1 + \frac{\gamma^2}{8}$*. Our assumption* $\alpha_1 - \frac{\gamma^2}{2} > 0$ *now becomes* $1 - \frac{3\gamma^2}{8} > 0$*, i.e.* $\gamma < \sqrt{\frac{8}{3}}$*, or* $\kappa < \frac{8}{3}$*. It would be of great interest if one could simultaneously define the limit of the circle average of the GFF along its flow lines in this case.*

3. Proofs

We shall need the following modification theorem.

Proposition 3.1. [11, Proposition 2.1] *The circle average process*

$$F : D \times (0,1] \ni (x,\epsilon) \mapsto \Gamma(\rho_{x,\epsilon}) \in \mathbb{R}$$

has a modification $\widetilde{F}$ *such that for every* $0 < \eta < 1/2$ *and* $\eta_1, \eta_2 > 0$ *there exists* $M = M(\eta, \eta_1, \eta_2)$ *such that*

$$\left|\widetilde{F}(x,\epsilon_1) - \widetilde{F}(y,\epsilon_2)\right| \le M \left(\log \frac{1}{\epsilon_1}\right)^{\eta_1} \frac{|(x,\epsilon_1) - (y,\epsilon_2)|^{\eta}}{{\epsilon_1}^{\eta+\eta_2}}$$

for all $x, y \in D$ *and* $\epsilon_1, \epsilon_2 \in (0,1]$ *with* $1/2 \le \epsilon_1/\epsilon_2 \le 2$.

From now on we shall always use the above modification version of the process, so in particular all the functions $x \mapsto 2^{-n\gamma^2/2} e^{\gamma \Gamma(\rho_{x,2^{-n}})}$ that we integrate against are continuous.

3.1. **Proof of Lemma 1.2.** The proof follows the same lines as the proof of [1, Proposition 3.1], using the following von Bahr-Esseen inequality.

Theorem 3.2. [22, Theorem 2] *Let* $\{X_m : 1 \le m \le n\}$ *be a sequence of random variables satisfying*

$$\mathbb{E}\big(X_{m+1} \big| X_1 + \ldots + X_m\big) = 0, \quad 1 \le m \le n-1.$$

Then for $1 \le p \le 2$

$$\mathbb{E}\Big(\Big|\sum_{m=1}^{n} X_m\Big|^p\Big) \le 2 \sum_{m=1}^{n} \mathbb{E}\big(|X_m|^p\big).$$

Proof of Lemma 1.2 Let $n \ge 1$ be fixed. Let $\mathscr{S}_n$ be the family of regions in $\mathbb{R}^2$ obtained as non-empty intersections of D with the dyadic squares of side-lengths 2^{-n}. For $S \in \mathscr{S}_n$ denote by $\widetilde{S} = \{x \in D : \mathrm{dist}(x,S) < 2^{-n}\}$ the 2^{-n}-neighborhood of S in D. From (1.1) for $S \in \mathscr{S}_n$ we can write

$$\Gamma = \Gamma^{\widetilde{S}} + \Gamma_{\widetilde{S}}, \tag{3.1}$$

where $\Gamma^{\widetilde{S}}$ and $\Gamma_{\widetilde{S}}$ are two independent Gaussian processes on $\mathscr{M}$ with covariance function $G_{\widetilde{S}}$ and $G_D - G_{\widetilde{S}}$ respectively. We can also choose a version of the process such that $\Gamma^{\widetilde{S}}$ vanishes on all measures supported in $D\setminus\widetilde{S}$, and $\Gamma_{\widetilde{S}}$ restricted to $\widetilde{S}$ is harmonic, that is for any measure τ supported in $\widetilde{S}$,

$$\Gamma_{\widetilde{S}}(\tau)=\int_{\widetilde{S}} h_{\widetilde{S}}(x)\,\tau(dx),$$

where $h_{\widetilde{S}}(x)=\Gamma(\mu_{\widetilde{S},x}),\ x\in\widetilde{S}$, is harmonic, where $\mu_{\widetilde{S},x}$ is the exit distribution of $\widetilde{S}$ by a Brownian motion started from x. In particular, by harmonicity,

$$\Gamma(\rho_{x,2^{-n}})=\Gamma^{\widetilde{S}}(\rho_{x,2^{-n}})+\Gamma(\mu_{\widetilde{S},x}),\ x\in S, \tag{3.2}$$

where $\{\Gamma^{\widetilde{S}}(\rho_{x,2^{-n}}):x\in S\}$ and $\{\Gamma(\mu_{\widetilde{S},x}):x\in S\}$ are independent.

There exists an integer N independent of n such that the family $\mathscr{S}_n$ can be decomposed into N subfamilies $\mathscr{S}_n^1,\dots,\mathscr{S}_n^N$ such that for each $j=1,\dots,N$, the closures of $\widetilde{S}$ and $\widetilde{S}'$ are disjoint for all $S,S'\in\mathscr{S}_n^j$. From (1.6) and (1.7),

$$\begin{aligned}
Y_{t,n+1}-Y_{t,n} &= \int_D \big(2^{-(n+1)\frac{\gamma^2}{2}}\mathrm{e}^{\gamma\Gamma(\rho_{x,2^{-n-1}})}-2^{-n\frac{\gamma^2}{2}}\mathrm{e}^{\gamma\Gamma(\rho_{x,2^{-n}})}\big)\nu_t(dx)\\
&= \sum_{j=1}^N\sum_{S\in\mathscr{S}_n^j}\int_S \big(2^{-(n+1)\frac{\gamma^2}{2}}\mathrm{e}^{\gamma\Gamma(\rho_{x,2^{-n-1}})}-2^{-n\frac{\gamma^2}{2}}\mathrm{e}^{\gamma\Gamma(\rho_{x,2^{-n}})}\big)\nu_t(dx)\\
&= \sum_{j=1}^N\sum_{S\in\mathscr{S}_n^j}\int_S U_S(x)V_S(x)\,\nu_t(dx), \qquad (3.3)
\end{aligned}$$

where

$$U_S(x)=2^{-n\frac{\gamma^2}{2}}\mathrm{e}^{\gamma\Gamma(\mu_{\widetilde{S},x})}$$

and

$$V_S(x)=2^{-\frac{\gamma^2}{2}}\mathrm{e}^{\gamma\Gamma^{\widetilde{S}}(\rho_{x,2^{-n-1}})}-\mathrm{e}^{\gamma\Gamma^{\widetilde{S}}(\rho_{x,2^{-n}})}$$

using (3.2). SInce the families of regions $\{\mathscr{S}_n^j\}_{j=1}^N$ are disjoint, we may choose a version of the process such that the decompositions in (3.1) and (3.2) hold simultaneously for all $S\in\mathscr{S}_n^j$. Thus $\{\{U_S(x):x\in S\}:S\in\mathscr{S}_n^j\}$ and $\{\{V_S(x):x\in S\}:S\in\mathscr{S}_n^j\}$ are independent for each $j=1,\dots,N$, and $\{\{V_S(x):x\in S\}:S\in\mathscr{S}_n^j\}$ are mutually independent and centred. By first applying Hölder's inequlity to the sum over j in (3.3), then taking conditional expectation with respect to $\{\{V_S(x):x\in S\}:S\in\mathscr{S}_n^j\}$, then applying Theorem 3.2 (the von Bahr-Esseen inequality), and finally taking the expectation, we get for $1\le p\le 2$,

$$\mathbb{E}\big(|Y_{t,n+1}-Y_{t,n}|^p\big)\le 2N^{p-1}\sum_{j=1}^N\sum_{S\in\mathscr{S}_n^j}\mathbb{E}\left(\left|\int_S U_S(x)V_S(x)\,\nu_t(dx)\right|^p\right). \tag{3.4}$$

Using Hölder's inequality and Fubini's theorem,

$$\mathbb{E}\left(\left|\int_S U_S(x)V_S(x)\,\nu_t(dx)\right|^p\right) \le \nu_t(S)^{p-1}\int_S \mathbb{E}\big(U_S(x)^p|V_S(x)|^p\big)\,\nu_t(dx). \tag{3.5}$$

Using Hölder's inequality, (1.5) and (3.2), for $x \in S$,

$$\begin{aligned}
&\mathbb{E}\big(U_S(x)^p|V_S(x)|^p\big)\\
&\quad= \mathbb{E}\left(2^{-n\frac{\gamma^2 p}{2}}\mathrm{e}^{\gamma p\Gamma(\mu_{\tilde{S},x})}\left|2^{-\frac{\gamma^2}{2}}\mathrm{e}^{\gamma\Gamma^{\tilde{S}}(\rho_{x,2^{-n-1}})}-\mathrm{e}^{\gamma\Gamma^{\tilde{S}}(\rho_{x,2^{-n}})}\right|^p\right)\\
&\quad\le 2^{p-1}\mathbb{E}\left(2^{-n\frac{\gamma^2 p}{2}}\mathrm{e}^{\gamma p\Gamma(\mu_{\tilde{S},x})}\left(2^{-\frac{\gamma^2 p}{2}}\mathrm{e}^{\gamma p\Gamma^{\tilde{S}}(\rho_{x,2^{-n-1}})}+\mathrm{e}^{\gamma p\Gamma^{\tilde{S}}(\rho_{x,2^{-n}})}\right)\right)\\
&\quad= 2^{p-1}\mathbb{E}\left(2^{-(n+1)\frac{\gamma^2 p}{2}}\mathrm{e}^{\gamma p\Gamma(\rho_{x,2^{-n-1}})}+2^{-n\frac{\gamma^2 p}{2}}\mathrm{e}^{\gamma p\Gamma(\rho_{x,2^{-n}})}\right)\\
&\quad= 2^{p-1}\left(2^{(n+1)\frac{\gamma^2}{2}(p^2-p)}R(x,D)^{\frac{\gamma^2p^2}{2}}+2^{n\frac{\gamma^2}{2}(p^2-p)}R(x,D)^{\frac{\gamma^2p^2}{2}}\right)\\
&\quad= 2^{p-1}\big(2^{\frac{\gamma^2}{2}(p^2-p)}+1\big)2^{n\frac{\gamma^2}{2}(p^2-p)}R(x,D)^{\frac{\gamma^2p^2}{2}}.
\end{aligned}$$

Combining this with (3.4) and (3.5) and noting that, from (A1), $\nu_t(S)^{p-1} \le C_1^{p-1}|S|^{\alpha_1(p-1)} \le (C_1 2^{\alpha_1/2})^{p-1}2^{-n\alpha_1(p-1)}$, we conclude that

$$\begin{aligned}
\mathbb{E}\big(|Y_{t,n+1}-Y_{t,n}|^p\big) &\le C'_p 2^{-n\alpha_1(p-1)}2^{n\frac{\gamma^2}{2}(p^2-p)}\sum_{j=1}^N\sum_{S\in\mathscr{S}_n^j}\int_S R(x,D)^{\frac{\gamma^2p^2}{2}}\,\nu_t(dx)\\
&\le C'_p 2^{-n(\alpha_1-\frac{\gamma^2}{2}p)(p-1)}\int_D R(x,D)^{\frac{\gamma^2p^2}{2}}\,\nu_t(dx),\\
&\le C'_p 2^{-n(\alpha_1-\frac{\gamma^2}{2}p)(p-1)}\max_{x\in D}R(x,D)^{\frac{\gamma^2p^2}{2}}\|\nu\|
\end{aligned}$$

where $C'_p = 2^p(NC_1 2^{\alpha_1/2})^{p-1}\big(2^{\frac{\gamma^2}{2}(p^2-p)}+1\big)$, noting that $x \mapsto R(x,D)$ is bounded on the compact set $\overline{D}$ and $\max_{t\in\mathscr{T}}\|\nu_t\| \le \|\nu\|$. The lemma follows with $C_p = C'_p \max_{x\in D} R(x,D)^{\frac{\gamma^2p^2}{2}}\|\nu\|$. □

3.2. **Proof of Lemma 1.3.**

Proof. For $n \ge 1$ and $x \in D$ define

$$\overline{F}_n(x) = \gamma\Gamma(\rho_{x,2^{-n}}) - \frac{\gamma^2}{2}n\log 2.$$

For a given $x \in D$ the random variables $X_1 := \overline{F}_1(x)$, $X_k := \overline{F}_k(x)-\overline{F}_{k-1}(x)$, $k = 2,3,\dots$ are independent. Since $\mathbb{E}\big(\mathrm{e}^{\overline{F}_n(x)}\big) = R(x,D)^{\gamma^2/2}$ is a constant in n, and $\overline{F}_n(x) = X_1+\cdots+X_n$ is a sum of n independent random variables, the sequence $\{\mathrm{e}^{\overline{F}_n(x)} : n \ge 1\}$ is a positive martingale with respect to the filtration $\{\sigma(X_1,\dots,X_n) : n \geq 1\}$. Using Doob's martingale maximal inequality,

$$\begin{aligned}\mathbb{E}\left(\max_{1\le k\le n} \mathrm{e}^{q\overline{F}_k(x)}\right) &\le \left(\frac{q}{q-1}\right)^q \mathbb{E}\left(\mathrm{e}^{q\overline{F}_n(x)}\right)\\ &= \left(\frac{q}{q-1}\right)^q 2^{n\frac{\gamma^2}{2}(q^2-q)} R(x,D)^{\frac{\gamma^2 q^2}{2}}\\ &\le C_q 2^{n\frac{\gamma^2}{2}(q^2-q)}, \qquad (3.6)\end{aligned}$$

where $C_q = \left(\frac{q}{q-1}\right)^q \max_{x\in D} R(x,D)^{\gamma^2 q^2/2}$.

For $s,t\in\mathscr{T}$ with $d(s,t)\le r\le r_2$, (A2) implies

$$\sup_{u\in I_s\cap I_t} |f_s(u)-f_t(u)| \le C_2 r^{\alpha_2} \qquad (3.7)$$

and

$$\max\left\{\nu(I_s\setminus I_t), \nu(I_t\setminus I_s)\right\} \le \nu(I_s\Delta I_t) \le C_2 r^{\alpha_2'}. \qquad (3.8)$$

We need to estimate the difference between

$$Y_{s,n} = \int_{I_s} \mathrm{e}^{\overline{F}_n(f_s(u))}\,\nu(du) \quad \text{and} \quad Y_{t,n} = \int_{I_t} \mathrm{e}^{\overline{F}_n(f_t(u))}\,\nu(du).$$

For $u\in I_s\cap I_t$ and $n\ge 1$ let $t_{u,n}\in\overline{B_d(t,r)}$ be such that

$$\overline{F}_n(f_{t_{u,n}}(u)) = \inf_{s\in B_d(t,r)} \overline{F}_n(f_s(u)).$$

Define

$$Y_{s,n}^* = \int_{I_s\cap I_t} \mathrm{e}^{\overline{F}_n(f_s(u))}\,\nu(du), \quad Y_{t,n}^* = \int_{I_s\cap I_t} \mathrm{e}^{\overline{F}_n(f_t(u))}\,\nu(du)$$

and

$$Y_n^* = \int_{I_s\cap I_t} \mathrm{e}^{\overline{F}_n(f_{t_{u,n}}(u))}\,\nu(du).$$

Then

$$\begin{aligned}|Y_{s,n}-Y_{t,n}| &\le \int_{I_s\setminus I_t} \mathrm{e}^{\overline{F}_n(f_s(u))}\,\nu(du) + \int_{I_t\setminus I_s} \mathrm{e}^{\overline{F}_n(f_t(u))}\,\nu(du)\\ &\quad +|Y_{s,n}^*-Y_n^*| + |Y_{t,n}^*-Y_n^*|. \qquad (3.9)\end{aligned}$$

Firstly, using Jensen's inequality, Fubini's theorem, (3.6) and (3.8),

$$\begin{aligned}\mathbb{E}\left(\max_{1\le k\le n}\left(\int_{I_s\setminus I_t} \mathrm{e}^{\overline{F}_k(f_s(u))}\,\nu(du)\right)^q\right) &\le C_q 2^{n\frac{\gamma^2}{2}(q^2-q)} \nu(I_s\setminus I_t)^q\\ &\le C_2^q C_q r^{q\alpha_2'} 2^{n\frac{\gamma^2}{2}(q^2-q)}, \qquad (3.10)\end{aligned}$$

and similarly

$$\mathbb{E}\left(\max_{1\le k\le n}\left(\int_{I_t\setminus I_s} e^{\overline{F}_k(f_t(u))}\,\nu(du)\right)^q\right) \le C_2^q C_q r^{q\alpha_2'} 2^{n\frac{\gamma^2}{2}(q^2-q)}. \tag{3.11}$$

Secondly,

$$\begin{aligned}\left|Y_{s,n}^* - Y_n^*\right| &= \int_{I_s\cap I_t}\left(e^{\overline{F}_n(f_s(u))} - e^{\overline{F}_n(f_{t_{u,n}}(u))}\right)\nu(du)\\ &= \int_{I_s\cap I_t} e^{\overline{F}_n(f_s(u))}\left(1-e^{-(\overline{F}_n(f_s(u))-\overline{F}_n(f_{t_{u,n}}(u)))}\right)\nu(du).\end{aligned}$$

From (3.7) $0\le f_s(u)-f_{t_{u,n}}(u)\le C_2 r^{\alpha_2}$, so by Proposition 3.1, given $0<\eta<1/2$ and $\eta_1,\eta_2>0$, we can find constants $M\equiv M(\eta,\eta_1,\eta_2)$ such that

$$\overline{F}_n(f_s(u))-\overline{F}_n(f_{t_{u,n}}(u)) \le C_2^\eta M(n\log 2)^{\eta_1} 2^{n(\eta+\eta_2)} r^{\eta\alpha_2}.$$

Since $1-e^{-x}\le x$,

$$\left|Y_{s,n}^* - Y_n^*\right| \le C_2^\eta M(n\log 2)^{\eta_1} 2^{n(\eta+\eta_2)} r^{\eta\alpha_2} Y_{s,n}^*.$$

By using similar estimates to (3.10) and (3.11) for $Y_{s,n}^*$ and $Y_{t,n}^*$we get

$$\mathbb{E}\left(\max_{1\le k\le n}\left|Y_{s,k}^* - Y_k^*\right|^q\right) \le C_{q,\eta}'(n\log 2)^{q\eta_1} 2^{qn(\eta+\eta_2)} r^{q\eta\alpha_2} 2^{n\frac{\gamma^2}{2}(q^2-q)}, \tag{3.12}$$

and

$$\mathbb{E}\left(\max_{1\le k\le n}\left|Y_{t,k}^* - Y_k^*\right|^q\right) \le C_{q,\eta}'(n\log 2)^{q\eta_1} 2^{qn(\eta+\eta_2)} r^{q\eta\alpha_2} 2^{n\frac{\gamma^2}{2}(q^2-q)}. \tag{3.13}$$

where $C_{q,\eta}' = C_q C_2^{q\eta} M^q \|\nu\|^q < \infty$.

Finally, using Hölder's inequality in (3.9) and incorporating (3.10), (3.11) (3.12) and (3.13),

$$\begin{aligned}\mathbb{E}\Big(\max_{1\le k\le n}|Y_{s,k}-Y_{t,k}|^q\Big) \le & 4^{q-1}\Big(2C_q C_2^q r^{q\alpha_2'} 2^{n\frac{\gamma^2}{2}(q^2-q)}\\ &\qquad + 2C_{q,\eta}'(n\log 2)^{q\eta_1} 2^{qn(\eta+\eta_2)} r^{q\eta\alpha_2} 2^{n\frac{\gamma^2}{2}(q^2-q)}\Big),\end{aligned}$$

so by taking η_1, η_2 close to 0 there exists a constant $C_{q,\eta}$ such that

$$\mathbb{E}\Big(\max_{1\le k\le n}|Y_{s,k}-Y_{t,k}|^q\Big) \le C_{q,\eta} r^{q((\eta\alpha_2)\wedge\alpha_2')} 2^{nq(\frac{1}{2}+\frac{\gamma^2}{2}(q-1))}.$$

□

3.3. **Proof of Proposition 1.4.** The proof is along the lines of that of the Kolmogorov-Chentsov continuity theorem: we invoke Lemma 1.3 to control $Y_{t,n}$ as $t \in \mathscr{T}$ varies and Lemma 1.2 to compare the approximations to Y_t given by $Y_{t,n}$ for consecutive circle averages of radii 2^{-n}.

Proof. Take $p > 1$ close enough to 1 so that $\alpha_1 - \frac{\gamma^2}{2}p > 0$ then take an integer ℓ large enough so that $\ell(\alpha_1 - \frac{\gamma^2}{2}p)(p-1) - \alpha_3 := \delta_1 > 0$. From (A3) and 1.2, for $j = 0, \dots, \ell - 1$,

$$\begin{aligned}
&\mathbb{E}\Big(\max_{t\in\mathscr{T}_n}|Y_{t,j+(n+1)\ell} - Y_{t,j+n\ell}|^p\Big)\\
&\qquad \le \sum_{t\in\mathscr{T}_n}\left(\sum_{k=0}^{\ell-1}\mathbb{E}\big(|Y_{t,j+n\ell+k+1} - Y_{t,j+n\ell+k}|^p\big)^{1/p}\right)^p\\
&\qquad \le C_3 2^{n\alpha_3}\ell^p C_p \sum_{k=0}^{\ell-1} 2^{-(j+n\ell+k)(\alpha_1-\frac{\gamma^2}{2}p)(p-1)}\\
&\qquad \le C2^{-n\delta_1}, \qquad (3.14)
\end{aligned}$$

where $C = C_3\ell^p C_p\big(1-2^{-(\alpha_1-\frac{\gamma^2}{2}p)(p-1)}\big)^{-1}$.

Fix $0 < \eta < 1/2$ and write $\eta' = (\eta\alpha_2)\wedge\alpha_2'$. Choose $q > 1$ large enough so that $\eta' q - \alpha_3 > 0$, then choose an integer ζ large enough such that $2^\zeta \ge C_3$ and $\zeta(\eta' q - \alpha_3) - \ell q\big(\frac{1}{2} + \frac{\gamma^2}{2}(q-1)\big) := \delta_2 > 0$. For $n \ge 1$ let

$$\mathscr{P}_n = \big\{(s,t) \in \mathscr{T}_n \times \mathscr{T}_n : d(s,t) \le 2^\zeta 2^{-n}\big\}.$$

From (A3) there exists a constant C_ζ such that $\#\mathscr{P}_n \le C_\zeta 2^{n\alpha_3}$. By Lemma 1.2, for $n \ge 1$ satisfying $2^{-\zeta(n-1)} \le r_2$ and taking $r = 2^{-\zeta(n-1)}$ in (1.9),

$$\begin{aligned}
&\mathbb{E}\Big(\max_{(s,t)\in\mathscr{P}_{n\zeta}}\max_{1\le k\le n\ell}|Y_{s,k} - Y_{t,k}|^q\Big)\\
&\qquad \le C_\zeta 2^{n\zeta\alpha_3} C_{q,\eta} 2^{-\zeta q\eta'} 2^{-n\zeta q\eta'} 2^{n\ell q(\frac{1}{2}+\frac{\gamma^2}{2}(q-1))}\\
&\qquad \le C' 2^{-n\delta_2}, \qquad (3.15)
\end{aligned}$$

where $C' = C_\zeta C_{q,\eta} 2^{-\zeta q\eta'}$.

Choose $\beta > 0$ such that both $\delta_1 - \beta p > 0$ and $\delta_2 - \beta q > 0$. Using Markov's inequality and (3.14) and (3.15),

$$\mathbb{P}\left(\max_{j=0,\dots,\ell-1}\max_{t\in\mathscr{T}_n}|Y_{t,j+(n+1)\ell} - Y_{t,j+n\ell}| > 2^{-n\beta}\right) \le \ell C 2^{-n(\delta_1-\beta p)}$$

and

$$\mathbb{P}\left(\max_{(s,t)\in\mathscr{P}_{n\zeta}}\max_{1\le k\le n\ell}|Y_{s,k} - Y_{t,k}| > 2^{-n\beta}\right) \le C' 2^{-n(\delta_2-\beta q)},$$

provided $2^{-\zeta(n-1)} \le r_2$. By the Borel-Cantelli lemma, with probability 1 there exists a random integer N such that, for all $n \ge N$, both

$$\max_{j=0,\dots,\ell-1} \max_{t\in\mathscr{T}_n} |Y_{t,j+(n+1)\ell} - Y_{t,j+n\ell}| \le 2^{-n\beta} \tag{3.16}$$

and

$$\max_{(s,t)\in\mathscr{P}_{n\zeta}} \max_{1\le k\le n\ell} |Y_{s,k} - Y_{t,k}| \le 2^{-n\beta}. \tag{3.17}$$

Fixing such an N and $n \ge N+1$, as well as $j \in \{0,\dots,\ell-1\}$, we will prove by induction on M that for all $M \ge n$, and all $s,t \in \mathscr{T}_{M\zeta}$ with $d(s,t) \le C_3 2^{-n\zeta}$,

$$\max_{0\le k\le M-1} |Y_{s,j+k\ell} - Y_{t,j+k\ell}| \le 2^{-n\beta} + 2\sum_{k=n}^{M-1} (2^{-(k+1)\beta} + 2^{-k\beta}). \tag{3.18}$$

To start the induction, if $s,t \in \mathscr{T}_{n\zeta}$ then $(s,t) \in \mathscr{P}_{n\zeta}$ (as $C_3 \le 2^{\zeta}$), so by (3.17),

$$\max_{0\le k\le n-1} |Y_{s,j+k\ell} - Y_{t,j+k\ell}| \le 2^{-n\beta},$$

which is (3.18) when $M = n$ (with the summation null).

Now suppose that (3.18) holds for some $M \ge n$. Let $s,t \in \mathscr{T}_{(M+1)\zeta}$ with $d(s,t) \le C_3 2^{-n\zeta}$. By (A3), there exist $s_*, t_* \in \mathscr{T}_{M\zeta}$ with $d(s,s_*) \le 2^{-M\zeta} = 2^{\zeta}2^{-(M+1)\zeta}$ and $d(t,t_*) \le 2^{-M\zeta} = 2^{\zeta}2^{-(M+1)\zeta}$, as well as $d(s_*,t_*) \le C_3 2^{-n\zeta}$. Thus $(s,s_*),(t,t_*) \in \mathscr{P}_{(M+1)\zeta}$. This gives, by considering the cases $1 \le k \le M-1$ and $k = M$ in the maximum separately, for all $j \in \{0,\dots,l-1\}$,

$$\begin{aligned}
&\max_{0\le k\le M} |Y_{s,j+k\ell} - Y_{t,j+k\ell}| \\
&\quad\le \max_{0\le k\le M-1} |Y_{s_*,j+k\ell} - Y_{t_*,j+k\ell}| \\
&\qquad + \max_{0\le k\le M} |Y_{s,j+k\ell} - Y_{s_*,j+k\ell}| + \max_{0\le k\le M} |Y_{t_*,j+k\ell} - Y_{t,j+k\ell}| \\
&\qquad + |Y_{s_*,j+M\ell} - Y_{s_*,j+(M-1)\ell}| + |Y_{t_*,j+M\ell} - Y_{t_*,j+(M-1)\ell}| \\
&\quad\le 2^{-n\beta} + 2\sum_{k=n}^{M-1} (2^{-(k+1)\beta} + 2^{-k\beta}) + 2\cdot 2^{-(M+1)\beta} + 2\cdot 2^{-M\beta},
\end{aligned}$$

using (3.17) and (3.16). Thus (3.18) is true with M replaced by $M+1$, completing the induction.

Letting $M \to \infty$ in (3.18) and summing the geometric series we get that for all $s,t \in \mathscr{T}_* = \bigcup_{n\ge1} \mathscr{T}_n$ with $d(s,t) \le C_3 2^{-n\zeta}$,

$$\sup_{k\ge1} |Y_{s,k} - Y_{t,k}| \le C'' 2^{-n\beta},$$

where C'' depends only on ℓ and β.

For $s,t\in\mathscr{T}_*$ with $d(s,t)\le C_3 2^{-(N+1)\zeta}$ there exists a least $n\ge N+1$ such that $C_3 2^{-(n+1)\zeta}\le d(s,t)\le C_3 2^{-n\zeta}$. Noting that $2^{-n\zeta}=2^{\zeta}2^{-(n+1)\zeta}\le 2^{\zeta}C_3^{-1}d(s,t)$,

$$\sup_{k\ge 1}|Y_{s,k}-Y_{t,k}|\le C''2^{-n\beta}=C''(2^{-n\zeta})^{\beta/\zeta}\le C'''d(s,t)^{\beta'}, \tag{3.19}$$

where $\beta'=\beta/\zeta$ and $C'''=C''(2^{\zeta}C_3^{-1})^{\beta'}$. Finally, to extend (3.19) from $\mathscr{T}_*$ to $\mathscr{T}$, we use the continuity of $t\mapsto Y_{t,n}$ for $n\ge 1$ and the fact that $\mathscr{T}_*$ is dense in $\mathscr{T}$. Inequality (1.10) follows by renaming constants appropriately.

It remains to estimate the Hölder exponent β. From the requirements in the proof on $\delta_1,\delta_2>0$, β' in (3.19) can be taken arbitrarily close to $\min\{\beta_1,\beta_2\}$ where

$$\beta_1=\eta'-\frac{\alpha_3}{q}-\frac{\ell\big(\frac12+\frac{\gamma^2}{2}(q-1)\big)}{\zeta}\quad\text{and}\quad\beta_2=\frac{\ell\big(\alpha_1-\frac{\gamma^2}{2}p\big)(p-1)-\alpha_3}{p\zeta},$$

for any valid choice of p,q,ζ and ℓ, that is subject to $1<p<\frac{2\alpha_1}{\gamma^2}$ and $1\le\frac{\alpha_3}{\eta'}<q$ with ζ and ℓ sufficiently large. Imposing the constraint $\zeta=q^2\frac{\gamma^2\ell}{2\alpha_3}$ for some such q,

$$\beta_1=\eta'-2\frac{\alpha_3}{q}+\Big(1-\frac{1}{\gamma^2}\Big)\frac{\alpha_3}{q^2}\quad\text{and}\quad\beta_2=\frac{\alpha_3}{q^2}\frac{\big(\frac{2\alpha_1}{\gamma^2}-p\big)(p-1)}{p}-\frac{\alpha_3^2}{q^2}\frac{1}{p\ell\frac{\gamma^2}{2}}.$$

Letting $\ell\to\infty$, so that also $\zeta\to\infty$ ensuring that $2^{\zeta}>C_3$,

$$\beta_1=\eta'-2\frac{\alpha_3}{q}+\Big(1-\frac{1}{\gamma^2}\Big)\frac{\alpha_3}{q^2}\quad\text{and}\quad\beta_2=\frac{\alpha_3}{q^2}\frac{\big(\frac{2\alpha_1}{\gamma^2}-p\big)(p-1)}{p}.$$

With the further constraint $p=\sqrt{\frac{2\alpha_1}{\gamma^2}}>1$,

$$\beta_1=\eta'-2\frac{\alpha_3}{q}+\Big(1-\frac{1}{\gamma^2}\Big)\frac{\alpha_3}{q^2}\quad\text{and}\quad\beta_2=\frac{\alpha_3}{q^2}\left(\sqrt{\frac{2\alpha_1}{\gamma^2}}-1\right)^2.$$

Then $\beta_1\nearrow\alpha_2$ and $\beta_2\searrow 0$ as $1<q\to\infty$, hence the maximum of $\min\{\beta_1,\beta_2\}$ subject to these constraints occurs when q is such that $\beta_1=\beta_2$, that is when

$$q^2\frac{\eta'}{\alpha_3}-2q+\Big(1-\frac{1}{\gamma^2}\Big)=\left(\sqrt{\frac{2\alpha_1}{\gamma^2}}-1\right)^2.$$

This yields

$$q=\frac{\alpha_3+\sqrt{\alpha_3^2+\eta'\alpha_3\Big(\frac{(2\alpha_1+1)}{\gamma^2}-\frac{2\sqrt{2\alpha_1}}{\gamma}\Big)}}{\eta'},$$

giving

$$\beta_1 = \beta_2 = \left(\frac{\eta'}{\sqrt{\alpha_3} + \sqrt{\alpha_3 + \eta' \left(\frac{(2\alpha_1 + 1)}{\gamma^2} - \frac{2\sqrt{2\alpha_1}}{\gamma} \right)}} \right)^2 \left(\sqrt{\frac{2\alpha_1}{\gamma^2}} - 1 \right)^2 .$$

So the exponent β' can be any number smaller than this common value with $\eta' = (\eta\alpha_2) \wedge \alpha_2'$ for η arbitrarily close to $1/2$. □

3.4. **Proof of Theorem 1.7.** Theorem 1.7 follows easily from the following lemma which is proved afterwards. As in the proof of Lemma 1.2 ,let $\mathscr{S}_n$ be the family of regions in $\mathbb{R}^2$ obtained as non-empty intersections of D with the dyadic squares of side-lengths 2^{-n}.

Lemma 3.3. *For $p > 1$ such that $\alpha_1 - \frac{\gamma^2}{2} p > 0$ there exists a constant C_p such that for all $S \in \mathscr{S}_n$,*

$$\mathbb{E}\left(\widetilde{\nu}(S)^p\right) \le C_p 2^{-n(\alpha_1 - \frac{\gamma^2}{2} p)(p-1)} \nu(S).$$

Proof of Theorem 1.7 For $\kappa > 0$ define

$$E_n(\kappa) := \{S \in \mathscr{S}_n : \widetilde{\nu}(S) > 2^{-n\kappa}\}.$$

Then

$$\begin{aligned} \widetilde{\nu}(E_n(\kappa)) &= \sum_{S \in \mathscr{S}_n} \mathbf{1}_{\{\widetilde{\nu}(S) > 2^{-n\kappa}\}} \widetilde{\nu}(S) \\ &\le \sum_{S \in \mathscr{S}_n} 2^{n\kappa(p-1)} \widetilde{\nu}(S)^{(p-1)} \widetilde{\nu}(S) \\ &= 2^{n\kappa(p-1)} \sum_{S \in \mathscr{S}_n} \widetilde{\nu}(S)^p. \end{aligned}$$

From Lemma 3.3,

$$\mathbb{E}\left(\widetilde{\nu}(E_n(\kappa))\right) \le C_p \nu(D) 2^{-n(\alpha_1 - \frac{\gamma^2}{2} p - \kappa)(p-1)}.$$

For all $\kappa < \alpha_1 - \frac{\gamma^2}{2} p$, the Borel-Cantelli lemma implies that, almost surely $\widetilde{\nu}(S_n(x)) \le 2^{-n\kappa}$ for all sufficiently large n for $\widetilde{\nu}$-almost all x, where $S_n(x)$ is the dyadic square in $\mathscr{S}_n$ containing x. Thus, almost surely,

$$\dim_H \widetilde{\nu} \ge \kappa,$$

for all $\kappa < \alpha_1 - \frac{\gamma^2}{2} p$. □

Proof of Lemma 3.3 Recall the identity from (3.2), that

$$\Gamma(\rho_{x,2^{-n}}) = \Gamma^{\widetilde{S}}(\rho_{x,2^{-n}}) + \Gamma(\mu_{\widetilde{S},x}), \ x \in S, \tag{3.20}$$

for $S \in \mathscr{S}_n$, where $\Gamma^{\widetilde{S}}$ is a GFF on $\widetilde{S} = \{x \in D : \mathrm{dist}(x,S) < 2^{-n}\}$, and $\{\Gamma^{\widetilde{S}}(\rho_{x,2^{-n}}) : x \in S\}$ and $\{\Gamma(\mu_{\widetilde{S},x}) : x \in S\}$ are independent. This implies

$$\widetilde{\nu}(dx) = \mathrm{e}^{\gamma\Gamma(\mu_{\widetilde{S},x})}\,\widetilde{\nu}^{\widetilde{S}}(dx),\ x \in S,$$

where $\widetilde{\nu}^{\widetilde{S}}$ is the Liouville quantum measure obtained from $\Gamma^{\widetilde{S}}$ acting on $\nu|_{\widetilde{S}}$. By Hölder's inequality and independence we have

$$\begin{aligned}
\mathbb{E}(\widetilde{\nu}(S)^p) &= \mathbb{E}\Big(\Big(\int_S \mathrm{e}^{\gamma\Gamma(\mu_{\widetilde{S},x})}\,\widetilde{\nu}^{\widetilde{S}}(dx)\Big)^p\Big)\\
&\le \mathbb{E}\Big(\widetilde{\nu}^{\widetilde{S}}(S)^{p-1}\int_S \mathrm{e}^{p\gamma\Gamma(\mu_{\widetilde{S},x})}\,\widetilde{\nu}^{\widetilde{S}}(dx)\Big)\\
&= \mathbb{E}\Big(\widetilde{\nu}^{\widetilde{S}}(S)^{p-1}\int_S \mathbb{E}\big(\mathrm{e}^{p\gamma\Gamma(\mu_{\widetilde{S},x})}\big)\,\widetilde{\nu}^{\widetilde{S}}(dx)\Big)\\
&\le \max_{x\in S}\mathbb{E}\big(\mathrm{e}^{p\gamma\Gamma(\mu_{\widetilde{S},x})}\big)\mathbb{E}\big(\widetilde{\nu}^{\widetilde{S}}(S)^p\big). \qquad (3.21)
\end{aligned}$$

From (3.20), independence and (1.5),

$$\mathbb{E}\big(\mathrm{e}^{p\gamma\Gamma(\mu_{\widetilde{S},x})}\big) = \Big(\frac{R(x,D)}{R(x,\widetilde{S})}\Big)^{\frac{\gamma^2p^2}{2}},$$

Recall (1.4), that

$$\mathrm{dist}(x,\partial D) \le R(x,D) \le 4\,\mathrm{dist}(x,\partial D). \qquad (3.22)$$

Since $\mathrm{dist}(x,\widetilde{S}) \ge 2^{-n}$,

$$\max_{x\in S}\mathbb{E}\big(\mathrm{e}^{p\gamma\Gamma(\mu_{\widetilde{S},x})}\big) \le (4|D|)^{\frac{\gamma^2p^2}{2}}2^{n\frac{\gamma^2p^2}{2}}. \qquad (3.23)$$

To estimate the second term in (3.21), for $m \ge n$ write

$$Y_m^{\widetilde{S}} = \int_S 2^{-m\frac{\gamma^2}{2}}\mathrm{e}^{\gamma\Gamma^{\widetilde{S}}(\rho_{x,2^{-m}})}\nu(dx).$$

By Minkowski's inequality,

$$\mathbb{E}\big(\widetilde{\nu}^{\widetilde{S}}(S)^p\big)^{\frac{1}{p}} \le \mathbb{E}\big((Y_n^{\widetilde{S}})^p\big)^{\frac{1}{p}} + \sum_{m=n}^{\infty}\mathbb{E}\big(|Y_{m+1}^{\widetilde{S}} - Y_m^{\widetilde{S}}|^p\big)^{\frac{1}{p}}. \qquad (3.24)$$

To estimate the first term of (3.24), we use Hölder's inequality, (1.5) and assumption on the measures of balls, to get

$$\begin{aligned}
\mathbb{E}\big((Y_n^{\widetilde{S}})^p\big) &\le 2^{-n\frac{\gamma^2p}{2}}\nu(S)^{p-1}\mathbb{E}\Big(\int_S \mathrm{e}^{p\gamma\Gamma^{\widetilde{S}}(\rho_{x,2^{-n}})}\nu(dx)\Big)\\
&\le 2^{-n\frac{\gamma^2p}{2}}C_1^{(p-1)}|S|^{\alpha_1(p-1)}2^{n\frac{\gamma^2p^2}{2}}\int_S R(x,\widetilde{S})^{\frac{\gamma^2p^2}{2}}\nu(dx)\\
&\le C_1^{p-1}2^{-n(\alpha_1-\frac{\gamma^2}{2}p)(p-1)}\int_S R(x,\widetilde{S})^{\frac{\gamma^2p^2}{2}}\nu(dx)
\end{aligned}$$

$$\leq \quad C_1^{p-1} 2^{-n(\alpha_1 - \frac{\gamma^2}{2}p)(p-1)} \max_{x\in S} R(x,\widetilde{S})^{\frac{\gamma^2 p^2}{2}} \nu(S).$$

For the summed terms in (3.24), by following the same lines as in the proof of Lemma 1.2 (working with the domain $\widetilde{S}$ instead of the domain D), we have for $m \geq n$,

$$\begin{aligned} \mathbb{E}\big(|Y^{\widetilde{S}}_{m+1} - Y^{\widetilde{S}}_m|^p\big) &\leq C_p 2^{-m(\alpha_1 - \frac{\gamma^2}{2}p)(p-1)} \int_S R(x,\widetilde{S})^{\frac{\gamma^2 p^2}{2}} \nu(dx) \\ &\leq C_p 2^{-m(\alpha_1 - \frac{\gamma^2}{2}p)(p-1)} \max_{x\in S} R(x,\widetilde{S})^{\frac{\gamma^2 p^2}{2}} \nu(S), \end{aligned}$$

where $C_p = 2^p (NC_1)^{p-1}\big(2^{\frac{\gamma^2}{2}(p^2-p)} + 1\big)$. Thus, from (3.24),

$$\begin{aligned} \mathbb{E}\big(\widetilde{\nu}^{\widetilde{S}}(S)^p\big)^{\frac{1}{p}} &\leq \mathbb{E}\big((Y^{\widetilde{S}}_n)^p\big)^{\frac{1}{p}} + \sum_{m=n}^{\infty} \Big[C_p 2^{-m(\alpha_1 - \frac{\gamma^2}{2}p)(p-1)} \max_{x\in S} R(x,\widetilde{S})^{\frac{\gamma^2 p^2}{2}} \nu(S)\Big]^{\frac{1}{p}} \\ &\leq C'_p \Big[2^{-n(\alpha_1 - \frac{\gamma^2}{2}p)(p-1)} \max_{x\in S} R(x,\widetilde{S})^{\frac{\gamma^2 p^2}{2}} \nu(S)\Big]^{\frac{1}{p}} \end{aligned}$$

where $C'_p = C_1^{(p-1)/p} + C_p^{1/p}\big/\big(1 - 2^{-(\alpha_1 - \frac{\gamma^2}{2}p)(p-1)/p}\big)$. Noting that $\mathrm{dist}(x,\widetilde{S}) \leq (\sqrt{2}/2+1)2^{-n}$ and applying (3.22) again, we deduce that

$$\mathbb{E}\big(\widetilde{\nu}^{\widetilde{S}}(S)^p\big) \leq C''_p 2^{-n(\alpha_1 - \frac{\gamma^2}{2}p)(p-1)} 2^{-n\frac{\gamma^2 p^2}{2}} \nu(S), \tag{3.25}$$

where $C''_p = C'_p 2^{-(\alpha_1 - \frac{\gamma^2}{2}p)(p-1)/p} (\sqrt{2}/2+1)^{\frac{\gamma^2 p^2}{2}}$. Incorporating estimates (3.23) and (3.25) in (3.21) we conclude that

$$\mathbb{E}\big(\widetilde{\nu}(S)^p\big) \leq C''_p (4|D|)^{\frac{\gamma^2 p^2}{2}} 2^{-n(\alpha_1 - \frac{\gamma^2}{2}p)(p-1)} \nu(S).$$

□

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