MINIMAL GENERATING SETS OF DIRECTED ORIENTED REIDEMEISTER MOVES

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Abstract. Polyak proved that the set $\{\Omega 1a, \Omega 1b, \Omega 2a, \Omega 3a\}$ is a minimal generating set of oriented Reidemeister moves. One may distinguish between *forward* and *backward* moves, obtaining 32 different types of moves, which we call *directed oriented Reidemeister moves*. In this article we prove that the set of 8 directed Polyak moves $\{\Omega 1a^{\uparrow}, \Omega 1a^{\downarrow}, \Omega 1b^{\uparrow}, \Omega 1b^{\downarrow}, \Omega 2a^{\uparrow}, \Omega 2a^{\downarrow}, \Omega 3a^{\uparrow}, \Omega 3a^{\downarrow}\}$ is a minimal generating set of directed oriented Reidemeister moves. We also specialize the problem, introducing the notion of a *L-generating set* for a link L. The same set is proven to be a minimal L-generating set for any link L with at least 2 components. Finally, we discuss knot diagram invariants arising in the study of K-generating sets for an arbitrary knot K, emphasizing the distinction between *ascending* and *descending* moves of type $\Omega 3$.

1. Introduction

A knot or link in $\mathbb{R}^3$ can be represented by its *diagram*, which is a generic projection of the knot or link on $\mathbb{R}^2$, admitting no singularities, triple points and non-transversal double points, together with a decoration of the double points indicating the choice of *overcrossings* and *undercrossings*. The theorem of Reidemeister [8] states that two diagrams represent the same link if and only if they can be connected by a sequence of *Reidemeister moves* of three distinct types $\Omega 1, \Omega 2$ and $\Omega 3$ (see Figure 1).

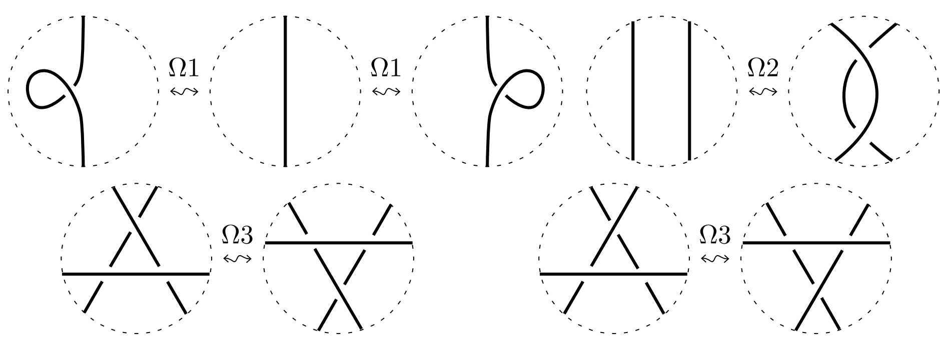

Figure 1. Reidemeister moves.

Considering oriented diagrams (diagrams of oriented knots or links), one obtains 16 different types of *oriented Reidemeister moves* (see Figures 2, 3, 4). Polyak proved in [7] that the set $\{\Omega 1a, \Omega 1b, \Omega 2a, \Omega 3a\}$ is sufficient to obtain all oriented Reidemeister moves. Moreover, he showed that there is no smaller (in terms of the

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number of elements) set of oriented Reidemeister moves. This finding reduces the procedure of checking whether a function of a link diagram is in fact a link invariant to examining changes of the function under only 4 types of moves. A similar study has been carried out by Kim, Joung and Lee [5] for Yoshikawa moves on surface-link diagrams. However, they have not proved that any of the generating sets they found is minimal.

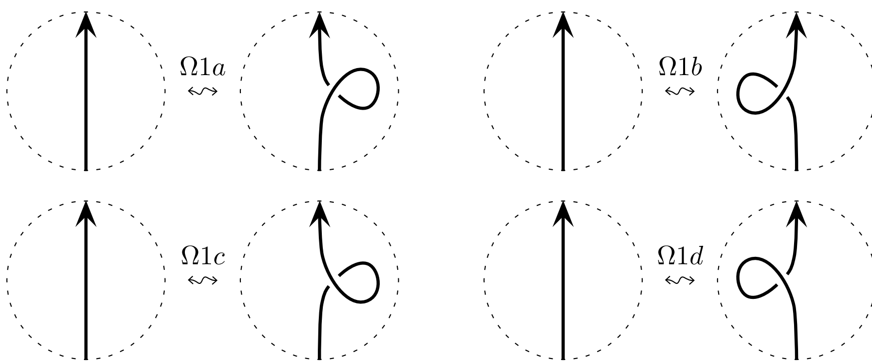


FIGURE 2. Oriented moves of type Ω1.

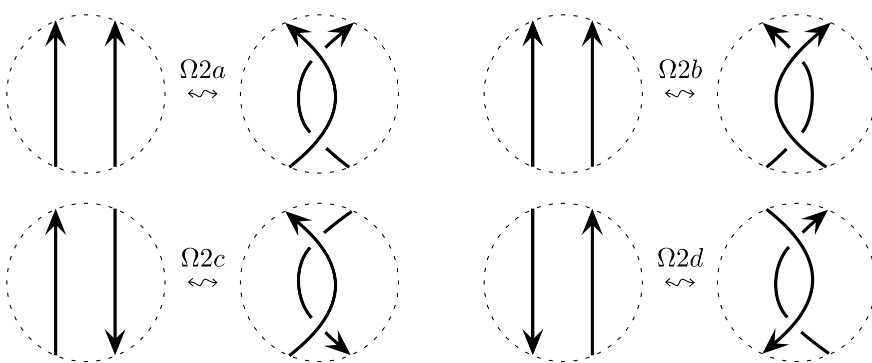


FIGURE 3. Oriented moves of type Ω2.

We rephrase Polyak's result introducing the notion of a generating set of moves:

Definition 1 (generating set of moves)**.** A set A of moves on oriented tangle diagrams is called *tangle-generating* (shortly, *generating*) if for any two tangle diagrams T_1, T_2 representing the same tangle, one can obtain T_2 from T_1 using moves from A.

Tangles are more general objects than knots and links and in particular diagrams of oriented Reidemeister moves may be considered tangles. The theorem of Reidemeister generalizes for tangles: the set of all oriented Reidemeister moves is tangle-generating. Therefore a set A is tangle-generating if and only if every oriented Reidemeister move (in both directions) may be obtained using moves from A . Thus we will drop the *tangle-* prefix and call such sets *generating*. Moreover,

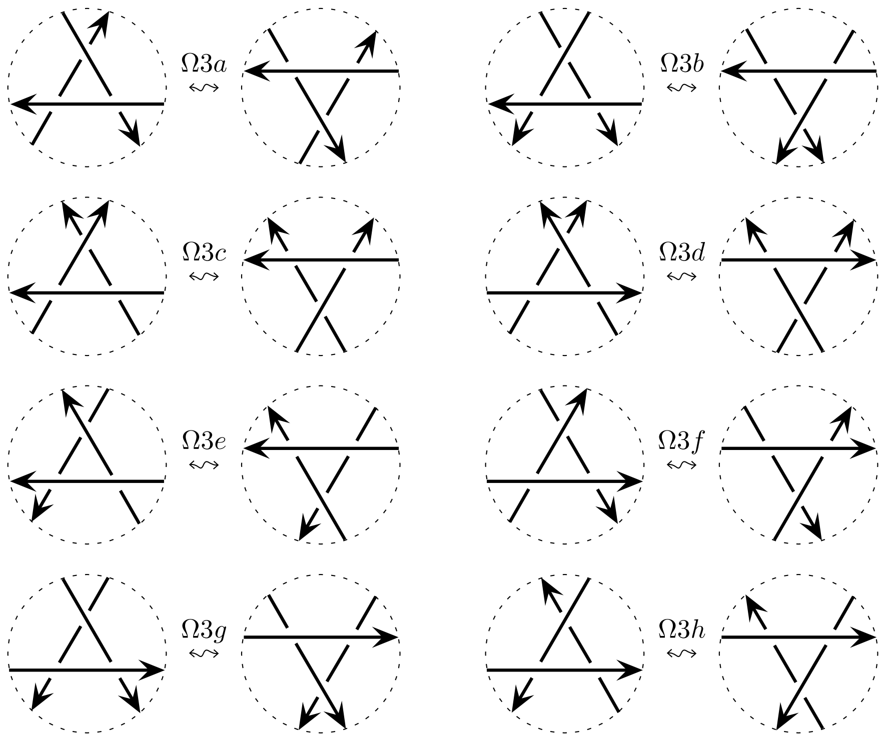


FIGURE 4. Oriented moves of type $\Omega3$.

the result of Polyak may be phrased as follows: the set $\{\Omega1a, \Omega1b, \Omega2a, \Omega3a\}$ is a minimal (with respect to size) generating subset of oriented Reidemeister moves.

We now generalize the problem, considering *directed oriented Reidemeister moves*, that is, distinguishing between *forward* and *backward* moves.

Definition 2 (directed oriented Reidemeister moves)**.** We will call a Reidemeister move of type $\Omega1$ or $\Omega2$ *forward* if it increases the number of crossings and *backward* if it decreases the number of crossings.

For an $\Omega3$ move, let us call the triangle formed by the three crossings in the $\Omega3$ move diagram the *vanishing triangle*. There is an ordering of its sides coming from the fact that they belong to distinct strands, and we order them top-middle-bottom. This ordering gives us an orientation of the vanishing triangle. Now let n be the number of its sides on which this orientation coincides with the orientation of the diagram. Let $q = (-1)^n$. Any $\Omega3$ move changes q since it changes n by ±1 or ±3. We define forward moves to be precisely those that change $q = -1$ to $q = +1$.

Forward moves are presented in Figures 2, 3 and 4, when considering them as moves from the diagram to the left to the diagram to the right. We denote forward moves using $\uparrow$ and backward using $\downarrow$, e.g. $\Omega1a^{\uparrow}$, $\Omega2c^{\downarrow}$ or $\Omega3h^{\uparrow}$.

These notions are motivated by the definitions of *positive* and *negative* moves on plane curves introduced by Arnold [1], but slightly modified, as suggested by Östlund [6]. Moreover, in Subsection 2.2 we present an equivalent definition of forward and backward moves of type $\Omega3$.

This way we obtain 32 distinct moves. Motivated by Polyak's work, we seek to find a minimal generating subset of these.

The only known results concerning this problem are direct consequences of results concerning generating sets of oriented Reidemeister moves: a set A of oriented Reidemeister moves is generating if and only if the set of both forward and backward types of moves from A is generating. In particular, Polyak's results imply that the set $\{\Omega 1a^{\uparrow}, \Omega 1a^{\downarrow}, \Omega 1b^{\uparrow}, \Omega 1b^{\downarrow}, \Omega 2a^{\uparrow}, \Omega 2a^{\downarrow}, \Omega 3a^{\uparrow}, \Omega 3a^{\downarrow}\}$ which we call *(directed) Polyak moves* is generating, and every generating subset of directed oriented moves consists of at least 4 moves. These results are not sharp: potentially, there could be a smaller generating set, in particular a proper subset of Polyak moves could be generating.

We prove that this is not the case:

Theorem 3 (minimal generating set)**.** *The set of directed Polyak moves*

$$\{\Omega 1a^{\uparrow}, \Omega 1a^{\downarrow}, \Omega 1b^{\uparrow}, \Omega 1b^{\downarrow}, \Omega 2a^{\uparrow}, \Omega 2a^{\downarrow}, \Omega 3a^{\uparrow}, \Omega 3a^{\downarrow}\}$$

is a minimal generating set of oriented directed Reidemeister moves.

More generally, any generating subset of directed oriented Reidemeister moves must contain:

(1) *at least one move from each of the sets* $\{\Omega 1a^{\uparrow}, \Omega 1d^{\downarrow}\}$, $\{\Omega 1a^{\downarrow}, \Omega 1d^{\uparrow}\}$, $\{\Omega 1b^{\uparrow}, \Omega 1c^{\downarrow}\}$, $\{\Omega 1b^{\downarrow}, \Omega 1c^{\uparrow}\}$,
(2) *at least one forward* ($\Omega 2^{\uparrow}$) *and backward* ($\Omega 2^{\downarrow}$) *move of type* $\Omega 2$,
(3) *at least one forward* ($\Omega 3^{\uparrow}$) *and backward* ($\Omega 3^{\downarrow}$) *move of type* $\Omega 3$.

Polyak [7] showed the existence of 4-element sets of oriented Reidemeister moves which satisfy the conditions above, but are not generating. Therefore these conditions are not sufficient to determine whether a set is generating.

To prove that some set is not generating, it suffices to prove that it is not L-generating for some L:

Definition 4 (L-generating set)**.** Let L be a link. A set A of moves is *L-generating*, if any two diagrams L_1, L_2 of L are connected by a sequence of moves from A.

Indeed, if A is generating, then it is L-generating for any link L. To prove Theorem 3 we show the following:

Theorem 5 ($\Omega 1$ in L-generating sets)**.** *For any link L, any L-generating subset of directed oriented Reidemeister moves contains at least:*

- 1 *move from the set* $\{\Omega 1a^{\uparrow}, \Omega 1d^{\downarrow}\}$,
- 1 *move from the set* $\{\Omega 1a^{\downarrow}, \Omega 1d^{\uparrow}\}$,
- 1 *move from the set* $\{\Omega 1b^{\uparrow}, \Omega 1c^{\downarrow}\}$,
- 1 *move from the set* $\{\Omega 1b^{\downarrow}, \Omega 1c^{\uparrow}\}$.

Theorem 6 ($\Omega 2$ in L-generating sets, for non-knot L)**.** *For any link L with at least* 2 *components, any L-generating subset of directed oriented Reidemeister moves contains at least* 1 *move of type* $\Omega 2^{\uparrow}$ *and* 1 *move of type* $\Omega 2^{\downarrow}$.

Theorem 7 ($\Omega 3$ in L-generating sets, for non-knot L)**.** *For any link L with at least* 2 *components, any L-generating subset of directed oriented Reidemeister moves contains at least* 1 *move of type* $\Omega 3^{\uparrow}$ *and* 1 *move of type* $\Omega 3^{\downarrow}$.

In fact, we answer the question of finding a minimal L-generating set for any link L which is not a knot.

It would be interesting to know if a similar result holds for K-generating sets when K is a knot. This problem seems to be much harder to solve and therefore we reduce it to the question whether the set of directed Polyak moves has K-generating subsets. Theorem 5 readily implies

Corollary 8 ($\Omega 1$ in L-generating subsets of Polyak moves)**.** *For any link L, any L-generating subset of directed Polyak moves contains moves $\Omega 1a^{\uparrow}, \Omega 1a^{\downarrow}, \Omega 1b^{\uparrow}$ and $\Omega 1b^{\downarrow}$.*

A similar result holds for moves of type $\Omega 2$:

Theorem 9 ($\Omega 2$ in L-generating subsets of Polyak moves)**.** *For any link L, any L-generating subset of directed Polyak moves contains moves $\Omega 2a^{\uparrow}$ and $\Omega 2a^{\downarrow}$.*

We also present partial results concerning moves of type $\Omega 3$, distinguishing between *ascending* and *descending* moves of type $\Omega 3$ (see Definition 24).

On the other hand, for any link L, the set $\{\Omega 1a, \Omega 1b, \Omega 2a, \Omega 3a\}$ is a minimal generating subset of (undirected) oriented Reidemeister moves. Indeed, Hagge [2] proved that for any knot K (and therefore for any link, too) there exist two diagrams K_1, K_2 of K such that one cannot obtain K_2 from K_1 without using moves of type $\Omega 2$, and there exist diagrams K_3, K_4 of K such that K_4 cannot be obtained from K_3 without using moves of type $\Omega 3$. These, together with Theorem 5, (proof of which mirrors the proof of Lemma 3.1 from [7]), proves that any L-generating subset of oriented Reidemeister moves contains at least 2 moves of type $\Omega 1$, 1 move of type $\Omega 2$ and 1 move of type $\Omega 3$.

The article begins with the proofs of Theorems 5, 6 and 7 in Section 2, from which Theorem 3 follows. The key ingredient to the proof of Theorem 7 is the introduction of the invariants CI and OCI, which are thoroughly studied. In Section 3 we study knot diagram invariants and their changes under Polyak moves, emphasizing the difference between ascending and descending moves of type $\Omega 3$. An invariant HNP, which is a special case of an invariant defined by Hass and Nowik, [4] is introduced and discussed. Moreover, families of invariants defined by Östlund [6], distinguishing between ascending and descending moves, are briefly recalled.

The results presented in the article were obtained as a part of author's Master's thesis at the University of Warsaw. The author would like to thank his advisor, Maciej Borodzik, for his insight and patience.

2. Minimal Generating Sets

In this section we prove Theorem 3 by proving Theorems 5, 6, 7.

2.1. $\Omega 1$ and $\Omega 2$ moves.

The following proof mirrors the proof of Lemma 3.1 from [7].

Proof of Theorem 5. The writhe n and the winding number c of a link diagram do not change under Reidemeister moves of type $\Omega 2$ and $\Omega 3$. Consider their sum $w_+ = n + c$ and difference $w_- = n - c$.

Invariant	$\Omega 1a^{\uparrow}$	$\Omega 1b^{\uparrow}$	$\Omega 1c^{\uparrow}$	$\Omega 1d^{\uparrow}$
n =writhe	$+1$	$+1$	-1	-1
c =winding number	-1	$+1$	-1	$+1$
$w_+ = n + c$	0	$+2$	-2	0
$w_- = n - c$	$+2$	0	0	-2

TABLE 1. Changes of the writhe and the winding number with respect to Polyak moves.

Notice w_+ increases only under $\Omega 1b^{\uparrow}$ and $\Omega 1c^{\downarrow}$ moves. Consider a diagram D of a link L and a diagram D' obtained from D by an $\Omega 1b^{\uparrow}$ move. Then $w_+(D') - w_+(D) = 2$, so any set of Reidemeister moves which transforms D into D' has to include at least one of the moves $\Omega 1b^{\uparrow}$ or $\Omega 1c^{\downarrow}$. Therefore any L-generating set of moves contains one of these. Moreover $w_+(D) - w_+(D') = -2$, and therefore any L-generating set of moves contains at least one of the moves $\{\Omega 1b^{\downarrow}, \Omega 1c^{\uparrow}\}$.

A similar argument for w_- shows that any L-generating set of moves contains at least one move from $\{\Omega 1a^{\uparrow}, \Omega 1d^{\downarrow}\}$ and from $\{\Omega 1a^{\downarrow}, \Omega 1d^{\uparrow}\}$. □

Proof of Theorem 6. $\Omega 1$ and $\Omega 3$ moves preserve the number of crossings between different components of the link diagram. The same is true for $\Omega 2$ moves between strands of the same link component.

On the other hand, any $\Omega 2^{\uparrow}$ move between strands belonging to different components of the link creates 2 such crossings and any $\Omega 2^{\downarrow}$ move between strands of distinct components cancels 2 such crossings. Let L be a link with at least 2 components (i.e. not a knot). Since any diagram of such link L admits an $\Omega 2^{\uparrow}$ move, therefore any L-generating set contains a move of type $\Omega 2^{\uparrow}$ and a move of type $\Omega 2^{\downarrow}$. □

2.2. $\Omega 3$ moves.

Definition 10. Denote by $\mathcal{C}_d(D)$ the set of crossings of different components of a diagram D.

For $p \notin \gamma(S^1)$, denote by $\mathrm{Ind}_{\gamma}(p)$ the index of a point $p \in \mathbb{R}^2$ with respect to a curve $\gamma : S^1 \to \mathbb{R}^2$.

For $p \in \mathcal{C}(D)$, denote by $\mathrm{sgn}(p) \in \{-1, +1\}$ the *sign* of the crossing p.

By a *changing disc* of a (oriented, directed oriented) Reidemeister move we mean the disc in the plane the move takes place in, as depicted in Figures 2, 3, 4 above.

Definition 11 (crossing index of a diagram)**.** Let D be a diagram of a 3-component link. For each crossing $p \in \mathcal{C}_d(D)$, define its *crossing index* as

$$CI(p) = \mathrm{sgn}(p) \cdot \mathrm{Ind}_{\gamma}(p),$$

where γ is

the component of the link diagram that does not pass through p.

Now set the *crossing index* of D to be

$$CI(D) = \sum_{p \in \mathcal{C}_d(D)} CI(p).$$

Finally, let D be a diagram of any n-component link, where $n \neq 3$. Let $D_1, \ldots, D_n$ denote the components of D. We define the crossing index of D to be

$$CI(D) = \sum_{1\le i<j<k\le n} CI(D|_{i,j,k}),$$

where $D|_{i,j,k}$ denotes a diagram obtained from D by forgetting all components other than D_i, D_j and D_k.

Remark 12. This invariant is a variation of Vassiliev's index-type invariants of ornaments [9].

We may give an equivalent, more direct definition of CI.

Definition 13 (overcrossing and undercrossing curve)**.** Let D be a diagram and p be one of its crossings. Denote by γ_p^o the curve of the diagram passing through p as an overcrossing. Denote by γ_p^u the curve of the diagram passing through p as an undercrossing.

Proposition 14 (alternative description of CI)**.** *Let D be a n-component link diagram and $\gamma_1, \dots, \gamma_n$ be the curves of the components of its diagram. Then*

$$CI(D) = \sum_{p\in\mathcal{C}_d(D)} \sum_{\substack{1\le i\le n,\\ \gamma_i\ne\gamma_p^o, \gamma_i\ne\gamma_p^u}} \operatorname{sgn}(p)\cdot \operatorname{Ind}_{\gamma_i}(p).$$

Proof. For $n = 3$, the formula coincides with the definition of CI for 3-component links. Using this fact, by the definition of CI for arbitrary n we obtain:

$$\begin{aligned}
CI(D) &= \sum_{1\le i<j<k\le n}\left(\sum_{p\in\mathcal{C}_d(D|_{i,j})}\operatorname{sgn}(p)\operatorname{Ind}_{\gamma_k}(p) + \sum_{p\in\mathcal{C}_d(D|_{j,k})}\operatorname{sgn}(p)\operatorname{Ind}_{\gamma_i}(p) + \sum_{p\in\mathcal{C}_d(D|_{i,k})}\operatorname{sgn}(p)\operatorname{Ind}_{\gamma_j}(p)\right)\\
&= \sum_{\substack{1\le i<j\le n,\\ 1\le k\le n,\\ k\ne i, k\ne j}} \sum_{p\in\mathcal{C}_d(D|_{i,j})}\operatorname{sgn}(p)\operatorname{Ind}_{\gamma_k}(p)\\
&= \sum_{p\in\mathcal{C}_d(D)} \sum_{\substack{1\le k\le n,\\ \gamma_k\ne\gamma_p^o, \gamma_k\ne\gamma_p^u}} \operatorname{sgn}(p)\cdot\operatorname{Ind}_{\gamma_k}(p).
\end{aligned}$$

where $D|_{i,j}$ denotes the diagram obtained from D by forgetting all components other than D_i and D_j. This finishes the proof. □

Proposition 15. *The quantity CI is invariant under moves of type $\Omega1$ and $\Omega2$, and under moves of type $\Omega3$ which involve at least two strands of the same component. It increases by 1 under $\Omega3^\uparrow$ moves involving three strands of different components.*

Proof. It follows from the construction of CI for arbitrary link diagram that it is sufficient to prove the claim for diagrams of 3-component links. Therefore we assume that D is a diagram of a 3-component link.

Any Reidemeister move does not change indices of points outside the changing disc. It also does not change signs of crossings outside the changing disc. Therefore it does not change $CI(p)$ for any crossing p outside the changing disc. It suffices to check how these moves change $CI(p)$ for the crossings inside the changing disc.

An $\Omega1$ move does not create or cancel any crossings between distinct components of a link. Therefore $\Omega1$ moves do not change $CI(D)$.

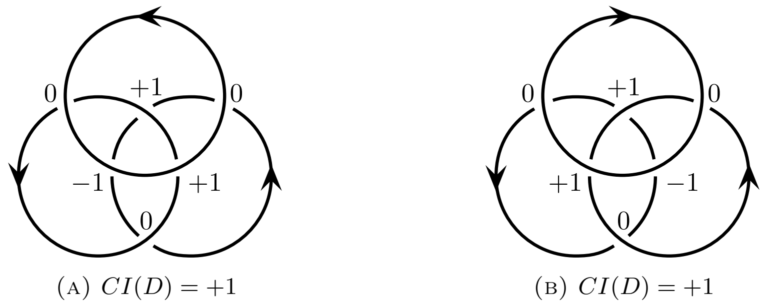


FIGURE 5. Calculations of $CI(p)$, $CI(D)$ for example diagrams of a 3-component unlink. Crossing indices $CI(p)$ are shown next to the crossings.

An $\Omega 2$ move creates or cancels two crossings. If two strands of the $\Omega 2$ move belong to the same component of the link, then the move does not create or cancel any crossings between different components, so it preserves $CI(D)$. If two strands of the $\Omega 2$ move belong to different components of the link, then both crossings that are created or cancelled are of different signs (one positive and one negative) and of the same index with respect to the third component, since they can be connected by a curve that does not intersect the other component. Therefore the contributions of both crossings to $CI(D)$ cancel, so $CI(D)$ is preserved by $\Omega 2$ moves.

An $\Omega 3$ move, in general, does not change the set of crossings of the diagram, but only changes the placement of three crossings involved. For a crossing p of two components involved in this move, its sign does not change, but its index with respect to the third component may change (see Figure 6).

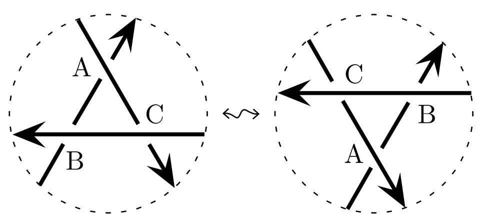


FIGURE 6. Corresponding crossings of $\Omega 3a$ move.

An $\Omega 3$ move that involves three strands of the same component does not change any crossings between different components, and so leaves $CI(D)$ unchanged.

An $\Omega 3$ move that involves two strands of the same component and one strand of any other component involves only crossings between these two components. Since it does not change their indices with respect to the third component (all points in the changing disc have the same index with respect to that component), it does not change $CI(D)$.

Consider an $\Omega 3$ move that involves three strands of different components. For a crossing p involved in this move, let γ be the diagram component not passing through p, and S be the strand taking part in the $\Omega 3$ move contained in γ. The $\Omega 3$ move changes the index of p with respect to γ by $+1$ if the move shifts p from the right to the left of strand S and by -1 if the move shifts p from the left to the right of S. In the first case, $CI(p)$ changes by $+1$ if crossing p is positive and by -1 if it is negative. In the second case, $CI(p)$ changes by -1 if crossing p is positive and by $+1$ it it is negative.

For $\Omega 3$ moves involving 3 different components, depicted in Figure 7, the signs of the crossings (diagrams to the left) and changes of $CI(p)$ for the crossings (diagrams to the right) are written down. Summing all the changes of $CI(p) = \operatorname{sgn}(p)\operatorname{Ind}_\gamma(p)$ for the three crossings of a move, it follows that $CI(D)$ changes by $+1$ for moves of type $\Omega 3^\uparrow$ and by -1 for moves of type $\Omega 3^\downarrow$.

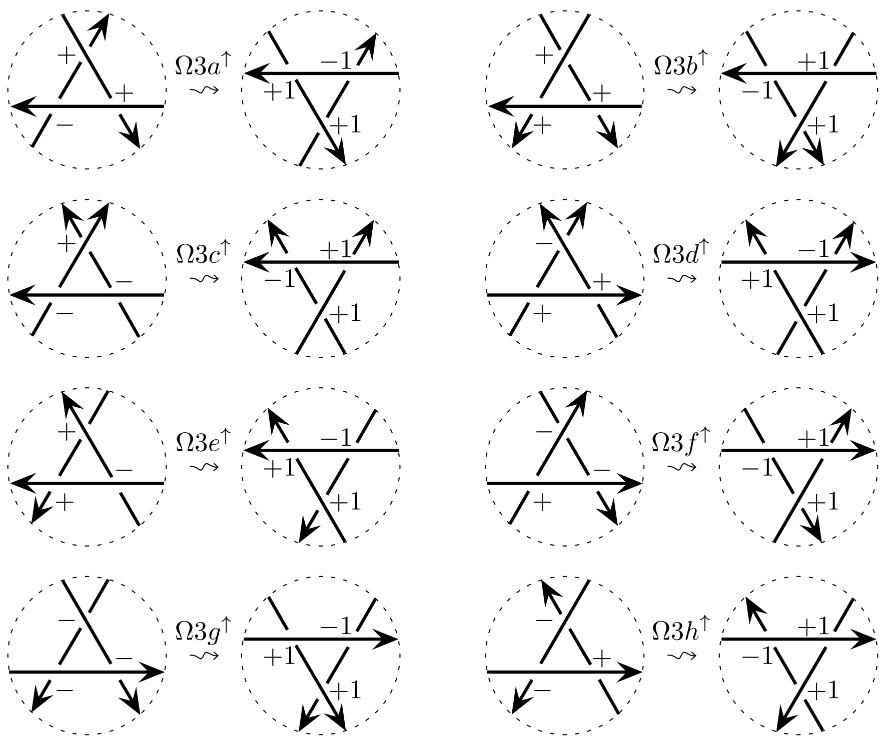


FIGURE 7. Signs (to the left) and changes of $CI(p)$ (to the right) for corresponding crossings of moves of type $\Omega 3$.

□

Proof of Theorem 7 for links of at least 3 components. Let L be a link diagram with at least 3 components. Having any diagram of L, by an appropriate sequence of Reidemeister moves one can obtain a diagram D of L which admits a $\Omega 3^\uparrow$ move involving 3 different components, by first making strands of 3 different components bound one of the regions of the plane, and then making $\Omega 2$ moves to obtain a diagram admitting an $\Omega 3a^\uparrow$ move, as in Figure 8.

The CI of the diagram differs by 1 from the CI of the diagram obtained after performing the $\Omega 3$ move. It follows that any L-generating set contains at least one move of type $\Omega 3^\uparrow$ and one of type $\Omega 3^\downarrow$. □

Remark 16. The above consideration yields an alternative characterization of forward and backward $\Omega 3$ moves. For an $\Omega 3$ move, one may complete its diagram to

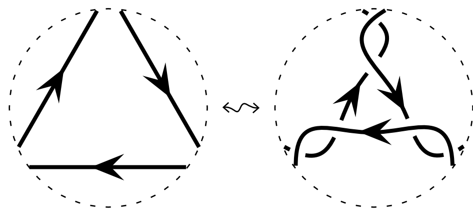


FIGURE 8. Preparing a diagram admitting an $\Omega 3a$ move.

a move on a link diagram such that the three strands of the move diagram belong to different components of that link. The change of CI of the obtained diagrams due to the move does not depend on the chosen completion as we have already shown, so $\Omega 3^{\uparrow}$ moves may be defined to be precisely the ones that increase CI of a diagram obtained this way by 1.

The invariant CI is zero for any 2-component link diagram, so one may still ask whether both forward and backward $\Omega 3^{\uparrow}$ moves are needed for 2-component link diagrams. Therefore we proceed to introduce another diagram invariant that distinguishes forward and backward $\Omega 3$ moves.

Definition 17 (half-index)**.** Let $\gamma : S^1 \to \mathbb{R}^2$ be an immersed curve and let $p \in \gamma(S^1)$ be a point which is not a double point of γ. Then define $\mathrm{hInd}_{\gamma}(p)$ to be the mean of two numbers: the index of a point to the left of γ close to p and the index of a point to the right of γ close to p.

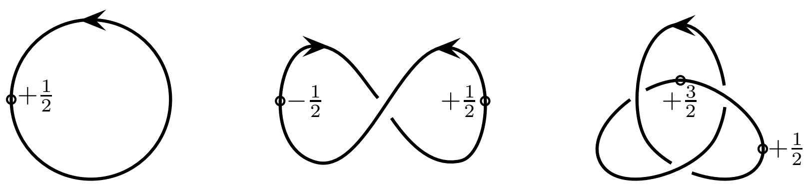


FIGURE 9. Examples of half-indices of points with respect to the underlying curves of knot diagrams.

Definition 18 (overcrossing index)**.** Let D be a diagram of a link. For a crossing $p \in \mathcal{C}_d(D)$, we define its *overcrossing index* as

$$OCI(p) = \mathrm{sgn}(p) \cdot \mathrm{hInd}_{\gamma_p^o}(p).$$

Recall that γ_p^o denotes the component of the diagram that contains the overcrossing of p.

Now define the *overcrossing index* of D to be

$$OCI(D) = \sum_{p \in \mathcal{C}_d(D)} OCI(p).$$

Proposition 19. *The quantity OCI in invariant under $\Omega 1$ and $\Omega 2$ moves, under $\Omega 3$ moves involving 3 strands of the same component and under $\Omega 3$ moves involving 3 strands of different components.*

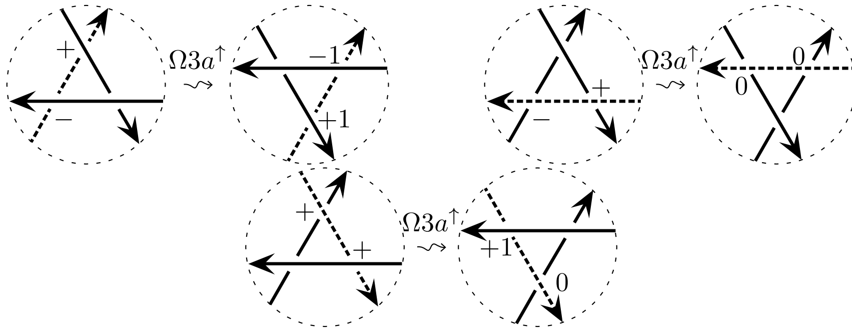


FIGURE 10. Signs (to the left) and changes of $OCI(p)$ (to the right) for corresponding crossings of different components for an $\Omega 3a^{\uparrow}$ move. The solid lines belong to one link component.

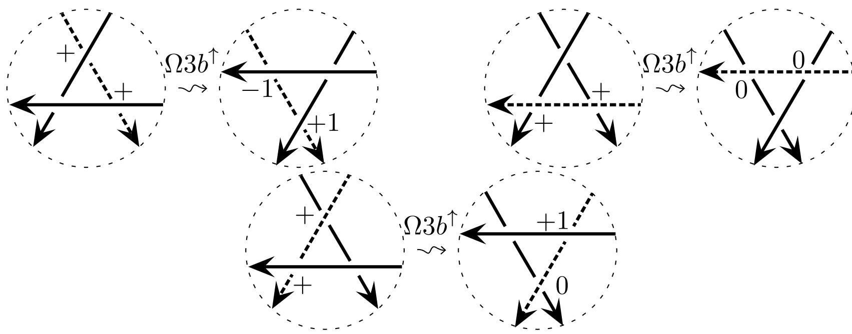


FIGURE 11. Signs (to the left) and changes of $OCI(p)$ (to the right) for corresponding crossings of different components for an $\Omega 3b^{\uparrow}$ move. The solid lines belong to one link component.

It increases by 0 *or* 1 *under an* $\Omega 3^{\uparrow}$ *move involving* 2 *strands of one link component and* 1 *strand of another link component, depending on which strands belong to the same component. Precisely, for such moves it increases by:*

- 0 *when top and middle strands are of the same component,*
- 0 *when middle and bottom strands are of the same component,*
- 1 *when top and bottom strands are of the same component.*

Proof. As before, it suffices to check the values of $OCI(p)$ for the crossings in the changing discs of Reidemeister moves.

Invariance under $\Omega 1$, $\Omega 2$ and $\Omega 3$ moves involving only one component of the link follows from the same argument as given for CI.

Invariance under $\Omega 2$ moves involving different components of the diagram follows from the same argument as given for CI since both crossings involved in such move share the same overcrossing curve, and since they have opposite signs, their $OCI(p)$ cancel.

$\Omega 3$ moves involving strands of 3 different components leave both signs and half-indices of corresponding crossings in the changing disc unchanged, so do not change $OCI(D)$.

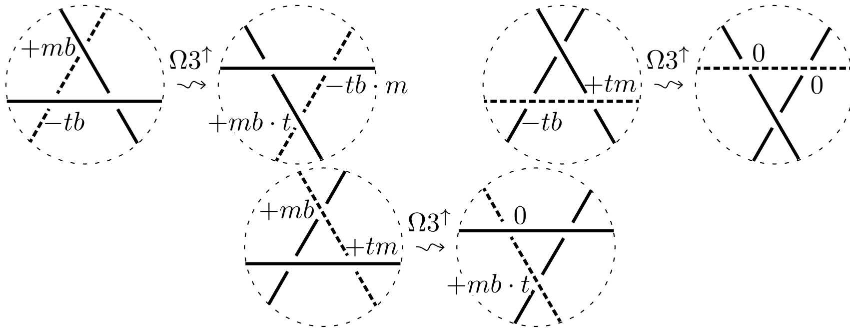


FIGURE 12. Signs (to the left) and changes of $OCI(p)$ (to the right) for corresponding crossings of different components for $\Omega 3a^{\uparrow}$, $\Omega 3d^{\uparrow}$, $\Omega 3e^{\uparrow}$, $\Omega 3g^{\uparrow}$ moves. The solid lines belong to one link component.

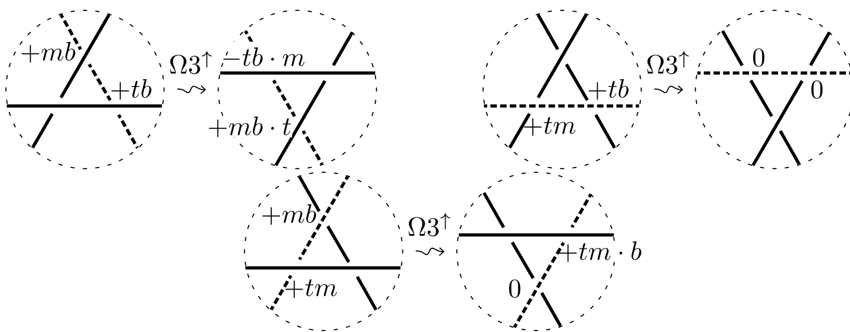


FIGURE 13. Signs (to the left) and changes of $OCI(p)$ (to the right) for corresponding crossings of different components for $\Omega 3b^{\uparrow}$, $\Omega 3c^{\uparrow}$, $\Omega 3f^{\uparrow}$, $\Omega 3h^{\uparrow}$ moves. The solid lines belong to one link component.

An $\Omega 3$ move involving strands of 2 different components have 2 crossings between different components in their changing discs. Consider one of these crossings, p. After the $\Omega 3^{\uparrow}$ move, the half-index of p with respect to γ_p^o stays unchanged if the strand S not passing through p belongs to γ_p^u. Otherwise it increases by 1 if the move shifts p from the right to the left of strand S, or decreases by 1 if the move shifts p from the left to the right of strand S. Then, to calculate the change of $OCI(p)$ we multiply the change of this half-index by the sign of the crossing.

For such $\Omega 3$ move, three cases are to be considered:

(a) top and middle strands are in the same component,
(b) middle and bottom strands are in the same component,
(c) top and bottom strands are in the same component.

Figure 10 summarizes signs (to the left) and changes of $OCI(p)$ (to the right) in these cases for an $\Omega 3a^{\uparrow}$ move. Figure 11 gives the same information for an $\Omega 3b^{\uparrow}$ move.

Now, take any $\Omega 3^{\uparrow}$ move from $\Omega 3a^{\uparrow}$, $\Omega 3d^{\uparrow}$, $\Omega 3e^{\uparrow}$ and $\Omega 3g^{\uparrow}$. The diagrams of these moves differ only by orientations of strands. Let t (resp. m, b) be equal to $+1$ if the orientation of top (resp. middle, bottom) strand coincides with the orientation of top (resp. middle, bottom) strand for an $\Omega 3a^{\uparrow}$ move and -1 otherwise. Figure 12 summarizes signs (to the left) and changes of $OCI(p)$ (to the right) for the three cases of a move of type $\Omega 3a^{\uparrow}$, $\Omega 3d^{\uparrow}$, $\Omega 3e^{\uparrow}$ or $\Omega 3g^{\uparrow}$. Analogous information is contained in Figure 13 for moves of type $\Omega 3b^{\uparrow}$, $\Omega 3c^{\uparrow}$, $\Omega 3f^{\uparrow}$, $\Omega 3h^{\uparrow}$ (t, m, b depend of the orientations of strands relative to the $\Omega 3b^{\uparrow}$ move).

Summing changes of $OCI(p)$ for crossings of these diagrams, it follows that in the first two cases $OCI(D)$ remains unchanged. In the third case, when the top and bottom strands belong to one component, $OCI(D)$ changes by $t \cdot m \cdot b$. Checking values of t, m, b for all $\Omega 3^{\uparrow}$ moves it follows that $tmb = 1$ for any $\Omega 3^{\uparrow}$ move. Indeed, each of the diagrams of moves $\Omega 3d^{\uparrow}, \Omega 3e^{\uparrow}, \Omega 3g^{\uparrow}$ has exactly two strands with orientations opposite to orientations of corresponding strands in $\Omega 3a^{\uparrow}$ move, and similar conclusion applies for $\Omega 3c^{\uparrow}, \Omega 3f^{\uparrow}, \Omega 3h^{\uparrow}$ with respect to $\Omega 3b^{\uparrow}$. □

Proof of Theorem 7. If a link L has at least 2 components, a suitable sequence of Reidemeister moves leads to a diagram D, part of which looks like the left diagram of Figure 8, with the bottom and the left strand belonging to the same component and the strand to the right belonging to another component. By conducting three moves of type $\Omega 2$ as in Figure 8 we obtain a diagram admitting an $\Omega 3a^{\uparrow}$ move that increases $OCI(D)$ by 1. □

Remark 20. In a similar way one can define the *undercrossing index* UCI of a diagram. Repeating the steps of the proof of Proposition 19 one can show that UCI changes exactly in the same way as OCI, so the difference $OCI - UCI$ is a link invariant. One can directly check that the difference is invariant under changes of crossings, and is zero on an unknot diagram. It follows that $OCI = UCI$.

Remark 21. Hayashi, Hayashi and Nowik constructed in [3] a family of unlink diagrams D_n and proved that the number of moves needed to separate both components of D_n is greater or equal to $(n^2 + 14n - 13)/16$, and the number of moves needed to obtain a diagram without crossings from D_n is greater or equal to $(n^2 + 10n - 13)/4$. But $OCI(D_n) = -n^2/4$ for n even and $OCI(D_n) = -(n^2 - 1)/4$ for n odd, so it follows that one needs at least $(n^2 - 1)/4$ moves (of very specific type, as described in Proposition 19) to separate components of D_n.

3. Polyak moves

3.1. $\Omega 2$ moves.

Proof of Theorem 9. Notice moves of type $\Omega 1a$ and $\Omega 1b$ do not change the number of negative crossings, n_-. This quantity is invariant under $\Omega 3$ moves, too.

On the other hand, $\Omega 2a^{\uparrow}$ increases n_- by 1, and $\Omega 2a^{\downarrow}$ decreases n_- by 1. Therefore, having two diagrams D_1, D_2 of a knot K, D_2 being obtained from D_1 by an $\Omega 2a^{\uparrow}$ move, we have $n_-(D_2) - n_-(D_1) = 1$, so one cannot get D_2 from D_1 using directed Polyak moves without $\Omega 2a^{\uparrow}$ and one cannot get D_1 from D_2 using directed Polyak moves without $\Omega 2a^{\downarrow}$. □

3.2. **Ascending and descending $\Omega 3$ moves.** We recall the definition of a diagram invariant introduced by Hass and Nowik in [4]. Let D be a knot diagram and p one of its crossings. Denote by D_p the link diagram obtained by smoothing the crossing p as shown in Figure 14. Let $\mathcal{C}_+(D)$ (resp. $\mathcal{C}_-(D)$) be the set of all positive (resp. negative) crossings of D.

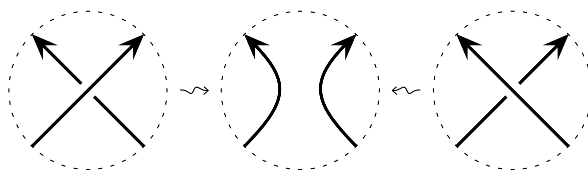


FIGURE 14. Smoothing positive and negative crossings.

Definition 22. Let ϕ be a two-component link invariant with values in a set S. Define a diagram invariant

$$I_\phi(D) = \sum_{p \in \mathcal{C}_+(D)} X_{\phi(D_p)} + \sum_{p \in \mathcal{C}_-(D)} Y_{\phi(D_p)}, \tag{1}$$

with values in $G(S) = \bigoplus_{s \in S} (\mathbb{N}X_s \oplus \mathbb{N}Y_s)$, where we consider X_s, Y_s to be formal variables representing generators of $\bigoplus_{s \in S} \mathbb{N}^2$.

We will call it the *Hass–Nowik invariant*. In their paper [4] Hass and Nowik calculated how this invariant, taken with $\phi = \mathrm{lk}$ (the linking number), changes with respect to Reidemeister moves.

For moves we are interested in, changes of the invariant are summarized in the table below (following [4]):

Move	Change
$\Omega 1a^\uparrow$	X_0
$\Omega 1b^\uparrow$	X_0
$\Omega 2a^\uparrow$	$X_n + Y_{n+1}$
$\Omega 3a^\uparrow$	$\pm(Y_n - Y_{n-1})$

TABLE 2. Changes of I_{lk} with respect to Polyak moves.

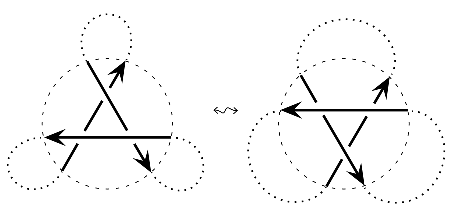
Here both n and $+$ or $-$ sign for $\pm$ depend on the part of the diagram outside the changing disc.

Definition 23. Denote by HNP the diagram invariant defined as a composition of I_{lk} and a semigroup homomorphism $\bigoplus_{n \in \mathbb{Z}} (\mathbb{N}X_n \oplus \mathbb{N}Y_n) \to \mathbb{Z}$ mapping $X_n \mapsto -n$, $Y_n \mapsto n - 1$. More explicitly,

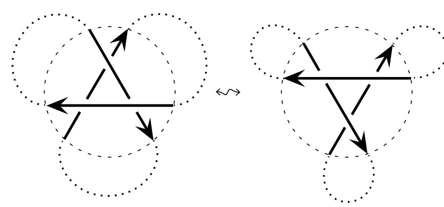
$$HNP(D) = \sum_{C \in \mathcal{C}_+(D)} \mathrm{lk}(D_C) - \sum_{C \in \mathcal{C}_-(D)} (\mathrm{lk}(D_C) - 1) \tag{2}$$

Considering the changes of I_{lk} under Polyak moves as written in Table 2, we notice that HNP is invariant under $\Omega 1a, \Omega 1b$ and $\Omega 2a$ moves and changes by ± 1 under $\Omega 3a$ moves. Carefully investigating the change of I_{lk} under $\Omega 3a$ moves we can distinguish between two different situations.

Definition 24 (ascending and descending moves)**.** We will call an $\Omega 3$ move on an oriented knot diagram to be *ascending* (resp. *descending*), if the order of three strands involved in the move when traversing the knot, in the direction of orientation, is from bottom to top (resp. top to bottom), as shown (schematically) in Figure 15a (resp. Figure 15b).



(A) An ascending $\Omega 3a$ move.



(B) A descending $\Omega 3a$ move.

FIGURE 15. Ascending and descending $\Omega 3a$ moves.

Remark 25. Östlund [6] calls forward ascending and backward descending $\Omega 3$ moves *positive*, and forward descending and backward ascending $\Omega 3$ moves *negative*.

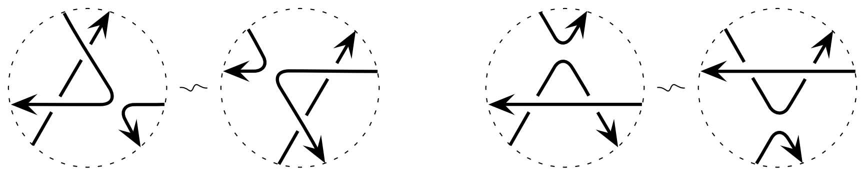
We denote an ascending or a descending move by adding an appropriate subscript to the move name, e.g. $\Omega 3a_a^{\uparrow}$ for an ascending $\Omega 3a^{\uparrow}$ move or $\Omega 3a_d^{\uparrow}$ for a descending one.

Proposition 26. *I_{lk} changes by $Y_n - Y_{n-1}$ under an $\Omega 3a_a^{\uparrow}$ move and by $-Y_n + Y_{n-1}$ under an $\Omega 3a_d^{\uparrow}$ move, for some $n \in \mathbb{Z}$.*

Proof. If we smooth a diagram D at crossing p, then the value of any link invariant on the smoothing does not depend on Reidemeister moves performed on the smoothed diagram D_p. What follows is that performing any Reidemeister move on a knot diagram D does not change either signs $\mathrm{sgn}(p)$ or values of $\phi(D_p)$ for any crossing p outside of the changing disc of this Reidemeister move. Therefore, in order to calculate the change of I_{lk}, it suffices to check the values of ϕ on diagrams obtained by smoothing the crossings involved in the move.

An $\Omega 3a^{\uparrow}$ move does not create or cancel crossings, or change signs of any crossings, but moves them in a particular way, giving a correspondence between crossings before and after performing the move, as depicted in Figure 6. We will distinguish these three crossings by strands that pass through them: top and middle, middle and bottom, or bottom and top.

Smoothing the crossing of top and middle strand we obtain isotopic links before and after the $\Omega 3a^{\uparrow}$ move (as seen in Figure 16a). The same is true for the crossing of middle and bottom strand (Figure 16b). The situation is different when considering top and bottom strands' crossing. Smoothing before and after the $\Omega 3a^{\uparrow}$ move we obtain two distinct links.

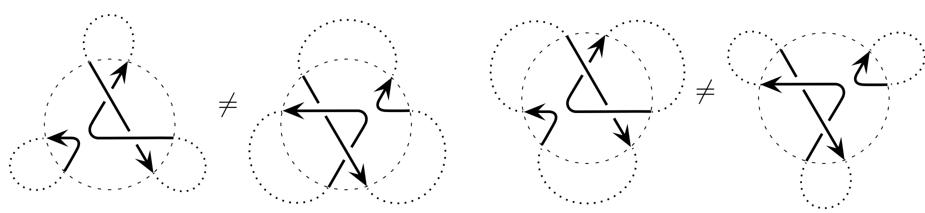


(A) Smoothings of crossings of top and middle strands in $\Omega 3a$ move diagrams.

(B) Smoothings of crossings of middle and bottom strands in $\Omega 3a$ move diagrams.

FIGURE 16. Isotopic smoothings of corresponding crossings taking part in an $\Omega 3a$ move.

For an ascending move, the middle (straight) strand and the upper-right strand of the smoothing (as seen in Figure 17a) belong to the same component and the lower-left strand belongs to the other component. The linking number of the smoothing, which is equal to some number n, increases by 1 since the two other crossings are positive and while before the move (and after smoothing) these were crossings between strands of one of the components, after the move they become crossings between different components of the link. The crossing of the top and bottom strand contributes Y_n to I_{lk} before the move and Y_{n+1} after the move. This, up to a shift of n by 1, proves the first part of the proposition.



(A) Smoothings of crossings top and bottom strands for an ascending $\Omega 3a$ move.

(B) Smoothings of crossings top and bottom strands for a descending $\Omega 3a$ move.

FIGURE 17. Nonisotopic smoothings of corresponding crossings taking part in an $\Omega 3a$ move.

For a descending move, the middle strand and the lower-left strand of the smoothing belong to one link component and the upper-right strand to the other component (Figure 17b). Similarly, in this case 2 positive crossings between these components become crossings between strands of the same link component. Therefore in this case the linking number of this smoothing decreases after performing an $\Omega 3a^{\uparrow}$ move. Before this move the top and bottom strands' crossing contributes Y_n to I_{lk} and after the move it contributes Y_{n-1} to I_{lk}, and the proposition follows. □

Corollary 27. *The quantity HNP increases by* 1 *under an* $\Omega 3a_a^{\uparrow}$ *move, decreases by* 1 *under an* $\Omega 3a_d^{\uparrow}$ *move, and is invariant with respect to* $\Omega 1a$*,* $\Omega 1b$ *and* $\Omega 2a$ *moves.*

Proof. It follows from evaluating changes of I_{lk} given in Proposition 26 and in Table 2 via map $X_n \mapsto -n$ and $Y_n \mapsto n-1$. □

This gives a partial answer to our problem:

Corollary 28. *Any knot-generating subset of*

$$\{\Omega 1a^{\uparrow}, \Omega 1a^{\downarrow}, \Omega 1b^{\uparrow}, \Omega 1b^{\downarrow}, \Omega 2a^{\uparrow}, \Omega 2a^{\downarrow}, \Omega 3a_a^{\uparrow}, \Omega 3a_a^{\downarrow}, \Omega 3a_d^{\uparrow}, \Omega 3a_d^{\downarrow}\}$$

(i.e. directed Polyak moves with distinct ascending and descending moves) contains at least one move from the set $\{\Omega 3a_a^{\uparrow}, \Omega 3a_d^{\downarrow}\}$ *and one move from the set* $\{\Omega 3a_a^{\downarrow}, \Omega 3a_d^{\uparrow}\}$.

The terms *ascending* and *descending* with regard to $\Omega 3$ moves are taken from the work of Östlund [6]. In his paper, Östlund defines three families of knot diagram invariants, namely A_n, D_n for $n \geq 4$ and W_n for $n \geq 3$ and n odd.

He proves that

Proposition 29 ([6]). *A_n, D_n and W_n are invariant with respect to $\Omega 1$ and $\Omega 2$ moves. Moreover, A_n is invariant with respect to descending $\Omega 3$ moves and D_n is invariant with respect to ascending $\Omega 3$ moves.*

Then he considers the figure eight knot diagram and its inverse

, showing that both A_4 and D_4 take different values on these two diagrams, and deduces that

Theorem 30. *Figure eight knot diagram cannot be transformed into its inverse without the use of both ascending and descending $\Omega 3$ moves.*

It follows that

Corollary 31. *Let K be the figure eight knot. Any K-generating subset of*

$$\{\Omega 1a^{\uparrow}, \Omega 1a^{\downarrow}, \Omega 1b^{\uparrow}, \Omega 1b^{\downarrow}, \Omega 2a^{\uparrow}, \Omega 2a^{\downarrow}, \Omega 3a_a^{\uparrow}, \Omega 3a_a^{\downarrow}, \Omega 3a_d^{\uparrow}, \Omega 3a_d^{\downarrow}\}$$

contains at least one move from the set $\{\Omega 3a_a^{\uparrow}, \Omega 3a_a^{\downarrow}\}$ *and one move from the set* $\{\Omega 3a_d^{\uparrow}, \Omega 3a_d^{\downarrow}\}$.

Still, having both $\Omega 3a_a^{\uparrow}$ and $\Omega 3a_d^{\uparrow}$ moves (or $\Omega 3a_a^{\downarrow}$ and $\Omega 3a_d^{\downarrow}$) is sufficient to meet both this condition and the condition presented in Corollary 28. Therefore the question of necessity of containing both $\Omega 3a^{\uparrow}$ and $\Omega 3a^{\downarrow}$ in K-generating subsets of directed Polyak moves remains open (even in the case of the figure eight knot).

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