Analysis in temporal regime of dispersive invisible structures designed from transformation optics

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A simple invisible structure made of two anisotropic layers is analyzed theoretically in temporal regime. The frequency dispersion is introduced and analytic expression of the transient part of the field is derived for large times when the structure is illuminated by a causal excitation. This expression shows that the limiting amplitude principle applies with transient fields decaying as the power $-3/4$ of the time. The quality of the cloak is then reduced at short times and remains preserved at large times. The one-dimensional theoretical analysis is supplemented with full-wave numerical simulations in two-dimensional situations which confirm the effect of dispersion.

I. INTRODUCTION

In 2006, Pendry et al. [1] and Leonhardt [2] independently showed the possibility of designing an invisibility cloak for electromagnetic radiation by blowing up a hole in optical space and hiding an object inside this invisible region. These proposals have been validated by microwave experiments [3]. However, these cloaks suffer from on an inherent narrow bandwidth as their transformational optics design leads to singular tensors with values on the frontier of the invisibility region. To remove the cloak's singularity, Kohn et al. proposed to blow up a small ball instead of a point [4], and in this way the cloak's singularity is removed, at the cost of adding a medium inside the invisibility region [4].

In the present letter, this regularized version of Pendry's transform is implemented for the design of the most simple system of invisible layers in the time-harmonic framework. In this way, infinities are avoided in the material parameters of the cloak which consists of two homogeneous anisotropic slabs. In addition frequency dispersion is considered, which is a required model for metamaterials whenever the permittivity (or permeability) is lower than that of vacuum (i.e. when the phase velocity is greater than c or negative). The effect of dispersion is analyzed with electromagnetic sources with sinusoidal time dependence that are switched on at an initial time. Such illumination has been originaly used by L. Brillouin [5] in homogeneous dispersive media, and more recently in the case of the negative index flat lens [6–8]. Dispersion in metamaterials has already been addressed not only in the case of the flat lens [6–10], but also in the case of cylindrical invisibility cloaks [11].

The originality of the present approach is to consider a simple invibility system made of two layers allowing analytic calculations. In addition, a new method is presented in detail in order to investigate the transient regime in this situation. The main idea is to exploit an integral expression of the time dependent electromgnetic field with no branch cut. The derivation of the transient regime shows that the electromagnetic field includes contributions generated by the singular values of the permittivity and permeability (zeros and infinities). Next, the limiting amplitude principle is considered to show that cloaking can be addressed in temporal regime after the transient regime. These results are supplemented by numerical simulations in the case of a two-dimensional cylindrical cloak, where the presence of additional modes is confirmed in the transient regime.

II. A SYSTEM OF INVISIBLE LAYERS

We start with the definition of a system of invisible layers. Let (x_1, x_2, x_3) be a cartesian coordinate system which specifies each vector $\boldsymbol{x}$ in the space $\mathbb{R}^3$. At the oscillating frequency ω, the electric field amplitude $\boldsymbol{E}(\boldsymbol{x})$ is governed in free space by the Helmholtz equation

$$-\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{x}) + \omega^2\mu_0\varepsilon_0\,\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{0}\,, \qquad (1)$$

where ε_0 and μ_0 are the vacuum permittivity and permeability. The invisible layered structure is then deduced using the one dimensional coordinate transform $\boldsymbol{x}\to\boldsymbol{x}'$ defined by (see Fig. 1)

$$\begin{aligned}
x_1' &= \frac{a}{\alpha}\,x_1 && 0\le \boldsymbol{x}_1\le\alpha\,,\\
x_1' &= a + \frac{b-a}{b-\alpha}\,(x_1-\alpha) && \alpha\le\boldsymbol{x}_1\le b\,,\\
x_1' &= x_1 && \boldsymbol{x}_1\le 0\,,\quad b\le x_1\,,
\end{aligned} \qquad (2)$$

where $a<\alpha<b$ are positive numbers, $x_2' = x_2$ and $x_3' = x_3$ being invariant. The effect of this geometric transform is to map the layer $0\le x_1\le\alpha$ onto the layer $0\le x_1'\le a$ (denominated hereafter as layer A), and the layer $\alpha\le x_1\le b$ onto $a\le x_1'\le b$ (denominated as layer B). Note that such geometric transform, adapted from [12], regularizes the original transform for an invisibility cloak proposed in [1]. The corresponding transformation is applied to the Helmholtz equation (1) which becomes

$$-\boldsymbol{\nabla}'\times\mu^{-1}(x_1')\boldsymbol{\nabla}'\times\boldsymbol{E}'(\boldsymbol{x}') + \varepsilon(\boldsymbol{x}_1')\,\omega^2\mu_0\varepsilon_0\,\boldsymbol{E}'(\boldsymbol{x}') = \boldsymbol{0}\,, \qquad (3)$$

where it is obtained that the tensors of relative permittivity and permeability are both equal to the tensor

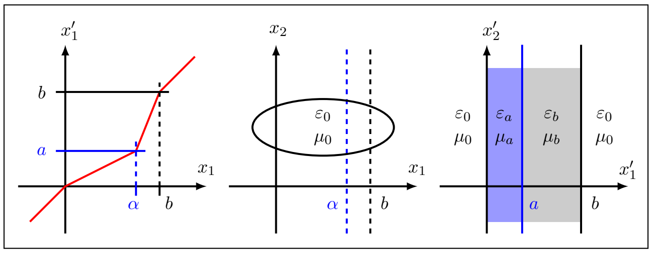

FIG. 1. Coordinate transform for invisible layers. Left: change of coordinate $x_1 \to x_1'$. Center: free space before coordinate transform. Right: invisible set of layers after coordinate transform.

$\nu \equiv \varepsilon = \mu$ taking constant values in each layer:

$$\begin{cases} \varepsilon(x_1') = \mu(x_1') = \nu(x_1') = \nu_a & \text{if} \quad 0 \le x_1' \le a\,, \\ \varepsilon(x_1') = \mu(x_1') = \nu(x_1') = \nu_b & \text{if} \quad a \le x_1' \le b\,, \\ \varepsilon(x_1') = \mu(x_1') = \nu(x_1') = 1 & \text{if} \quad x_1' \le 0\,, \quad b \le x_1'\,. \end{cases} \tag{4}$$

The constant values in layers A abd B are given by

$$\nu_{a,b} = \begin{bmatrix} \nu_{a,b}^{\perp} & 0 & 0 \\ 0 & \nu_{a,b}^{\parallel} & 0 \\ 0 & 0 & \nu_{a,b}^{\parallel} \end{bmatrix}, \tag{5}$$

where the components parallel and perpendicular to the plane interfaces, respectively denoted by the superscripts $\parallel$ and $\perp$, are

$$\nu_a^{\perp} = 1/\nu_a^{\parallel} = a/\alpha\,, \qquad \nu_b^{\perp} = 1/\nu_b^{\parallel} = (b-a)/(b-\alpha)\,. \tag{6}$$

The transformed Helmholtz equation (3) can be reduced to a set of two independent scalar equations using the invariances under the translations and rotations in the plane (x_2', x_3'), namely the symmetries of the geometry. After a Fourier decomposition from (x_2', x_3') to (k_2', k_3'), equation (3) becomes

$$\frac{\partial}{\partial x}\frac{1}{\nu^{\parallel}(x)}\frac{\partial U}{\partial x}(x) - \frac{k^2}{\nu^{\perp}(x)}U(x) + \frac{\omega^2}{c^2}\nu^{\parallel}(x)U(x) = 0\,, \tag{7}$$

for $U(x)$, the (Fourier transformed) electric field component along direction $(-k_3, k_2)$. Here, x denotes x_1', k^2 is $k_2^2 + k_3^2$ (with $k_2 = k_2'$ and $k_3 = k_3'$), c is the light velocity in vacuum $1/\sqrt{\varepsilon_0\mu_0}$, and functions $\nu^{\parallel}(x)$ and $\nu^{\perp}(x)$ are the components of $\nu(x)$ respectively parallel and perpendicular to the plane interfaces. Notice that, since $\varepsilon = \mu$, the second scalar equation derived from the Helmholtz equation is fully identical to (7), except that $U(x)$ should be the (Fourier transformed) magnetic field component along direction $(-k_3', k_2')$ [or $(-k_3, k_2)$].

In this letter, the system is analyzed using a transfer matrix formalism [13]. The equation (7) is formulated as

$$\frac{\partial}{\partial x}F(x) = -iM(x)F(x)\,, \tag{8}$$

where

$$F = \begin{bmatrix} U \\ \dfrac{i}{\nu^{\parallel}}\dfrac{\partial U}{\partial x} \end{bmatrix}, \quad M = \begin{bmatrix} 0 & \nu^{\parallel} \\ \dfrac{\omega^2}{c^2}\nu^{\parallel} - \dfrac{k^2}{\nu^{\perp}} & 0 \end{bmatrix}. \tag{9}$$

The transfer matrices T_a and T_b, associated to layers A and B, defined by $F(a) = T_a F(0)$ and $F(b) = T_b F(a)$, are given by

$$T_a = \exp[-iM_0\alpha]\,, \quad T_b = \exp[-iM_0(b-\alpha)]\,, \tag{10}$$

the matrix M_0 being the value taken by the matrix $M(x)$ in vacuum, i.e. when $\nu^{\parallel}(x) = \nu^{\perp}(x) = 1$. This implies that the transfer matrix $T_bT_a = \exp[-iM_0 b]$, associated with layers A and B, is exactly the same as the one of a vacuum layer of thickness b. Hence the system of layers A and B is invisible.

Nevertheless, as pointed out by V. Veselago when he introduced negative index materials [14], causality principle and passivity require for permittivity and permeability to be frequency dispersive when they take relative value below unity [15, 16]. According to this requirement, frequency dispersion is introduced in the components of ν_a and ν_b with value below unity, assuming the simple Drude-Lorentz model [16]:

$$\begin{aligned} \nu_a^{\perp}(\omega) &= 1 - \frac{\Omega_a^2}{\omega^2 - \omega_a^2}\,, & \Omega_a^2 &= \frac{\alpha - a}{\alpha}\,(\omega_0^2 - \omega_a^2)\,, \\ \nu_b^{\parallel}(\omega) &= 1 - \frac{\Omega_b^2}{\omega^2 - \omega_b^2}\,, & \Omega_b^2 &= \frac{\alpha - a}{b - a}\,(\omega_0^2 - \omega_b^2)\,. \end{aligned} \tag{11}$$

Under this assumption, the functions $\nu_a^{\perp}(\omega)$ and $\nu_b^{\parallel}(\omega)$ take the appropriate values for the invisibility requirement at $\omega = \omega_0$. Notice that the resonance frequencies ω_a and ω_b must be smaller than the operating frequency ω_0 in order to ensure that the oscillator strengths Ω_a^2 and Ω_b^2 are positive. For frequencies different from ω_0, the system has no reason to be invisible.

III. ANALYSIS IN TEMPORAL REGIME

The effect of dispersion is analyzed using illumination with sinusoidal time-dependence oscillating at ω_0 and switched on at an initial time. Such a "causal" incident field, originaly used by L. Brillouin [5] and more recently in [6–8], is assumed to be in normal incidence for the sake of simplicity. Hence the following current source is considered:

$$S(x,t) = S_0\,\delta(x - x_0)\theta(t)\,\sin[\omega_0 t]\,, \tag{12}$$

where δ is the Dirac function, $\theta(t)$ the step function (it equals 0 if $t < 0$ and 1 otherwise), and S_0 the constant component of the source parallel to the field component $U(x)$. In the domain of complex frequencies $z = \omega + i\eta$, the electric field radiated in vacuum by this source is

$$U_0(x,z) = \frac{S_0\mu_0 c}{2}\,\frac{\omega_0}{z^2 - \omega_0^2}\,\exp\left[iz|x - x_0|/c\right]. \tag{13}$$

Note that the positive imaginary part η has been added to the frequency ω in order to ensure a correct definition of the Fourier transform with respect to time of the

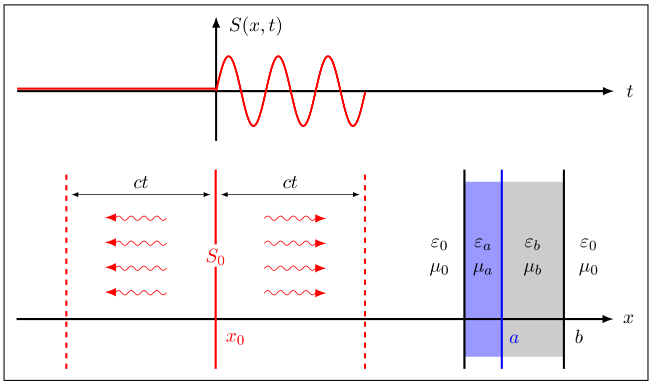

FIG. 2. Excitation of the system. Top: Causal current source with sinusoidal time dependence. Bottom: Field radiated by the causal source and illuminating the invisible layers.

source (12). The time dependent incident field radiated in vacuum is, with $z = \omega + i\eta$,

$$\begin{aligned} E_0(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} d\omega\, \exp[-izt]\, U_0(x,z) \\ &= -\frac{S_0\mu_0 c}{2}\,\theta(t-|x-x_0|/c)\,\sin[\omega_0(t-|x-x_0|/c)]\,. \end{aligned} \tag{14}$$

The next steps are to compute the time dependent field transmitted throught the system, and to analyze the behavior of the filed when the time t tends to infinity. According to the limiting amplitude principle, the solution has an asymptotic behavior corresponding to the time harmonic frame oscillating at the frequency ω_0. Let $T(\omega)$ be the transmission coefficient of the system made of layers A and B. Then, the time dependent electric field is, for $x > b$,

$$E_T(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} d\omega\, \exp[-izt]\, U_0(0,z)\, T(z)\, \exp[iz(x-b)/c]\,. \tag{15}$$

At this stage, it is stressed that, for a fixed incident angle, the transmission coefficient $T(z)$ does not contain any square root of the permittivities and permeabilities of the layered system and of the complex frequency z. This remarkable property is the key of the present technique, since it removes all branch cuts in the evaluation of the integral of the transmitted field. This is a breakthrough in comparison with the method used by L. Brillouin for the analysis of wave propagation in dispersive media [5]. The expression of the transmitted field is thus given by the sum of the contributions from all the poles in the function $f(z)$ under the integral in (15) (using residus).

The poles of the factor $U_0(0,z)$ at $z = \pm\omega_0$ [see Eq. (13)] provide the contribution at the operating frequency ω_0,

$$\begin{aligned} E_T^{(0)}(x,t) =& -\frac{S_0\mu_0 c}{2}\,\theta(t-\{x-x_0+\alpha-a\}/c) \\ &\times \sin[\omega_0(t-\{x-x_0\}/c)]\,, \end{aligned} \tag{16}$$

corresponding to the time harmonic solution for which the system is invisible. Note that this contribution vanishes for times such that ct is smaller than $x-x_0+\alpha-a = x+|x_0|+\alpha-a > x+|x_0|$, instead of $x-x_0 = x+|x_0|$. This is not suprising since the dispersion has not been taken into account in both parallel permittivity and permeability $\varepsilon_a^{\parallel} = \mu_a^{\parallel} = \nu_a^{\parallel} > 1$ of layer A: hence the corresponding delay $(\alpha-a)/c$ is retrieved in the expression above.

The poles of the transmission coefficient are determined via

$$T(z) = \exp[iz\{\alpha + (b-a)\nu_b^{\parallel}(z)\}/c]\,. \tag{17}$$

The resulting expression of the transmission coefficient is relatively simple since there perfect impedance matching holds at every interface in the structure defined from transformation optics. Indeed, the permittivity and permeability take the same values in each layer. Replacing $\nu_b^{\parallel}(z)$ by the dispersive model (11) yields

$$T(z) = \exp[iz(\alpha+b-a)/c]\,\exp\left[-i\,\frac{z(b-a)}{c}\,\frac{\Omega_b^2}{z^2-\omega_b^2}\right]. \tag{18}$$

Thus the transmission coefficient has two isolated singularities at $z = \pm\omega_b$. The resulting contribution can be estimated for large values of the relative time

$$\tau = t - \frac{x-x_0+\alpha-a}{c} \gg \beta = \frac{(b-a)\Omega_b^2}{2\omega_b^2 c}\,. \tag{19}$$

In appendix, it is shown that

$$\begin{aligned} E_T^{(b)}(x,t) \underset{\tau/\beta\to\infty}{\approx} &- 2S_0\mu_0\pi c\,\frac{\omega_0\omega_b}{\omega_b^2-\omega_0^2}\,\sqrt{\beta/\tau}\,\theta(\tau) \\ &\times J_1(2\omega_b\beta\sqrt{\tau/\beta})\cos[\omega_b(\tau+2\beta)]\,, \end{aligned} \tag{20}$$

where J_1 is the Bessel function given by (A9). Using the asymptotic form $J_1(u) \approx \sqrt{2/(\pi u)}\cos[u-3\pi/4]$ of the Bessel function provides an explicite expression for large $\tau \gg \beta$. The resulting contribution in the electric field is

$$\begin{aligned} E_T^{(b)}(x,t) \underset{\tau/\beta\to\infty}{\approx} &- 2S_0\mu_0 c\,\frac{\omega_0\omega_b\sqrt{\pi}}{\omega_b^2-\omega_0^2}\,(\omega_b\beta)^{1/4}\,\theta(\tau)\,(\omega_b\tau)^{-3/4} \\ &\times \cos\left[2\omega_b\beta\sqrt{\tau/\beta}-3\pi/4\right]\cos[\omega_b(\tau+2\beta)]\,. \end{aligned} \tag{21}$$

This expression shows that the second contribution has a first factor oscillating at the frequency ω_b and a second factor with more complex oscillating behavior with argument $\Omega_b\sqrt{2(b-a)\tau/c}$. More importantly, the amplitude of this contribution decreases like $(\omega_b\tau)^{-3/4}$, and thus the total transmitted electric field

$$E_T(x,t) \underset{\tau/\beta\to\infty}{\approx} -\frac{S_0\mu_0 c}{2}\,\theta(\tau)\,\sin[\omega_0(\tau+\{\alpha-a\}/c)] \tag{22}$$

tends to the field radiated in vacuum (14) for time quantity τ long enough. Hence the limiting amplitude principle applies here, unlike for the perfect lens [6, 8].

The situation where small absorption is included can be considered. In this case the resonance frequencies $\pm\omega_b$

are replaced by $\pm\omega_b - i\gamma$ with $\gamma > 0$ in (11) while Ω_b remains positive. Then, the main change in the second contribution (21) is the presence of the additional factor $\exp[-\gamma\tau]$ which makes the permanent regime (purely oscillating at the operating frequency ω_0) easier to handle. Notice that the argument of the Bessel function, independent of ω_b, remains purely real and thus absorption does not affect the behavior governed by this function. Finally, it is stressed that the introduction of small absorption affects the transmission coefficient at the operating frequency ω_0 by an attenuation of $\exp[-\gamma(b-a)/c]$, which results in a signature of the "invisible" structure.

In non normal incidence, expressions become much more complicated since reflections occur at the different interfaces. However, it is clear that the term $-k^2/\nu_a^\perp$ in (9) leads to a singularity at the frequency ω_p for which $\nu_a^\perp$ vanishes:

$$\nu_a^\perp(\omega_p) = 0\,, \quad \omega_p = \sqrt{\omega_a^2 + \Omega_a^2}\,. \tag{23}$$

This singularity generates an additional contribution at the frequency ω_p, as well as the singularity at ω_b. It is found that both singularities $\nu \to 0$ and $\nu \to \infty$ lead to additional contributions of the field in temporal regime. This result confirms the well-known difficulties associated with cloak's singularities [12].

IV. TWO-DIMENSIONAL CYLINDRICAL CLOAK

The analytical results obtained in this paper are numerically tested in the case of a cylindrical cloak designed using homogenization techniques [17, 18]. The considered cloak is a concentric multi-layered structure of inner radius R_1 and outer radius $R_2 = 2R_1$, consisting of 20 homogeneous layers of equal thickness $R_1/20$ and made of non dispersive dielectrics (see table I in Supplemental Material for the values of the relative permittivities, the relative permeability being unity).

The left panel of Fig. 3 shows that the cylindrical cloak perfectly works in time harmonic regime oscillating at the frequency $\omega_0 = 2\pi c/\lambda_0$, where $\lambda_0 = R_2/2$. It is stressed that a purely dielectric structure is used for this 2D cloak, and thus the interfaces between the different concentric layers are subject to reflections which produce effective dispersion. Hence, it is expected to observe an effect of dispersion even if all the dielectric layers are non dispersive [13]. The right panel of Fig. 3 shows the longitudinal magnetic field amplitude when the cloak is illuminated by the causal incident field given by Eq. (12) and Fig. 2.

The cloaking effect appears to be of similar quality in both panels of Fig. 3. However, we would now like to analyze the magnetic field at short times. In Fig. 4, one can see that cylindrical modes are exited in the multi-layers when the incident front wave reaches the cloak (left), what produces a superluminal concentric wave. Notice that these modes can propagate in the cloak faster than

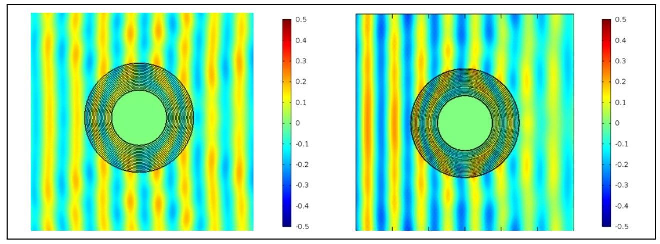

FIG. 3. Magnetic field in the presence of the cylindrical cloak when illuminated by a time harmonic plane wave (left) and by the causal incident field given by Eq. (12) and Fig. 2 (right).

the front wave in vacuum since the frequency dispersion is not introduced in the dielectrics, especially those with index values below unity. The cylindrical modes excited in the multi-layers then radiate cylindrical waves outside the cloak, as evidenced by the right panel in Fig. 4, which explains the tiny perturbation of the field observed on right panel of Fig. 3 (the field perturbation is smoothed down at long times, in agreement with the analytical part). In addition, Fig. 4 shows a picture of the

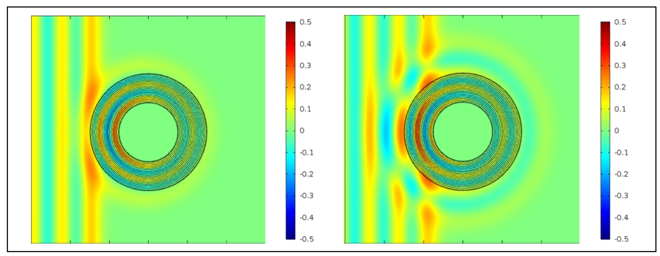

FIG. 4. Magnetic field in the presence of the cylindrical cloak when illuminated by the causal incident field at two time steps in the transient regime. Cylindrical modes inside the cloak generate a supraluminal concentric wave.

transient part of the field produced by the causal source. Here, we take benefit of the supra-luminal propagation of the modes in the cloak to observe that the radiated transient part is almost isotropic. It can be deduced that the radial dependence of this transient part resembles the Hankel function H_0, which does not correspond to the function J_1 found by A. Sommerfeld and L. Brillouin [5], and exhibited in the present Eqs. (A8,A10). It is stressed that there is no contradiction since the J_1 dependence is clearly related to the Drude-Lorentz model of the dispersion, while the transient field around the 2D cloak is related to the effective dispersion produced by the multilayered geometry. Nevertheless, one can conclude that both situations considered in this letter show that the quality of cloaking is reduced at short times under illumination by a causal incident field.

V. CONCLUSION

A new method to analyze propagation of electromagnetic waves in dispersive media has been proposed. The major ideas are to consider a layered structure to eliminate branch cuts, and an invisible structure (with $\varepsilon = \mu$) to eliminate reflections in normal incidence. In this situation, the transient regime can be highlighted and, especially, an explicit expression is obtained in the long time limit. As a result the amplitude of the transient part decreases like $(t - x/c)^{-3/4}$. Hence the technique proposed in this letter brings new elements to the method used by Brillouin [5], where wavefronts (forerunners) can be simply exhibited.

The analysis of the transient regime in the situation of the invisible structure has shown that the singularities of the permittivity and permeability generate additional contributions to the electric field. However, in normal incidence, the contributions vanish in the long time limit, thus cloaking can be addressed after the transient regime. Finally, numerical simulations for a two-dimensional cylindrical layered cloak confirm the effect of dispersion, which affects the quality of cloaking at short times when it is illuminated by a causal incident field.

The proposed method opens new possibilities for investigating transient regime of dispersive systems, notably structures designed from transformation optics like cloaks, concentrators and rotators.

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Appendix A: Estimate of the transient field

The contribution $E_T^{(b)}(x,t)$ of the two isolated singularities at $z = \pm\omega_b$ in the integral expression (15) is estimated for large values of the relative time τ [given by (19)] after the front wave. These two singularities are present in the transmission coefficient $T(z)$ given by (18). Decomposing the ratio $z/(z^2 - \omega_b^2)$ in simple poles, the whole function under the integral in (15) can be formulated as

$$f(z) = f_\pm(z)\, \exp\left[-i\, \frac{(b-a)\Omega_b^2/(2c)}{z - (\pm\omega_b)}\right], \tag{A1}$$

where $f_\pm(z)$ are analytic around $\pm\omega_b$. Let $\xi = z - (\pm\omega_b)$, then the functions $f_\pm$ and exponential can be expanded in power series around $\xi = 0$:

$$f(z) = \sum_{q\in\mathbb{N}} \frac{f_\pm^{(q)}(\pm\omega_b)}{q!}\, \xi^q \sum_{p\in\mathbb{N}} \frac{[(b-a)\Omega_b^2/(2ic)]^p}{p!}\, \xi^{-p}\,, \tag{A2}$$

where $f_\pm^{(q)}(\pm\omega_b)$ is the derivative of order q of $f_\pm(z)$ evaluated at $\pm\omega_b$. Thanks to the convergence of the series, the terms of this product can be arranged in order to obtain the coefficients of the poles ξ^{-1}, and thus the residus $\mathrm{Res}(\pm\omega_b)$ of the function $f(z)$ at $z = \pm\omega_b$:

$$\mathrm{Res}(\pm\omega_b) = \sum_{p\in\mathbb{N}\backslash\{0\}} \frac{f_\pm^{(p-1)}(\pm\omega_b)}{(p-1)!} \frac{[(b-a)\Omega_b^2/(2ic)]^p}{p!} . \tag{A3}$$

Notice that it can be checked that the series above converges as well as the series expansion of the exponential function. Hence the residus $\mathrm{Res}(\pm\omega_b)$ are well-defined.

Using that the complex conjugated of $f(z)$ is $\overline{f(z)} = f(-\overline{z})$, the contribution of the singularities at $\pm\omega_b$ in the time dependent transmitted field is

$$E_T^{(b)}(x,t) = \theta(t - \{x - x_0 + \alpha - a\}/c)\,\mathrm{Imag}\left\{4\pi\mathrm{Res}(\omega_b)\right\} . \tag{A4}$$

The exact calculation of this second contribution, corresponding to the transient regime, cannot be performed in general. However, the (x,t) dependence can be analyzed from the one of $f_\pm(z)$ which can be expressed as

$$f_\pm(z) = g_\pm(z)\exp[-iz\tau] , \quad \tau = t - (x - x_0 + \alpha - a)/c . \tag{A5}$$

where the functions $g_\pm(z)$ are (x,t) independent, and the time quantity τ defines the arrival of the signal (from $\tau = 0$). Denoting $\beta = (b-a)\Omega_b^2/(2\omega_b^2 c)$ and recalling that $\xi = z - (\pm\omega_b)$, the function (A1) becomes

$$f(z) = g_\pm(\xi \pm \omega_b)\exp[-i(\pm\omega_b)\tau]\ \exp[-i(\tau\xi + \omega_b^2\beta/\xi)]. \tag{A6}$$

Then the residus can be expressed as

$$\mathrm{Res}(\pm\omega_b) = \frac{1}{2i\pi}\int_{|\xi|=d} d\xi\, f(\xi \pm \omega_b) \tag{A7}$$

as soon as the functions $g_\pm(z)$ are analytic in the disks of radius d and centered at $\pm\omega_b$. In particular, this expression can be estimated for τ tending to infinity. Let the radius of the disks set to $d = \omega_b\sqrt{\beta/\tau}$ and the complex number $\xi = \omega_b\sqrt{\beta/\tau}\exp[i\phi]$. For $\tau/\beta \to \infty$, the functions $g_\pm(\xi \pm \omega_b) \approx g_\pm(\pm\omega_b)$ and the residus can be approached by

$$\begin{aligned}\mathrm{Res}(\pm\omega_b) \approx& \frac{1}{2i\pi} g_\pm(\pm\omega_b)\exp[-i(\pm\omega_b)\tau]\, i\omega_b\sqrt{\beta/\tau} \\ &\times \int_{[0,2\pi]} d\phi \exp[i\phi - i2\omega_b\sqrt{\beta\tau}\cos\phi] .\end{aligned} \tag{A8}$$

Using the integral representation of the Bessel function

$$J_1(u) = -\frac{1}{2i\pi}\int_{[0,2\pi]} d\phi \exp[i\phi - iu\cos\phi] , \tag{A9}$$

it is deduced that, for $\tau/\beta \to \infty$,

$$\mathrm{Res}(\pm\omega_b) \approx -ig_\pm(\pm\omega_b)\exp[-i(\pm\omega_b)\tau]\sqrt{\beta/\tau}\ J_1(2\omega_b\sqrt{\beta\tau}) . \tag{A10}$$

Appendix B: Opto-geometric parameters of the layered cloak

Table I gives the values of dielectric permittivity of layers which are non-magnetic and of identical thickness $R_1/20$. Note that, according to causality principle, all layers with relative permittivity lower than 1 are necessarily dispersive. This requirement is not considered in the numerical simulations, which explains the supraluminal propagation of the electromagnetic field.

layer	1	2	3	4	5	6	7
$\varepsilon/\varepsilon_0$	0.0012	8.0	0.02	8.0	0.07	8.0	0.12
layer	8	9	10	11	12	13	14
$\varepsilon/\varepsilon_0$	8.0	0.18	8.0	0.24	8.0	0.3	8.0
layer	15	16	17	18	19	20	
$\varepsilon/\varepsilon_0$	0.38	8.0	0.44	8.0	0.5	8.0	

TABLE I. Relative permittivity values of the layered cloak from the inside (layer 1) to the outside (layer 20). One layer in two has the constant value 8.0.