NILSYSTEMS AND ERGODIC AVERAGES ALONG PRIMES

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Dedicated to Vitaly Bergelson on the occasion of his 65^{th} birthday

Abstract. A celebrated result by Bourgain and Wierdl states that ergodic averages along primes converge almost everywhere for L^p-functions, $p > 1$, with a polynomial version by Wierdl and Nair. Using an anti-correlation result for the von Mangoldt function due to Green and Tao we observe everywhere convergence of such averages for nilsystems and continuous functions.

1. Introduction

Nilsystems enjoy remarkable algebraic and ergodic properties making them an important class of systems in the classical ergodic theory, see Auslander, Green, Hahn [1], Green [16], Parry [29, 30] and Leibman [24]. During the years, their relevance for norm convergence of multiple ergodic averages was noted by many authors such as Conze, Lesigne [8], Furstenberg, Weiss [10], Host, Kra [21], Lesigne [27], Ziegler [35]. When finally, motivated by Gowers' uniformity norms introduced in [15], the structure theory of characteristic factors for multiple ergodic averages was developed by Host and Kra [22] and later by Ziegler [36] via an alternative method, nilsystems became fundamental in modern ergodic theory. For further developments involving nilsystems and nilsequences we refer to e.g. Bergelson, Host, Kra [2], Bergelson, Leibman, Lesigne [4], Bergelson, Leibman [3], Leibman [25, 26], Frantzikinakis [11], Host, Kra [23], Chu [7], Eisner, Zorin-Kranich [9], Zorin-Kranich [38].

In their study of arithmetic progressions in the primes, Green and Tao [18, 19, 17], partially together with Ziegler [20], have developed a powerful quantitative theory of Gowers' uniformity norms, nilsequences and their orthogonality to the von Mangoldt and Möbius functions. Their results have found applications back in ergodic theory, see e.g. Frantzikinakis, Host, Kra [13, 14], Wooley, Ziegler [34], Bergelson, Leibman, Ziegler [5], Frantzikinakis, Host [12]. This note is one more example of such an application.

An important ergodic property of nilsystems is that single and multiple ergodic averages converge *everywhere* for such systems. We extend this property to polynomial ergodic averages along primes, motivated by the celebrated result on almost everywhere convergence of ergodic averages along primes by Bourgain [6] and Wierdl [32], see also Thouvenot [31] and Zorin-Kranich [37], and its polynomial generalisation by Wierdl [33] and Nair [28]. For the definition of a polynomial sequence see Section 2.

Key words and phrases. Ergodic averages along primes, nilsystems, everywhere convergence.

Theorem 1.1. *Let G/Γ be a nilmanifold, $g:\mathbb{N}\to G$ be a polynomial sequence and $F\in C(G/\Gamma)$. Then the averages*

$$\frac{1}{\pi(N)}\sum_{p\in\mathbb{P},p\le N}F(g(p)x)$$

converge for every $x\in G/\Gamma$. Moreover, if G is connected and simply connected, $g(n)=g^n$ and the system $(G/\Gamma,\mu,g)$ is ergodic, then the limit equals $\int_X F d\mu$ and is uniform in x.

Our argument is similar to (but simpler than) the one in Wooley, Ziegler [34] in the context of the norm convergence of multiple polynomial ergodic averages along primes.

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2. Preliminaries and the W-trick

Let G be an s-step Lie group and Γ be a discrete cocompact subgroup of G. The homogeneous space G/Γ together with the Haar measure μ is called an *s-step nilmanifold*. For every $g\in G$, the left multiplication by g is an invertible μ-preserving transformation on G/Γ, and the triple $(G/\Gamma,\mu,g)$ is called a *nilsystem*. For a continuous function F on G/Γ, the sequence $(F(g^nx))_{n\in\mathbb{N}}$ is called a *(basic linear) nilsequence* as introduces by Bergelson, Host, Kra [2]. A nilsequence in their definition is a uniform limit of basic nilsequences (being allowed to come from different systems and functions). Note that the property of Cesáro summability along primes is preserved by uniform limits, so Theorem 1.1 implies in particular that every nilsequence is Cesáro summable along primes.

Rather than linear sequences (g^n), following Leibman [24], Green, Tao [17] and Green, Tao, Ziegler [20], we will consider polynomial sequences $(g(n))$, where $g:\mathbb{N}\to G$ is called a *polynomial sequence* if it is of the form $g(n)=g_1^{p_1(n)}\cdot\ldots\cdot g_m^{p_m(n)}$ for some $m\in\mathbb{N}$, $g_1,\ldots,g_m\in G$ and some integer polynomials $p_1,\ldots,p_m$. For an abstract equivalent definition see [17]. A sequence of the form $(F(g(n)x))$ for a continuous function F on G/Γ is called a *polynomial nilsequence*. Although this notion seems to be more general than the one of linear basic nilsequences, it is not, see the references at the beginning of the proof of Theorem 1.1 in the following section.

Note that a nilsequence does not determine G, Γ, F etc. uniquely, giving room for reductions. For example, we can assume without loss of generality that $x=\mathrm{id}_G\Gamma$. Moreover, denoting by G^0 the connected component of the identity in G, since we are only interested in the orbit of x under $g(n)$, we can assume without loss of generality that $G=\langle G_0,g_1,\ldots,g_m\rangle$.

We use the notations $o_{a,b}(1)$ and $O_{a,b}(1)$ to denote a function which converges to zero or is bounded, respectively, for fixed parameters a,b uniformly in all other parameters.

We now introduce the W-trick as in Green and Tao [18]. Consider

$$\Lambda'(n) := \begin{cases} \log n & \text{if } n \in \mathbb{P}, \\ 0 & \text{otherwise.} \end{cases}$$

For $\omega \in \mathbb{N}$ define

$$W = W_\omega := \prod_{p\in\mathbb{P}, p\le\omega} p$$

and for $r < W$ coprime to W define the modified Λ'-function by

$$\Lambda'_{r,\omega}(n) := \frac{\phi(W)}{W}\Lambda'(Wn + r), \quad n \in \mathbb{N},$$

where ϕ denotes the Euler totient function.

The key to our result is the following anti-correlation property of $\Lambda'_{r,\omega}$ with nilsequences due to Green and Tao [18] conditional to the "Möbius and nilsequences conjecture" proven by them later in [19]. Here, $\omega : \mathbb{N} \to \mathbb{N}$ is an arbitrary function with $\lim_{N\to\infty} \omega(N) = \infty$ satisfying $\omega(N) \le \frac{1}{2} \log\log N$ for all large $N \in \mathbb{N}$. Note that the corresponding function $W : \mathbb{N} \to \mathbb{N}$ is then $O(\log^{1/2} N)$.

Theorem 2.1. *(Green-Tao* [18, Prop. 10.2]*) Let $\omega(\cdot)$ and $W(\cdot)$ be as above, G/Γ be an s-step nilmanifold with a smooth metric, G being connected and simply connected, and let $F(g^n x)$ be a bounded nilsequence on G/Γ with Lipschitz constant M. Then*

$$\max_{r<W(N),(r,W(N))=1} \frac{1}{N}\sum_{n=1}^{N}(\Lambda'_{r,\omega(N)}(n) - 1)F(g^n x) = o_{M,G/\Gamma,s}(1)$$

as $N \to \infty$.

An immediate corollary is the following, cf. [14, p. 5].

Corollary 2.2. *Let G/Γ be an s-step nilmanifold with a smooth metric, G being connected and simply connected, and let $F(g^n x)$ be a bounded nilsequence on G/Γ with Lipschitz constant M. Then*

$$\max_{r<W,(r,W)=1} \left|\frac{1}{N}\sum_{n=1}^{N}(\Lambda'_{r,\omega}(n) - 1)F(g^n x)\right| = o_{M,G/\Gamma,s}(1),$$

where one first takes $\limsup_{N\to\infty}$ and then $\lim_{\omega\to\infty}$.

Proof. Define for $\omega, N \in \mathbb{N}$

$$a_\omega(N) := \max_{r\in W,(r,W)=1} \left|\frac{1}{N}\sum_{n=1}^{N}(\Lambda'_{r,\omega}(n) - 1)F(g^n x)\right|$$

and assume that the claimed convergence does not hold. Then there exist $\varepsilon > 0$ and a subsequence (ω_j) of $\mathbb{N}$ so that

$$\limsup_{N\to\infty} a_{\omega_j}(N) > \varepsilon \quad \text{for all } j \in \mathbb{N}.$$

In particular there exists a subsequence (N_j) of $\mathbb{N}$ such that $a_{\omega_j}(N_j) > \varepsilon$ for every $j \in \mathbb{N}$.

Define now the function $\omega : \mathbb{N} \to \mathbb{N}$ by

$$\omega(N) := \omega_j \quad \text{if } N \in [N_j, N_{j+1})$$

which grows sufficiently slowly if (N_j) grows sufficiently fast. Then we have

$$a_{\omega(N_j)}(N_j) = a_{\omega_j}(N_j) > \varepsilon$$

contradicting Theorem 2.1 which states $\lim_{N\to\infty} a_{\omega(N)}(N) = 0$. Note that this argument respects the claimed uniformity in F, g and x. □

3. Proof of Theorem 1.1

We first need several standard simple facts.

Lemma 3.1. *(See, e.g., [13]) For a bounded sequence $(a_n) \subset \mathbb{C}$ one has*

$$\lim_{N\to\infty} \left| \frac{1}{\pi(N)} \sum_{p\in\mathbb{P}, p\le N} a_p - \frac{1}{N}\sum_{n=1}^{N} \Lambda'(n) a_n \right| = 0.$$

Lemma 3.2. *Let $(b_n) \subset \mathbb{C}$ satisfy $b_n = o(n)$ and let $W \in \mathbb{N}$. Then the following assertions hold.*

(a) *If $\left(\frac{1}{WN}\sum_{n=1}^{WN} b_n\right)_{N\in\mathbb{N}}$ converges, then so does $\left(\frac{1}{N}\sum_{n=1}^{N} b_n\right)_{N\in\mathbb{N}}$.*

(b) *If (b_n) is supported on the primes, then*

$$\frac{1}{WN}\sum_{n=1}^{WN} b_n = \frac{1}{W}\sum_{r<W,(r,W)=1} \frac{1}{N}\sum_{n=1}^{N} b_{Wn+r} + o_W(1). \tag{1}$$

Proof. (a) is clear.

(b) The growth condition implies

$$\frac{1}{WN}\sum_{n=1}^{WN} b_n = \frac{1}{WN}\sum_{r=1}^{W}\sum_{n=0}^{N-1} b_{Wn+r} = \frac{1}{W}\sum_{r=1}^{W}\frac{1}{N}\sum_{n=1}^{N} b_{Wn+r} + o_W(1).$$

If (b_n) is supported on the primes, (1) follows. □

The following property of connected nilsystems is well known.

Lemma 3.3. *Let $X := G/\Gamma$ be a connected nilsystem with Haar measure μ and $g \in G$. Then (X, μ, g) is ergodic if and only if (X, μ, g) is totally ergodic.*

Proof. Since ergodicity of a nilsystem is equivalent to the ergodicity of its Kronecker factor (also called maximal factor-torus, or "horizontal" torus) $G/([G,G]\Gamma)$, see Leibman [24], we can assume without loss of generality that X is a compact connected abelian group.

Let (X, μ, g) be ergodic, $m \in \mathbb{N}$ and let $F \in L^2(X, \mu)$ be an g^m-invariant function, i.e., $F(g^m x) = F(x)$ for every $x \in X$. Consider the Fourier decomposition

$$F = \sum_{\chi\in\widehat{X}} c_\chi \chi.$$

By the assumption we have

$$F = \sum_{\chi \in \widehat{X}} c_\chi (\chi(g))^m \chi.$$

By the uniqueness of the decomposition we obtain

$$c_\chi = c_\chi (\chi(g))^m \quad \forall \chi \in \widehat{X}.$$

Assume that $c_\chi \neq 0$. Then $(\chi(g))^m = 1$, i.e., $\chi(g)$ is an m^{th} root of unity. Since (X, μ, g) is ergodic, $\{g^n : n \in \mathbf{Z}\}$ is dense in X. Since χ is a character and X is connected, $\chi(g)$ has to be equal to 1 - otherwise X would have two clopen components $\overline{\{g^n : m_0 | n\}}$ and $\overline{\{g^n : m_0 \nmid n\}}$, where m_0 is the smallest period of $\chi(g)$. Thus $F = c_1 1$ and (X, μ, g) is totally ergodic. □

Proof of Theorem 1.1. As mentioned above, we can assume that $x = \mathrm{id}_G \Gamma \in G^0$, where G^0 is the connected component of the identity in G, and $G = \langle G_0, g_1, \dots, g_m \rangle$.

Every polynomial nilsequence can be represented as a linear nilsequence on a larger nilmanifold, see Leibman [24, Prop. 3.14], Chu [7, Prop. 2.1 and its proof] and, in the context of connected groups, Green, Tao, Ziegler [20, Prop. C.2]. Thus we can assume that $g(n) = g^n$ for some $g \in G$.

By the argument in Wooley, Ziegler [34, p. 17], the nilsequence $(F(g^n x))$ can be written as a finite sum of (linear) nilsequences coming from a connected, simply connected Lie group. Thus we can assume without loss of generality that G is connected and simply connected.

We first assume that F is Lipschitz and define $b_n := \Lambda'(n) F(g^n x)$. To show convergence of

$$\frac{1}{\pi(N)} \sum_{p \in \mathbb{P}, p \leq N} F(g^p x), \tag{2}$$

by Lemmata 3.1 and 3.2(a) it is enough to find $W \in \mathbb{N}$ so that

$$\frac{1}{WN} \sum_{n=1}^{WN} b_n \tag{3}$$

is a Cauchy sequence.

Indeed, for every $\omega \in \mathbb{N}$

$$
\begin{aligned}
\frac{1}{WN}\sum_{n=1}^{WN} b_n &= \frac{1}{W}\sum_{r<W,(r,W)=1}\frac{1}{N}\sum_{n=1}^{N} b_{Wn+r} + o_W(1)\\
&= \frac{1}{\phi(W)}\sum_{r<W,(r,W)=1}\frac{1}{N}\sum_{n=1}^{N}\Lambda'_{r,\omega}(n)F(g^{Wn+r}x) + o_W(1)\\
&= \frac{1}{\phi(W)}\sum_{r<W,(r,W)=1}\frac{1}{N}\sum_{n=1}^{N}(\Lambda'_{r,\omega}(n)-1)F(g^{Wn+r}x)\\
&\quad +\frac{1}{\phi(W)}\sum_{r<W,(r,W)=1}\frac{1}{N}\sum_{n=1}^{N}F(g^{Wn+r}x) + o_W(1) \qquad (4)\\
&=: I(N) + II(N) + o_W(1).
\end{aligned}
$$

Let $\varepsilon > 0$ and take a large ω such that $\limsup_{N\to\infty}|I(N)| < \varepsilon$ which exists by Corollary 2.2. Since the sequence $(F(g^{Wn+r}x))_{n\in\mathbb{N}}$ is Cesáro convergent for every r, see Leibman [24] and Parry [29, 30], there is $N_0 \in \mathbb{N}$ such that $|II(N_1)-II(N_2)| < \varepsilon$ whenever $N_1, N_2 > N_0$. Thus the averages (3) form a Cauchy sequence implying convergence of (2).

Take now $F \in C(G/\Gamma)$ arbitrary, $x \in G/\Gamma$ and $\varepsilon > 0$. By the uniform continuity of F there exists $G \in C(G/\Gamma)$ Lipschitz with $\|F - G\|_\infty \le \varepsilon$. We then have

$$
\begin{aligned}
&\left|\frac{1}{\pi(N)}\sum_{p\in\mathbb{P},p\le N}F(g^px) - \frac{1}{\pi(M)}\sum_{p\in\mathbb{P},p\le M}F(g^px)\right|\\
\le &\left|\frac{1}{\pi(N)}\sum_{p\in\mathbb{P},p\le N}F(g^px) - \frac{1}{\pi(N)}\sum_{p\in\mathbb{P},p\le N}G(g^px)\right|\\
&+\left|\frac{1}{\pi(N)}\sum_{p\in\mathbb{P},p\le N}G(g^px) - \frac{1}{\pi(M)}\sum_{p\in\mathbb{P},p\le M}G(g^px)\right|\\
&+\left|\frac{1}{\pi(M)}\sum_{p\in\mathbb{P},p\le M}G(g^px) - \frac{1}{\pi(M)}\sum_{p\in\mathbb{P},p\le M}F(g^px)\right|\\
\le &\ 2\varepsilon + \left|\frac{1}{\pi(N)}\sum_{p\in\mathbb{P},p\le N}G(g^px) - \frac{1}{\pi(M)}\sum_{p\in\mathbb{P},p\le M}G(g^px)\right|
\end{aligned}
$$

which is less than 3ε for large enough N, M by the above, finishing the argument.

The last assertion of the theorem follows analogously from the decomposition (4) using Lemma 3.3, the fact that a nilsystem is ergodic if and only if it is uniquely ergodic, see Parry [29, 30], and the uniform convergence of Birkhoff's ergodic averages to the space mean for uniquely ergodic systems. The last step (for non-Lipschitz functions) should be modified by showing that the difference $\frac{1}{\pi(N)}\sum_{p\in\mathbb{P},p\le N}F(g^px) - \frac{1}{N}\sum_{n=1}^{N}F(g^nx)$ converges to zero. $\square$

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