ON INFINITE SERIES CONCERNING ZEROS OF BESSEL FUNCTIONS OF THE FIRST KIND

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Abstract. A relevant result by Sneddon on an identity on series involving the zeros of Bessel functions of the first kind is derived by an alternative method based on Laplace transforms. Our method leads to a Dirichlet series in time that, once integrated, allows us to simply recover Sneddon's result.

Discussion and Main Results

As discussed in our previous paper [2], there are strong hints about the possibility of computing the sum of the infinite series

$$S_\nu = \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu+1)}\,, \quad \nu > -1\,, \tag{1}$$

where $j_{\nu,n}$ stands for the nth positive zero of the Bessel function of the first kind J_ν. The proof is now provided in this note, completely based on the *Laplace transform method* and on the related *Limiting Theorems*. This result was first derived by Sneddon in 1960 (see [4]).

Consider the following function defined in the Laplace domain,

$$\widetilde{F}_\nu(s) = \frac{2(\nu+1)}{\sqrt{s}} \frac{I_{\nu+1}(\sqrt{s})}{I_\nu(\sqrt{s})}\,, \quad \nu > -1\,. \tag{2}$$

Inverting the Laplace transform we get the function $F_\nu(t)$ (in the time domain)

$$F_\nu(t) = 4\nu \sum_{n=1}^{\infty} \exp\left(-j_{\nu,n}^2\, t\right)\,, \tag{3}$$

where $t > 0$.

Proof. Firstly, consider the power series representation for the Modified Bessel functions of the First Kind (see e.g. [1]):

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k}\,. \tag{4}$$

This implies that the complex function $\widetilde{F}_\nu(s)$ is regular in $s=0$ and it does not have any branch cuts.

In the second place, we can obtain the function $F_\nu(t)$ by means of the Bromwich Integral:

$$F_\nu(t) = \frac{1}{2\pi i} \int_{Br} \widetilde{F}_\nu(s)\, e^{st}\, ds\,. \tag{5}$$

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Indeed, $\widetilde{F}_\nu(s)$ has poles such that:

$$I_\nu(\sqrt{s}) = 0\,.$$

Now, if we rename $\sqrt{s}$ as $\sqrt{s} = -i\lambda$, then

$$I_\nu(\sqrt{s}) = 0 \quad \Longleftrightarrow \quad J_\nu(\lambda) = 0\,.$$

For $\nu > -1$ the zeros of J_ν are all simple and real (see e.g. [1]). Therefore, the simple poles of $\widetilde{F}_\nu(s)$ are located on the negative real axis, and they are given by

$$s_n = -j_{\nu,\,n}^2 \tag{6}$$

with $n = 1, 2, \ldots$
From the previous statements, we can then conclude that:

$$F_\nu(t) = \sum_{s_n} \mathcal{R}es \left\{ \widetilde{F}_\nu(s)\, e^{st} \right\}_{s=s_n} = \sum_{n=1}^{\infty} \mathcal{R}es \left\{ \frac{2(\nu+1)}{\sqrt{s}} \frac{I_{\nu+1}(\sqrt{s})}{I_\nu(\sqrt{s})}\, e^{st} \right\}_{s=s_n}\,. \tag{7}$$

It is quite straightforward to see that

$$\begin{aligned} \mathcal{R}es \left\{ \frac{2(\nu+1)}{\sqrt{s}} \frac{I_{\nu+1}(\sqrt{s}) e^{st}}{I_\nu(\sqrt{s})} \right\}_{s=s_n} &= \lim_{s \to s_n} (s - s_n)\, \frac{2(\nu+1)}{\sqrt{s}} \frac{I_{\nu+1}(\sqrt{s}) e^{st}}{I_\nu(\sqrt{s})} = \\ &= 4(\nu+1) \exp(s_n t)\,. \end{aligned} \tag{8}$$

Thus,

$$F_\nu(t) = \sum_{n=1}^{\infty} \mathcal{R}es \left\{ \widetilde{F}_\nu(s)\, e^{st}\,;\, s = -j_{\nu,\,n}^2 \right\} = 4(\nu+1) \sum_{n=1}^{\infty} \exp\left(-j_{\nu,\,n}^2\, t\right)\,. \tag{9}$$

□

Now, it is easy to see that the time representation of the function F_ν in Eq. (3) is a *Generalized Dirichlet Series* (see [3]). In particular, the series in Eq. (3) is convergent for $t > 0$.

Proof. Consider a generalized Dirichlet series:

$$f(z) = \sum_{n=1}^{\infty} a_n \exp(-\alpha_n z)\,, \qquad z \in \mathbb{C}\,. \tag{10}$$

In general, we have that the abscissa of convergence and the abscissa of absolute convergence would be different, i.e. $\sigma_c \neq \sigma_a$, but they will satisfy the following condition:

$$0 \leqslant \sigma_a - \sigma_c \leqslant d = \limsup_{n\to\infty} \frac{\ln n}{\alpha_n}\,. \tag{11}$$

If $d = 0$, then

$$\sigma \equiv \sigma_c = \sigma_a = \limsup_{n\to\infty} \frac{\ln |a_n|}{\alpha_n}\,. \tag{12}$$

In the case of our concern, $a_n = 1$ and $\alpha_n = j_{\nu-1,\,n}^2 \neq 0$. Then, we have to understand the behavior of the coefficients $j_{\nu-1,\,n}$ for $n \gg 1$.
Considering the asymptotic expansion:

$$J_\nu(x) \overset{x \gg 1}{\sim} \sqrt{\frac{2}{\pi x}}\, \cos\left(x - \frac{(2\nu+1)\pi}{4} \right) + o(x^{-3/2})\,, \tag{13}$$

we can eventually get to the following conclusion:

$$J_\nu(j_{\nu,\,n}) = 0\,, \;\text{ for } n \gg 1 \quad \Longrightarrow \quad j_{\nu,\,n} \propto n\,, \;\text{ for } n \gg 1\,. \tag{14}$$

Thus,

$$\frac{\ln n}{\alpha_n} = \frac{\ln n}{j^2_{\nu,n}} \overset{n\gg 1}{\sim} \frac{\ln n}{n^2} \overset{n\to\infty}{\longrightarrow} 0\,, \tag{15}$$

from which we deduce that $d = 0$.
Finally,

$$\sigma \equiv \sigma_c = \sigma_a = \limsup_{n\to\infty} \frac{\ln |a_n|}{\alpha_n} = 0\,, \tag{16}$$

being $a_n = 1$.
This result allows us to conclude that the series in Eq. (3) converges for $t > 0$. □

Now, let us define a function $G(t)$ such that

$$s\,\widetilde{G}_\nu(s) = \widetilde{F}_\nu(s)\,, \tag{17}$$

also providing that $G(0+) = 0$, for sake of simplicity. Under these assumption we can easily deduce that

$$G'_\nu(t) = F_\nu(t)\,, \tag{18}$$

where the prime denotes the time derivative. Therefore,

$$G_\nu(t) = \int_0^t F_\nu(u)\,du\,. \tag{19}$$

As shown in [2], Lebesgue's dominated convergence theorem allows us to integrate term by term the Dirichlet series defining $F_\nu(t)$. Then, we get

$$G_\nu(t) = 4(\nu+1) \sum_{n=1}^{\infty} \left[\frac{1}{j^2_{\nu,n}} - \frac{1}{j^2_{\nu,n}} \exp\left(-j^2_{\nu,n}\,t\right) \right]\,, \tag{20}$$

whose limit is

$$\lim_{t\to+\infty} G_\nu(t) = 4(\nu+1) \sum_{n=1}^{\infty} \frac{1}{j^2_{\nu,n}}\,. \tag{21}$$

On the other hand, taking profit of the so called *Final Value Theorem*, we get

$$\lim_{t\to+\infty} G_\nu(t) = \lim_{s\to 0} s\,\widetilde{G}_\nu(s) = \lim_{s\to 0} \widetilde{F}_\nu(s) = 1\,. \tag{22}$$

Proof. Considering the asymptotic behavior of $I_\nu(z)$ for $z \to 0$, *i.e.*

$$I_\nu(z) = \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu + O\left(z^{\nu+2}\right)\,, \tag{23}$$

we get

$$\widetilde{F}_\nu(s) = \frac{2(\nu+1)}{\sqrt{s}} \frac{I_{\nu+1}(\sqrt{s})}{I_\nu(\sqrt{s})} = 1 + O(s) \tag{24}$$

as $s \to 0$. □

Then, combining the results in Eq. (21) and (22) we immediately get

$$S_\nu = \sum_{n=1}^{\infty} \frac{1}{j^2_{\nu,n}} = \frac{1}{4(\nu+1)}\,, \quad \nu > -1\,, \tag{25}$$

which agrees with the result for the the zeros of Bessel functions shown in [4].

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