# A first-order approach to conformal gravity

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January 5, 2016

**Abstract**

We investigate whether a spontaneously-broken gauge theory of the group $SU(2,2)$ may be a genuine competitor to General Relativity. The basic ingredients of the theory are an $SU(2,2)$ gauge field $A_\mu$ and a Higgs field $W$ in the adjoint representation of the group with the Higgs field producing the symmetry breaking $SU(2,2) \rightarrow SO(1,3) \times SO(1,1)$. The action for gravity is polynomial in $\{A_\mu, W\}$ and the field equations are first-order in derivatives of these fields. The new $SO(1,1)$ symmetry in the gravitational sector is interpreted in terms of an emergent scale symmetry and the recovery of conformalized General Relativity and fourth-order Weyl conformal gravity as limits of the theory- following imposition of Lagrangian constraints- is demonstrated. Maximally symmetric spacetime solutions to the full theory are found and stability of the theory around these solutions is investigated; it is shown that regions of the theory's parameter space describe perturbations identical to that of General Relativity coupled to a massive scalar field and a massless one-form field. The coupling of gravity to matter is considered and it is shown that actions for all fields are naturally gauge-invariant, polynomial in fields and yield first-order field equations; no auxiliary fields are introduced. Familiar Yang-Mills and Klein-Gordon type Lagrangians are recovered on-shell in the General-Relativistic limit of the theory. In this formalism, the General-Relativistic limit and the breaking of scale invariance appear as two sides of the same coin and it is shown that the latter generates mass terms for Higgs and spinor fields.

## 1 Introduction

The prevailing classical theory of gravity remains Einstein's General Relativity. In General Relativity, the gravitational field is described solely by a field $e^I \equiv e^I_\mu dx^\mu$, the *co-tetrad*. The index $I$ denotes that the one-form $e^I$ is in the fundamental representation of the Lorentz group $SO(1,3)$; the action for General Relativity- the Einstein-Hilbert action- possesses an invariance under local Lorentz transformations represented by matrices $\Lambda^I{}_J(x)$ with $e^I \rightarrow \Lambda^I{}_J e^J$ [1]. This field is sufficient to describe the inherent dynamics of gravity and the coupling of gravity to known matter fields: scalar fields, gauge fields, and fermionic fields; the latter are spinorial representations of the group $SL(2,C)$ which is the double-cover of $SO(1,3)$ and so the inclusion of fermions into gravitational theory implies that most correctly the local symmetry of General Relativity is that of $SL(2,C)$. In General Relativity the coupling of gravity to each of these fields requires use of the *tetrad* $(e^{-1})_I \equiv e^\mu_I \partial_\mu$, where $e^\mu_I$ is the matrix inverse of $e^I_\mu$. Therefore the coupling of gravity to all matter in General Relativity is non-polynomial. In the absence of fermionic fields, it suffices to use a composite object called the *metric tensor* $g_{\mu\nu}$, which is defined as follows:

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$$g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu \tag{1}$$

where $\eta_{IJ} = \mathrm{diag}(-1,1,1,1)$ is the invariant matrix of $SO(1,3)$. Coupling to scalar fields and gauge fields then requires use of the inverse metric $g^{\mu\nu}$. As was discovered by Cartan, an elegant reformulation of General Relativity is provided by introducing a new, independent field $\omega^I{}_J \equiv \omega_\mu{}^I{}_J dx^\mu$- the *spin connection*- alongside $e^I$ in the description of gravity. In these variables, the action for gravity is given by the Palatini action:

$$S_P[\omega, e] = \int \epsilon_{IJKL} e^I e^J \left( d\omega^{KL} + \omega^K{}_M \omega^{ML} \right) \tag{2}$$

where multiplication of differential forms with one another is via the wedge product. Gravity from this perspective is known as Einstein-Cartan gravity[1]. This action is invariant under local $SO(1,3)$ transformations if $\omega^I{}_J$ transforms as a gauge field for those transformations. Indeed we can see that the terms containing $\omega^I{}_J$ in (2) combine to form the curvature two-form $R^I{}_J$ of a non-Abelian gauge field. The equation of motion for $\omega^I{}_J$ is as follows:

$$de^I + \omega^I{}_J e^J = 0 \tag{3}$$

Unlike the equations of motion of gauge fields in particle physics, this equation is algebraic in $\omega^I{}_J$. Using the solution for $\omega^I{}_J(e)$ in the $e^I$ equation of motion yields the Einstein field equations. Alternatively, the solution for $\omega^I{}_J(e)$ may be inserted back into the action, resulting in the Einstein-Hilbert action of General Relativity, which- upon the addition of a boundary term- yields the Einstein field equations upon variation. Similarly, $\omega^I{}_J$ as an independent field may be used to write the action of a fermionic field coupled to gravity in a polynomial manner. Due to the coupling to $\omega^I{}_J$, fermionic currents can act as a source in (3); one may still solve for $\omega^I{}_J$ algebraically but it now depends on $e^I$ and the fermionic fields. Inserting the solution back into the action yields terms quartic in the fermionic fields, over and above terms usually present when describing gravity coupled to fermions in General Relativity [2, 3, 4, 5].

Thus, the Einstein-Cartan approach results in a simplification of some actions involving the gravitational field and introduces structure (the $SO(1,3)$ gauge field $\omega^I{}_J$) reminiscent of the Yang-Mills fields of particle physics. However, $e^I$ has no counterpart amongst non-gravitational fields. It possesses a spacetime index but does not transform as a gauge field. Intriguingly though, there exists a re-writing of the Palatini action with appears a step closer to commonality with the ingredients of particle physics. The idea, originally due to MacDowell and Mansouri [6] (closely resembling earlier work due to Cartan [7]), is to enlarge the gauge group of gravity from $SO(1,3)$ to $SO(2,3)$ or $SO(1,4)$ (henceforth collectively referred to as $SO(1,4)|SO(2,3)$). The trick is to imagine there there exists structure in a hypothetical $SO(1,4)|SO(2,3)$ gauge theory to break the symmetry down to the $SO(1,3)$ of the Einstein-Cartan theory. For example, this could be accomplished via a gravitational Higgs field $V^A$ in the fundamental representation of $SO(1,4)|SO(2,3)$ achieving a non-vanishing expectation value for its norm $V^2 \equiv \eta_{AB} V^A V^B$. This norm should be positive/spacelike for $SO(1,4)$ and negative/timelike for $SO(2,3)$. Then we may choose a gauge where $V^A = \ell \delta^A_4$ (where $\ell$ is a constant); the residual gauge transformations that leave this explicit form of $V^A$ invariant are those of $SO(1,3)$, and we may decompose the $SO(1,4)|SO(2,3)$ gauge field $A^A{}_B \equiv A_\mu{}^A{}_B dx^\mu$ as follows:

$$A^A{}_B = \begin{pmatrix} A^I{}_J & A^I{}_4 \\ A^4{}_I & 0 \end{pmatrix} \tag{4}$$

We see then that there is a new field in the formalism: $A^I{}_4$. This is a one-form field that will transform homogeneously under the residual $SO(1,3)$ transformations, precisely like $e^I$!

[1] Within Einstein-Cartan gravity one may additionally consider polynomial Lagrangians $\epsilon_{IJKL} e^I e^J e^K e^L$ and $e_I e_J (d\omega^{IJ} + \omega^I{}_K \omega^{KJ})$ which correspond to the cosmological constant term and the Holst term respectively. The Holst term only produces a non-zero contribution to field equations when the spin connection couples to other fields.

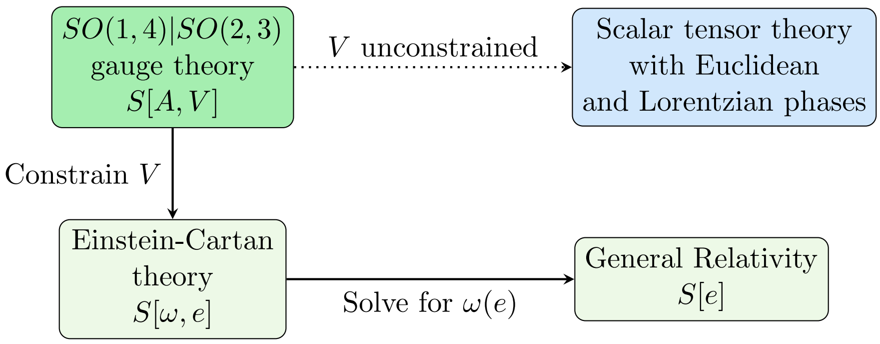


Figure 1: Diagram depicting known results of $SO(1,4)|SO(2,3)$ gravity and relation to General Relativity.

There exists a simple action principle due to Stelle and West [8] that- post symmetry breaking- contains the Palatini action. For concreteness we look at the case of the group $SO(1,4)$ and the action is given by:

$$S_{SW}[A,V,\lambda] \quad = \quad \int \alpha \epsilon_{ABCDE} V^E F^{AB} F^{CD} + \lambda (V_A V^A - \ell^2) \tag{5}$$

Where $\alpha$ is a constant and the four-form field $\lambda$ is introduced entirely to enforce the constraint that $V_A V^A$ is constant. To illustrate the relation of this action to Einstein-Cartan theory, we may enforce the fixed-norm constraint at the level of the action, choosing a gauge where $V^A = \ell \delta^A_4$, and identifying $A^I{}_J = \omega^I{}_J$, $A^I{}_4 = e^I/\ell$ we have:

$$A^{AB} \stackrel{*}{=} \begin{pmatrix} \omega^{IJ} & \frac{e^I}{\ell} \\ -\frac{e^I}{\ell} & 0 \end{pmatrix} \tag{6}$$

$$S_{SW}[\omega, e] \stackrel{*}{=} -\frac{2\alpha}{\ell} \int \epsilon_{IJKL} \left( e^I e^J R^{KL} - \frac{1}{2\ell^2} e^I e^J e^K e^L - \frac{\ell^2}{2} R^{IJ} R^{KL} \right) \tag{7}$$

where the notation $\stackrel{*}{=}$ means that something holds in a specified gauge (here the gauge where $V^A = \ell \delta^A_4$. We see then that the Palatini action (plus a cosmological constant term and a topological term quadratic in $R^{IJ}$) can be recovered from a spontaneously-broken gauge theory[2].

The four-form $\lambda$ is a simple way to achieve a non-vanishing norm for $V^A$ but is not necessary as an ingredient of the theory. Indeed, it has been found that there exist polynomial actions solely in terms of the set $\{A^A{}_B, V^A\}$ that dynamically yield a non-vanishing expectation value of $V^2$ [10, 11]. These theories possess a rich phenomenology with the dynamics of the scalar $V^2$ acting as a potential source of cosmic inflation or quintessence, even facilitating more exotic behaviour such as cosmological changes of the signature of the four-dimensional metric [12].

Therefore General Relativity (and scalar-tensor extensions thereof) can arise as a limit of a spontaneously-broken gauge theory based on $SO(1,4)|SO(2,3)$. However, the presence of a new degree of freedom $V^2$ in gravitation means the theory is more general than a re-casting of the Einstein-Cartan theory. This situation is summarized briefly in Figure 1. Research into the link between $SO(1,4)|SO(2,3)$ groups and gravity and matter is ongoing [13, 14, 15, 16, 17, 18].

There are, however, issues with a description of gravity based on $SO(1,4)|SO(2,3)$. Although it is possible to couple matter fields to gravity in a fashion consistent with the gauge principle [19], we nevertheless encounter a somewhat unnatural structure when coupling gravity to other Yang-Mills fields. In this paper we consider an alternative to the above $SO(1,4)|SO(2,3)$, approach wherein these issues are resolved. Specifically we will develop a description of gravity

[2]See [9] for a more detailed discussion of these steps towards regarding gravity as a gauge theory

as a spontaneously-broken gauge theory based on the group $SU(2,2)$, a group which contains both $SO(1,4)$ and $SO(2,3)$ as sub-groups. This group has a matrix representation as the set of all $4\times 4$ complex matrices $U^{\alpha}{}_{\beta}$ of unit determinant that satisfy

$$h_{\alpha\alpha'} = h_{\beta\beta'}U^{\beta}{}_{\alpha}U^{*\beta'}{}_{\alpha'}, \quad h_{\alpha\alpha'} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{8}$$

where $I$ is the $2\times 2$ identity matrix.

The group $SU(2,2)$ is the double cover of the orthogonal group $SO(2,4)$ which itself is the double cover of the *conformal group* $C(1,3)$ of coordinate transformations which preserve a metric of signature $(-,+,+,+)$ up to an overall multiplicative function. Instead of breaking the symmetry down to the $SO(1,3)$ symmetry of General Relativity, our Higgs sector will recover a symmetry $SO(1,3)\times SO(1,1)$. The additional $SO(1,1)$ symmetry now present in gravitation sector will be found to be related to Weyl/scale invariance.

Let us illustrate this more concretely. If there exists structure in the theory to break $SU(2,2)\to SO(1,3)\times SO(1,1)$ then we may decompose the $SO(2,4)\simeq SU(2,2)$ connection as follows:

$$A^{AB} \overset{*}{=} \begin{pmatrix} 0 & \frac{1}{2}(e^I+f^I) & c \\ -\frac{1}{2}(e^I+f^I) & \omega^{IJ} & -\frac{1}{2}(e^I-f^I) \\ -c & \frac{1}{2}(e^I-f^I) & 0 \end{pmatrix} \tag{9}$$

where we now use indices $A,B,C,\ldots$ to denote indices in the fundamental representation of $SO(2,4)$. Here we identify $\omega^{IJ}$ as the spin-connection, $c$ as a gauge field for the group $SO(1,1)$ and what appear to be *two* frame fields $e^I$ and $f^I$. Under an $SO(1,1)\subset SO(2,4)$ transformation we have the following transformation properties:

$$e^I\to e^{\alpha(x)}e^I, \quad f^I\to e^{-\alpha(x)}f^I, \quad c\to c+d\alpha \tag{10}$$

By analogy with (6) and (7) we may immediately speculate as to the kind of theory that may result following symmetry breaking $SU(2,2)\to SO(1,3)\times SO(1,1)$. We will construct the $SU(2,2)$ gauge theory from an action first-order and polynomial in fields. We may then expect that in the limit where all dynamics due to the Higgs field is 'frozen' (much like $V^2=\ell^2$ in the Stelle-West case), that we will recover the most general $SO(1,3)\times SO(1,1)$ invariant action that can be built from the ingredients (9). It is straightforward to write down such an action, it takes the following form:

$$S[\omega,c,e,f] = \int \alpha\epsilon_{IJKL}e^If^JR^{KL} + \beta e_If_JR^{IJ} + \gamma\epsilon_{IJKL}e^If^Je^Kf^L + \mu e_If^Ie_Jf^J + \xi e_If^Idc \tag{11}$$

Where $\{\alpha,\beta,\mu,\xi\}$ are constants and we have not included possible boundary terms. Due to the symmetries (10), this action defines a scale-invariant theory of gravity that we may call *Conformal Einstein-Cartan theory*. Remarkably- as we shall see in more detail later in the paper- in the absence of matter its solution space contains that of General Relativity with positive/negative cosmological constant, corresponding to cases where dynamically the frames $e^I$ and $f^I$ become aligned or anti-aligned i.e. $f^I=\pm\Omega^2(x)e^I$.

As a theory in and of itself, the Conformal Einstein-Cartan theory provides an elegant and simple scale-invariant generalization of the Einstein-Cartan theory (and hence of General Relativity). However, we note again that the fields $e^I$ and $f^I$ do not have analogues within Yang-Mills theory: they are one-forms but do not transform as Yang-Mills gauge fields. In this paper we shall focus on a different description in which gravity- at the fundamental level- is described in the same language as Yang-Mills theory: a $SO(2,4)\simeq SU(2,2)=Spin(2,4)$ gauge connection $A^A{}_B\equiv A_\mu{}^A{}_Bdx^\mu$ and a Higgs field $W^A{}_B$ in the adjoint representation of $SO(2,4)$. In contrast to prior attempts to construct theories of gravity based on the group $SU(2,2)$, our approach will be to place no constraints on the fields of the theory- no object is required to be 'invertible', constant, or non-zero. In addition, no metric tensor is needed to write down our actions, nor do our actions make use of the Hodge dual operator. Instead, all actions will be required to be gauge-invariant and polynomial in the fields and their exterior derivatives and all (classical) behaviour of the fields will result from their field equations. As a simple consequence, all field equations in the Lagrangian formalism will fundamentally first-order partial differential equations[3]. This might seem alarming but we will see that the Hodge-dual structure of standard Yang-Mills fields and Higgs fields pops up naturally on-shell for the simple actions we study in this paper. This marks a departure for conventional formulations in which both gravity and bosonic fields are governed in the Lagrangian formalism by second-order differential equations. All physical fields are taken to be differential forms on a four-dimensional manifold, though they may have internal indices in representations of non-gravitational groups. No non-dynamical fields will allowed in the action: we only allow invariants of the groups such as the invariants $h_{\alpha'\alpha}$ and $\epsilon_{\alpha\beta\delta\gamma}$ of $SU(2,2)$.

By comparison, the gravitational actions based on the Einstein-Cartan fields $\{\omega^I{}_J, e^I\}$ or $SO(1,4)|SO(2,3)$ gauge gravity fields $\{A^A{}_B, V^A\}$ are polynomial in the Lagrangian formulation and produce first-order field equations; these properties persist when including coupling to fermionic matter. This is rather different from the conventional Lagrangians describing other Yang-Mills fields and Higgs fields which produce second-order equations of motion e.g. consider the action for a Yang-Mills field $B = B_\mu dx^\nu$ with curvature $F = dB + BB = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$ (internal indices suppressed):

$$S_{YM} = -\frac{1}{4}\int g^{\alpha\gamma}g^{\beta\delta}\mathrm{Tr}(F_{\alpha\beta}F_{\gamma\delta})\sqrt{-g}dx^4 \tag{12}$$

where Tr denotes contraction of internal adjoint indices using the $\mathcal{G}$ Killing metric. This action is clearly polynomial in $B$ but non-polynomial in gravitational fields because we must use the *inverse* metric $g^{\mu\nu}$

$$g^{\mu\nu} = \frac{4\epsilon^{\alpha\beta\gamma\mu}\epsilon^{\epsilon\zeta\eta\nu}g_{\alpha\epsilon}g_{\beta\zeta}g_{\gamma\eta}}{\epsilon^{\varrho\chi\kappa\lambda}\epsilon^{\xi\rho\sigma\tau}g_{\varrho\xi}g_{\chi\rho}g_{\kappa\sigma}g_{\lambda\tau}} \tag{13}$$

When one couples these matter fields to gravity in Einstein-Cartan theory, this non-polynomial coupling appears in the stress-energy tensor of matter and so the gravitational field equations become fundamentally non-polynomial in fields unless one introduces auxiliary non-dynamical fields which have the sole purpose of re-writing (12) as a first-order theory. Indeed, a popular modification to the Einstein-Cartan gravity theory has been to allow terms non-polynomial in $e^I$ into the action of pure-gravity, much as they are present in the Einstein-Cartan matter sector. This allows one to construct non-topological terms quadratic in the $SO(1,3)$ curvature $R^{IJ}(\omega)$. This approach typically is referred to as *Poincaré gauge theory* and it presents a wealth of new phenomenology in the gravitational sector, notably the propagation of $\omega^{IJ}$ itself via its own field equations (as opposed to simply being solvable for $\omega^{IJ} = \omega^{IJ}(e)$ as in the Einstein-Cartan case) [20, 21, 22, 23, 24].

For gravity based on $SO(1,4)|SO(2,3)$, it becomes possible to naturally base the dynamics of Higgs fields in a first order formalism. Essentially a Higgs field is taken to be a field in the fundamental representation of the gravity group as well as whatever representation it may belong to for the internal gauge group $\mathcal{G}$ (e.g. $SU(2)\times U(1)$ for the electroweak field). The fundamental representation of $SO(1,4)|SO(2,3)$ is five-dimensional and the Higgs field has five 'gravity' components- four of these are found on-shell to be related to derivatives of the Higgs field whilst the final component contains the familiar Higgs field. However, as mentioned above, coupling to other Yang-Mills either still requires the introduction of auxiliary fields or of exotic transformation properties of $B$ that depend on the dynamics of the gravitational Higgs field

[3]The generation of higher-order partial derivatives is made impossible by a polynomial Lagrangian and Bianchi identities $DF = 0$ and $DDV = FV$ where $F$ is a curvature two-form of some gauge connection $A$; $D$ is the associated gauge covariant exterior derivative, and $V$ is a field in some representation of the gauge group

[19]. We see it as a comparative advantage of the $SU(2,2)$ approach that it- as we shall see later- allows for a simple first order formalism for *all* matter fields.

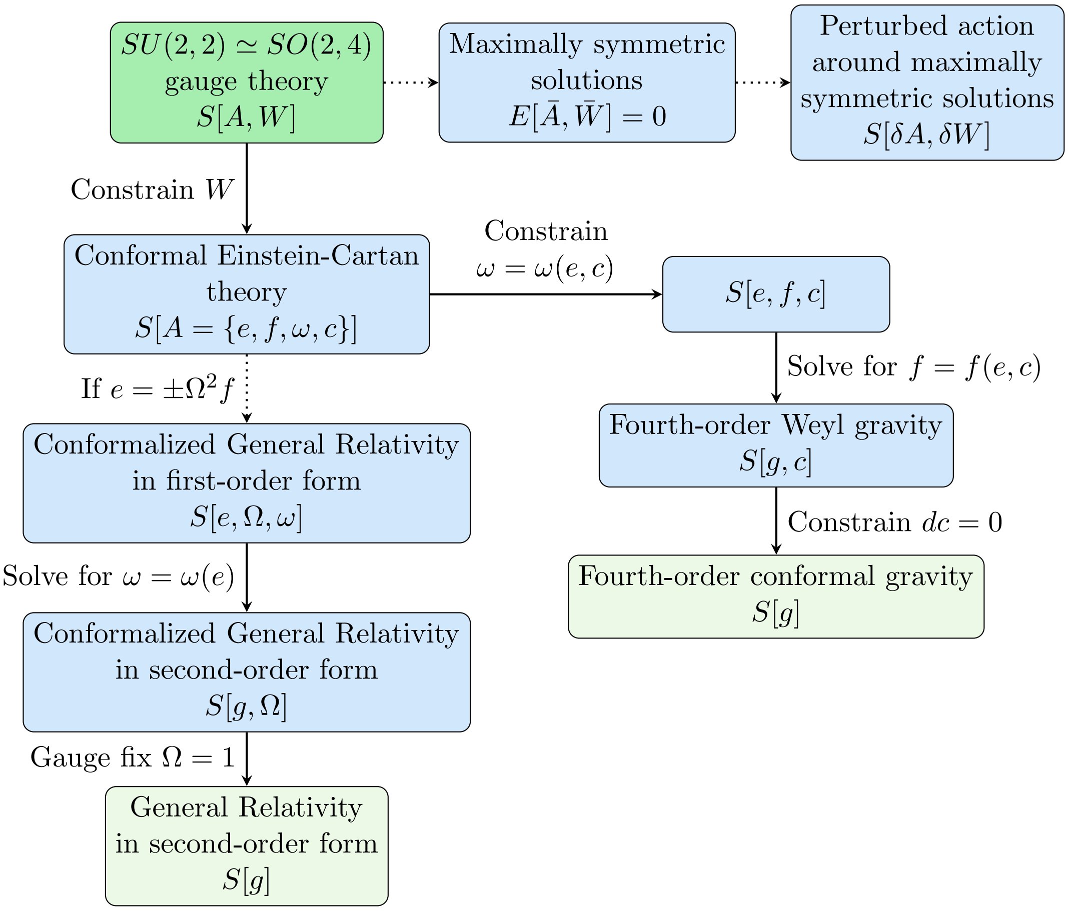


Figure 2: Diagram depicting known results of $SU(2,2) \simeq SO(2,4)$ gravity and relation to other gravitational models. Dotted paths denote those taken with $W^{AB}$ entirely unconstrained.

The outline of the paper is as follows: In Section 2 we present our proposed actions and discuss the field content of the theory. In Section 3 we consider the analog of the Stelle-West approach for these theories, that is: we see what theory emerges by constraining all the degrees of freedom present in the Higgs fields. We will show that this constrained theory indeed corresponds to Conformal Einstein-Cartan theory and we shall show that this theory possesses a limit where it corresponds to ‘conformalized General Relativity’ i.e. a scale-invariant theory of a scalar field (‘dilaton’) conformally coupled to a metric tensor $g_{\mu\nu}$- in the ‘scale gauge’ where this field is a constant, the theory is identical to General Relativity. Furthermore we will show how by further constraining the theory, we recover fourth-order conformal gravity i.e. gravity with action

$$S_C[g] = -\alpha \int \mathcal{C}_{\alpha\beta\delta\gamma}\mathcal{C}^{\alpha\beta\delta\gamma}\sqrt{-g}d^4x \tag{14}$$

where $\alpha$ is a constant and $\mathcal{C}_{\alpha\beta\delta\gamma}$ is the Weyl tensor built with metric $g_{\mu\nu}$ and Christoffel symbols $\Gamma^{\alpha}_{\beta\delta}$.

In Section 4 we return to the full theory and show that there exist maximally symmetric solutions to the theory i.e. solutions interpretable as spacetimes possessing ten Killing vectors each. In Section 5 we examine the nature of small perturbations around these solutions, establishing conditions for linear stability. To aid the reader, the flow chart Figure 2 summarizes the structure of these sections and some results therein. In Section 6 we discuss how one can couple the gravitational fields to matter fields in a simple and elegant fashion. This construction makes essential use of the conformal group. All actions are polynomial and yield first order partial differential equations but yield on-shell the familiar second order equations for Yang-Mills fields and Higgs fields. It is suggested that Yang-Mills fields for a symmetry group $\mathcal{G}$ are necessarily accompanied by scalar fields in the adjoint representation of $\mathcal{G}$. In Section 7 we discuss the relation of the work presented in this paper to previous approaches in the literature and in Section 8 we discuss the paper's results and present conclusions.

## 2 Gravitational action

Though we will take gravity to be a gauge theory of $SU(2,2)$, for calculation purposes it is rather convenient in the gravitational sector to work with representations of $SO(2,4)$. The machinery to move from representations of $SU(2,2)$ to $SO(2,4)$ is discussed in detail in section 6.3 when we discuss the coupling of gravity to fermions, which we take to be fields in the fundamental representation of $SU(2,2)$.

A group element of $SO(2,4)$ may be represented as a matrix $\Lambda^A{}_B$ with unit-determinant and satisfying

$$\eta_{AB} = \eta_{CD}\Lambda^C{}_A\Lambda^D{}_B, \quad \eta_{AB} = \mathrm{diag}(-1,-1,1,1,1,1) \tag{15}$$

A field in the fundamental representation of $SO(2,4)$ is a six-component vector $U^A$ and a field in the adjoint representation takes of the form of an antisymmetric matrix $Y^{AB} = -Y^{BA}$, where indices are lowered and raised with $\eta_{AB}$ and its inverse.

We seek to build a model describing gravitation from a locally $SO(2,4)$-invariant polynomial action. The gravitational fields will be an $SO(2,4)$ gauge field $A^{AB} = A_\mu{}^{AB}dx^\mu$ and a field $W^{AB}$ in the adjoint representation. The field $W^{AB}$ can always be put in the following 'block-diagonal' form by appropriate $SO(2,4)$ transformations:

$$W^{AB} = \begin{pmatrix} t_1\Sigma & 0 & 0 \\ 0 & t_2\Sigma & 0 \\ 0 & 0 & t_3\Sigma \end{pmatrix}, \quad \Sigma \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{16}$$

If $t_1 = t_2 = 0$ (labeling indices on their rows and columns by $I, J, K, \cdots = 0,1,2,3$) and $t_3 \neq 0$ (labeling indices on its rows and columns by $a, b, c, \cdots = -1, 4$) then this form of $W^{AB}$ will be invariant under the $SO(2,4)$ transformations $\Lambda^I{}_J$ and $\Lambda^a{}_b$. If $\mathrm{sign}(\eta_{-1-1}) \neq \mathrm{sign}(\eta_{44})$ then- by implication- $\Lambda^a{}_b$ represent hyperbolic rotations/boosts in the $(-1,4)$ plane whilst $\Lambda^I{}_J$ are transformations generated by the Lorentz group subgroup of $SO(2,4)$. Thus with $t_3 \neq 0$, $t_1, t_2 = 0$, the residual symmetry is $SO(1,3) \times SO(1,1)$.

Useful quantities are the curvature two-form $F^{AB}$ and covariant derivative of $W^{AB}$, $DW^{AB}$, given as follows:

$$F^{AB} = dA^{AB} + A^{AC}A_C{}^B \tag{17}$$

$$DW^{AB} = dW^{AB} + A^{AC}W_C{}^B + A^{BC}W^A{}_C \tag{18}$$

We will look to consider the most general locally $SO(2,4)$-invariant and diffeomorphism-invariant action polynomial in $\{A^{AB}, W^{AB}\}$. To do this we may build differential four-forms from $F^{AB}$ and $DW^{AB}$, thus guaranteeing that the Lagrangian is coordinate independent. To further enforce local $SO(2,4)$ invariance, we will look to contract away all free $SO(2,4)$ indices for which we can in principle additionally use the scalar $W^{AB}$ and the $SO(2,4)$ invariants the matrix $\eta_{AB}$ and the completely antisymmetric symbol $\epsilon_{ABCDEF}$:

$$S[A,W] = \int a_{ABCD}F^{AB}F^{CD} + b_{ABCD}DW^A{}_E DW^{EB}F^{CD} + c_{ABCD}DW^A{}_E DW^{EB}DW^C{}_F DW^{FD} \tag{19}$$

where:

$$\begin{aligned}
a_{ABCD} &\equiv a_1\epsilon_{ABCDEF}W^{EF} + a_2 W_{AE}W^{E}{}_{D}\eta_{BC} + a_3 W_{AB}W_{CD} \\
&\quad + a_4\eta_{AC}\eta_{BD} + a_5\eta_{AC}W_{BD} \qquad (20)\\
b_{ABCD} &\equiv b_1\epsilon_{ABCDEF}W^{EF} \qquad (21)\\
c_{ABCD} &\equiv c_1\epsilon_{ABCDEF}W^{EF} \qquad (22)
\end{aligned}$$

The coefficients $\{a_i, b_i, c_1\}$ may in general depend on $SO(2,4)$ invariants built from $W^{AB}$ and the group invariants $\eta_{AB}$, $\epsilon_{ABCDEF}$. In this paper, as a first approach to the theory, we will take these coefficients to be constant numbers but it is conceivable that functional dependences on such invariants cannot be consistently neglected. Though the action contains terms quadratic in $F^{AB}$ and quartic in $DW^{AB}$, the wedge-product structure guarantees that *components* of these fields appear at most linearly in the action. As mentioned in the previous section, the generation of higher-order partial derivatives in the equations of motion is made impossible by a polynomial Lagrangian and Bianchi identities $DF^{AB} = 0$ and '$DDV = FV$'. Therefore, as the Lagrangian is at most linear in derivatives of any component, the equations of motion are at most first order in derivatives. The action may look very unfamiliar and so in the next section we will initially look at a simpler theory emerges when we 'freeze' all the degrees of freedom in the field $W^{AB}$.

Finally we note that though we will use $W^{AB}$ in the following sections, we may alternatively (and equivalently) use an entirely antisymmetric field $Y_{ABCD}$, where the two are related via

$$Y_{ABCD} = \frac{1}{2}\epsilon_{ABCDEF}W^{EF} \qquad (23)$$

For example the equivalent of '$a_1$' term in the gravitational Lagrangian would then take the form:

$$\int a_1 Y_{ABCD}F^{AB}F^{CD} \qquad (24)$$

with the required symmetry breaking happening when $Y_{IJKL} \neq 0$ ($W_{ab} \neq 0$) and $Y_{abKL} = 0$ ($W_{IJ} = 0$) whilst we may adopt $Y_{IJKa}$ ($W_{Ia} \stackrel{*}{=} 0$) as a gauge choice.

# 3 Conformal Einstein-Cartan theory

We first discuss a theory which emerges when the degrees of freedom of $W^{AB}$ in the model (19) are completely frozen by means of constraints imposed at the level of the action. Recall that the theory recovered by the same process of freezing the Higgs degree of freedom for the $SO(1,4)|SO(2,3)$ gauge theories resulted in the Einstein-Cartan theory (or equivalently General Relativity). We shall show that the corresponding theory for gravity based on the gauge group $SU(2,2) \simeq C(1,3)$ is in some senses a natural conformal generalization of the Einstein-Cartan theory.

Assuming the block-diagonal form of $W^{AB}$ from (31), degrees of freedom in $W^{AB}$ can be characterized in terms of three $SO(2,4)$-invariant quantities:

$$\begin{aligned}
\mathcal{C}_1 &= \epsilon_{ABCDEF}W^{AB}W^{CD}W^{EF} = 6t_1t_2t_3 \qquad (25)\\
\mathcal{C}_2 &= W_{AB}W^{AB} = -2\left(\eta_{00}\eta_{11}(t_1)^2 + \eta_{22}\eta_{33}(t_2)^2 + \eta_{-1-1}\eta_{44}(t_3)^2\right) \qquad (26)\\
\mathcal{C}_3 &= W_{AB}W^{B}{}_{C}W^{CD}W_{D}{}^{A} = 2\left((t_1)^4 + (t_2)^4 + (t_3)^4\right) \qquad (27)
\end{aligned}$$

If it is enforced $\mathcal{C}_1 = 0$ and $\mathcal{C}_3 = (\mathcal{C}_2)^2$, this implies that two out of $\{t_1, t_2, t_3\}$ are zero. For example, we can choose $t_3 = 0$ (the label '3' is completely arbitrary at this point) from $\mathcal{C}_1 = 0$. Subsequently, the condition $(\mathcal{C}_2)^2 - \mathcal{C}_3 = 0$ takes the form:

$$-8\eta_{00}\eta_{11}\eta_{22}\eta_{33}(t_1)^2(t_2)^2 = 0 \tag{28}$$

Hence we can choose $t_2 = 0$. Now if we further require $\mathcal{C}_2 > 0$ we have the condition:

$$-2\eta_{00}\eta_{11}(t_1)^2 > 0 \tag{29}$$

Therefore $\eta_{00}\eta_{11} < 0$ and so we have the breaking $SO(1,1)\times SO(1,3)$. If we then add on the following four-form Lagrange-multiplier constraints

$$S_\lambda[W^{AB},\lambda_1,\lambda_2,\lambda_3] = \int \lambda_1\mathcal{C}_1 + \lambda_2\left(\mathcal{C}_2 - \bar{\phi}^2\right) + \lambda_3\left(\mathcal{C}_3 - (\mathcal{C}_2)^2\right) \tag{30}$$

then the constraints will be enforced via the field equations obtained by varying $\lambda_i$. We may instead enforce the constraints at the level of the action, and so $W^{AB}$ may be assumed to take the form at the level of the action:

$$W^{AB} \stackrel{*}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \bar{\phi}\epsilon^{ab} \end{pmatrix} \tag{31}$$

where we use indices $a, b, c\ldots$ as indices in the fundamental representation of $SO(1,1)$ and we use the convention $\epsilon_{-14} = 1$, $\epsilon^{-14} = -1$ and $\bar{\phi}$ is a constant. Given the application of these constraints, there are no longer any degrees of freedom for $W^{AB}$ left in the action (19); the action is now a functional only of $A^{AB}$, a general ansatz for which is given in this gauge by:

$$A^{AB} = \begin{pmatrix} \omega^{IJ} & E^{Ia} \\ -E^{Ia} & c\epsilon^{ab} \end{pmatrix} \tag{32}$$

The one-form field $\omega^{IJ}$ is the Lorentz-group spin connection, while the one-form $c$ is a connection for the group $SO(1,1)$. The 'off-diagonal' components $E^{Ia}$ look much less familiar; they transform homogeneously under the remnant $SO(1,3)\times SO(1,1)$ symmetry and appear in the $SO(2,4)$ covariant object $DW^{AB}$ as follows:

$$DW^{AB} = \begin{pmatrix} 0 & \bar{\phi}E^I{}_c\epsilon^{ca} \\ -\bar{\phi}E^I{}_c\epsilon^{ca} & 0 \end{pmatrix} \tag{33}$$

Now we write down the total constrained form of (19) in the 'preferred gauge', making use of the following results:

$$F^{AB} = \begin{pmatrix} R^{IJ} - E^I{}_a E^{Ja} & D^{(\omega+c)}E^{Ia} \\ -D^{(\omega+c)}E^{Ia} & dc\epsilon^{ab} - E_J{}^a E^{Jb} \end{pmatrix}, \quad DW^A{}_C DW^{CB} = \begin{pmatrix} \bar{\phi}^2 E^I{}_d E^{Jd} & 0 \\ 0 & 0 \end{pmatrix} \tag{34}$$

where $R^{IJ} \equiv d\omega^{IJ} + \omega^I{}_K\omega^{KJ}$ is the $SO(1,3)$ curvature two-form. The action then becomes:

$$\begin{aligned} S[\omega,c,E] = \int & 2\bar{\phi}(2a_1 - \bar{\phi}^2 b_1)\epsilon_{IJKL}E^I{}_a E^{Ja}R^{KL} - 2\bar{\phi}(a_1 - b_1\bar{\phi}^2 + c_1\bar{\phi}^4)\epsilon_{IJKL}E^I{}_a E^{Ja}E^K{}_b E^{Lb} \\ & + a_2\bar{\phi}^2 R_{IJ}E^{Ja}E^I{}_a + \bar{\phi}^2(a_2 + 2a_3)dc\epsilon^{cd}E_{Jc}E^J{}_d \\ & + a_2\bar{\phi}^2 E_{Jd}E^J{}_c E_K{}^c E^{Kd} + a_3\bar{\phi}^2\epsilon_{ab}\epsilon_{cd}E_J{}^a E^{Jb}E_K{}^c E^{Kd} \\ & + 2\bar{\phi}^2(a_2 + 2a_3)dcdc - 2\bar{\phi}a_1\epsilon_{IJKL}R^{IJ}R^{KL} \end{aligned} \tag{35}$$

where we have used the result $DDE^{Ia} = R^I{}_J E^{Ja} + dc\epsilon^a{}_b E^{Ib}$. To make further progress, we make the following general ansatz for $E^{Ia}$:

$$E^{Ia} = \frac{1}{2}(e^I + f^I)U^a + \frac{1}{2}(e^I - f^I)V^a \tag{36}$$

where $\eta_{ab}U^aU^b = 1$ and $V^a \equiv \epsilon^a{}_b U^b$. Here $U^a$ and $V^a$ should be regarded as an arbitrary choice of basis of the two-dimensional vector space and not a new fundamental field. As such, $e^I$ and $f^I$ as defined in (36) are $SO(1,1)$-invariant (and so are not the same fields as the $\{e^I, f^I\}$ of (9)). Applying this ansatz to (35) yields:

$$\begin{aligned} S[\omega, c, e, f] &= \int \frac{1}{32\pi\tilde{G}}\left(\epsilon_{IJKL}\left(e^I f^J R^{KL} - \frac{\tilde{\Lambda}}{6} e^I f^J e^K f^L\right) - \frac{2}{\gamma} e^I f^J R_{IJ}\right) \\ &\quad + \xi\left(\frac{1}{2} e_I f^I e_J f^J - dce^I f_I\right) + \mathcal{C}_1 \epsilon_{IJKL} R^{IJ} R^{KL} + \mathcal{C}_2 dcdc \end{aligned} \tag{37}$$

where

$$\begin{aligned} \frac{1}{16\pi\tilde{G}}(\bar{\phi}) &= 4\bar{\phi}(2a_1 - b_1\bar{\phi}^2), \quad \tilde{\Lambda}(\bar{\phi}) = \frac{6(a_1 - b_1\bar{\phi}^2 + c_1\bar{\phi}^4)}{(2a_1 - \bar{\phi}^2 b_1)}, \quad \gamma(\bar{\phi}) = \frac{4(2a_1 - b_1\bar{\phi}^2)}{a_2\bar{\phi}} \\ \xi(\bar{\phi}) &= (a_2 + 2a_3)\bar{\phi}^2, \quad \mathcal{C}_1(\bar{\phi}) = -2\bar{\phi}a_1, \quad \mathcal{C}_2(\bar{\phi}) = 2(a_2 + 2a_3)\bar{\phi}^2 \end{aligned}$$

The action (37)- up to boundary terms (those with $\mathcal{C}_i$ coefficients)- indeed corresponds to the action (11) i.e. Conformal Einstein-Cartan theory is indeed the theory that emerges in the limit in which all degrees of freed of $W^{AB}$ are frozen out and $W^{AB}$ is constrained to break $SO(2,4) \to SO(1,3) \times SO(1,1)$. As expected, the action is manifestly locally Lorentz invariant and invariant under local $SO(1,1)$ gauge transformations $c \to c + d\theta(x)$. The action additionally possesses invariance under local dilations of $e^I$ and $f^I$ of opposite weight:

$$e^I \to e^{\alpha(x)} e^I, \quad f^I \to e^{-\alpha(x)} f^I \tag{38}$$

By appearance, the theory (37) resembles the Einstein-Cartan theory but instead has a *pair* of frame fields $\{e^I, f^I\}$ with a local scale symmetry under opposite rescalings. This theory is indeed none other than the Conformal Einstein-Cartan theory proposed in equation (11). Indeed, if there exist solutions when $e^I = \pm f^I$ then the first three terms in (37) become terms familiar from the Einstein-Cartan theory: Palatini, cosmological, and Holst terms respectively. The terms proportional to the coefficient $\xi$ vanish in this limit and so represent new behaviour, whilst the final terms quadratic in curvature are boundary terms and do not contribute to the equations of motion.

## 3.1 General-Relativistic limit

By conducting small variations of (37) one may straightforwardly obtain the equations of motion. Remarkably, solutions to these equations of motion exist for the ansatz $f^I \propto e^I$ i.e.

$$f^I = \pm\Omega^2(x) e^I \tag{39}$$

We now show that- as suggested above- such solutions constitute a General-Relativistic limit of the theory. In this limit, we find from combining the $e^I$ and $f^I$ field equations that

$$dc = 0 \tag{40}$$

and the field $c$ disappears from the system of field equations. The remaining field equations are equivalent to those obtained from the following action:

$$S[\omega, e, \Omega] = \int \frac{\Omega^2}{32\pi\tilde{G}}\left(\epsilon_{IJKL}\left(\pm e^I e^J R^{KL} - \frac{\Omega^2\tilde{\Lambda}}{6} e^I e^J e^K e^L\right) \mp \frac{2}{\gamma} R_{IJ} e^I e^J\right)$$

This action is invariant under local rescalings $e^I \to e^{\alpha(x)} e^I$, $\Omega \to e^{-\alpha(x)}\Omega$. Then, varying with respect to $\omega$ and solving for $\omega(e, \Omega)$, eliminating it from the action, we recover the following, second-order action:

$$S[g,\Omega] = \int d^4x\sqrt{-g}\frac{1}{16\pi G_0(\bar{\phi})}\left(\Omega^2 R + 6g^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega - 2\Lambda_0(\bar{\phi})\Omega^4\right) \tag{41}$$

where $g_{\mu\nu} \equiv \eta_{IJ}e^I_\mu e^J_\nu$, $R$ is the Ricci scalar according to the Christoffel symbols $\Gamma^\mu_{\nu\rho}$ and

$$\frac{1}{16\pi G_0}(\bar{\phi}) \;=\; 4\bar{\phi}(2a_1 - b_1\bar{\phi}^2), \quad \Lambda_0(\bar{\phi}) = \frac{6(a_1 - b_1\bar{\phi}^2 + c_1\bar{\phi}^4)}{(2a_1 - \bar{\phi}^2 b_1)}$$

Thus we see in the second-order formalism a kinetic term for $\Omega$ (sometimes this field is referred to as a dilaton) emerges. We may utilize the gauge freedom to locally rescale $\Omega$: if we assume that the action is an integration over regions where $\Omega \neq 0$ then a convenient gauge choice is $\Omega = 1$, in which case he action reduces to:

$$S[g] = \int \frac{1}{16\pi G_0(\bar{\phi})}\left(R - 2\Lambda_0(\bar{\phi})\right)\sqrt{-g}d^4x \tag{42}$$

This is the Einstein-Hilbert action of General Relativity. Therefore the equations of motion obtained in the case $e^I \propto f^I$ are equivalent[4] to those of General Relativity[5]. Because of this, we refer to the theory (41) as *conformalized General Relativity*; if we instead had begun from the Einstein-Hilbert action (42) we could recover (41) via a local conformal rescaling of $g_{\mu\nu}$ using $\Omega(x)$.

We must now ask how a theory like General Relativity, with an inbuilt scale, can emerge from a theory where it is not clear that there is an inbuilt scale. Frequently in the literature an additional Higgs scalar $\Phi(x)$ is introduced alongside $g_{\mu\nu}$ [29] to so that locally scale-invariant actions can be built from $\{\Phi, g_{\mu\nu}\}$ in the manner that $\{\Omega, g_{\mu\nu}\}$ combine to yield the locally scale-invariant action (41). The scalar $\Phi$ would have dimensions of length or mass and would this set a specific scale at each point in spacetime. However, locally in regions where $\Phi(x) \neq 0$ we may then readily impose a gauge $\Phi(x) = cst.$ in which the theory would no-longer be manifestly scale-invariant. To some this introduction of scale invariance and then its immediate elimination might seem a bit contrived.

To that end we wish to point out that the breaking of scale invariance in our proposed theory is not aided by the introduction by any additional fundamental Higgs fields but is instead intimately related to the General-Relativistic limit of the theory. The appearance of a a length scale is a feature of a specific subclass of solutions, namely the General-Relativistic solutions characterized by the condition $e^I \propto f^I$. At an extreme, as the equations of motion following from the action (19) are polynomial and each term is at least cubic in $\{A^{AB}, W^{AB}\}$ then there exist solutions where $\{A^{AB} = 0, W^{AB} = 0\}$ and the entire $SO(2,4)$-invariance is retained and there is no definable scale. One may imagine situations where initial data close to this scaleless solution evolves towards the General-Relativitistic limit; in this sense we may think of scale spontaneously emerging in the theory.

We may characterize the condition $e^I \propto f^I$ in a different fashion. Above we introduced the arbitrary basis $\{U^a,\ V^a \equiv \epsilon^a{}_b U^b, \eta_{ab}U^aU^b = 1\}$ for the $SO(1,1) \in SO(2,4)$ vector space indexed by $a$ via the ansatz (36). The General-Relativistic limit is the limit in which a preferred basis $\{U^a_{(P)}, V^a_{(P)}, \eta_{ab}U^a_{(P)}U^b_{(P)} = 1\}$ of the $SO(1,1)$ vector space emerges. This basis is defined by one or two independent possibilities

$$U^a_{(P)}A^I{}_a = 0 \tag{43}$$

or

$$V^a_{(P)}A^I{}_a = 0 \tag{44}$$

[4]We note that due to the presence of second-order partial derivatives of the metric tensor in the Einstein-Hilbert action it is necessary to introduce an additional topological term- the *Gibbons-Hawking* term- for the variational principle to work out correctly.

[5]For a different approach to General Relativity and scale invariance see [25, 26, 27, 28].

For example in the case of the condition (43) holding, we have-using (36) and (43)- that

$$f^I \quad = \quad -\frac{\left(U^a U_{(P)a} + V^a U_{(P)a}\right)}{\left(U^a U_{(P)a} - V^a U_{(P)a}\right)} e^I \equiv -\Omega^2 e^I \tag{45}$$

If, alternatively, the condition (44) holds then $f^I = +\Omega^2 e^I$.

In some respects this is similar to the breaking of rotational invariance for a ferromagnet: Although a preferred direction $\overrightarrow{X}$ appears as a property of low temperature solutions this direction $\overrightarrow{X}$ makes no appearance in the fundamental equations of motion. Similarly, a preferred direction in the $SO(1,1)$ space- $U^a_{(P)}$- appears for General-Relativistic solutions but its existence is a property of some solutions rather than a basic constituent of the theory. Neither can such a preferred vector $U^a_{(P)}$ be defined uniquely outside the General-Relativistic limit. Thus, the General-Relativistic limit and the breaking of scale invariance are in our approach two sides of the same coin.

## 3.2 Relation to Weyl and fourth-order conformal gravity

In the previous section we consider a theory recovered by 'freezing' all the degrees of freedom in $W^{AB}$ to take a specific form; the resulting theory possessed a local $SO(1,3)$ invariance as well as independent rescaling invariances under $e^I \to e^{\alpha(x)} e^I$, $f^I \to e^{-\alpha(x)} f^I$, $c \to c - d\beta(x)$. An interesting property of this theory is that beginning from the field equations for the set $\{e^I, f^I, \omega^I{}_J, c\}$ we may solve algebraically for *any* one of $\{e^I, f^I, \omega^I{}_J\}$ and eliminate it from the action. In Appendix B, this is explicitly illustrated for the following action, solving for $f^I$ using its own field equation:

$$S[\omega, c, e, f] = \int \alpha \epsilon_{IJKL} e^I f^J R^{KL} + \beta \epsilon_{IJKL} e^I e^J f^K f^L + \gamma e^I f_I dc \tag{46}$$

Inserting the solution for $f^I$ into the action one obtains:

$$S[\omega, c, e] = \frac{1}{\beta} \int \frac{\alpha^2}{4} \epsilon_{IJKL} C^{IJ} C^{KL} - 6\gamma^2 dc * dc + 11\alpha\gamma e_K R^K dc - \frac{\alpha^2}{4} \epsilon_{IJKL} R^{IJ} R^{KL} \tag{47}$$

where $*$ is the Hodge dual operator built from the field $e^I$, $R^J$ is the Ricci one-form $R^J \equiv ((e^{-1})_I \lrcorner R^{IJ})$ and

$$C^{IJ} \quad \equiv \quad R^{IJ} - 6e^I \left( R^J - \frac{R}{6} e^J \right) \tag{48}$$

where $R$ is the Ricci scalar. We now consider the effect of placing an additional constraint on this theory. Recall that in the Einstein-Cartan theory, the equation $D^{(\omega)} e^I = 0$ was the equation of motion for $\omega^I{}_J$ and allowed one to solve for $\omega^I{}_J$ and eliminate it from the action principle. Now consider the following generalisation of this equation:

$$D^{(\omega+c)} e^I \quad = \quad de^I + \omega^I{}_J e^J + c e^I = 0 \tag{49}$$

This equation is invariant under a more restricted group of symmetry transformations of the Conformal Einstein-Cartan theory: that when $\alpha(x) = \beta(x)$. This equation is *not* the equation of motion for $\omega^I{}_J$ one would get by varying with respect to $\omega^I{}_J$ for this theory but we may enforce it via a Lagrangian constraint. Doing so, we may now use (49) to solve for $\omega^I{}_J(e, c)$ and substitute this solution into the action (47). The resulting action is a functional only of $e^I$ (appearing variously via $\{e_I, (e^{-1})^{I\mu}, R^{IJ}(e)\}$) and $c$ (via $dc$):

$$S[e,c] \quad = \quad \frac{1}{\beta}\int \frac{\alpha^2}{4}\epsilon_{IJKL}\mathcal{C}^{IJ}\mathcal{C}^{KL} - 6\gamma^2 dc * dc + 11\alpha\gamma e_K R^K dc - \frac{\alpha^2}{4}\epsilon_{IJKL}R^{IJ}R^{KL} \quad (50)$$

This is the action for *fourth-order Weyl gravity* which, as the name suggests, yields field equations containing fourth-derivatives of fields. If we further constrain $dc = 0$ then we recover the action:

$$S[e] = \frac{1}{\beta}\int \frac{\alpha^2}{4}\epsilon_{IJKL}C^{IJ}(e)C^{KL}(e) - \frac{\alpha^2}{4}\epsilon_{IJKL}R^{IJ}(e)R^{KL}(e) \quad (51)$$

The quantity $\mathcal{C}^{IJ}(e)$ is the Weyl two-form, related to the Weyl tensor $\mathcal{C}_{\mu\nu\alpha\beta} = \mathcal{C}_{\mu\nu}{}^{IJ}e_{I\alpha}e_{J\beta}$ via:

$$\mathcal{C}^{IJ} = \frac{1}{2}\mathcal{C}_{\mu\nu}{}^{IJ}dx^\mu dx^\nu \quad (52)$$

The action (51) is thus proportional to that of *fourth-order conformal gravity* plus a boundary term quadratic in $R^{IJ}$. Hence, fourth-order conformal gravity can be recovered from the original $SU(2,2)$ gauge theory via the implementation of a number of constraints. Indeed, this was the result found by Kaku, Townsend, and Van Nieuwenhuizen [30]. Their approach was essentially the same as the steps discussed in this section: i.e. fourth-order conformal gravity was recovered by implicitly constraining the symmetry breaking fields that break $SO(2,4) \to SO(1,3) \times SO(1,1)$ and explicitly constraining the spin-connection $\omega^I{}_J$. For the particular action they considered (specifically the action (46) with $\gamma = 0$) $dc$ vanishes automatically from the action and did not need to be constrained to vanish. The relation of this approach of recovering Weyl gravity from a gauge theory of gravity to Cartan's conception of geometry has recently been discussed in detail[6] [32].

It has become somewhat common lore that fourth-order Weyl gravity is the gauge theory of the conformal group [33, 34] . However, if one is looking to cast gravity as a gauge theory akin to those of particle physics, why completely freeze all the degrees of freedom in the symmetry breaking fields? The analogue in electroweak theory would be an insistence that $\varphi^\dagger\varphi$ for the electroweak Higgs $\varphi$ were fixed to be a constant- this would force a non-vanishing expectation value for $\varphi$ much as a non-vanishing expectation value for $W^{AB}$ was achieved in the above approach. The discovery of the Higgs boson demonstrates that in that case it would be incorrect to apply such constraints; should gravity be any different? Even allowing this, why then further constrain $\omega^I{}_J$ to take a solution that would not generally follow from $\omega^I{}_J$'s equation of motion [35]?

# 4 Vacuum solutions of the full theory

We now 'un-freeze' the field $W^{AB}$. Its behaviour will now be dictated entirely by its own equations of motion in conjunction with those of other fields.

In this section we will demonstrate that there exist simple solutions to the theory in which the field $W^{AB}$ has a non-vanishing, constant expectation value and one may interpret the accompanying spacetime geometry as being de Sitter or anti de Sitter space. We will look for solutions where $W^{AB}$ takes the following form:

$$W^{AB} \stackrel{*}{=} \begin{pmatrix} 0 & 0 \\ 0 & \phi\epsilon^{ab} \end{pmatrix} \quad (53)$$

where recall that $a, b, c, \ldots$ are $SO(1,1)$ indices and the gauge-fixing condition $W^{Ia} = 0$ has been imposed. We will focus on searching for solutions where

[6] The link between Cartan geometry and conformal physics has previously been investigated in the case of $2+1$ spacetime dimensions [31]

$$\phi = \bar{\phi} = \text{cst.} \tag{54}$$

As detailed in the previous section, this form of $W^{AB}$ (even if $\phi$ were not constant) breaks the original $SO(2,4)$ symmetry of the theory down to $SO(1,3) \times SO(1,1)$ if the signature of $\eta_{ab}$ is $(-,+)$; we will assume this to be the case. Clearly, the existence of solutions satisfying this condition does not indicate that they are dynamically favoured. In this paper our analysis will be limited to establishing linear stability of them with respect to small perturbations.

Now we turn to the form of the connection $A^{AB}$. Given the above symmetry breaking, a general ansatz would be:

$$A^{AB} = \begin{pmatrix} \omega^{IJ} & E^{Ia} \\ -E^{Ia} & c\epsilon^{ab} \end{pmatrix} \tag{55}$$

The one-form field $\omega^{IJ}$ is the Lorentz-group spin connection, while the one-form $c$ is a connection for the group $SO(1,1)$. The 'off-diagonal' components $E^{Ia}$ look much less familiar. By way of interpretation, let's look at its contribution to an $SO(2,4)$ covariant object $DW^{AB}$:

$$DW^{AB} = \begin{pmatrix} 0 & \phi E^{I}{}_{c}\epsilon^{ca} \\ -\phi E^{I}{}_{c}\epsilon^{ca} & 0 \end{pmatrix} \tag{56}$$

This looks rather like there are two 'frame-fields' $(E^I_{-1}, E^I_4)$. Can one or more tensors be constructed that may play the role of spacetime metric? Up to multiplicative powers of $W^{AB}W_{AB}$, two rank-two tensors can be constructed:

$$G^{(1)}_{\mu\nu} = \eta_{AB} D_\mu W^{A}{}_{C} D_\nu W^{CB} \stackrel{*}{=} \phi^2 \eta_{ab} E^a_{\mu I} E^{Ib}_\nu \tag{57}$$

$$G^{(2)}_{\mu\nu} = W_{AB} D_\mu W^{A}{}_{C} D_\nu W^{CB} \stackrel{*}{=} \phi^3 \epsilon_{ab} E^a_{\mu I} E^{Ib}_\nu \tag{58}$$

The first tensor $G^{(1)}_{\mu\nu}$ is symmetric in $\mu$ and $\nu$ whilst $G^{(2)}_{\mu\nu}$ is anti-symmetric and hence is a spacetime two-form. Considered as a metric tensor, the tensor $G^{(1)}_{\mu\nu}$ is a little more exotic than the metric tensor $g_{\mu\nu} \equiv \eta_{IJ} e^I_\mu e^J_\nu$ of the Einstein-Cartan theory. In that theory, even if there exist regions where $e^I_\mu$ is zero, the metric will (following chosen convention) either always of 'signature' $(-,+,+,+)$ or $(+,-,-,-)$. In the case of $G^{(1)}_{\mu\nu}$ though, the indefinite sign of $\eta_{ab}$ means that the tensor could pass through zero from a region of being non-zero with signature $(-,+,+,+)$ to one with $(+,-,-,-)$, this being more than convention. However, even within Einstein-Cartan theory the possibility of $g_{\mu\nu} = 0$ in any region of spacetime is a controversial one with no known observational signatures [36]; if $G^{(1)}_{\mu\nu}$ can avoiding vanishing then it should be just as much of a metric tensor as $g_{\mu\nu}$.

We will assume henceforth that the two-form $G^{(2)}_{\mu\nu}$ is zero in this limit and that $DW^{AB}$ takes the following form:

$$DW^{AB} = \begin{pmatrix} 0 & e^I u^a \\ -e^I u^a & 0 \end{pmatrix} \tag{59}$$

where $u^a$ may in principle be timelike or spacelike with respect to the metric $\eta_{ab} = \text{diag}(-1,1)$ (i.e. $u^2 = \eta_{ab}u^a u^b < 0$ or $\eta_{ab}u^a u^b > 0$). We identify $e^I$ with the field $e^I$ in the arbitrary basis (36) and so using that expression and (55) we can see that the ansatz (59) implies that:

$$\begin{aligned} A^{Ia} = E^{Ia} = -\frac{1}{\phi} v^a e^I &= \frac{1}{2}(e^I + f^I)U^a + \frac{1}{2}(e^I - f^I)V^a \\ &= \frac{1}{2}\left((1 \mp \Omega^2)U^a_{(P)} + (1 \pm \Omega^2)V^a_{(P)}\right) e^I \end{aligned} \tag{60}$$

where we have allowed for the collective cases $f^I = \mp\Omega^2 e^I$ (depending on whether $A^{Ia}U_{(P)a} = 0$ or $A^{Ia}V_{(P)a} = 0$ respectively). Therefore

$$v^a = -\frac{\phi}{2}\left((1\mp\Omega^2)U^a_{(P)} + (1\pm\Omega^2)V^a_{(P)}\right) \tag{61}$$

and so

$$u^2 = -v^2 = \pm\phi^2\Omega^2 \tag{62}$$

Therefore when $u^2 > 0$ this corresponds to the ansatz $f^I = -\Omega^2 e^I$ and when $u^2 < 0$ this corresponds to the ansatz $f^I = +\Omega^2 e^I$. From (59) we can see that clearly there is the flexibility to locally rescale $e^I \to e^{\alpha(x)}e^I$ and $u^a \to e^{-\alpha(x)}u^a$ with affecting the ansatz for $DW^{AB}$. We can see from equation (62) that rescaling of $u^a$ away from unit norm corresponds to a choice of $\Omega^2$ different from $\Omega^2 = 1/\phi^2$. For the remainder of this paper we will allow for the possibility that $u^a$ may be timelike or spacelike but we will make the 'scale gauge' fixing choice $|u^2| = 1$ (equivalently $\Omega^2 = 1/\phi^2$); this will simplify calculations.

The $SO(2,4)$ curvature two-form takes the following form:

$$F^{AB} = \begin{pmatrix} R^{IJ} + \frac{u^2}{\phi^2}e^I e^J & -\frac{1}{\phi}\left(D^{(c)}v^a e^I + v^a D^{(\omega)}e^I\right) \\ -F^{Ia} & dc\epsilon^{ab} \end{pmatrix} \tag{63}$$

where $D^{(c)}$ and $D^{(\omega)}$ are the $SO(1,1)$ and $SO(1,3)$ covariant derivatives respectively. We look for solutions where

$$F^{Ia} = 0 \tag{64}$$

For such such solutions, the condition (64) implies that

$$\begin{aligned} \frac{1}{2u^2}du^2 e^I + D^{(\omega)}e^I &\equiv \frac{1}{2u^2}du^2 e^I + de^I + \omega^I{}_J e^J = 0 & (65) \\ u_a\epsilon^{ab}D^{(c)}u_b &= u_a\epsilon^a{}_b du^b + u^2 c = 0 & (66) \end{aligned}$$

From equations (65) and (66) we then have that

$$\begin{aligned} \omega_\mu{}^{IJ} &= 2e^{\nu[I}\partial_{[\mu}e^{J]}_{\nu]} + e_{\mu K}e^{\nu I}e^{\alpha J}\partial_{[\alpha}e^K_{\nu]} & (67) \\ c_\mu &= -\frac{1}{u^2}u_a\epsilon^a{}_b\partial_\mu u^b & (68) \end{aligned}$$

From (68) we have that:

$$dc = -d\left(\frac{1}{u^2}\right)u_a\epsilon^a{}_b du^b - \frac{1}{u^2}\epsilon_{ab}du^a du^b - \frac{1}{u^2}u_a\epsilon^a{}_b ddu^b \tag{69}$$

The first term disappears due to the choice $u^2 = \pm 1$ and the final term disappears due to the identity $dd = 0$ on any differential form. Additionally the second term disappears because we may globally adopt an $SO(1,1)$ gauge where $u^a = \delta^a_4$ (for $u^2 = 1$) or $u^a = \delta^a_{-1}$ (for $u^2 = -1$), in which case the second term $\epsilon_{ab}du^a du^b$ disappears and so $dc = 0$ given our ansatz. Furthermore we propose that the $SO(1,3)$ curvature takes the following form:

$$R^{IJ} = \lambda e^I e^J \tag{70}$$

where $\lambda$ is a constant. Given $\omega^I{}_J(g)$, this implies that the solution for $g_{\mu\nu} \equiv \eta_{IJ}e^I_\mu e^J_\nu$ will be the metric de Sitter space ($\lambda > 0$) or anti-de Sitter space ($\lambda < 0$). Therefore in summary the $SO(2,4)$ curvature is assumed to *on-shell* take the following, simple form:

$$F^{AB} = \begin{pmatrix} \left(\lambda + \frac{u^2}{\phi^2}\right)e^I e^J & 0 \\ 0 & 0 \end{pmatrix} \tag{71}$$

From the $A^{Jb}$ equations of motion we find that only $\{a_1, b_1, c_1\}$ terms of (19) offer a non-vanishing contribution and that these equations of motion provide a value for the cosmological constant $\lambda$ in terms of parameters $\{a_1, b_1, c_1\}$ and $\bar{\phi}^2$:

$$\lambda = -\frac{u^2}{\bar{\phi}^2}\frac{\left(a_1 - b_1\bar{\phi}^2 + c_1\bar{\phi}^4\right)}{\left(a_1 - \frac{\bar{\phi}^2 b_1}{2}\right)} \tag{72}$$

A solution for the value of $\bar{\phi}$ may be obtained by looking at the equations of motion obtained by varying $W^{ab}$. Again, only $\{a_1, b_1, c_1\}$ terms yield a non-vanishing contribution, and the equation reads:

$$\left(2a_1\xi^2 - 6b_1\xi\bar{\phi}^2 + 10c_1\bar{\phi}^4\right)u^4 = 0 \tag{73}$$

where $\xi = 1 + \lambda\bar{\phi}^2/u^2$. If we now assume that $u^2 \neq 0$, equations (72) and (73) may be combined to obtain the following equation:

$$0 = \bar{\phi}\frac{\left(b_1^2 - 4a_1c_1\right)\left(-5a_1 + 3b_1\bar{\phi}^2 - c_1\bar{\phi}^4\right)}{\left(b_1\bar{\phi}^2 - 2a_1\right)} \tag{74}$$

This equation may be seen as a defining equation for $\bar{\phi}^2$ -assuming that the special case $b_1^2 - 4a_1c_1 = 0$ does not apply- i.e. we may solve it to find $\bar{\phi}^2(a_1, b_1, c_1)$. This restricts the $\{a_1, b_1, c_1\}$ parameter space to values where real, positive solutions for $\bar{\phi}^2$ exist. If we use this result in (72) we recover the simple relation

$$\lambda = \frac{4u^2}{\bar{\phi}^2} \tag{75}$$

Thus, $u^2$ determines the sign of the cosmological constant $\lambda$. We see then that extremely simple solutions to the theory exist, just as in the case of General Relativity, and that many of the degrees of freedom of the theory do not contribute within these solutions (e.g. it is assumed that $W^{IJ} = 0$).

An important point is that the existence of a constant solution for the field $\phi$ depended on our ansatz $R^{IJ} = \lambda e^I e^J$. This is because the $(a, b)$ component of the $W^{AB}$ equation of motion contains a term proportional to $a_1\epsilon_{IJKL}R^{IJ}R^{KL}$; only when $R^{IJ} = \lambda e^I e^J$ does this term appear on the same footing as terms structurally similar to potential terms i.e. for a potential $\chi(\phi)$, contributions of the form $(d\chi/d\phi)\epsilon_{IJKL}e^I e^J e^K e^L$. If $R^{IJ}$ is of a more general form then static solutions for $\phi$ likely will not always exist. This is highly reminiscent of the case of models of gravity based on the groups $SO(1,4)|SO(2,3)$ wherein it was rather more common for $\phi$ to 'roll' down an effective potential rather than be static [37].

We now check the stability of the maximally symmetric solutions against small perturbations in all the degrees of freedom of the theory.

# 5 Perturbations

We now consider perturbations around the above de Sitter/anti-de Sitter solutions. The aim is to expand the Lagrangian for the theory (incorporating all terms) around the background solutions up to quadratic order. This will tell us whether there exist instabilities in the maximally-symmetric solutions to the theory. The analogue of this in General Relativity would be investigating the stability of de Sitter and anti-de Sitter space with respect to small perturbations in the co-tetrad/metric. In the model based on the action (19) there are new gravitational fields over and above those in General Relativity and so it is important to check whether the effect they have on stability.

At the level of perturbations we can retain the gauge fixing $W^{Ia} \stackrel{*}{=} 0$ and we will use bars above quantities to denote that they are background quantities (e.g. $\bar{A}^{AB}$ denotes the background form of $A^{AB}$). As for the case of the background solutions, we fix the local rescaling symmetry amongst the background pair $\{u^a, \bar{e}^I\}$ so that $|u^2| = 1$. A completely general ansatz for small perturbations is as follows:

$$\delta W^{AB} = \bar{\phi}\begin{pmatrix} \delta H^{IJ} & 0 \\ 0 & \delta\alpha\epsilon^{ab} \end{pmatrix}, \quad \delta A^{AB} = \begin{pmatrix} \delta\omega^{IJ} & -\frac{v^a}{\bar{\phi}}\delta n^I - \frac{u^a}{\bar{\phi}}\delta h^I \\ -\delta A^{Ia} & \delta c\epsilon^{ab} \end{pmatrix} \tag{76}$$

We may insert this ansatz into the Lagrangian four-form for the theory to obtain a Lagrangian quadratic in smallness. The perturbed total Lagrangian $\delta L$ decomposes into two independent parts: a part that depends on the variables $\{\delta n^I, \delta\omega^{IJ}, \delta\alpha\}$ and a part that depends on the variables $\{\delta h^I, \delta c, \delta H^{IJ}\}$:

$$\delta L = \delta L_{\{0,2\}}(\delta n, \delta\omega, \delta\alpha) + \delta L_1(\delta h, \delta c, \delta H) \tag{77}$$

As will be seen, the labels $\{0, 2\}$ and 1 refer to the spin of the perturbations from each part of the Lagrangian when cast into a second-order form. We first concentrate on $\delta L_{\{0,2\}}$. It is extremely useful to make a variable redefinition to help cast terms in a more familiar way: after calculation, we find amongst the terms the following contribution to the perturbed Lagrangian:

$$\epsilon_{IJKL}\bar{e}^I D^{(\bar{\omega})}\delta\omega^{KL}\left(2u^2\bar{e}^J\delta\alpha\left(b_1\bar{\phi} - 2\left(\frac{\lambda\bar{\phi}^2}{u^2} - 1\right)\frac{a_1}{\bar{\phi}}\right) + 4u^2\left(b_1\bar{\phi} - \frac{2a_1}{\bar{\phi}}\right)(\bar{e}^I\delta\alpha + \delta n^I)\right) \tag{78}$$

If we define the following variables:

$$\begin{aligned} \bar{\theta}^I &= \sqrt{2u^2(b_1\bar{\phi} - 2\frac{a_1}{\bar{\phi}})}\bar{e}^I \\ \delta\theta^I &= \sqrt{2u^2(b_1\bar{\phi} - 2\frac{a_1}{\bar{\phi}})}\delta n^I + \left(\frac{d\left(\sqrt{2u^2(b_1\bar{\phi} - 2\frac{a_1}{\bar{\phi}})}\bar{\phi}\right)}{d\bar{\phi}} - 2\frac{a_1}{\sqrt{2u^2(b_1\bar{\phi} - 2\frac{a_1}{\bar{\phi}})}}\lambda\bar{\phi}\right)\bar{e}^I\delta\alpha \end{aligned}$$

then the terms (78) become:

$$2\epsilon_{IJKL}\bar{\theta}^I\delta\theta^J D^{(\bar{\omega})}\delta\omega^{KL} + \epsilon_{IJKL}\bar{\theta}^I\bar{\theta}^J\delta\omega^K{}_M\delta\omega^{ML} \tag{79}$$

This is simply equivalent to the perturbation to the Einstein-Palatini action $\epsilon_{IJKL}e^Ie^JR^{KL}(\omega)$ around a background $\theta^I = \bar{\theta}^I$. All dependency upon $\delta\alpha$ has disappeared. We can make further progress by decomposing the spin connection perturbation $\delta\omega^{IJ}$ into a 'torsion-free' part $\delta\tilde{\omega}^{IJ}$ and the contorsion $\delta C^{IJ}$:

$$\delta\omega^{IJ} = \delta\tilde{\omega}^{IJ} + \delta C^{IJ} \tag{80}$$

where $\delta\tilde{\omega}^{IJ}(\delta\theta, \partial\delta\theta)$ is the solution to the equation:

$$d\delta\theta^I + \bar{\omega}^I{}_J\delta\theta^J + \delta\tilde{\omega}^I{}_J\bar{\theta}^J = 0 \tag{81}$$

It may be shown then that

$$\begin{aligned} \delta L_{\{0,2\}} = \; & 2\epsilon_{IJKL}\bar{\theta}^I\delta\theta^J D^{(\bar{\omega})}\delta\tilde{\omega}^{KL}(\theta) + \epsilon_{IJKL}\bar{\theta}^I\bar{\theta}^J\delta\tilde{\omega}^K{}_M(\theta)\delta\tilde{\omega}^{ML}(\theta) - 4\chi\epsilon_{IJKL}\delta\theta^I\delta\theta^J\bar{\theta}^K\bar{\theta}^L \\ & + \epsilon_{IJKL}\bar{\theta}^I\bar{\theta}^J\delta C^K{}_M\delta C^{ML} + u^2\frac{a_2}{\mu^2}\left(\delta C^{IK}\delta C_K{}^J - 2\left(\frac{1}{\mu}\frac{d\mu}{d\bar{\phi}}\bar{\phi} - 4a_1\bar{\phi}\chi + 1\right)d\delta\alpha\delta C^{IJ}\right)\bar{\theta}_I\bar{\theta}_J \\ & - \left(\frac{1}{2}\frac{d^2\chi}{d\bar{\phi}^2} - 64\chi^3a_1^2\right)\bar{\phi}^2(\delta\alpha)^2\epsilon_{IJKL}\bar{\theta}^I\bar{\theta}^J\bar{\theta}^K\bar{\theta}^L \end{aligned} \tag{82}$$

where

$$\mu \equiv \sqrt{2u^2(b_1\bar{\phi} - 2\frac{a_1}{\bar{\phi}})}, \quad \chi \equiv \frac{1}{2}\frac{\left(\frac{a_1}{\bar{\phi}^3} - \frac{b_1}{\bar{\phi}} + c_1\bar{\phi}\right)}{\left(b_1\bar{\phi} - 2\frac{a_1}{\bar{\phi}}\right)^2} \tag{83}$$

We may now vary with respect to $\delta C^{IJ}$; we find that the resulting equation of motion is an algebraic equation with which we can actually solve for $\delta C^{IJ}$ in terms of $\partial\delta\alpha$. This solution may be re-inserted into the Lagrangian to eliminate $\delta C^{IJ}$ from the action, yielding:

$$\delta L_{\{0,2\}} = \delta L_2 + \delta L_0 \tag{84}$$

$$\begin{aligned}\delta L_2(\delta e) &= 2\epsilon_{IJKL}\bar{\theta}^I\delta\theta^J D^{(\bar{\omega})}\delta\tilde{\omega}^{KL}(\theta) + \epsilon_{IJKL}\bar{\theta}^I\bar{\theta}^J\delta\tilde{\omega}^K{}_M(\theta)\delta\tilde{\omega}^{ML}(\theta) \\ &\quad -4\chi\epsilon_{IJKL}\delta\theta^I\delta\theta^J\bar{\theta}^K\bar{\theta}^L\end{aligned} \tag{85}$$

$$\begin{aligned}\delta L_0(\delta\alpha) &\quad -\left(\frac{a_2^2}{8\mu^4 + 2a_2^2}\right)\left(\frac{1}{\mu}\frac{d\mu}{d\bar{\phi}}\bar{\phi} - 4a_1\bar{\phi}\chi + 1\right)^2\left(\partial_M\delta\alpha\partial^M\delta\alpha\right)\epsilon_{IJKL}\bar{\theta}^I\bar{\theta}^J\bar{\theta}^K\bar{\theta}^L \\ &\quad -\left(\frac{1}{2}\frac{d^2\chi}{d\bar{\phi}^2} - 64\chi^3a_1^2\right)\bar{\phi}^2(\delta\alpha)^2\epsilon_{IJKL}\bar{\theta}^I\bar{\theta}^J\bar{\theta}^K\bar{\theta}^L\end{aligned} \tag{86}$$

We see then that this part of the perturbed Lagrangian decomposes into two independent pieces. The first piece, $\delta L_2$, depends solely on $\delta\theta$; it is equivalent to the perturbed Lagrangian of General Relativity against a background maximally symmetric spacetime with co-tetrad $\bar{\theta}^I$ and cosmological constant $\Lambda = 12\chi$. Therefore, perturbations include spin-2 gravitational wave solutions, much as in General Relativity.

The second piece, $\delta L_0$ depends solely on perturbations $\delta\alpha$. The first term is a correct-sign kinetic term for $\delta\alpha$, whereas the second term acts as a mass term for the field, as one would expect for perturbations around a solution $\phi = $ cst.. However, note that the term is not due solely to the 'second derivative of a potential' (here $d^2\chi/d\bar{\phi}^2$) term as one might expect in a theory of a scalar field in curved spacetime- there is an additional term due to the $a_1$ in the action- this reflects the direct coupling between $\phi$ and a term quadratic in spacetime curvature (this corresponds to the $\mathcal{C}_1$ term of (37) with $\phi$ 'un-frozen'). The sign of this mass-squared term is not of definite sign for all $\{a_1, b_1, c_1\}$; its sign is given by:

$$\text{sign}\left(\frac{d^2\chi}{d\bar{\phi}^2} - 128\chi^3a_1^2 = \frac{40a_1b_1\bar{\phi}^2 - 100a_1^2 - 3b_1^2\bar{\phi}^4}{(b_1\bar{\phi}^3 - 2a_1\bar{\phi})^3}\right) \tag{87}$$

where recall that the background value $\bar{\phi}$ is a function of $\{a_1, b_1, c_1\}$. Hence, positivity of effective mass-squared places a restriction upon the parameter space $\{a_1, b_1, c_1\}$. Recall our earlier field redefinition $\bar{\theta}^I = \mu\bar{e}^I$; this implies that we must have that $\mu \equiv \sqrt{2u^2(b_1\bar{\phi} - 2\frac{a_1}{\bar{\phi}})}$ is real and so

$$2u^2\bar{\phi}\left(b_1 - 2\frac{a_1}{\bar{\phi}^2}\right) > 0 \tag{88}$$

This implies that the sign of the mass-squared is

$$\text{sign}\left(u^2\left(40a_1b_1\bar{\phi}^2 - 100a_1^2 - 3b_1^2\bar{\phi}^4\right)\right) \tag{89}$$

The sign of the cosmological constant in the background is given by sign$(u^2)$ and so if we take a positive cosmological constant ($u^2 > 0$) then (89) implies that oscillations of the scalar field $\delta\alpha$ are only stable if the combination of $\{a_1, b_1, \bar{\phi}\}$ in parenthesis is positive:

$$\left(40a_1b_1\bar{\phi}^2 - 100a_1^2 - 3b_1^2\bar{\phi}^4\right) > 0 \tag{90}$$

We can immediately see that if $a_1 = 0$ then the degree of freedom $\delta\alpha$ is not stable: there would exist exponentially growing solutions for the field $\delta\alpha$ which implies that the background solution would not be stable. Therefore the presence of $a_1$ and $b_1$ terms together can ensure positivity of the effective mass-squared of $\delta\alpha$, yielding small oscillations of $\delta\alpha$ as solutions to its perturbative equation of motion.

We now look at the remaining part of the perturbed Lagrangian: $\delta L_1$. Making use of the background relation $\lambda = 4u^2/\bar{\phi}^2$, this part of the Lagrangian simplifies to:

$$\begin{aligned}\delta L_1(\delta H, \delta h, \delta c) &= 4\left(*\delta H - \xi_2 \delta N\right) d\delta c + 6\xi_2 \delta N \delta N + 2\xi_2 d\delta c d\delta c + \frac{5}{4}\xi_4 \delta H * \delta H - \xi_4 \nu \delta N * \delta H \\ &\quad + \xi_3\left(-2d\delta c + 2\delta N + \frac{10}{\nu}\delta H\right)^2 \end{aligned} \tag{91}$$

where

$$\begin{aligned}\delta H &\equiv \frac{1}{2}\delta H_{MN}\bar{\theta}^M\bar{\theta}^N, \quad \delta N = \frac{u^2}{\mu}\bar{\theta}_I \delta h^I \\ \xi_2 &\equiv \bar{\phi}^2 a_2, \quad \xi_3 \equiv \bar{\phi}^2 a_3, \quad \xi_4 \equiv \frac{32}{\mu^4 \bar{\phi}}(5b_1 - 2c_1\bar{\phi}^2), \quad \nu \equiv \frac{\mu^2\bar{\phi}^2}{u^2}\end{aligned}$$

Furthermore we have defined the Hodge star/Hodge dual operator on a two-form $T$ as:

$$*T \equiv \frac{1}{4}\epsilon_{IJKL}T^{IJ}\bar{\theta}^K\bar{\theta}^L \tag{92}$$

To clean up notation we have defined the two-forms $\delta H$ (which comes from $\delta W^{AB}$) and $\delta N$ (which comes from $\delta A^{AB}$). These fields appear algebraically in the action and we may actually solve for each of them in terms of $d\delta c$ and $*d\delta c$. We first note a perhaps surprising structural feature of $\delta L_1$. If the perturbation to $\delta W^{AB}$ is switched-off then we recover:

$$\delta L_1(\delta H = 0, \delta h, \delta c) = -4\xi_2 \delta N d\delta c + 6\xi_2 \delta N \delta N + 2\xi_2 d\delta c d\delta c + \xi_3\left(-2d\delta c + 2\delta N\right)^2 \tag{93}$$

The resulting equation of motion for $\delta N$ reveals that $\delta N$ is simply proportional to $\delta c$. Using this solution back in $\delta L_1$ we recover:

$$\delta L_1(\delta H = 0, \delta c) \propto d\delta c d\delta c \tag{94}$$

This is a boundary term (the equation of motion obtained by varying $\delta c$ is the identity $dd\delta c = 0$) and so we see that in the absence of the perturbation to the Higgs field $W^{AB}$, there is no corresponding second-order dynamics for $\delta c$! Allowing for a non-zero $\delta H$, the end result of eliminating $\delta N$ and $\delta H$ is rather complicated, but is of the general form:

$$\delta L_1 = -f_1(\xi_i, \nu, \bar{\phi}) d\delta c * d\delta c + f_2(\xi_i, \nu, \bar{\phi}) d\delta c d\delta c \tag{95}$$

We see the first term is a Maxwell-type kinetic term for the field $\delta c$. The 'right sign' of such a term is $-d\delta c * d\delta c$.

$$\delta L_1 = -\frac{20}{\xi_4}\left(\frac{(-12\xi_2 + \xi_2\xi_4\nu)^2}{(30\xi_2)^2 + (\xi_4\nu^2)^2}\right) d\delta c * d\delta c + f_2(\xi_i, \nu, \bar{\phi}) d\delta c d\delta c \tag{96}$$

and in the limit $\xi_2 \to 0$:

$$\delta L_1 = -\frac{20^2}{\xi_4}\left(\frac{(-4\xi_3+\xi_4\xi_3\nu)^2}{(100\xi_3)^2+(\xi_4\nu^2)^2}\right)d\delta c * d\delta c + f_2(\xi_i,\nu,\bar{\phi})d\delta c d\delta c \tag{97}$$

In both limits, the sign of the term in front of $d\delta c * d\delta c$ is given by the sign of $\xi_4$, and so we require in these limits that:

$$\frac{(5b_1 - 2c_1\bar{\phi}^2)}{\bar{\phi}} > 0 \tag{98}$$

In conclusion then, the spectrum of perturbations around the maximally symmetric background solutions is that of General Relativity (via the field $\delta\theta^I$), a massive scalar field (via $\delta\alpha$), and a massless one-form field (via $\delta c$). The latter two fields may be stable in the sense of having right-sign mass-squared term and right-sign kinetic terms for a subregion of the $\{a_1, b_1, c_1\}$ parameter space. Collectively then the constraints on the parameter space are:

$$u^2\bar{\phi}\left(b_1 - 2\frac{a_1}{\bar{\phi}^2}\right) > 0 \tag{99}$$

$$\bar{\phi}\left(5b_1 - 2c_1\bar{\phi}^2\right) > 0 \tag{100}$$

$$40a_1b_1\bar{\phi}^2 - 100a_1^2 - 3b_1^2\bar{\phi}^2 > 0 \tag{101}$$

Recalling that $\bar{\phi} = \bar{\phi}(a_1, b_1, c_1)$, we may use the background field equations to express $c_1 = c_1(a_1, b_1, \bar{\phi})$ and express the above conditions as:

$$u^2\bar{\phi}\left(b_1\bar{\phi}^2 - 2a_1\right) > 0 \tag{102}$$

$$\bar{\phi}\left(b_1\bar{\phi}^2 - 10a_1\right) < 0 \tag{103}$$

$$40a_1b_1\bar{\phi}^2 - 100a_1^2 - 3b_1^2\bar{\phi}^2 > 0 \tag{104}$$

From the constraint (104) we have that

$$b_1\bar{\phi}^2 < 10a_1 < 3b_1\bar{\phi}^2 \tag{105}$$

The constraint (103) then implies that $\bar{\phi} > 0$ whilst (102) provides no further constraint on the parameter space.

# 6 Coupling to matter

We now consider the coupling of matter fields when gravity is described by the pair $\{A^{AB}, W^{AB}\}$. We will focus on the limit where the matter fields do not back-react on the gravitational field. This is the analogue of considering the formulation of matter fields for a fixed background frame $e^I$ in General Relativity. We will show that when gravity is regarded as a spontaneously-broken theory of $SO(2,4) \simeq SU(2,2)$ then all known matter fields can be cast in a first-order formalism without introducing auxiliary fields. The second-order formulations of matter gauge fields and matter gauge fields will be shown to be artefacts of some of the first-order degrees of freedom being eliminated from the action principle by the presence of algebraic solutions for them from their own equation of motion. We have already seen evidence of this possibility in the previous section wherein the perturbation $\delta\omega^{IJ}$ to the spin-connection could be solved for in terms of $\partial\delta e$ and $\partial\delta\alpha$. Similarly, the perturbation $\delta W^{IJ}$ was found to be fixed in terms of $\partial\delta c$.

First we will show how familiar kinetic terms are recovered for matter gauge fields, scalar Higgs fields, and spinor fields. Following this we will look at the type of potential terms that may be constructed for Higgs and spinor fields. Throughout the section we will take the limit of the gravitational theory where

$$\begin{aligned}
W^{ab} &= \bar{\phi}\epsilon^{ab} \qquad (\bar{\phi} = cst.) &(106)\\
W^{IJ} &= 0 &(107)\\
DW^{Ia} &= e^I u^a &(108)\\
\omega_\mu{}^{IJ} &= 2e^{\nu[I}\partial_{[\mu}e_{\nu]}^{J]} + e_{\mu K}e^{\nu I}e^{\alpha J}\partial_{[\alpha}e_{\nu]}^K &(109)\\
c_\mu &= -\frac{1}{u^2}u_a\epsilon^a{}_b\partial_\mu u^b &(110)
\end{aligned}$$

This form for the fields $\{A^{AB}, W^{AB}\}$ coincides with that taken for the maximally symmetric solutions of Section 4 but more generally we can assume that- much as for the Conformal Einstein-Cartan theory- this is a good approximation to the form the fields will take in any limit where the dynamics of the gravitational field is approximately that of General-Relativity (as is undoubtedly the case on Earth). Therefore we will refer to this specific assumed form of the gravitational fields as the General-Relativistic limit of the full theory.

## 6.1 Adjoint Higgs, non-gravitational gauge fields

The result that part of a Higgs field (namely $\delta W^{IJ}$) enabled the recovery of a familiar second-order kinetic term for $\delta c$ is rather surprising and we will now show the same behaviour is repeated in the matter sector: it will be shown that the presence of a scalar field in the adjoint representation of a group $\mathcal{G}$ is necessary to yield familiar second-order dynamics of the gauge fields of that group.

Consider the $SO(2,4)$ model of gravity coupled to matter in the adjoint representation of some Lie group $\mathcal{G}$. We will take the field to be in the adjoint representation of $\mathcal{G}$ *and* of the gravitational gauge group $SO(2,4)$ i.e. we consider a field with index structure

$$\Phi^A{}_B{}^i{}_j$$

where $i, j, k, \ldots$ are $\mathcal{G}$ indices in the fundamental representation. This is an immediate departure from models such as the Einstein-Cartan theory wherein spacetime scalar fields have no 'gravitational' indices. Henceforth for notational compactness we will suppress $\mathcal{G}$ indices and in the General-Relativistic limit we may express $\Phi^{AB}$ as follows:

$$\Phi^{AB} = \begin{pmatrix} \Phi^{IJ} & \frac{1}{u^2}Y^I u^a + \frac{1}{v^2}Z^I v^a \\ -\Phi^{Ia} & \Phi\epsilon^{ab} \end{pmatrix} \tag{111}$$

Each of $\{\Phi, Y^I, Z^I, \Phi^{IJ}\}$ then transform in the adjoint representation of $\mathcal{G}$ but, respectively, as Lorentz scalar, Lorentz vectors, and anti-symmetric Lorentz tensor. Consider then the following locally $SO(2,4) \times \mathcal{G}$ invariant action, at most quadratic in $\Phi$ and $\mathcal{F} \equiv d\mathcal{A} + \mathcal{A}\mathcal{A}$, where $\mathcal{A}_\mu$ is the $\mathcal{G}$ group gauge field:

$$S = -\int \epsilon_{ABCDEF} DW^A{}_G DW^{GB} \mathrm{Tr}\left(W^{CD}\Phi^{EF}\mathcal{F} + DW^{CD}\Phi^E{}_G D\Phi^{GF}\right) \tag{112}$$

where the trace denotes contraction of internal adjoint indices using the $\mathcal{G}$ Killing metric and the $D$ operators are the full $SO(2,4) \times \mathcal{G}$ covariant derivatives. In the General-Relativistic limit, it will be useful to decompose the effect of $D$ on $\Phi^{AB}$ into the effect of the $SO(1,3) \times SO(1,1) \times \mathcal{G}$ covariant derivative $\mathcal{D}$ on $\Phi^{AB}$ plus any additional terms that may exist. It may be calculated that:

$$D\Phi^{AB} = \begin{pmatrix} \mathcal{D}\Phi^{IJ} + \frac{2}{\phi}e^{[I}Z^{J]} & \mathcal{D}\Phi^{Ia} + \frac{1}{\phi}e^I u^a \Phi + \frac{1}{\phi}v^a g_J \Phi^{IJ} \\ -D\Phi^{Ia} & \mathcal{D}\Phi^{ab} + \frac{2}{\phi u^2}v^{[a}u^{b]}g_J Y^J \end{pmatrix} \tag{113}$$

In this limit, the above action becomes (making the traces over $\mathcal{G}$ indices implicit for a bit to clean up notation):

$$S = \int ( u^2 Y^M \mathcal{D}_M \Phi + \frac{u^2}{2\bar{\phi}} Y^M Y_M - 2\frac{1}{\bar{\phi}}\Phi\Phi$$
$$+ u^2 \Phi^{MN}\mathcal{D}_M Z_N - \frac{3}{2}\frac{u^2}{\bar{\phi}} Z^M Z_M + \frac{1}{4\bar{\phi}}\Phi^{MN}\Phi_{MN} + \frac{\bar{\phi}u^2}{6}\Phi^{MN}\mathcal{F}_{MN} )\, \epsilon_{IJKL} e^I e^J e^K e^L$$

where we have used results from Appendix A to identify some terms as equal to one another. Varying with respect to $Y^I$ and $\Phi^{IJ}$ we obtain the equations of motion:

$$0 = \mathcal{D}_I \Phi + \frac{1}{\bar{\phi}} Y_I$$
$$0 = u^2 \mathcal{D}_{[I} Z_{J]} + \frac{1}{2\bar{\phi}}\Phi_{IJ} + \frac{\bar{\phi}u^2}{6}\mathcal{F}_{IJ}$$

This enables us to solve for $Y_I$ and $\Phi_{IJ}$ in terms of other fields; we may then use these expressions in the action, eliminating these fields from the variational principle:

$$S = \int ( -\frac{u^2\bar{\phi}}{2}\mathcal{D}^I\Phi\mathcal{D}_I\Phi - \frac{2}{\bar{\phi}}\Phi\Phi$$
$$-\frac{\bar{\phi}^3}{36}\left(\mathcal{F}^{IJ} + \frac{6}{\bar{\phi}}\mathcal{D}^{[I}Z^{J]}\right)\left(\mathcal{F}_{IJ} + \frac{6}{\bar{\phi}}\mathcal{D}_{[I}Z_{J]}\right) - \frac{3}{2}\frac{u^2}{\bar{\phi}} Z^I Z_I )\, \epsilon_{IJKL} e^I e^J e^K e^L$$

We may now redefine the $\mathcal{G}$ gauge field $\mathcal{B}_\mu$, creating a new field:

$$\tilde{\mathcal{B}}_\mu \equiv \mathcal{B}_\mu + \frac{12}{\bar{\phi}} Z_\mu \tag{114}$$

where $Z_\mu \equiv e^I_\mu Z_I$. Hence:

$$\mathcal{F}_{\mu\nu} + \frac{6}{\bar{\phi}}\mathcal{D}_{[\mu}Z_{\nu]} = \tilde{\mathcal{F}}_{\mu\nu} - \frac{288}{\bar{\phi}^2} Z_{[\mu}Z_{\nu]} \tag{115}$$

Then, varying with respect to $Z_I$ we obtain an equation polynomial in $Z_I$. A solution to this equation is $Z_I = 0$; adopting this solution and inserting this back into the action yields:

$$S = \int \mathrm{Tr}\left( -\frac{\bar{\phi}^3}{36}\tilde{\mathcal{F}}^{MN}\tilde{\mathcal{F}}_{MN} - \frac{u^2\bar{\phi}}{2}\mathcal{D}^M\Phi\mathcal{D}_M\Phi - \frac{2}{\bar{\phi}}\Phi\Phi\right) \epsilon_{IJKL} e^I e^J e^K e^L \tag{116}$$

We can easily cast this into the metric formalism, defining $g_{\mu\nu} \equiv \eta_{IJ} e^I_\mu e^J_\nu$ we may write (116) as:

$$S = -24 \int \mathrm{Tr}\left( \frac{\bar{\phi}^3}{36}\tilde{\mathcal{F}}^{\mu\nu}\tilde{\mathcal{F}}_{\mu\nu} + \frac{u^2\bar{\phi}}{2}\mathcal{D}^\mu\Phi\mathcal{D}_\mu\Phi + \frac{2}{\bar{\phi}}\Phi\Phi\right)\sqrt{-g}d^4x \tag{117}$$

Therefore the second-order action for a gauge field $\tilde{B}$ and massive adjoint Higgs field $\Phi$ is recovered from the first-order, polynomial action (112) built from the pair $\{\mathcal{B}, \Phi^{AB}\}$. The mass term for $\Phi$ arises from the coupling of $\Phi^{AB}$ to the full $SU(2,2)\times\mathcal{G}$-covariant derivative $D$. This is rather like the origin of mass terms for gauge bosons in gauge theory coming via covariant derivatives of Higgs fields, but here because the 'gauge boson' is the spacetime frame $e^I$ itself, the more direct interpretation is that such a term provides a mass for the Higgs field itself. Interestingly, the kinetic terms for $\tilde{\mathcal{B}}$ and $\Phi$ are of the correct relative sign only if $u^2 = 1$.

### 6.2 Fundamental representation Higgs

We now describe the coupling of gravity to a Higgs field valued in the fundamental representation of $\mathcal{G}$. We will look to construct a first-order formalism for a field $\varphi^A$ which is in the fundamental representation of $\mathcal{G}$ *and* $SO(2,4)$ i.e.

$$\varphi^{Ai}$$

where we use $i,j,k,\ldots$ for indices in the fundamental representation of $\mathcal{G}$. For concreteness we will look at the case where $\mathcal{G}=SU(N)$. This enables us to construct the field $\phi_i^{\dagger A}\equiv(\phi^{Ai'})^*\delta_{i'i}$ where $\delta_{i'i}$ is the $SU(N)$ invariant matrix. Generalizations to other groups such as $SO(N)$ is straightforward, requiring there instead use of the $SO(N)$ invariant matrix in the kinetic term. Consider the following action:

$$S=\int \epsilon_{ABCDEF}DW^A{}_G DW^{GB}DW^{CD}\varphi^{\dagger E}D\varphi^F \tag{118}$$

Again it is useful to decompose the full $SU(2,2)\times\mathcal{G}$-covariant derivative into the $SO(1,3)\times SO(1,1)\times\mathcal{G}$ covariant derivative $\mathcal{D}$ and additional terms:

$$D\varphi^I = \mathcal{D}\varphi^I-\frac{1}{\bar{\phi}}e^I v_a\varphi^a \tag{119}$$

$$D\varphi^a = \mathcal{D}\varphi^a+\frac{1}{\bar{\phi}}v^a g_I\varphi^I \tag{120}$$

As before, we restrict ourselves to the General-Relativistic limit, and decompose $\varphi^A$ as follows:

$$\varphi^A=\begin{pmatrix}\varphi^I\\ \frac{1}{u^2}\varpi u^a+\frac{1}{v^2}\varphi v^a\end{pmatrix}$$

and the action may be shown to take the form:

$$\begin{aligned}S &= \int -2u^2\epsilon_{IJKL}e^Ie^Je^K(\varphi^L\mathcal{D}\varphi^\dagger+\varphi^{\dagger L}\mathcal{D}\varphi)+\frac{2u^4}{\bar{\phi}}\epsilon_{IJKL}e^Ie^Je^K\varphi^{\dagger L}e_M\varphi^M\\ &\quad -2u^2\frac{\varphi^\dagger\varphi}{\bar{\phi}}\epsilon_{IJKL}e^Ie^Je^Ke^L\\ &= \int -4u^2\epsilon_{IJKL}e^Ie^Je^K\varphi^{\dagger L}e_M(\mathcal{D}^M\varphi-\frac{u^2}{2\bar{\phi}}\varphi^M)-2u^2\frac{\varphi^\dagger\varphi}{\bar{\phi}}\epsilon_{IJKL}e^Ie^Je^Ke^L\end{aligned}$$

Note that in the General-Relativistic limit, the field $\varpi$ disappears from the action. Varying with respect to $\Phi^{\dagger L}$ we recover:

$$0 = \epsilon_{IJKL}e^Ie^Je^Ke_M(\mathcal{D}^M\varphi-\frac{u^2}{2\bar{\phi}}\varphi^M)-\frac{u^2}{2\bar{\phi}}\epsilon_{IJKM}e^Ie^Je^K\varphi^M\mathcal{E}_L \tag{121}$$

We can solve this equation to yield: $\varphi^M=\frac{\bar{\phi}}{u^2}\mathcal{D}^M\varphi$. Inserting this back into the action yields:

$$S = \int -\frac{\bar{\phi}}{2}\left(\mathcal{D}^M\varphi^\dagger\mathcal{D}_M\varphi\right)\epsilon_{IJKL}e^Ie^Je^Ke^L-2u^2\frac{\varphi^\dagger\varphi}{\bar{\phi}}\epsilon_{IJKL}e^Ie^Je^Ke^L \tag{122}$$

$$= -24\int\left(\frac{\bar{\phi}}{2}\left(\mathcal{D}^\mu\varphi^\dagger\mathcal{D}_\mu\varphi\right)+2u^2\frac{\varphi^\dagger\varphi}{\bar{\phi}}\right)\sqrt{-g}d^4x \tag{123}$$

Therefore in the General-Relativistic limit we recover the action for a massive scalar field. The mass-squared of the scalar field is real if $u^2=1$. Though we have focused on the case where the scalar field is in the fundamental representation of $\mathcal{G}$ (and of $SO(2,4)$), we do not see an obstruction to recovering second-order dynamics from similar actions to (118) for Higgs fields that may exist in other representations of $\mathcal{G}$ as long as $\mathcal{G}$ possesses structure so that $SU(2,2)\times\mathcal{G}$ invariant actions can be constructed.

## 6.3 Spinor fields

So far we have dealt exclusively with real representations of the group $SO(2,4)$. To incorporate spinorial matter into the theory it is necessary to make use of the complex representations of $SU(2,2)$. To recap, the group $SU(2,2)$ has a matrix representation as the set of $4\times 4$ matrices $U^{\alpha}{}_{\beta}$ with unit determinant that preserve the Hermitian matrix

$$h_{\alpha'\alpha} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{124}$$

where $I$ is the $2\times 2$ unit matrix. We can then define a four-dimensional complex vector space $\mathbb{C}^{(2,2)}$ (we use the notation $\chi^{\alpha}$ to denote a vector in this space) and primed indices denote vectors belonging to the conjugate space $\mathbb{C}^{*(2,2)}$ e.g. if a vector $\chi^{\alpha}$ transforms as $U^{\alpha}{}_{\beta}\chi^{\beta}$ then a vector $\beta^{\alpha'}$ transforms as $(U^{*})^{\alpha'}{}_{\beta'}\beta^{\beta'}$. The space $\mathbb{C}^{(2,2)}$ possesses the symmetric inner product $(,)$:

$$(\beta,\chi) \equiv \frac{1}{2}h_{\alpha'\alpha}\left((\beta^{*})^{\alpha'}\chi^{\alpha} + (\chi^{*})^{\alpha'}\beta^{\alpha}\right) \tag{125}$$

We may additionally consider spaces of 'forms'. For example consider $\bigwedge^2 \mathbb{C}^{(2,2)}$, the space of antisymmetric matrices $\xi^{\alpha\beta} = -\xi^{\beta\alpha}$. This is a six-dimensional complex vector space equipped with symmetric inner product $\langle,\rangle$:

$$\langle\mu,\xi\rangle \equiv \frac{1}{2}h_{\alpha'\alpha}h_{\beta'\beta}\left((\mu^{*})^{\beta'\alpha'}\xi^{\alpha\beta} + (\xi^{*})^{\beta'\alpha'}\mu^{\alpha\beta}\right) \tag{126}$$

By definition the completely antisymmetric symbol $\epsilon_{\alpha\beta\gamma\delta}$ is invariant under $SU(2,2)$ transformations. The presence of this invariant symbol allows us to decompose elements according how they transform under operation by the antisymmetric symbol: a 'real' matrix $\tilde{\xi}^{\alpha\beta}$ is one that satisfies:

$$\tilde{\xi}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}(\tilde{\xi}^{*})^{\gamma\delta} \tag{127}$$

where indices on $(\xi^{*})^{\alpha'\beta'}$ have been raised with $h^{\alpha\alpha'}$. The space of matrices satisfying (127) thus has six real dimensions and we may express a given $\xi^{\alpha\beta}$ in terms of a basis $\sigma_{A}{}^{\alpha\beta}$ :

$$\tilde{\xi}^{\alpha\beta} = \tilde{\xi}^{A}\sigma_{A}{}^{\alpha\beta} \tag{128}$$

where the coefficients $\tilde{\xi}_{A}$ are real numbers and:

$$\sigma_{A\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\sigma_{A}^{*\gamma\delta} \tag{129}$$

An explicit set of $\sigma_{A\alpha\beta}$ are:

$$\sigma_{(-1)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \sigma_{(0)} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$
$$\sigma_{(1)} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \sigma_{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$
$$\sigma_{(3)} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_{(4)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{130}$$

It can be checked then that

$$\frac{1}{2}\left(\sigma_A^{*\alpha\gamma}\sigma_{B\gamma\beta} + \sigma_B^{*\alpha\gamma}\sigma_{A\gamma\beta}\right) = \eta_{AB}\delta^{\alpha}{}_{\beta} \tag{131}$$

where $\eta_{AB} = \text{diag}(-1,-1,1,1,1,1)$ is the invariant matrix of $SO(2,4)$. Indeed, we have that:

$$\left\langle \tilde{\mu}, \tilde{\xi} \right\rangle = \frac{1}{2}\tilde{\mu}^A\tilde{\xi}^B h_{\alpha'\alpha} h_{\beta'\beta}\left(\sigma_A^{*\beta'\alpha'}\sigma_B^{\alpha\beta} + \sigma_B^{*\beta'\alpha'}\sigma_A^{\alpha\beta}\right) = 4\eta_{AB}\tilde{\mu}^A\tilde{\xi}^B \tag{132}$$

We can write an $SU(2,2)$ group element as $U^{\alpha}{}_{\beta}$ as:

$$U^{\alpha}{}_{\beta} = \left(e^{i\theta_j T^j}\right)^{\alpha}{}_{\beta} \tag{133}$$

The unitarity of $U$ implies that the generators $(T^j)^{\alpha}{}_{\beta}$ must then satisfy $(T^j)_{\alpha'\alpha} = (T^{j\dagger})_{\alpha'\alpha}$ i.e. they are Hermitian when indices have been lowered with $h_{\alpha'\alpha}$. Consider the set of fifteen matrices:

$$(j_{AB})^{\alpha}{}_{\beta} = \frac{i}{4}\left(\sigma_A^{*\alpha\gamma}\sigma_{B\gamma\beta} - \sigma_B^{*\alpha\gamma}\sigma_{A\gamma\beta}\right) \tag{134}$$
$$= \frac{i}{2}\left(\sigma_A^{*\alpha\gamma}\sigma_{B\gamma\beta} - \eta_{AB}\delta^{\alpha}{}_{\beta}\right) \tag{135}$$

where we have used the result (131). It can be checked that the matrices $j_{AB}$ satisfy the following Lie algebra:

$$[j^{AB}, j^{CD}] = i\left(\eta^{BC}j^{AD} - \eta^{AC}j^{BD} - \eta^{BD}j^{AC} + \eta^{AD}j^{BC}\right) \tag{136}$$

This indeed is the Lie algebra of $SU(2,2)$ and $SO(2,4)$. To put things on a more familiar footing, the generators $j_{AB}$ are explicitly given by:

$$j^{-1,I} = \frac{1}{2}\gamma^I, \quad j^{4,I} = \frac{1}{2}\gamma_5\gamma^I \tag{137}$$
$$j^{IJ} = \frac{1}{4i}[\gamma^I, \gamma^J], \quad j^{-1,4} = i\gamma^5 \tag{138}$$

where

$$\gamma^I = \begin{pmatrix} 0 & \Sigma^I \\ \bar{\Sigma}^I & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{139}$$

where $\Sigma^I = (1, \Sigma^i)$, $\bar{\Sigma}^I = (1, -\Sigma^i)$, where $\Sigma^i$ are the Pauli matrices. We see then that $j^{IJ}$ generate Lorentz transformations and under these transformations an $SU(2,2)$ spinor $\chi^\alpha$ transforms like a four-component representation of $Spin(1,3)$ i.e. a *Dirac* spinor. We can additionally relate an $SU(2,2)$ $\mathcal{A}^\alpha{}_\beta$ connection to the $SO(2,4)$ connection $A^A{}_B$ used so far:

$$\mathcal{A}^\alpha{}_\beta = \frac{1}{2} A_{AB} j^{AB\alpha}{}_\beta \tag{140}$$

The $SU(2,2)$-covariant derivative of $\chi^\alpha$ is then as follows:

$$\begin{aligned} D\chi^\alpha &= d\chi^\alpha - i\mathcal{A}^\alpha{}_\beta \chi^\beta && (141) \\ &= d\chi^\alpha - \frac{i}{2} A_{AB} j^{AB\alpha}{}_\beta \chi^\beta \chi^\beta && (142) \end{aligned}$$

If $\chi$ additionally belongs to a representation of a group $\mathcal{G}$ then we can decompose the derivative $D$ into the $SO(1,3) \times SO(1,1) \times \mathcal{G}$ covariant derivative $\mathcal{G}$ and additional pieces:

$$\begin{aligned} D\chi^\alpha &= \mathcal{D}\chi^\alpha - \frac{i}{2} A_{-1J} j^{-1J\alpha}{}_\beta \chi^\beta - \frac{i}{2} A_{4J} j^{4J\alpha}{}_\beta \chi^\beta && (143) \\ &= \mathcal{D}\chi^\alpha - \frac{i}{2} A_{-1J} \gamma^{J\alpha}{}_\beta \chi^\beta - \frac{i}{2} A_{4J} (\gamma^5 \gamma^J)^\alpha{}_\beta \chi^\beta && (144) \end{aligned}$$

Recall that $A^{Ia} = -\frac{1}{\phi} g^I v^a$. If $u^2 = 1$ then we may choose a gauge where $u^a = \delta^a_4$, hence $v_a = -\delta^a_{-1}$ and $A^{I,-1} = \frac{1}{\phi} g^I$, $A^{I4} = 0$ hence:

$$D\chi^\alpha = \mathcal{D}\chi^\alpha - \frac{i}{2\phi} g_J \gamma^{J\alpha}{}_\beta \chi^\beta \tag{145}$$

We can now write down an action for a spinor field coupled to gravity:

$$S = i \int \epsilon_{ABCDEF} DW^A{}_G DW^{GB} DW^{CD} \bar{\chi} j^{EF} D\chi \tag{146}$$

where $\bar{\chi}_\alpha \equiv (\chi^*)^{\alpha'} h_{\alpha'\alpha}$ and we implicitly use any invariant group structure from $\mathcal{G}$ necessary for (146) to be $\mathcal{G}$ invariant. In the General-Relativistic limit this action takes the form:

$$S = -i \int 4\epsilon_{IJKL} e^I e^J e^K \bar{\chi} j^{La} v_a D\chi \tag{147}$$

($v_a = \delta^{-1}_a$) and so $j^{La} v_a = j^{L,-1} = -\frac{1}{2}\gamma^L$, hence:

$$\begin{aligned} S &= i \int 2\epsilon_{IJKL} e^I e^J e^K \bar{\chi} \gamma^L D\chi \\ &= i \int 2\epsilon_{IJKL} e^I e^J e^K \bar{\chi} \gamma^L (\mathcal{D}\chi - \frac{i}{2\phi} e_M \gamma^M \chi) \\ &= i \int 2\epsilon_{IJKL} e^I e^J e^K (\bar{\chi} \gamma^L \mathcal{D}\chi + \frac{i}{2\phi} e^L \bar{\chi}\chi) \\ &= \int 2i\epsilon_{IJKL} e^I e^J e^K \bar{\chi} \gamma^L \mathcal{D}\chi - \frac{1}{\phi} \bar{\chi}\chi \epsilon_{IJKL} e^I e^J e^K e^L && (148) \end{aligned}$$

Note that in this $SO(1,1)$ gauge we can see from equation (110) that $c \stackrel{*}{=} 0$. Thus we see that in the General-Relativistic limit, the action (146) reduces to the action for a massive Dirac spinor covariant under local $Spin(1,3) \times \mathcal{G}$ transformations.

## 6.4 Chirality and mass

We have seen then that it is possible to recover second-order dynamics for gauge and Higgs fields from a first-order perspective. Additionally, we see that mass terms for Higgs fields $\Phi$, $\varphi$, and spinor fields $\chi$ appear quite naturally in the context of first-order $SU(2,2)\times\mathcal{G}$-invariant actions via the $SU(2,2)$ covariant derivative.

As spinorial representations of $SU(2,2)$, fermionic fields are necessarily four-component Dirac spinor fields. Compare this to the standard model of particle physics wherein fermions are two-component Weyl spinors. Though prior to the standard model it was thought that the left-handed electron $e_L$ and right-handed electron $e_R$ were indeed two parts of a single Dirac spinor $\Psi=(e_L,e_R)$, it is now known that this is not the correct structure. Rather, $e_R$ is a Weyl spinor and an $SU(2)$ *singlet* whilst $e_L$ is part of an $SU(2)$ *doublet* Weyl spinor (the electron-neutrino), and so $e_L$ couples to $SU(2)$ gauge fields directly via the covariant derivative, whereas $e_R$ does not. A parity transformation is taken to interchange $e_R\leftrightarrow e_L$. Clearly the standard model Lagrangian cannot be invariant under this transformation. In this sense the standard model is referred to as being a chiral/parity-violating theory.

Focusing on the $SU(2)$ singlet $e_R$, from the $SU(2,2)$ perspective there are no two-dimensional spinorial representations this field must be part of a Dirac spinor $\chi=(\mathcal{E}_L,e_R)$ where $\mathcal{E}_L$ has the same hypercharge as $e_R$ but transforms as a left-handed Weyl spinor under Lorentz transformations (these defined post-symmetry breaking by $W^{AB}$). The field $\mathcal{E}_L$ cannot be identified with the left-handed electron because it possesses the wrong $SU(2)$ index structure. Indeed, the left-handed electron-neutrino doublet will be part of another $SU(2,2)$ spinor additionally containing a right-handed $SU(2)$ doublet with the same $SU(2)\times U(1)$ hypercharges but transforming as a right-handed Weyl spinor. Where is these new fields in nature?

If the action for the dynamics of $\chi=\{\mathcal{E}_L,e_R\}$ were described by (148) then we have a symmetry under interchange of $\mathcal{E}_L\leftrightarrow e_R$; thus $\mathcal{E}_L$ would have the same mass as $e_R$. The field $\mathcal{E}_L$ has not been observed in nature and so if the $SU(2,2)$ approach is a viable model of gravitation there must be an explanation for this. If it is the case that $\mathcal{E}_L$ is simply too massive to have been detected yet, then there must exist additional terms in the spinor action that break the $\mathcal{E}_L\leftrightarrow e_R$ symmetry. Consider the following action involving $\chi=\{\mathcal{E}_L,e_R\}$:

$$\begin{aligned} S &= \int \epsilon_{ABCDEF}\bar{\chi}j^{AB}\chi DW^{C}{}_{G}DW^{GD}DW^{E}{}_{H}DW^{HF} \\ &\to -2i\int \bar{\chi}\gamma_5\chi\epsilon_{IJKL}e^Ie^Je^Ke^L \end{aligned} \tag{149}$$

Under $\mathcal{E}_L\leftrightarrow e_R$, the action (149) becomes the minus of its original value whereas the action (148) is unchanged. If $\chi$ were described by the combined actions (148) and (149) then its action will not transform homogeneously under $\mathcal{E}_L\leftrightarrow e_R$ and generally $\mathcal{E}_L$ and $e_R$ will have different masses. Of course this some way from showing that it is to be expected that 'mirror' fields like $\mathcal{E}_L$ to be unobserved whilst retaining the physics of the standard model at lower energies, but clearly there is structure in the gravitational sector ($W\sim\gamma_5$) that will typically prevent the familiar field $e_R$ and unfamliar field $\mathcal{E}_L$ having the same mass.

Another source of fermion mass should be via Yukawa-type interactions with a Higgs field such as $\varphi^A$. Consider a gauge group $\mathcal{G}$, assumed to have a matrix representation. Using $i,j,k,\ldots$ for indices in the fundamental representation of $\mathcal{G}$ (e.g. the fundamental representation Higgs of $SU(5)$ would have the $SU(2,2)\times\mathcal{G}$ index structure $\varphi^{Ai}$) then we may write down an action coupling two separate $SU(2,2)$ spinors $P^{i\alpha}$ and $E^{\alpha}$. Consider the following action:

$$\begin{aligned} S &= \int \epsilon_{ABCDEF}\varphi_i^{*E}P^{i\alpha}\sigma^F_{\alpha\beta}E^\beta DW^A{}_G DW^{GB}DW^C{}_H DW^{HD} \\ &\rightarrow \int \epsilon_{ab}v^a\varphi_i^* P^{i\alpha}\sigma^b_{\alpha\beta}E^\beta\epsilon_{IJKL}e^Ie^Je^Ke^L \\ &\stackrel{*}{=} -\int \varphi_i^* P^{i\alpha}C_{\alpha\beta}E^\beta\epsilon_{IJKL}e^Ie^Je^Ke^L \\ &\equiv -\int \varphi_i^* L^{\dagger i\beta'}h_{\beta'\beta}E^\beta\epsilon_{IJKL}e^Ie^Je^Ke^L \end{aligned} \tag{150}$$

where we have assumed $u^2 = 1$, $C_{\alpha\beta}$ is the $Spin(1,3)$-invariant charge conjugation matrix recovered from projecting $\sigma^a_{\alpha\beta}$ along $u_a$, and we have defined the spinor $L^{i\beta}$ via

$$L^{i'\beta} = P^{\dagger i'\alpha'}C_{\alpha'\beta'}h^{\beta'\beta} \tag{151}$$

For example, focusing specifically on the case where $\mathcal{G}$ is the electroweak group $SU(2)\times U(1)$, we see this is a Yukawa-type mass term for a left-handed field $SU(2)$-doublet within $L^{i'\beta}$ coupled to a right-handed $SU(2)$ singlet within $E^\beta$ along with an identical term for a right-handed field $SU(2)$-doublet within $L^{i'\beta}$ coupled to a left-handed $SU(2)$ singlet within $E^\beta$.

The term (150) acts as a mass term only when the field $\varphi$ achieves a non-vanishing expectation value. Suppressing internal indices for compactness again, we now briefly consider how potential terms may be constructed for fields such as $\Phi^{AB}$, $\varphi^A$ such that $\Phi$ and $\varphi$ achieve non-vanishing expectation values, thus spontaneously breaking the symmetry $\mathcal{G}$.

In the General-Relativistic limit, scalars formed with the field $W^{AB}$ reduce to familiar terms e.g.

$$\mathrm{Tr}(W_{AB}W_{CD}\Phi^{AB}\Phi^{CD}) = 4\bar{\phi}^2\mathrm{Tr}(\Phi\Phi) \tag{152}$$
$$W_{AB}W^B{}_C\varphi^{\dagger A}\varphi^C = \bar{\phi}^2\varphi^\dagger\varphi \tag{153}$$

Hence, polynomial functions of these scalars appearing alongside terms which are proportional to the familiar spacetime volume four-form in the General-Relativistic limit (e.g. the $c_1$ term from the gravitational action) will enable the recovery of symmetry breaking potentials.

Therefore we see that symmetry-breaking potentials for Higgs fields and masses for Weyl fermions and their 'mirrors' may be recovered from an $SU(2,2)$ framework in the General-Relativistic limit.

# 7 Relation to other work

We now discuss a number of alternative approaches that have been made to recovering gravitational theory from $SU(2,2) \simeq SO(2,4)$ gauge theories.

The work by Kaku et al. [30] was discussed in some detail in Section 3.2 A somewhat different approach was pursued by Kerrick [38] who introduced fields $Y_{\alpha\beta} = -Y_{\beta\alpha}$ and $Z_{\alpha\beta} = -Z_{\beta\alpha}$ (linear combinations of that author's original fields $I_{\alpha\beta}$ and $\iota_{\alpha\beta}$) alongside the $SU(2,2)$ connection $A^\alpha{}_\beta$. Such fields were assumed to live in the six-dimensional space of matrices satisfying (127) and so are equivalent to introducing two $SO(2,4)$ vectors $\{Y^A, Z^A\}$. The author implicitly constrained $Y^AY_A = \text{cst.} > 0$ and $Z^AZ_A = \text{cst.} < 0$, thus breaking $SU(2,2) \rightarrow SO(1,3)$. An action quartic in these fields and quadratic in the curvature $F^{AB}$ was shown to be equivalent to General Relativity in the presence of a cosmological and Holst term.

Different again is the approach of Aros and Diaz [39] who discuss an $SO(2,4)$ gauge theory on a *five*-dimensional manifold. This is in some respects a higher-dimensional analogue of theories based on $SO(2,3)$ in four dimensions. The authors consider one of the five dimensions to have the topology of a circle and making the following ansatz for the *five*-dimensional $A^{AB} \equiv A_\mu{}^{AB}dx^\mu + A_y{}^{AB}dy$:

$$A^{AB} = \begin{pmatrix} 0 & \frac{1}{2}(e^I + f^I) & \xi d\phi \\ -\frac{1}{2}(e^I + f^I) & \omega^{IJ} & -\frac{1}{2}(e^I - f^I) \\ -\xi d\phi & \frac{1}{2}(e^I - f^I) & 0 \end{pmatrix} \tag{154}$$

where fields are taken to be independent of coordinates along the circular dimension but may depend on the remaining four coordinates. The authors then propose a Chern-Simons-type action (an integral over a five-form) with explicit $SO(2,4)$ symmetry breaking (e.g. the $SO(1,4)|SO(2,3)$ invariant $\epsilon_{ABCDE}$ is used in the action instead of the $SO(2,4)$ invariant $\epsilon_{ABCDEF}$ without the use of a Higgs field to accomplish this covariantly). After dimensional reduction (integration of the action over the circle) it is found that the resulting four-dimensional action is that of Conformal Einstein-Cartan theory coupled to a scalar field $\xi(x)$. This theory can be related fourth-order conformal gravity by adding constraints in the manner of Section 3.2 and freezing out the field $\xi(x)$. In higher dimensions yet, a novel approach to spacetime and gravity based on the group $SO(2,4)$ on eight-dimensional manifolds has been explored by Hazboun and Wheeler [40].

# 8 Discussion and conclusions

We now discuss the results contained in the paper and present our conclusions. In this paper we have presented an approach to gravity as a spontaneously-broken gauge theory based on the group $SU(2,2)$ using the the pair $\{A^{AB}, W^{AB}\}$. We saw that when the degrees of freedom in the field $W^{AB}$ were 'frozen' to a symmetry-breaking solution, that a conformal version of the Einstein-Cartan gravity emerged; furthermore, when the two-frame fields $e^I$ and $f^I$ were aligned or anti-aligned then conformalized General Relativity was recovered i.e. General Relativity conformally coupled to a dilaton field. It was then shown that fourth-order Weyl and conformal gravity could be obtained by further constraining the theory.

Next we focused on obtaining solutions for the theory when no constraints were placed on fields. Solutions corresponding to de Sitter space and anti-de Sitter space were found (corresponding to the cases $u^2 > 0$ and $u^2 < 0$ respectively). Perturbations around these spacetimes were considered and the Lagrangian quadratic in these perturbations was constructed. It was found that the spectrum of perturbations was that of General Relativity, a massive scalar field, and a massless one-form field, each decoupled from one another at this level of perturbations. Thus from a first-order gravitational theory, three types of second-order theory emerged in the perturbations: perturbations identical to the metric perturbations of General Relativity for a field $\delta\theta^I$ with dynamics provided by $\delta\omega^{IJ}$; scalar field perturbations described by a field $\delta\alpha$ with dynamics also provided by $\delta\omega^{IJ}$; Weyl field perturbations $\delta c$ with dynamics provided by $\delta W_{IJ}$.

It is important to stress that these results, although encouraging, are provisional and there is some way to go before it's seen whether a theory of gravity based on $SU(2,2)$ can reproduce the phenomenological success of General Relativity. In particular, we lack a clear understanding of whether there exists a convergence mechanism (beyond linear stability, which was demonstrated in this paper against maximally symmetric backgrounds) within the theory that somehow dynamically favours the General-Relativistic limit. Another central point of uncertainty seems to be that of the dynamics of $W^{AB}$. Though we found sets of parameters $\{a_1, b_1, c_1\}$ such that the 'scalar field' part $\phi$ of $W^{AB}$ was static in our background solutions and has a positive effective mass-squared at the level of perturbations, it is not entirely clear that it is necessary and that $\phi$ must behave this way for agreement with experiment; it is conceivable that the field may more naturally 'roll' down an effective potential as the universe evolves and so possess time variation. Indeed, for gravity based on the groups $SO(1,4)|SO(2,3)$ it was found that Peebles-Ratra rolling quintessence was rather easily recovered [12, 41].

We have additionally shown how it is possible to couple the $SU(2,2)$ gravitational fields to matter in polynomial Lagrangians that yield first-order field equations. As a consequence, time evolution of gravity and matter in the Lagrangian formulation is determined entirely by the values of fields at a moment in time. This brings to mind Zeno's 'Arrow Paradox' wherein it is suggested that the trajectory of a moving arrow can be broken down into a series of moments, and in each moment the arrow is 'static', perpetually contained within that moment. If the arrow is static in each moment, how does it move and how does it know where to move? In second-order Lagrangian theories, there is 'hidden' information encapsulated in each moment: that of the velocity of the arrow. Moving to a Hamiltonian formulation, the arena of reality becomes phase space wherein the information present in the moment includes the arrow's position and momentum. From the perspective of first-order Lagrangian formulations of field theories, all this information is present in the field configuration space itself. For example, for the adjoint Higgs field we found in Section 6.1 that:

$$\Phi^{AB} \quad \simeq \quad \begin{pmatrix} (\mathcal{F})^{IJ} & \mathcal{D}^I\Phi v^a \\ -\mathcal{D}^I\Phi v^a & \Phi\epsilon^{ab} \end{pmatrix} \tag{155}$$

where $\mathcal{D}^I\Phi = (e^{-1})^{\mu I}\mathcal{D}_\mu\Phi$ etc. At any moment in time, the field $\Phi^{AB}$ contains the information about the value of the adjoint Higgs $\Phi$, how the Higgs field is changing (via the time derivative within the spacetime derivative $\mathcal{D}$), and also how the gauge field for the internal group that $\Phi$ is in the adjoint representation of is changing (via the field strength $\mathcal{F}$).

In the table below we summarize the differences between gravity and how it couples to matter between Einstein-Cartan gravity (based on the group $SL(2,C)$) and the present model of gravity (based on the group $SU(2,2)$):

| Model | Gravity | Fund. Higgs | Adjoint Higgs | Gauge Fields | Spinors |
|---|---|---|---|---|---|
| $SL(2,C)$ | $\{e^I_\mu, \omega_\mu{}^I{}_J\}$ | $\varphi^i$ | $\phi^i{}_j$ | $\mathcal{B}_\mu{}^i{}_j$ | $Weyl$ |
| $SU(2,2)$ | $\{Y^A{}_B{}^C{}_D, A_\mu{}^A{}_B\}$ | $\varphi^{Ai}$ | $\{\phi^A{}_B{}^i{}_j, \mathcal{B}_\mu{}^i{}_j\}$ | | $Dirac$ |

where $i,j,k,\ldots$ are indices in the fundamental representation of the matter sector symmetry group $\mathcal{G}$. From this perspective, the ingredients of gauge theories are a *pair* $\{\phi^A{}_B{}^i{}_j, \mathcal{B}_\mu{}^i{}_j\}$ which from a second-order perspective describe Yang-Mills type dynamics of $\mathcal{B}$ alongside a massive adjoint Higgs field $\Phi$. The field $\Phi$ in principle need not obtain a non-vanishing vev and so it needn't break the gauge symmetry $\mathcal{G}$. It is interesting to note that gravity as a gauge theory couples to itself via precisely the same prescription: if we take the group $\mathcal{G}$ to be $SO(2,4)$ and identify $i,j,k,\ldots$ with $A,B,C,\ldots$ we recover the pair $\{\phi^A{}_B{}^C{}_D, \mathcal{B}_\mu{}^A{}_B\}$. Identifying $\phi^A{}_B{}^C{}_D = Y^A{}_B{}^C{}_D \equiv \frac{1}{2}\epsilon^A{}_B{}^C{}_D{}^{EF}W_{EF}$ and $\mathcal{B}_\mu{}^A{}_B = A_\mu{}^A{}_B$ leads to the fields describing gravity.

Clearly a challenge for an $SU(2,2)$ description of gravity is the prediction of additional fermions beyond those in the standard model of particle physics: the fact that all spinor fields are Dirac spinors implies that there exist fermions with the same hypercharges but opposite handedness as the observed standard model fermions- there is currently no evidence for the existence of these fermions. We saw in Section 6.4 that generally the left and right handed parts of an $SU(2,2)$ spinor $\chi^\alpha$ will not have the same mass, but whether there exists a successful mechanism for making the theory compatible with experiment is an open issue[7].

We note that the matter actions that we considered were at most linear in the covariant derivatives $D\phi^{AB}$ and $D\varphi^A$. What would be the effect of considering polynomial actions quadratic or cubic in these derivatives? Why not include such terms? Would alternative second-order scalar field theories such as those contained in the Horndeski formalism [43, 44, 45, 46] be recoverable in some cases? Additionally, if there is a background where the field $\phi$ in $W^{AB}$ possesses a time dependence, the 'metric' $\eta_{AB}D_\mu W^A{}_C D_\nu W^{CB}$ picks up an additional dependence upon $\partial_\mu\phi\partial_\nu\phi$- the new metric being then disformally related to the one in which $\phi$ is static. Disformal couplings of scalar fields to matter have been investigated [47, 48, 49, 50, 51] and it would be interesting to see if links can be made. Indeed, more generally it would be very useful to characterize the deviations from General Relativity that a theory based on the pair $\{A^{AB}, W^{AB}\}$ may result in to move towards comparison with cosmological data. For example, from the perspective of cosmological phenomenology it would be interesting to see to what extent the theory could fit into the formalism described within [52, 53].

[7] See [42] for a discussion about recent proposals on what may be a similar issue in the context of the fate of non-observed additional generations of particles in $SO(18)$ grand unification schemes

We now discuss some more speculative ideas based on the findings in the paper. Recall that the General-Relativistic limit of the Conformal Einstein-Cartan theory corresponded to the case where frames $e^I$ and $f^I$ were aligned or anti-aligned i.e. $f^I = \pm\Omega^2 e^I$, and this determined the sign of the cosmological constant for the resulting theory (in the absence of matter). It is interesting to wonder whether there could exist solutions where $f^I$ varies smoothly (presumably passing through 0 along the way) from being aligned with $e^I$ to anti-aligned with $e^I$; what would be the interpretation of such solutions? This behaviour may also be possible in the full $\{A^{AB}, W^{AB}\}$ theory.

In the unconstrained theory, though we have focused on finding maximally symmetric spacetime solutions (solutions with Lorentzian signature metric), in principle there may be very different phases of the theory depending on the symmetry breaking behaviour of the field $W^{AB}$. For instance, there can also exist forms of $W^{AB}$ that are preserved under $SO(4)\times SO(2)$, yielding a Euclidean theory of gravity with an additional local $SO(2)$ symmetry. It was found in the case of $SO(1,4)|SO(2,3)$ gravity a dynamic symmetry breaking field could transition between Lorentzian and Euclidean phases in simple cosmological models [37]. It would be interesting to see if this is also possible for the $SU(2,2)$ theory. If $W^{AB} = 0$ then the entire $SO(2,4)$ symmetry may be preserved though it is not clear whether an interpretation of a 'spacetime' theory is any longer possible there.

An important issue is whether big bang and and black hole singularities will still be present in this theory of gravity. It is quite possible that the new degrees of freedom beyond those in General Relativity (i.e. the Higgs field $W^{AB}$, and degrees of freedom in $A^{AB}$ outside of the General-Relativistic limit) would "kick in" in extreme situations and significantly modify the evolution. This would require the study of exact solutions to the full theory possibly with matter included.

It would also be interesting to see whether unification of gravity and the matter sector would be possible. For instance, the smallest groups containing $SU(2,2) \simeq SO(2,4)$ and a grand unification group $SO(10)$ as commuting subgroups are $SO(4,12)$ and $SO(2,14)$ [54]. Alternatively, the smallest group containing $SU(2,2)$ and $SU(5)$ as commuting subgroups is $SU(2,7)$. Denoting indices in the adjoint representation of the 'total unification' group as $\mathcal{A},\mathcal{B},\mathcal{C},\ldots$ we can speculate that the gravity/gauge fields part of the theory would be described by a pair $\{C^{\mathcal{A}}, \Phi^{\mathcal{AB}}\}$: a connection $C^{\mathcal{A}} \equiv C_\mu{}^{\mathcal{A}} dx^\mu$ (containing $A_\mu{}^A{}_B$ and $\mathcal{B}_\mu{}^i{}_j$) and a scalar Higgs field $\Phi^{\mathcal{AB}}$ (containing $Y^A{}_B{}^C{}_D$ and $\phi^A{}_B{}^i{}_j$), from which Lagrangians polynomial in $\Phi^{\mathcal{AB}}$, its covariant derivative, and the curvature of $C^{\mathcal{A}}$ can be constructed[8]. This is somewhat different than approaches based on $SO(3,11)$ which contain $SL(2,C) \simeq SO(1,3)$ and $SO(10)$ as commuting subgroups [55, 56, 57, 58]; these approaches typically involve fields aside from the $SO(3,11)$ connection possessing spacetime indices e.g. a co-tetrad in the fundamental representation of $SO(3,11)$ [55] or a field which can dynamically tend to a Hodge dual operator on two-forms [58].

## Acknowledgements

We thank James Bjorken, Johannes Noller, Jeffrey Hazboun, Andrew Randono, Andrew Tolley, Garrett Lisi, and Roberto Percacci for very useful discussions on first-order theories of gravity and unification. HW is indebted to Julian Barbour, Sean Gryb, Tim Koslowski, Flavio Mercati, Sebastiano Sonego, and Rafael Sorkin for many discussions about scale invariance in physics.

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# A Useful identities

$$\begin{aligned}
\epsilon_{IJKL}e^Ie^Je^Ke_N\Phi^{LM}\Phi_M{}^N &= -\frac{1}{8}\epsilon_{IJKL}e^Ie^Je^Ke^L\Phi^{MN}\Phi_{MN} \\
\epsilon_{IJKL}e^Ie^Je^Ke_MY^LY^M &= \frac{1}{4}\epsilon_{IJKL}e^Ie^Je^Ke^LY^MY_M \\
\epsilon_{IJKL}e^Ie^Je^KY^L\mathcal{D}\Phi &= \frac{1}{4}\epsilon_{IJKL}e^Ie^Je^Ke^LY^M\mathcal{D}_M\Phi \\
\epsilon_{IJKL}e^Ie^Je^K\Phi^{LM}\mathcal{D}Z_M &= \frac{1}{4}\epsilon_{IJKL}e^Ie^Je^Ke^L\Phi^{MN}\mathcal{D}_MZ_N
\end{aligned}$$

# B Solving for $f$

We start from the action

$$S[\omega,e,f,c] = \int \alpha\epsilon_{IJKL}e^If^JR^{KL} + \beta\epsilon_{IJKL}e^Ie^Jf^Kf^L + \gamma e^If_Idc \tag{156}$$

where $\{\alpha,\beta,\gamma\}$ are constants. Variation of the action yields the equations of motion

$$\omega: \quad 0 = \epsilon_{IJKL}d_\omega(e^Kf^L) \tag{157}$$
$$e: \quad 0 = \alpha\epsilon_{IJKL}f^JR^{KL} + 2\beta\epsilon_{IJKL}e^Jf^Kf^L + \gamma f_Idc \tag{158}$$
$$f: \quad 0 = \alpha\epsilon_{IJKL}e^JR^{KL} + 2\beta\epsilon_{IJKL}f^Je^Ke^L - \gamma e_Idc \tag{159}$$
$$c: \quad 0 = d(e^If_I) \tag{160}$$

Solving the $f$-equation we obtain

$$f^I = -\frac{3\alpha}{\beta}\left(R^I - \frac{R}{6}e^I\right) - \frac{\gamma}{\beta}e^I \lrcorner * dc. \tag{161}$$

Since $f^I$ appears algebraically in the action it is permissible to substitute this solution back into the action i.e. this operation does not alter the space of solutions. Carrying out this step yields

$$
\begin{aligned}
S = \frac{1}{\beta}\int & -\alpha\epsilon_{IJKL}e^I\left(3\alpha R^J - \frac{\alpha R}{2}e^J + \gamma e^J \lrcorner * dc\right)R^{KL} - \gamma e_I\left(3\alpha R^I - \frac{\alpha R}{2}e^I + \gamma e^I \lrcorner * dc\right)dc \\
& + \epsilon_{IJKL}e^I e^J\left(3\alpha R^K - \frac{\alpha R}{2}e^K + \gamma e^K \lrcorner * dc\right)\left(3\alpha R^L - \frac{\alpha R}{2}e^L + \gamma e^L \lrcorner * dc\right) \\
= \frac{1}{\beta}\int & -\alpha^2\epsilon_{IJKL}e^I\left(3R^J - \frac{R}{2}e^J + \frac{\gamma}{\alpha}e^J \lrcorner * dc\right)R^{KL} - 3\alpha\gamma e_I R^I dc \\
& + 9\alpha^2\epsilon_{IJKL}e^I e^J\left(R^K - \frac{R}{6}e^K\right)\left(R^L - \frac{R}{6}e^L\right) + 6\alpha\gamma\epsilon_{IJKL}e^I e^J\left(R^K - \frac{R}{6}e^K\right)e^L \lrcorner * dc \\
& + \gamma^2\left(\epsilon_{IJKL}e^I e^J e^K \lrcorner * dc e^L \lrcorner * dc - 2dc * dc\right) \qquad (162)
\end{aligned}
$$

where $*$ is the Hodge star operator on differential forms. Using the fact that $e^I \lrcorner * H = \frac{1}{2}\epsilon^I{}_{JKL}e^J H^{KL}$ with $H \equiv dc$ let us now take a closer look at the penultimate term:

$$
\begin{aligned}
& \epsilon_{IJKL}e^I e^J e^K \lrcorner * dc e^L \lrcorner * dc = \frac{1}{4}\epsilon_{IJKL}e^I e^J e^M e^P \epsilon^K{}_{MNO}\epsilon^L{}_{PQR}H^{QR}H^{NO} \\
& \sim \frac{e}{4}\epsilon_{IJKL}\varepsilon^{IJMP}\epsilon_{KMNO}\epsilon_{LPQR}H^{QR}H^{NO} = \frac{e}{2}(\delta^M_K\delta^P_L - \delta^P_K\delta^M_L)\epsilon^K{}_{MNO}\epsilon^L{}_{PQR}H^{QR}H^{NO} \\
& = \frac{e}{2}\delta^M_K\delta^P_L\epsilon^K{}_{MNO}\epsilon^L{}_{PQR}H^{QR}H^{NO} - \frac{e}{2}\delta^P_K\delta^M_L\epsilon^K{}_{MNO}\epsilon^L{}_{PQR}H^{QR}H^{NO} \\
& = \frac{e}{2}\epsilon_{KLNO}\epsilon^{KL}{}_{QR}H^{QR}H^{NO} = 2e\eta_{NR}\eta_{OQ}H^{QR}H^{NO} = -2eH^{IJ}H_{IJ} \qquad (163)
\end{aligned}
$$

Using $dc * dc \sim \frac{e}{2}H^{IJ}H_{IJ}$ we thus have

$$
\epsilon_{IJKL}e^I e^J e^K \lrcorner * dc e^L \lrcorner * dc = -4dc * dc \qquad (164)
$$

and the action becomes

$$
\begin{aligned}
S = \frac{1}{\beta}\int & -\alpha^2\epsilon_{IJKL}e^I\left(3R^J - \frac{R}{2}e^J + \frac{\gamma}{\alpha}e^J \lrcorner * dc\right)R^{KL} - 3\alpha\gamma e_I R^I dc \\
& + 9\alpha^2\epsilon_{IJKL}e^I e^J\left(R^K - \frac{R}{6}e^K\right)\left(R^L - \frac{R}{6}e^L\right) + 6\alpha\gamma\epsilon_{IJKL}e^I e^J\left(R^K - \frac{R}{6}e^K\right)e^L \lrcorner * dc \\
& - 6\gamma^2 dc * dc \qquad (165)
\end{aligned}
$$

Thus, the one-form $c$ looks like a gauge field with a standard Yang-Mills/Maxwell term $dc * dc$ alongside coupling of $dc$ to the curvature $R^{IJ}$. We then have

$$
\begin{aligned}
& \epsilon_{IJKL}e^I \frac{\gamma}{\alpha}e^J \lrcorner * dc R^{KL} = \frac{\gamma}{2\alpha}\epsilon_{IJKL}e^I \epsilon^J{}_{MNO}e^M H^{NO}R^{KL} \\
& = \frac{\gamma}{2\alpha}(\eta_{JO}\eta_{KM}\eta_{LN} + \eta_{JN}\eta_{KO}\eta_{LM} - \eta_{JO}\eta_{LM}\eta_{KN} - \eta_{JN}\eta_{LO}\eta_{KM})e^J e^M H^{NO}R^{KL} \\
& = \frac{\gamma}{\alpha}(\eta_{JO}\eta_{KM}\eta_{LN} + \eta_{JN}\eta_{KO}\eta_{LM})e^J e^M H^{NO}R^{KL} \\
& = \frac{2\gamma}{\alpha}H_{KJ}e^J e_L R^{KL} = \frac{2\gamma}{\alpha}e_K \lrcorner dc DT^K \qquad (166)
\end{aligned}
$$

and also

$$
\begin{aligned}
& 6\alpha\gamma\epsilon_{IJKL}e^I e^J\left(R^K - \frac{R}{6}e^K\right)e^L \lrcorner * dc = -3\alpha\gamma e^I e^J e^M \epsilon_{LIJK}\epsilon^L{}_{MNO}H^{NO}\left(R^K - \frac{R}{6}e^K\right) \\
& = 6\alpha\gamma e^I e^J H_{IJ}e_K R^K = 12\alpha\gamma e_K R^K dc \qquad (167)
\end{aligned}
$$

The action now becomes

$$\begin{aligned} S &= \frac{1}{\beta}\int -3\alpha^2\epsilon_{IJKL}e^I\left(R^J-\frac{R}{6}e^J\right)R^{KL}+9\alpha^2\epsilon_{IJKL}e^Ie^J\left(R^K-\frac{R}{6}e^K\right)\left(R^L-\frac{R}{6}e^L\right) \\ &\quad -2\alpha\gamma e_I\lrcorner dcR^{IJ}e_J+9\alpha\gamma e_KR^Kdc-6\gamma^2dc*dc \\ &= \frac{1}{\beta}\int \frac{\alpha^2}{4}\epsilon_{IJKL}\mathcal{C}^{IJ}\mathcal{C}^{KL}-6\gamma^2dc*dc \\ &\quad -2\alpha\gamma e_I\lrcorner dcR^{IJ}e_J+9\alpha\gamma e_KR^Kdc-\frac{\alpha^2}{4}\epsilon_{IJKL}R^{IJ}R^{KL} \end{aligned} \tag{168}$$

where the Weyl two-form has been defined as:

$$\mathcal{C}^{IJ}\equiv R^{IJ}-6e^I\left(R^J-\frac{R}{6}e^J\right) \tag{169}$$

We observe that the relative sign of 'Weyl-squared' term and Maxwell-type term in (168) are fixed. We can further develop (168) by noting that:

$$\begin{aligned} e_I\lrcorner(dcR^{IJ}e_J) &= e_I\lrcorner(0)=0 \\ &= (e_I\lrcorner dc)R^{IJ}e_J+dc(e_I\lrcorner R^{IJ})e_J+dcR^{IJ}e_I\lrcorner e_J \\ &= (e_I\lrcorner dc)R^{IJ}e_J+dcR^Je_J \end{aligned}$$

where we have used the fact that $R^J\equiv(e_I\lrcorner R^{IJ})$, $e_I\lrcorner e_J=\eta_{IJ}$, and $R^{IJ}\eta_{IJ}=0$. The action (168) can then be seen to reduce to:

$$S=\frac{1}{\beta}\int \frac{\alpha^2}{4}\epsilon_{IJKL}\mathcal{C}^{IJ}\mathcal{C}^{KL}-6\gamma^2dc*dc+11\alpha\gamma e_KR^Kdc-\frac{\alpha^2}{4}\epsilon_{IJKL}R^{IJ}R^{KL} \tag{170}$$