Signed tilings by ribbon L-shaped n-ominoes, n even, via Gröbner bases

Viorel Nitica[a]

[a]*Department of Mathematics, West Chester University, PA 19383, USA*

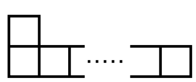
Abstract

We investigate signed tilings of rectangles by ribbon L-shaped n-ominoes, $n \geq 6$ even. We show that for $n = 6$ a rectangle has a signed tiling by ribbon L-shaped hexominoes if and only if one of the sides of the rectangle is divisible by 6. We show that a rectangle has a signed tiling by $\mathcal{T}_n$, $n \geq 8$ even, if and only if both sides of the rectangle are even and one of them is divisible by n, or if one of the sides is odd and the other side is divisible by $n\left(\frac{n}{2} - 2\right)$. Our proofs are based on the exhibition of explicit Gröbner bases. In particular, this paper shows that some of the regular tiling results in *V. Nitica, Every tiling of the first quadrant by ribbon L n-ominoes follows the rectangular pattern. Open Journal of Discrete Mathematics,* 5, *(2015) 11–25,* cannot be obtained from coloring invariants.

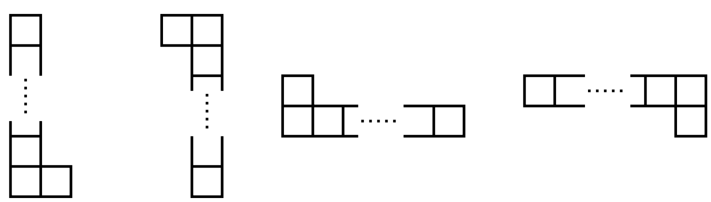
Keywords: polyomino; replicating tile; L-shaped polyomino; skewed L-shaped polyomino; signed tilings; Gröbner basis; tiling rectangles; coloring invariants

1. Introduction

In this article we study tiling problems for regions in a square lattice by certain symmetries of an L-shaped polyomino. Polyominoes were introduced by Golomb in [6] and the standard reference about this subject is the book *Polyominoes* [8]. The L-shaped polyomino we study is placed in a square lattice and is made out of $n, n \geq 3$, unit squares, or *cells*. See Figure 1a. In an $a \times b$ rectangle, a is the height and b is the base. We consider translations (only!) of the tiles shown in Figure 1b. They are ribbon L-shaped n-ominoes. A *ribbon polyomino* [12] is a simply connected polyomino with no two unit squares lying along a line parallel to the first bisector $y = x$. We denote the set of tiles by $\mathcal{T}_n$.



(a) An L n-omino with n cells.



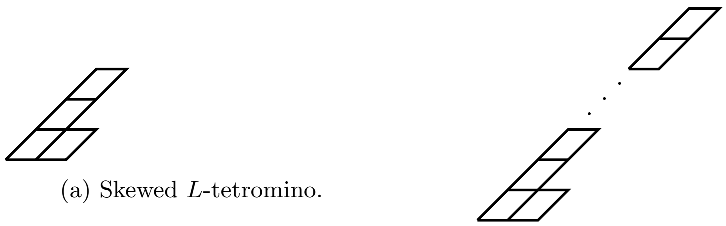
(b) The set of tiles $\mathcal{T}_n$.

Figure 1

Related papers are [3], [11], investigating tilings by $\mathcal{T}_n, n$ even. In [3] we look at tilings by $\mathcal{T}_n$ in the particular case $n = 4$. The starting point was a problem from recreational mathematics. We recall that a replicating tile is one that can make larger copies of itself. The order of replication is the number of initial tiles that fit in the larger copy. Replicating tiles were introduced by Golomb in [7]. In [9] we study replication of higher orders for several tiles introduced in [7]. In particular, we suggested that the skewed L-tetromino showed in Figure 2a is not replicating of order k^2 for any odd k. The question is equivalent to that of tiling a k-inflated copy of the straight L-tetromino using only the ribbon orientations of an L-tetromino. The question is solved in [3], where it is shown that the L-tetromino is not replicating of any odd order. This is a

Email address: `vnitica@wcupa.edu` (Viorel Nitica)

consequence of a stronger result: a tiling of the first quadrant by $\mathcal{T}_4$ always follows the rectangular pattern, that is, the tiling reduces to a tiling by 4×2 and 2×4 rectangles, each tiled in turn by two tiles from $\mathcal{T}_4$.



(a) Skewed L-tetromino.

(b) Skewed L n-omino.

Figure 2: Skewed polyominoes

The results in [3] are generalized in [11] to $\mathcal{T}_n, n$ even. The main result shows that any tiling of the first quadrant by $\mathcal{T}_n$ reduces to a tiling by $2 \times n$ and $n \times 2$ rectangles. An application is the characterization of all rectangles that can be tiled by $\mathcal{T}_n, n$ even: a rectangle can be tiled by $\mathcal{T}_n, n$ even, if and only if both sides are even and at least one side is divisible by n. One shows that these results are valid for any odd n. The rectangular pattern persists if one adds an extra 2×2 tile to $\mathcal{T}_n, n$ even. A rectangle can be tiled by the larger set of tiles if and only if it has both sides even. The main result also implies that a skewed L-shaped n-omino, n even, (see Figure 2b) is not a replicating tile of order k^2 for any odd k.

The discussion above shows that the limitation of the orientations of the tiles used in a tiling problem can be of interest, in particular when investigating tiling problems in a skewed lattice.

Signed tilings (see [4]) are also of interest. These are finite placements of tiles on a plane, with weights +1 or -1 assigned to each of the tiles. We say that they tile a region R if the sum of the weights of the tiles is 1 for every cell inside R and 0 for every cell elsewhere. The existence of a regular tiling clearly implies the existence of a signed tiling. Many times solving a tiling problem can be reduced to a coloring argument. It was shown in [4] that the most general argument of this type is equivalent to the existence of a signed tiling. Consequently, different conditions for regular versus signed tilings can be used to show that certain tiling arguments are stronger then coloring arguments. By looking at signed tilings of rectangles by $\mathcal{T}_n, n$ even, we show that some of the *hard* results in [11] cannot be obtained via coloring arguments.

A useful tool in the study of signed tilings is a Gröbner basis associated to the polynomial ideal generated by the tiling set. See Bodini and Nouvel [2]. One can associate to any cell in the square lattice a monomial in two variable x, y. If the coordinates of the lower left corner of the cell are (α, β), one associates $x^\alpha y^\beta$. This correspondence associates to any bounded tile a Laurent polynomial with all coefficients 1. The polynomial associated to a tile P is denoted by f_P. The polynomial associated to a tile translated by an integer vector (γ, δ) is the initial polynomial multiplied by the monomial $x^\gamma y^\delta$. If the region we tile is bounded and the tile set consists of bounded tiles, then the problem can be translated in the first quadrant via a translation by an integer vector, and one can work only with regular polynomials in $\mathbb{Z}[X, Y]$. See Theorem 9 below.

Signed tilings by ribbon L n-ominoes, n odd are studied in [10], where we show that a rectangle can be signed tiled by ribbon L n-ominoes, n odd, if and only if it has a side divisible by n.

The main results of the paper are the following:

Theorem 1. *A rectangle can be signed tiled by $\mathcal{T}_6$ if and only if one of the sides is divisible by* 6.

Due to the Gröbner basis that we find in Section 3, the proof of Theorem 1 is similar to the proof of [10, Theorem 1]. Theorem 1 shows the results in [10] for $n = 6$ cannot be obtained by coloring arguments.

Due to the Gröbner basis that we exhibit for $n = 6$ we also have:

Proposition 2. *A k-inflated copy of the ribbon L hexomino has a signed tiling by $\mathcal{T}_6$.*

Theorem 3. *A rectangle can be signed tiled by $\mathcal{T}_n$, $n \geq 8$ even, only in the following cases:*

1. *Both sides of the rectangle are even and one of them is is divisible by n.*
2. *One of the sides is odd and the other is divisible by $n\left(\frac{n}{2} - 2\right)$.*

The proof of Theorem 3 is shown in Section 4. Examples for Case 1 of Theorem 3 can be obtained from regular tilings that follows the rectangular pattern. An example for Case 2 is a signed tiling of a 16×1 rectangle by $\mathcal{T}_8$, shown in the Appendix.

Theorem 3 shows that some of the regular tiling results in [10] obtained for the tile set $\mathcal{T}_n, n \geq 8$ even, cannot be discovered using coloring arguments.

Due to the Gröbner basis that we exhibit for $n \geq 8, n$ even, we also have:

Proposition 4. *A k-inflated copy of the ribbon L n-omino, $n \geq 8$ even, has a signed tiling by $\mathcal{T}_n$ if and only if k is even.*

The proof of Proposition 4 is shown in Section 5.

Remark 1. 1) We recall that it is shown in [3] that a rectangle is signed tiled by $\mathcal{T}_4$ if and only if both of its sides are even and one side is divisible by 4.

2) The difference between the case $n = 6$ and the case $n \geq 8$ even is due to different Gröbner basis that can be used in each case.

2. Summary of Gröbner basis theory

Let $R[\underline{X}] = R[X_1, \ldots, X_k]$ be the ring of polynomials with coefficients in a principal ideal domain (PID) R. A *term* in the variables $x_1, \ldots, x_k$ is a power product $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_\ell^{\alpha_\ell}$ with $\alpha_i \in \mathbb{N}, 1 \leq i \leq \ell$; in particular $1 = x_1^0 \ldots x_\ell^0$ is a term. A term with an associated coefficient from R is called *monomial*. We endow the set of terms with the *total degree-lexicographical order*, in which we first compare the degrees of the monomials and then break the ties by means of lexicographic order for the order $x_1 > x_2 > \cdots > x_\ell$ on the variables. If the variables are only x, y and $x > y$, this gives the total order:

$$1 < y < x < y^2 < xy < x^2 < y^3 < xy^2 < x^2y < x^3 < y^4 < \cdots . \tag{1}$$

For $P \in R[\underline{X}]$ we denote by $HT(P)$ the leading term and by $HM(P)$ the highest monomial in P with respect to the above order. We denote by $HC(P)$ the coefficient of the leading monomial in P. We denote by $T(P)$ the set of terms appearing in P and by $M(P)$ the set of monomials in P.

For a given ideal $I \subseteq R[\underline{X}]$ an associated Gröbner basis is introduced as in Chapters 5, 10 in [1]).

If $G \subseteq R[\underline{X}]$ is a finite set, we denote by $I(G)$ the ideal generated by G in $R[\underline{X}]$.

Definition 1. Let $f, g, p \in R[\underline{X}]$. We say that f D-reduces to g modulo p and write $f \underset{p}{\rightarrow} g$ if there exists $m \in M(f)$ with $HM(p)|m$, say $m = m' \cdot HM(p)$, and $g = f - m'p$. For a finite set $G \subseteq R[\underline{X}]$, we denote by $\overset{*}{\underset{G}{\rightarrow}}$ the reflexive-transitive closure of $\underset{p}{\rightarrow}, p \in G$. We say that g is a normal form for f with respect to G if $f \overset{*}{\underset{G}{\rightarrow}} g$ and no further D-reduction is possible. We say that f is D-reducible modulo G if $f \overset{*}{\underset{G}{\rightarrow}} 0$.

If $f \overset{*}{\underset{G}{\rightarrow}} 0$, then $f \in I(G)$. The converse is also true if G is a Gröbner basis.

Definition 2. A D-Gröbner basis is a finite set G of $R[\underline{X}]$ with the property that all D-normal forms modulo G of elements of $I(G)$ equal zero. If $I \subseteq R[\underline{X}]$ is an ideal, then a D-Gröebner basis of I is a D-Gröebner basis that generates the ideal I.

Proposition 5. *Let G be a finite set of $R[\underline{X}]$. Then the following statements are equivalent:*

1. *G is a Gröebner basis.*
2. *Every $f \neq 0, f \in I(G)$, is D-reducible modulo G.*

We observe, nevertheless, that if R is only a (PID), the normal form associated to a polynomial f by a finite set $G \subseteq R[\underline{X}]$ is not unique. That is, the reminder of the division of f by G is not unique.

We introduce now the notions of S-polynomial and G-polynomial that allows to check if a given finite set $G \subseteq R[\underline{X}]$ is a Gröbner basis for the ideal it generates. As usual, *lcm* is the notation for the least common multiple and *gcd* is the notation for the greatest common divisor.

Definition 3. Let $0 \neq g_i \in R[\underline{X}], i = 1, 2$, with $HC(g_i) = a_i$ and $HT(g_i) = t_i$. Let $a = b_i a_i = \text{lcm}(a_1, a_2)$ with $b_i \in R$, and $t = s_i t_i = \text{lcm}(t_1, t_2)$ with $s_i \in T$. The the S-polynomial of g_1, g_2 is defined as:

$$S(g_1, g_2) = b_1 s_1 g_1 - b_2 s_2 g_2. \tag{2}$$

If $c_1, c_2 \in R$ such that $\gcd(a_1, a_2) = c_1 a_1 + c_2 a_2$. Then the G-polynomial of g_1, g_2 is defined as:

$$G(g_1, g_2) = c_1 s_1 g_1 + c_2 s_2 g_2. \tag{3}$$

Theorem 6. *Let G be a finite set of $R[\underline{X}]$. Assume that for all $g_1, g_2 \in G$, $S(g_1, g_2) \xrightarrow[G]{*} 0$ and $G(g_1, g_2)$ is top-D-reducible modulo G. Then G is a Gröbner basis.*

Assume now that R is an Euclidean domain with unique reminders (see [1, p. 463]). This is the case for the ring of integers $\mathbb{Z}$ if we specify reminders upon division by $0 \neq m$ to be in the interval $[0, m)$.

Definition 4. Let $f, g, p \in R[\underline{X}]$. We say that f E-reduces to g modulo p and write $f \xrightarrow[p]{} g$ if there exists $m = at \in M(f)$ with $HM(p)|t$, say $t = s \cdot HT(p)$, and $g = f - qsp$ where $0 \neq q \in R$ is the quotient of a upon division with unique reminder by $HC(p)$.

Proposition 7. *E-reduction extends D-reduction, i.e., every D-reduction step in an E-reduction step.*

Theorem 8. *Let R be an Euclidean domain with unique reminders, and assume $G \subseteq R[\underline{X}]$ is a D-Gröbner basis. Then the following hold:*

1. *$f \xrightarrow[G]{*} 0$ for all $f \in I(G)$, where $\xrightarrow[G]{*}$ denotes the E-reduction modulo G.*
2. *E-reduction modulo G has unique normal forms.*

The following result connect signed tilings and Gröbner bases. See [2] and [5] for a proof.

Theorem 9. *A polyomino P admits a signed tiling by translates of prototiles $P_1, P_2, \ldots, P_k$ if and only if for some (test) monomial $x^\alpha y^\beta$ the polynomial $x^\alpha y^\beta f_P$ is in the ideal generated in $\mathbb{Z}[X, Y]$ by the polynomials $f_{P_1}, \ldots, f_{P_k}$.*

Moreover, the set of test monomials $\mathcal{T} = \{x^\alpha\}$ can be chosen from any set $T \subseteq \mathbb{N}^n$ of multi-indices which is cofinal in $(\mathbb{N}^n, \leq)$.

3. Gröbner basis for $\mathcal{T}_n$, n even

We show Gröbner bases for the ideals generated by $\mathcal{T}_4, \mathcal{T}_6$, as these are different from the general case.

Proposition 10. *The set of polynomials:*

$$C_1(2) = x^2 + x + y + 1, \quad C_2(2) = y^2 + x + y + 1 \tag{4}$$

form a Gröbner basis for the ideal of polynomials generated by $\mathcal{T}_4$.

Proof. The polynomials corresponding to tiles in $\mathcal{T}_4$ are

$$C_1(2), C_2(2), xy^2 + xy + y^2 + x, x^2 y + xy + x^2 + y. \tag{5}$$

The last two polynomials can be generated by $C_1(2), C_2(2)$:

$$\begin{aligned} xy^2 + xy + y^2 + x &= -C_1(2) + (x+1)C_2(2) \\ x^2 y + xy + x^2 + y &= -C_2(2) + (y+1)C_1(2). \end{aligned} \tag{6}$$

It remains to show that the S-polynomial associated to $C_1(2), C_2(2)$ can be reduced. The leading term in $C_1(2)$ is x^2 and the leading term in $C_2(2)$ is y^2. One has:

$$S(C_1(2), C_2(2)) = y^2(x^2 + x + y + 1) - x^2(y^2 + y + 1 + x) = (x + y + 1)C_1(2) + (x + y + 1)C_2(2). \tag{7}$$

Proposition 11. *The set of polynomials:*

$$C_1(3) = x^3 + x^2 + x + y^2 + y + 1, \quad C_2(3) = y^3 + y^2 + y + x^2 + x + 1, \quad C_3(3) = xy - 1 \tag{8}$$

form a Gröbner basis for the ideal of polynomials generated by $\mathcal{T}_6$.

Proof. The polynomials associated to $\mathcal{T}_6$ are:

$$\begin{aligned} H_1(k) &= y^4 + y^3 + y^2 + y + 1 + x \\ H_2(k) &= y^4 + xy^4 + xy^3 + xy^2 + xy + x \\ H_3(k) &= y + x^4 + x^3 + x^2 + x + 1 \\ H_4(k) &= x^4y + x^3y + x^2y + xy + y + x^4. \end{aligned} \tag{9}$$

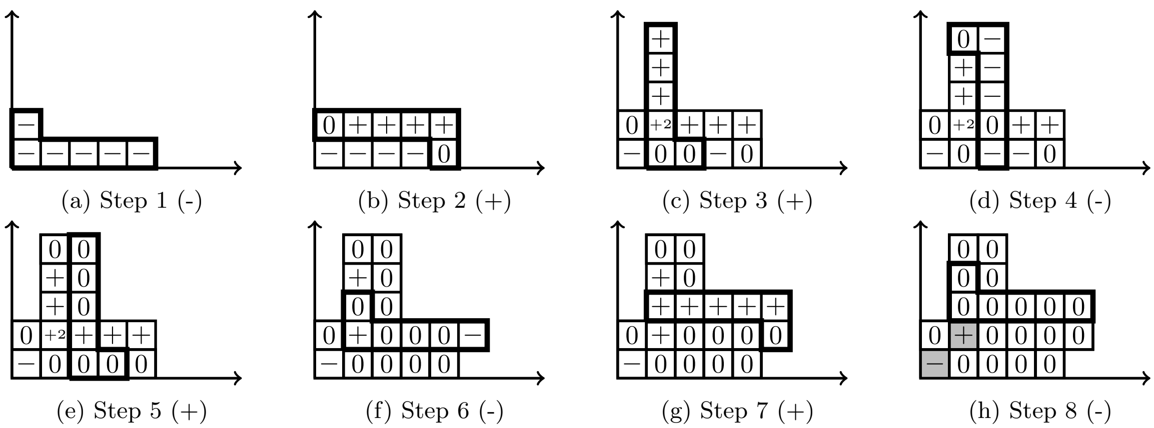


Figure 3: The polynomial $C_3(6)$ is generated by $\{H_1(6), H_2(6), H_3(6), H_4(6)\}$.

Similar to what is done in [10], the presence of $C_3(3)$ in the Gröbner basis allows to reduce the algebraic proofs to combinatorial considerations. We leave most of the details of this proof to the reader. The proof that $H_1(3), H_2(3), H_3(3), H_4(3)$ are in the ideal generated by $C_1(3), C_2(3), C_3(3)$ is similar to that of [10, Proposition 5]. The proof that $C_1(3), C_2(3), C_3(3)$ are in the ideal generated by $H_1(3), H_2(3), H_3(3), H_4(3)$ is similar to that of [10, Proposition 6]. A step by step geometric proof that $C_3(3)$ belongs to the ideal generated by $H_1(3), H_2(3), H_3(3), H_4(3)$ is shown in Figure 3.

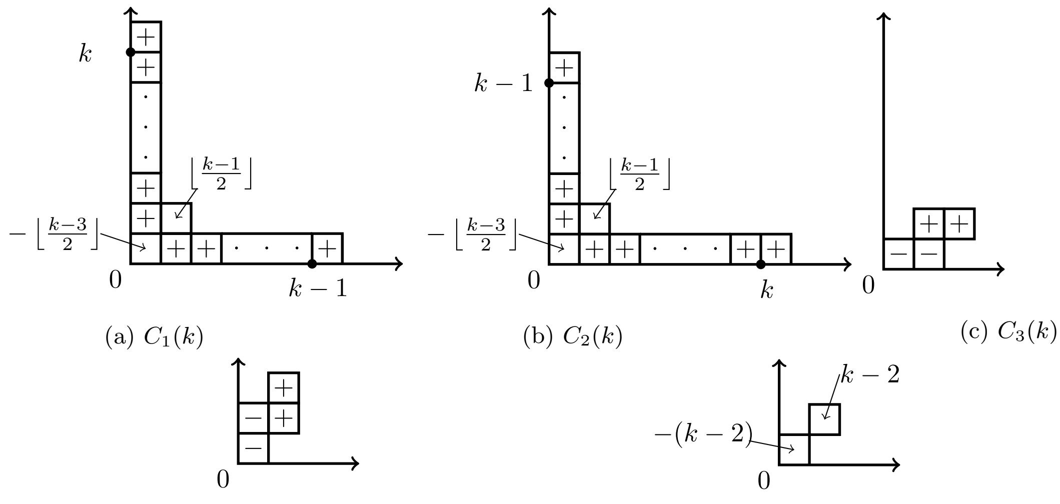
From now on $n \geq 8$. It is convenient and simplifies the notation to write $n = 2k$, where $k \geq 3$.

The polynomials (written in a condensed form) associated to the tiles in $\mathcal{T}_n$ are:

$$H_1(k) = \frac{y^{2k+1}-1}{y-1} + x, \; H_2(k) = y^{2k} + \frac{x(y^{2k+1}-1)}{y-1}, \; H_3(k) = y + \frac{x^{2k+1}-1}{x-1}, \; H_4(k) = \frac{y(x^{2k+1}-1)}{x-1} + x^{2k}. \tag{10}$$

We show in the rest of this section that a Gröebner basis for the ideal generated in $\mathbb{Z}[X,Y]$ by $H_1(k)$, $H_2(k)$, $H_3(k), H_4(k)$, is given by the polynomials (written in condensed from):

$$\begin{aligned} C_1(k) &= \frac{y^{k+1}-1}{y-1} + x \cdot \frac{x^{k-1}-1}{x-1} + \left\lfloor \frac{k-1}{2} \right\rfloor xy - \left\lfloor \frac{k-1}{2} \right\rfloor \\ C_2(k) &= \frac{x^{k+1}-1}{x-1} + y \cdot \frac{y^{k-1}-1}{y-1} + \left\lfloor \frac{k-1}{2} \right\rfloor xy - \left\lfloor \frac{k-1}{2} \right\rfloor \\ C_3(k) &= x^2y + xy - x - 1 \\ C_4(k) &= xy^2 + xy - y - 1 \\ C_5(k) &= (k-2)xy - (k-2), \end{aligned} \tag{11}$$



(a) $C_1(k)$ (b) $C_2(k)$ (c) $C_3(k)$

(d) $C_4(k)$ (e) $C_5(k)$

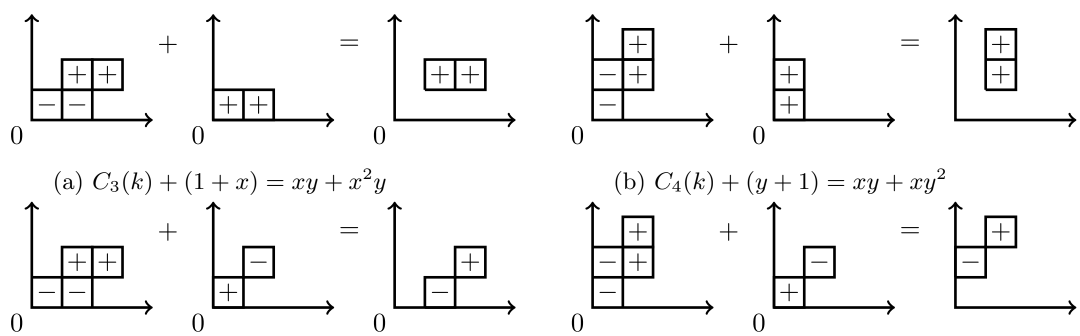
Figure 4: The Gröbner basis $\{C_1(k), C_2(k), C_3(k), C_4(k), C_5(k)\}$.

where $\lfloor x \rfloor$ is the integer part of x.

It is convenient visualize at the elements of the basis as tiles with cells labeled by integers, see Figure 4.

Proposition 12. *The polynomials $H_1(k), H_2(k), H_3(k), H_4(k)$ belong to the ideal generated by $C_1(k)$, $C_2(k)$, $C_3(k)$, $C_4(k), C_5(k)$.*

Proof. Due to the symmetry, it is enough to show that $H_1(k), H_2(k)$ belong to the ideal. The polynomials $C_3(k), C_4(k)$ allow to translate a block consisting of two cells labeled by the same sign adjacent horizontally, respectively adjacent vertically, along a vector parallel to the first bisector $y = x$. They also allow to translate horizontally or vertically a block of two cells adjacent at a vertex and labeled by different signs into a similar block. If the length of the translation is even, the signs stay the same. If the length of the translation is odd, all signs are changed. See Figure 5.



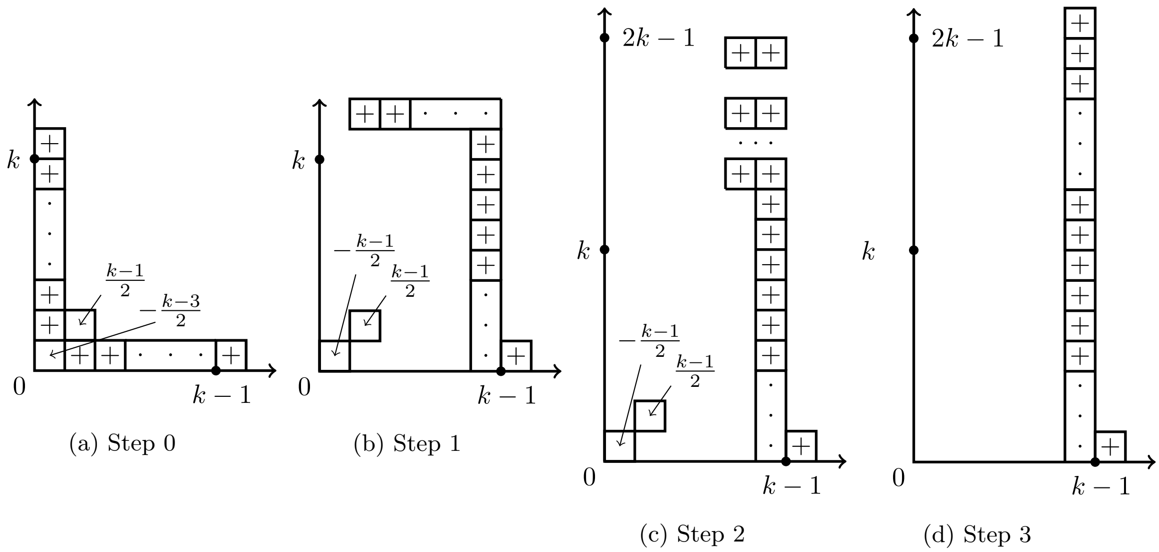
(a) $C_3(k) + (1 + x) = xy + x^2y$ (b) $C_4(k) + (y + 1) = xy + xy^2$

(c) $C_3(k) + (1 - xy) = -x + x^2y$ (d) $C_4(k) + (1 - xy) = -y + xy^2$

Figure 5: Tiles arithmetic.

We show how to build $H_1(k)$. There are two cases to be considered, k odd and k even.

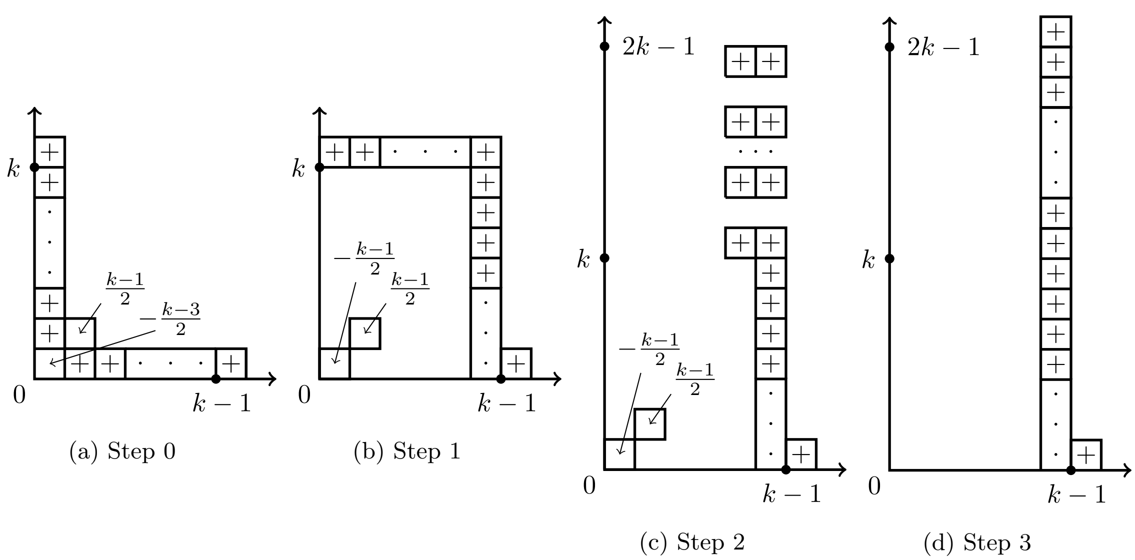
The steps of a geometric constructions for k odd are shown in Figure 6. To reach Step 1, we add several times multiples of $C_4(k)$, as in Figure 5, b). To reach Step 2, we add several times multiples of $C_3(k)$, as in Figure 5, a). To reach Step 3, first we subtract $C_5(k)$, then add several times multiples of $C_3(k), C_4(k)$ as in Figure 5, c), d). To obtain now $H_1(k)$ in the initial position, we multiply the tile in Step 3 by x^{k-2}, which will translate the tile $k - 2$ cells up, and then add multiples on $C_3(k), C_4(k)$, as in Figure 5, c), d).



(a) Step 0 (b) Step 1 (c) Step 2 (d) Step 3

Figure 6: Building $H_1(k), k$ odd, out of $\{C_1(k), C_2(k), C_3(k), C_4(k), C_5(k)\}$.

The steps of a geometric constructions for k even are shown in Figure 7. To reach Step 1, we add several times multiples of $C_4(k)$, as in Figure 5, b). To reach Step 2, we add several times multiples of $C_3(k)$, as in Figure 5, a). To reach Step 3, first we subtract $C_5(k)$, then add several times multiples of $C_3(k), C_4(k)$ as in Figure 5, c), d). To obtain now $H_1(k)$ in the initial position, we multiply the tile in Step 3 by x^{k-2}, which will translate the tile $k-2$ cells up, and then add multiples on $C_3(k), C_4(k)$, as in Figure 5, c), d).



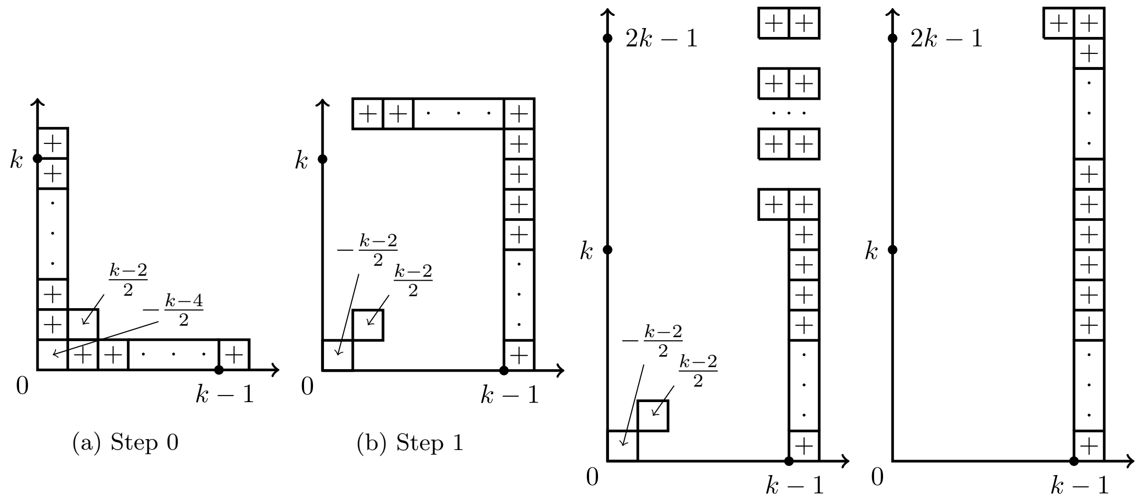
(a) Step 0 (b) Step 1 (c) Step 2 (d) Step 3

Figure 7: Building $H_1(k), k$ even, out of $\{C_1(k), C_2(k), C_3(k), C_4(k), C_5(k)\}$.

We show how to build $H_2(k)$. There are two cases to be considered, k odd and k even.

The steps of a geometric constructions for k odd are shown in Figure 8. To reach Step 1, we add several times multiples of $C_4(k)$, as in Figure 5, b). To reach Step 2, we add several times multiples of $C_3(k)$, as in Figure 5, a). To reach Step 3, first we subtract $C_5(k)$, then add several times multiples of $C_3(k), C_4(k)$ as in Figure 5, c), d). To obtain now $H_2(k)$ in the initial position, we multiply the tile in Step 3 by x^{k-2}, which will translate the tile $k-2$ cells up, and then add multiples on $C_3(k), C_4(k)$, as in Figure 5, c), d).

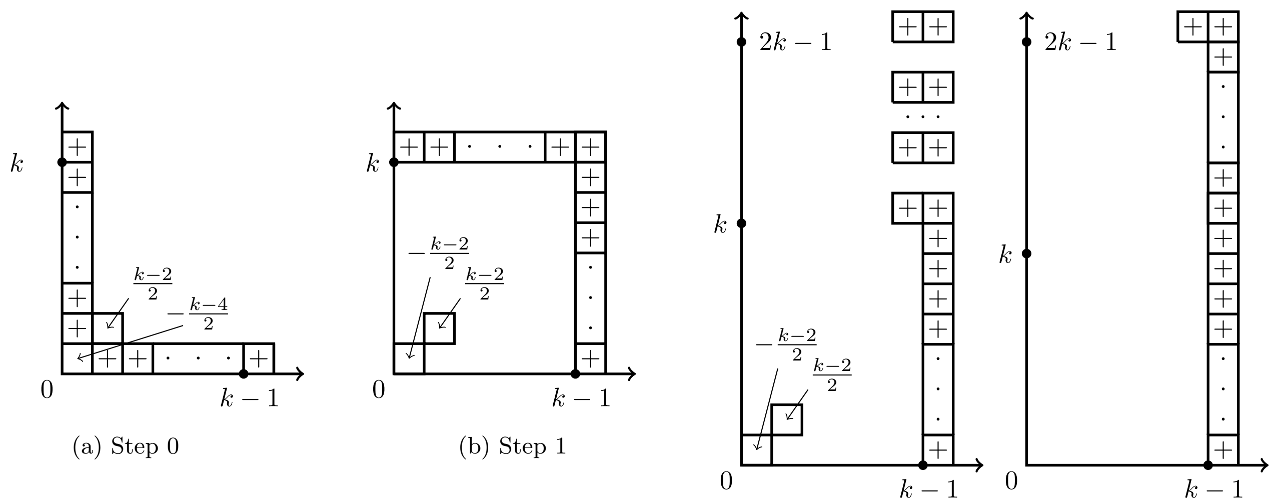
The steps of a geometric constructions for k even are shown in Figure 9. To reach Step 1, we add several times multiples of $C_4(k)$, as in Figure 5, b). To reach Step 2, we add several times multiples of $C_3(k)$, as in



(a) Step 0 (b) Step 1 (c) Step 2 (d) Step 3

Figure 8: Building $H_2(k), k$ odd, out of $\{C_1(k), C_2(k), C_3(k), C_4(k), C_5(k)\}$.

Figure 5, a). To reach Step 3, first we subtract $C_5(k)$, then add several times multiples of $C_3(k), C_4(k)$ as in Figure 5, c), d). To obtain now $H_2(k)$ in the initial position, we multiply the tile in Step 3 by x^{k-2}, which will translate the tile $k-2$ cells up, and then add multiples on $C_3(k), C_4(k)$, as in Figure 5, c), d).



(a) Step 0 (b) Step 1 (c) Step 2 (d) Step 3

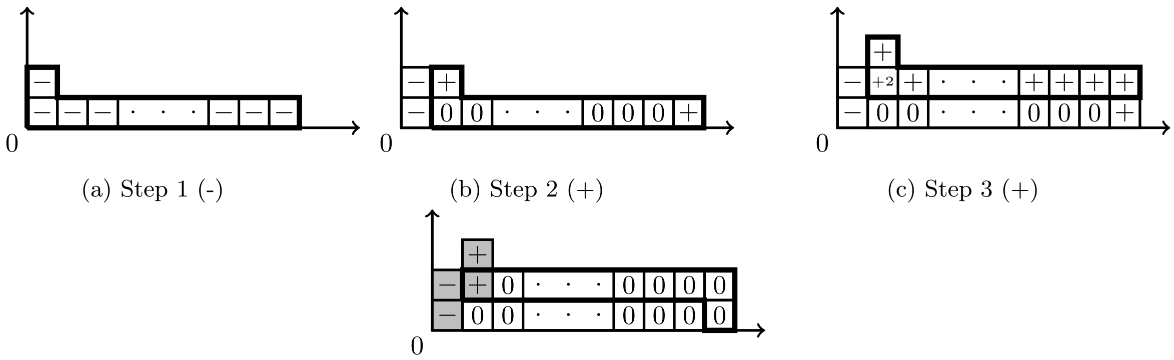
Figure 9: Building $H_2(k), k$ even, out of $\{C_1(k), C_2(k), C_3(k), C_4(k), C_5(k)\}$.

Proposition 13. *The polynomials $C_1(k), C_2(k), C_3(k)$, $C_4(k), C_5(k)$ belong to the ideal generated by $H_1(k)$, $H_2(k), H_3(k), H_4(k)$.*

PROOF. Due to the symmetry, it is enough to show that $C_1(k), C_3(k)$ and $C_5(k)$ belong to the ideal. We show how to generate $C_3(k), C_5(k)$ (and consequently $C_4(k)$). To generate $C_1(k)$ we can reverse the process in Proposition 12.

For $C_3(k)$, one has the following formula:

$$C_3(k) = (xy + x - 1)H_3(k) - xH_4(k). \tag{12}$$



(a) Step 1 (-) (b) Step 2 (+) (c) Step 3 (+)

(d) Step 4 (-)

Figure 10: Building $C_3(k)$ out of $\{H_1(k), H_2(k), H_3(k), H_4(k)\}$.

A geometric proof of (12) is shown in Figure 10.

To generate $C_5(k)$ we first show how to obtain a configuration in which all nontrivial cells, 4 of them, are located on the main diagonal. See Figure 11. Then we use the tiles arithmetic shown in Figure 12 to pull the cells in position $(k-1, k-1)$ and $(2k-2, 2k-2)$ in positions $(1,1)$ and $(2,2)$. This constructs $C_5(k)$.

Proposition 14. *The sets* $\{C_1(k), C_2(k), C_3(k), C_4(k), C_5(k)\}$ *and* $\{H_1(k), H_2(k), H_3(k), H_4(k)\}$ *generate the same ideal in* $\mathbb{Z}[X, Y]$.

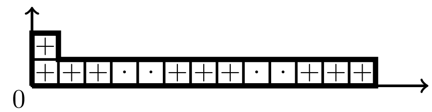
PROOF. This follows from Propositions 12, 13.

Proposition 15. *One has the following formulas:*

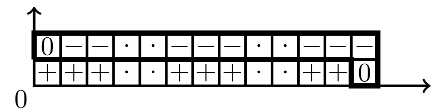
$$\begin{aligned}
S(C_1(k), C_2(k)) &= -y^{k-1}C_1(k) + x^{k-1}C_2(k) - y^{k-1}(1 + x^2 + \cdots + x^{k-3})C_3(k) \\
&\quad + x^{k-1}(1 + y^2 + \cdots + y^{k-3})C_4(k) + y^{k-1}C_5(k) - y^{k-1}\left\lfloor \frac{k-1}{2} \right\rfloor C_4(k) \\
&\quad - x^{k-1}C_5(k) + x^{k-1}\left\lfloor \frac{k-1}{2} \right\rfloor C_3(k), k \text{ odd} \\
S(C_1(k), C_2(k)) &= -y^{k-1}C_1(k) + x^{k-1}C_2(k) - y^{k-1}(1 + x^2 + \cdots + x^{k-3})C_3(k) \\
&\quad + x^{k-1}(1 + y^2 + \cdots + y^{k-3})C_4(k) + y^{k-1}C_5(k) - y^{k-1}\left\lfloor \frac{k-1}{2} \right\rfloor C_4(k) \\
&\quad - x^{k-1}C_5(k) + x^{k-1}\left\lfloor \frac{k-1}{2} \right\rfloor C_3(k), k \text{ even}
\end{aligned} \tag{13}$$

$$\begin{aligned}
S(C_1(k), C_3(k)) &= xC_2(k) - y^{k-2}C_4(k) + y^{k-2}C_3(k) + (xy^{k-4} + xy^{k-6} + \cdots + xy)C_4(k) \\
&\quad - xC_5(k) + \left\lfloor \frac{k-1}{2} \right\rfloor xC_3(k), k \text{ odd} \\
S(C_1(k), C_3(k)) &= xC_2(k) - y^{k-2}C_4(k) + y^{k-2}C_3(k) + (xy^{k-4} + xy^{k-6} + \cdots + xy^2 + x)C_4(k) \\
&\quad - xC_5(k) + \left\lfloor \frac{k-1}{2} \right\rfloor xC_3(k), k \text{ even}
\end{aligned} \tag{14}$$

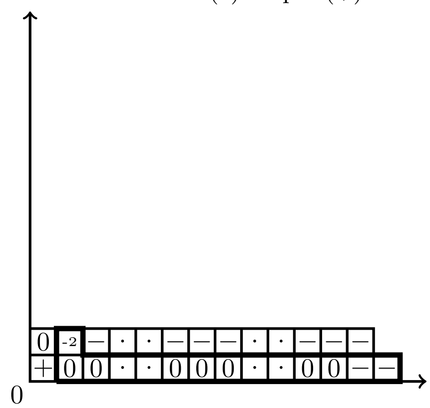
$$\begin{aligned}
S(C_1(k), C_4(k)) &= C_2(k) + (x^{k-4} + x^{k-6} + \cdots + x)C_4(k) - C_5(k) + \left\lfloor \frac{k-1}{2} \right\rfloor C_3(k), k \text{ odd} \\
S(C_1(k), C_4(k)) &= C_2(k) + (x^{k-4} + x^{k-6} + \cdots + x^2 + 1)C_4(k) - C_5(k) + \left\lfloor \frac{k-1}{2} \right\rfloor C_3(k), k \text{ even}
\end{aligned} \tag{15}$$



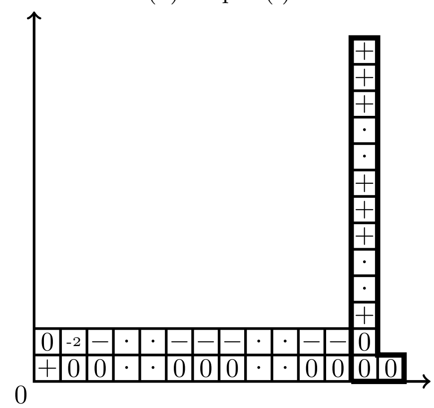
(a) Step 1 (+)



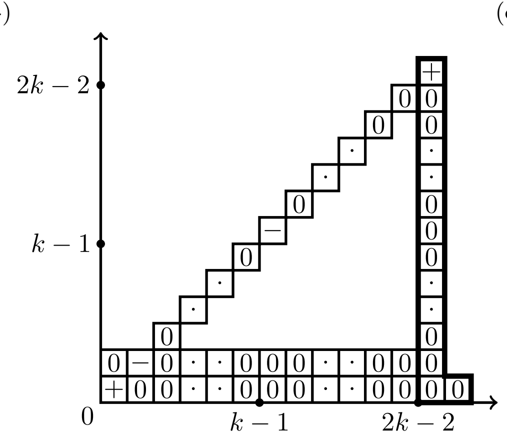
(b) Step 2 (-)



(c) Step 3 (-)



(d) Step 4 (+)



(e) Step 5

Figure 11: Building $C_5(k)$ out of $\{H_1(k), H_2(k), H_3(k), H_4(k)\}$.

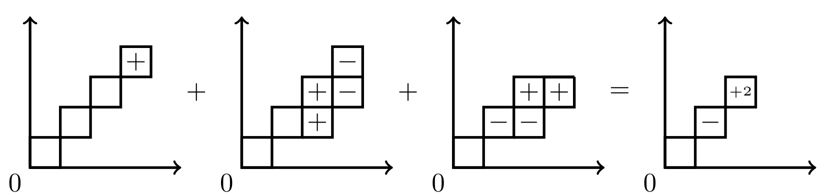


Figure 12: Tiles arithmetic: $x^3 + y^3 - x^2yC_4(k) + xyC_3(k) = 2x^2y^2 - xy$.

$$\begin{aligned}S(C_1(k),C_5(k)) &= (k-2)C_2(k)+(k-2)C_3(k)(1+y^2+\cdots+y^{k-3})+2\left\lfloor\frac{k-1}{2}\right\rfloor C_5(k)\\ &\quad+(k-2)\left\lfloor\frac{k-1}{2}\right\rfloor C_3(k), k\ odd\\ S(C_1(k),C_5(k)) &= (k-2)C_2(k)+(k-2)C_3(k)(y+y^3+\cdots+y^{k-3})+\left(2\left\lfloor\frac{k-3}{2}\right\rfloor+1\right)C_5(k)\\ &\quad+(k-2)\left\lfloor\frac{k-1}{2}\right\rfloor C_3(k), k\ even\end{aligned} \tag{16}$$

$$\begin{aligned}S(C_2(k),C_3(k)) &= C_1(k)+(y^{k-4}+y^{k-6}+\cdots+y)C_3(k)-C_5(k)+\left\lfloor\frac{k-1}{2}\right\rfloor C_4(k), k\ odd\\ S(C_2(k),C_3(k)) &= C_1(k)+(y^{k-4}+y^{k-6}+\cdots+y^2+1)C_3(k)-C_5(k)+\left\lfloor\frac{k-1}{2}\right\rfloor C_4(k), k\ even\end{aligned} \tag{17}$$

$$\begin{aligned}S(C_2(k),C_4(k)) &= yC_1(k)-x^{k-2}C_3(k)+x^{k-2}C_4(k)+(x^{k-4}y+x^{k-6}y+\cdots+xy)C_3(k)\\ &\quad-yC_5(k)+\left\lfloor\frac{k-1}{2}\right\rfloor yC_4(k), k\ odd\\ S(C_2(k),C_4(k)) &= yC_1(k)-x^{k-2}C_3(k)+x^{k-2}C_4(k)+(x^{k-4}y+x^{k-6}y+\cdots+x^2y+y)C_3(k)\\ &\quad-yC_5(k)+\left\lfloor\frac{k-1}{2}\right\rfloor yC_4(k), k\ even\end{aligned} \tag{18}$$

$$\begin{aligned}S(C_2(k),C_5(k)) &= (k-2)C_1(k)+(k-2)C_4(k)(1+x^2+\cdots+x^{k-3})+2\left\lfloor\frac{k-1}{2}\right\rfloor C_5(k)\\ &\quad+(k-2)\left\lfloor\frac{k-1}{2}\right\rfloor C_4(k), k\ odd\\ S(C_2(k),C_5(k)) &= (k-2)C_1(k)+(k-2)C_4(k)(y+y^3+\cdots+y^{k-3})+\left(2\left\lfloor\frac{k-3}{2}\right\rfloor+1\right)C_5(k)\\ &\quad+(k-2)\left\lfloor\frac{k-1}{2}\right\rfloor C_4(k), k\ even\end{aligned} \tag{19}$$

$$S(C_3(k),C_4(k)) = -C_3(k)+C_4(k) \tag{20}$$

$$\begin{aligned}S(C_3(k),C_5(k)) &= C_5(k)\\ S(C_4(k),C_5(k)) &= C_5(k),\end{aligned} \tag{21}$$

which are all given by D-reductions. Therefore, $\{c_1(k), C_2(k), C_3)k), C_4(k), C_5(k)\}$ *form a Gröbner basis.*

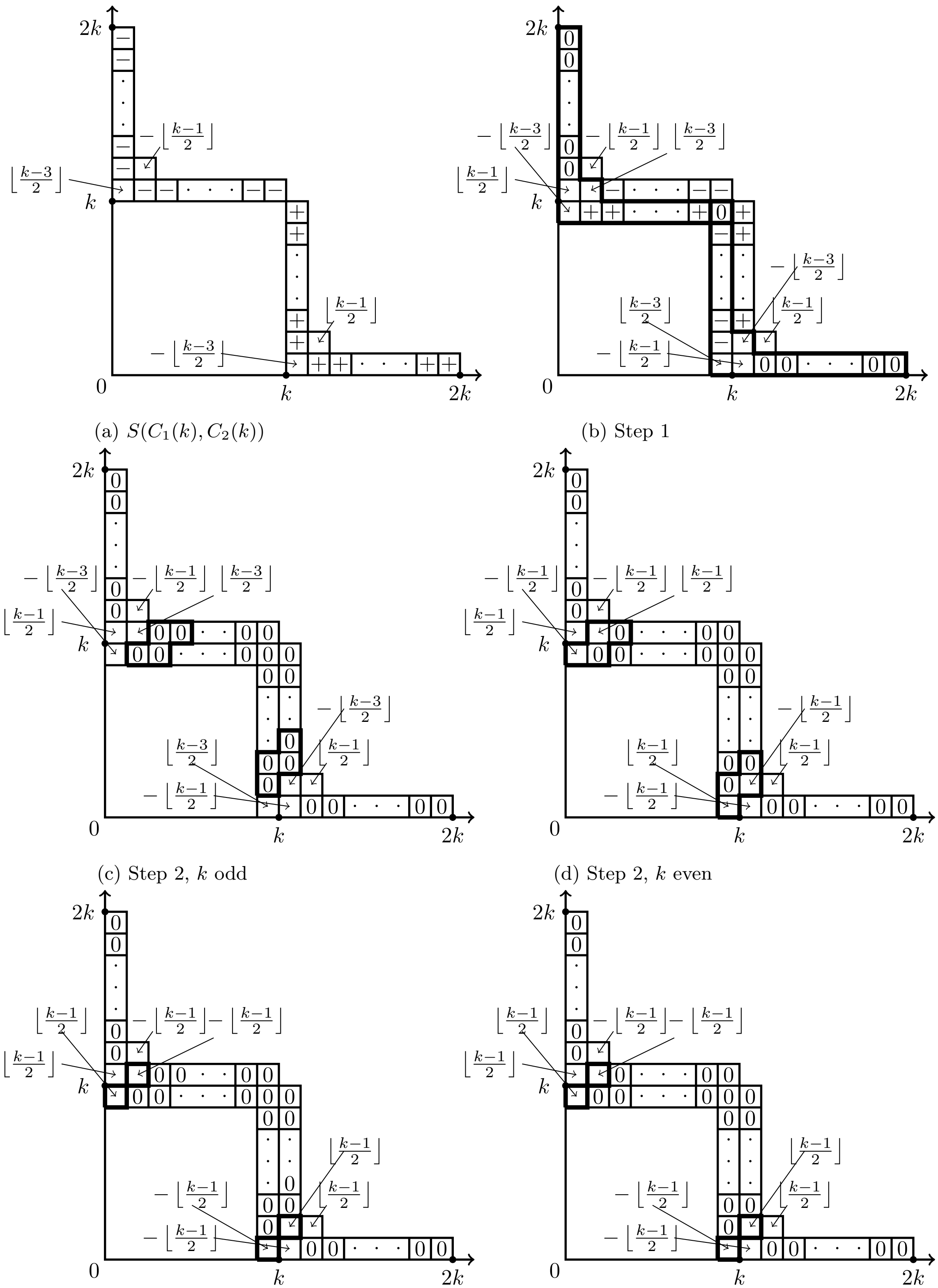
PROOF. We observe that we can always choose one of the coefficients c_1, c_2 in Definition 3 to be zero. So in order to check that we have a Gröbner basis, we do not need to use G-polynomials.

Due to the symmetry, some formulas above follows immediately from others: $S(C_2(k), C_3(k))$ follows from $S(C_1(k), C_4(k))$, $S(C_2(k), C_4(k))$ follows from $S(C_1(k), C_3(k))$, $S(C_2(k), C_5(k))$ follows from $S(C_1(k), C_5(k))$, and $S(C_4(k), C_5(k))$ follows from $S(C_3(k), C_5(k))$.

To prove the rest, we observe that the leading monomial in $C_1(k)$ is y^k, the leading monomial in $C_2(k)$ is x^k, the leading monomial in $C_3(k)$ is x^2y, the leading monomial in $C_4(k)$ is xy^2, and the leading monomial in $C_5(k)$ is $(k-2)xy$.

The D-reduction of $S(C_1(k), C_2(k))$ is shown in Figure 13. $S(C_1(k), C_2(k))$ consists of two disjoint symmetric tiles. The reduction of them is similar and it is shown in parallel in Figure 13. We start with

$$S(C_1(k), C_2(k)) = x^kC_1(k) - y^kC_2(k). \tag{22}$$



(a) $S(C_1(k), C_2(k))$

(b) Step 1

(c) Step 2, k odd

(d) Step 2, k even

(e) Step 3, k odd

(f) Step 3, k even

Figure 13: The D-reduction of $S(C_1(k), C_2(k))$.

The D-reduction of $S(C_1(k), C_3(k))$ is shown in Figure 14. We start with

$$S(C_1(k), C_3(k)) = x^2 C_1(k) - y^{k-1} C_3(k). \tag{23}$$

From Step 3 to Step 4 we subtract $(xy^{k-4}+xy^{k-6}+\cdots+xy)C_4(k)$ or $(xy^{k-4}+xy^{k-6}+\cdots+xy^2+x)C_4(k)$, depending on k odd or even. From Step 4 to Step 5 we use the following formula if k is odd:

$$(k-2) - \left\lfloor \frac{k-3}{2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor, \tag{24}$$

and we use the following formula if k is even:

$$(k-2) - \left\lfloor \frac{k-1}{2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor. \tag{25}$$

The D-reduction of $S(C_1(k), C_4(k))$ is shown in Figure 15. We start with

$$S(C_1(k), C_4(k)) = xC_1(k) - y^{k-2} C_4(k). \tag{26}$$

From Step 1 to Step 2 we subtract $(x^{k-4} + x^{k-6} + \cdots + x)C_4(k)$ or $(x^{k-4} + x^{k-6} + \cdots + x^2 + 1)C_4(k)$, depending on k odd or even. From Step 2 to Step 3 we use formula 25 if k is odd and 27 if k is even.

The D-reduction of $S(C_1(k), C_5(k))$ is shown in Figure 16. We start with

$$S(C_1(k), C_5(k)) = (k-2)xC_1(k) - y^{k-1} C_5(k). \tag{27}$$

To reach Step 1, we subtract $(k-2)C_2(k)$. To reach Step 2, we subtract $(1 + y^2 + \cdots + y^{k-3})C_4(k)$ if k is odd and $(y + \cdots + y^{k-3})C_4(k)$ if k is even. To reach Step 3, we add $2\left\lfloor \frac{k-1}{2} \right\rfloor C_5(k)$ if k is odd and $\left(2\left\lfloor \frac{k-3}{2} \right\rfloor + 1\right) C_5(k)$ if k is even.

The D-reduction of $S(C_3(k), C_4(k))$ is:

$$\begin{aligned} S(C_3(k), C_4(k)) &= yC_3(k) - xC_4(k) \\ = x^2y^2 + xy^2 - xy - y - (x^2y^2 + x^2y - xy - x) &= xy^2 - y - x^2y + x \\ &= -C_3(k) + C_4(k). \end{aligned} \tag{28}$$

The D-reduction of $S(C_3(k), C_5(k))$ is:

$$\begin{aligned} S(C_3(k), C_5(k)) &= (k-2)C_3(k) - xC_5(k) \\ &= (k-2)x^2y + (k-2)xy - (k-2)x - (k-2) - (k-2)x^2y + (k-2)x \\ &= C_5(k). \end{aligned} \tag{29}$$

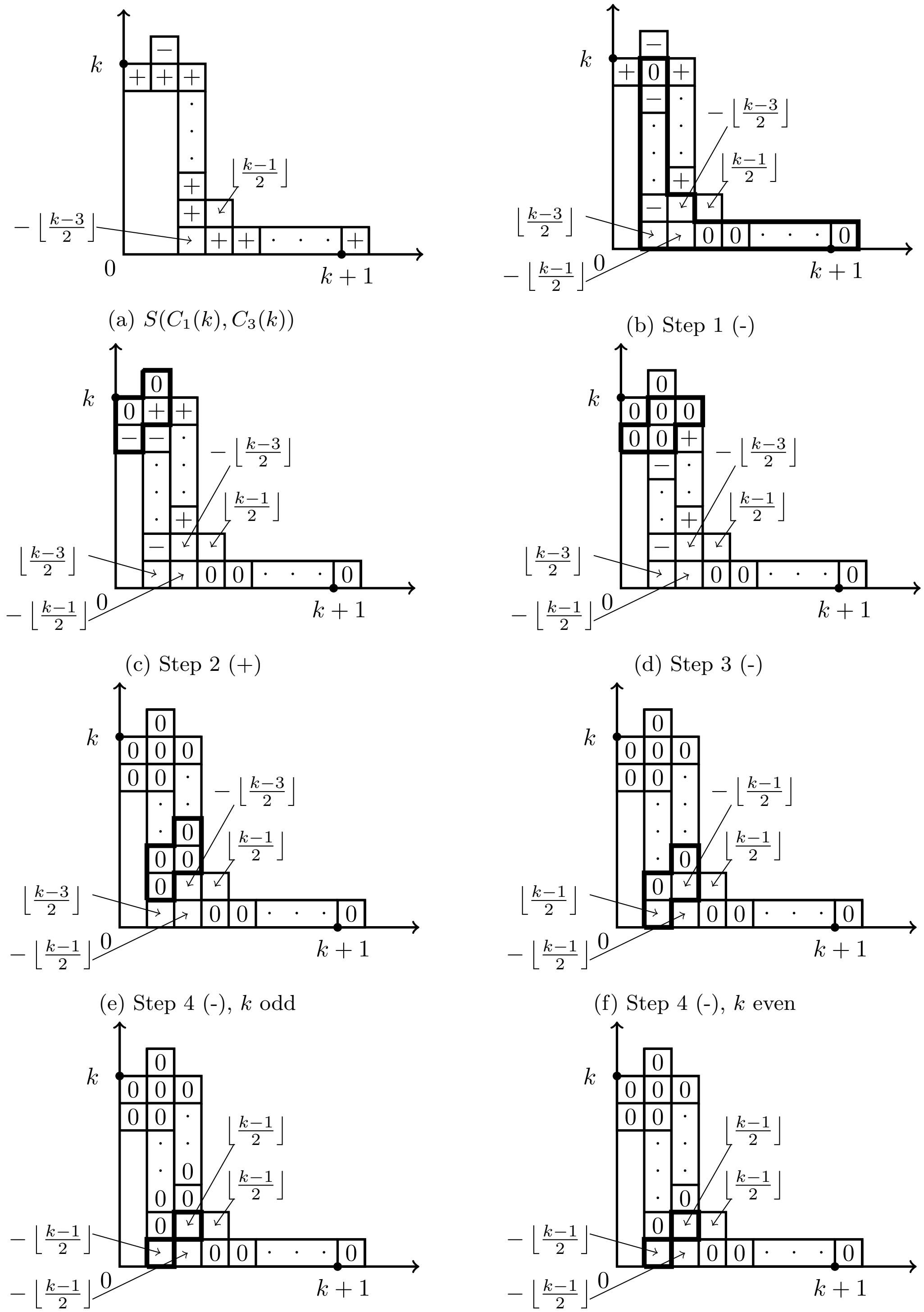
4. Proof of Theorem 3

Consider a $q \times p, q \geq p \geq 1$, rectangle. Using the presence of $C_3(k)$ and $C_4(k)$ in the Gröbner basis, the rectangle can be reduced to one of the configurations in Figure 17, a), b). Configuration b) appears when q, p are both even. The number of cells labeled by p is $q - p + 1$ in a) and $q - p$ in b).

In what follows the signed tile $B = xy - 1$ will play an important role. We recall that it can be moved horizontally/vertically as shown in Figure 5. The tile B does not belong to the ideal generated by $\mathcal{T}_n$. Other signed tile of interest in the sequel is $D = y^{n+1} + y^n + y^{n-1} + \cdots + y^2 + y + 1 - xy$, which is the concatenation of a vertical bar of length n and B. The tile $D = yH_1(k) - C_4(k)$ belongs to the ideal generated by $\mathcal{T}_n$.

Multiplying the polynomial associated to the rectangle by y^p, we can assume that the configurations in Figure 17 are at height $p-1$ above the x-axis. Using the tiles $C_3(k), C_4(k)$ and an amount of tiles B ($p/2$ if p is even and zero if p is odd), they can be reduced further to the configurations shown in Figure 17,c), d). We observe that b) is the sum of a) with $p/2$ copies of B.

Reducing further the configurations in Figure 17, c), d), with copies of D, the existence of a signed tiling for the $q \times p$ rectangle becomes equivalent to deciding when the following two conditions are both true:



(a) $S(C_1(k), C_3(k))$ (b) Step 1 (-)

(c) Step 2 (+) (d) Step 3 (-)

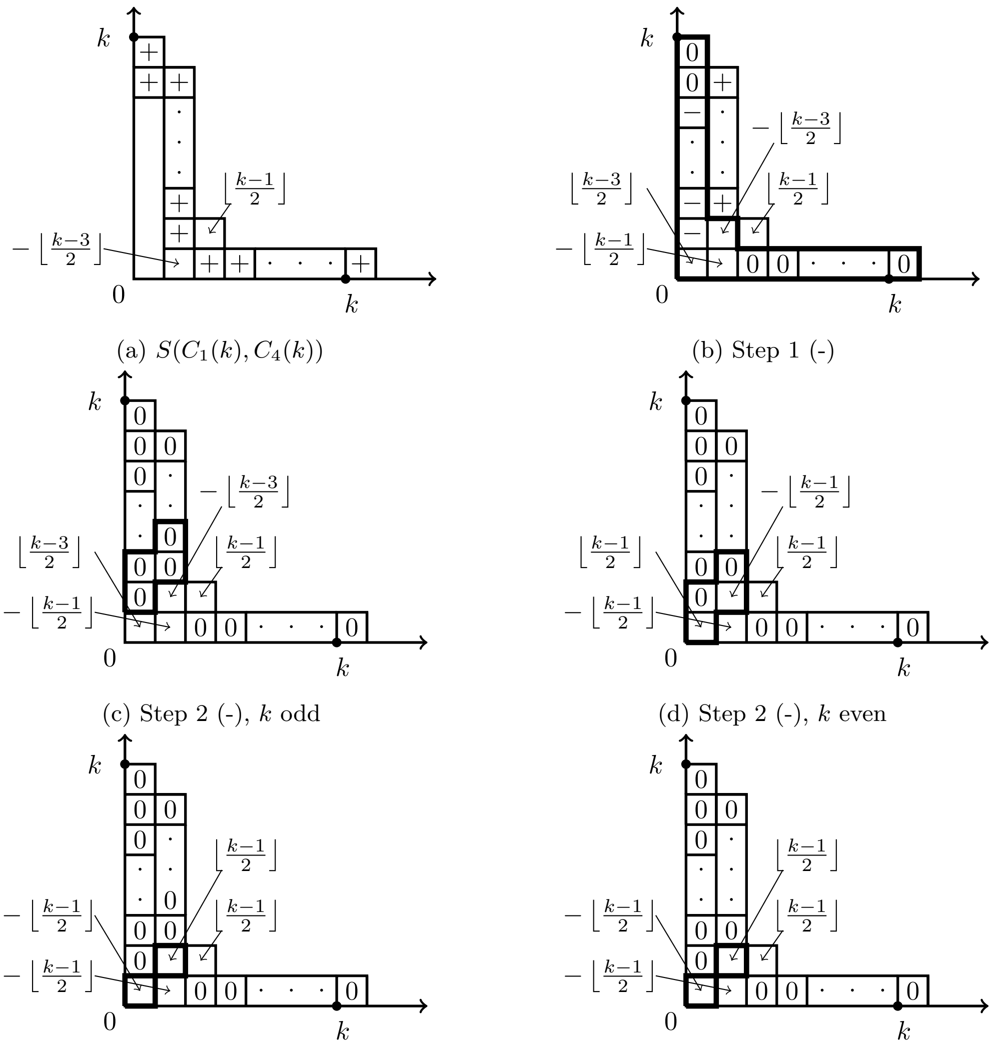
(e) Step 4 (-), k odd (f) Step 4 (-), k even

(g) Step 5 (+), k odd (h) Step 5 (+), k even

Figure 14: The D-reduction of $S(C_1(k), C_3(k))$.

1) The the polynomial $Q(x) = 1 + y + y^2 + \cdots + y^{n-1}$ divides:

$$P_{p,q}(y) = 1+2y+3y^2+\cdots+py^{p-1}+py^p+\cdots+py^{q-1}+(p-1)y^q+(p-2)y^{q+1}+\cdots+2y^{p+q-3}+y^{p+q-2}. \quad (30)$$



(a) $S(C_1(k), C_4(k))$ (b) Step 1 (-)

(c) Step 2 (-), k odd (d) Step 2 (-), k even

(e) Step 3 (-), k odd (f) Step 3 (-), k even

Figure 15: The D-reduction of $S(C_1(k), C_4(k))$.

2) The extra tiles B that appear while doing tile arithmetic for 1), including those from Figure 17, can be cancelled out by $C_5(k)$.

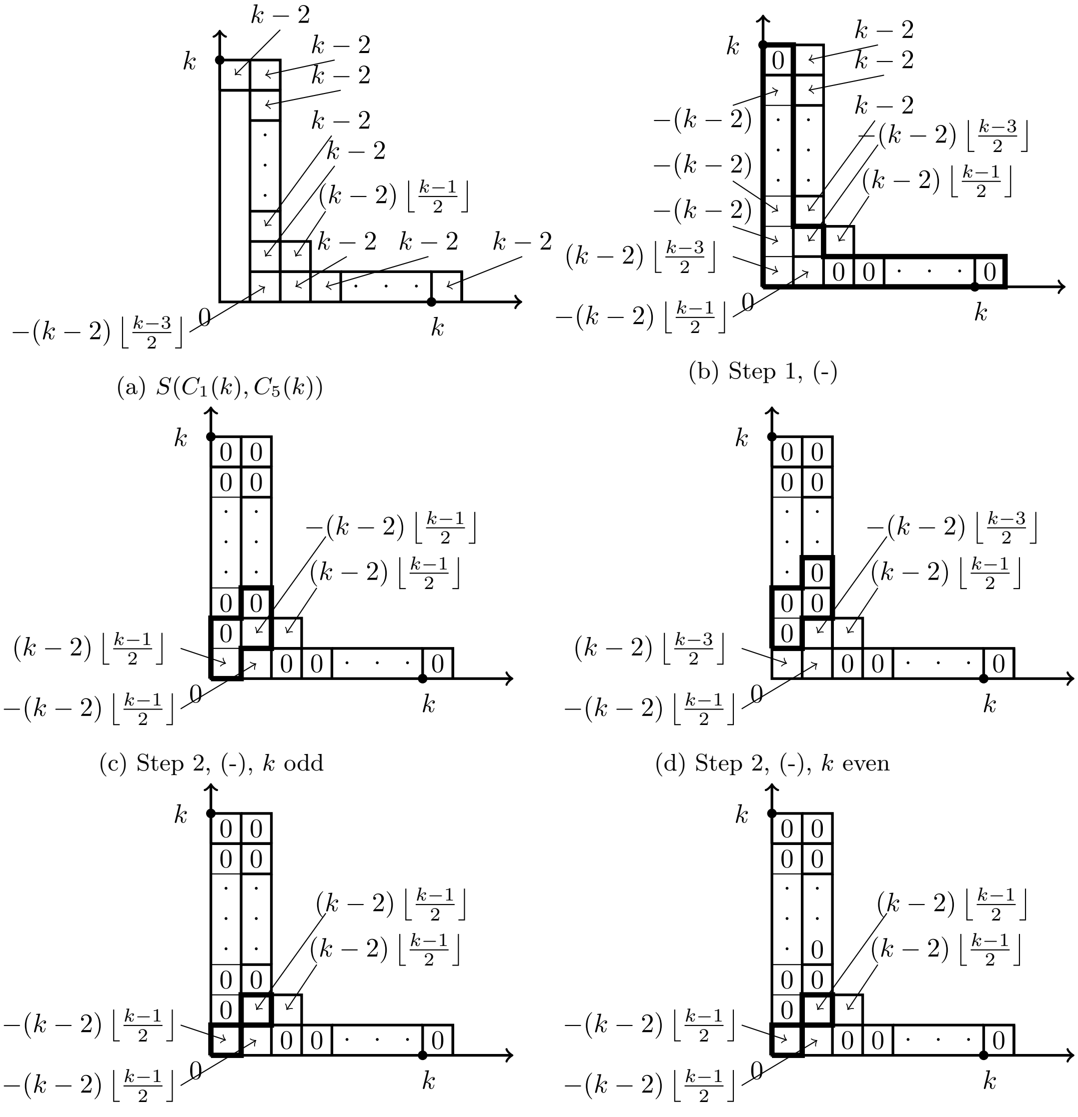
If $p+q-1 < n$, then $\deg Q > \deg P_{p,q}$, so divisibility does not hold. If $p+q-1 \geq n$, we look at $P_{p,q}$ as a sum of p polynomials with all coefficients equal to 1:

$$\begin{aligned} P_{p,q}(y) = 1 + y + y^2 + y^3 + \dots + y^{p-1} + y^p + \ldots + y^{q-1} + y^q + y^{q+1} + \dots + y^{p+q-4} + y^{p+q-3} + y^{p+q-2} \\ +y + y^2 + y^3 + \dots + y^{p-1} + y^p + \ldots + y^{q-1} + y^q + y^{q+1} + y^{p+q-4} + \dots + y^{p+q-3} \\ +y^3 + \dots + y^{p-1} + y^p + \ldots + y^{q-1} + y^q + y^{q+1} + \dots + y^{p+q-4} \\ \dots\dots\dots\dots \\ +y^p + \ldots + y^{q-1}. \end{aligned} \tag{31}$$

We discuss first 1) and show that it is true when p or q is divisible by n. Then, assuming this condition satisfied, we discuss 2).

1) Assume that $p+q-1 = nm + r, 0 \leq r < n$, and $p = ns + t, 0 \leq t < n$. The remainder $R_{p,q}(y)$ of the division of $P_{p,q}(y)$ by $Q(y)$ is the sum of the remainders of the division of the p polynomials above by $Q(y)$.

If r is odd, one has the following sequence of remainders, each remainder written in a separate pair of



(a) $S(C_1(k), C_5(k))$ (b) Step 1, (-)

(c) Step 2, (-), k odd (d) Step 2, (-), k even

(e) Step 3, (+), k odd (f) Step 3, (+), k even

Figure 16: The D-reduction of $S(C_1(k), C_5(k))$.

parentheses:

$$
\begin{aligned}
R_{p,q}(y) =& (1 + y + y^2 + \cdots + y^{r-1}) \\
&+ (y + y^2 + \cdots + y^{r-2}) \\
&+ (y^2 + \cdots + y^{r-3}) \\
&\quad \cdots\cdots\cdots \\
&+ (y^{\frac{r-1}{2}}) - (y^{\frac{r-1}{2}}) \\
&\quad \cdots\cdots\cdots \\
&- (y + y^2 + \cdots + y^{r-2}) \\
&- (1 + y + y^2 + \cdots + y^{r-1}) \\
&+ (y^{r+1} + y^{r+2} + \cdots + y^{n-3} + y^{n-2}) \\
&+ (y^{r+2} + \cdots + y^{n-3}) \\
&\quad \cdots\cdots\cdots \\
&+ (y^{\frac{r+n-1}{2}}) - (y^{\frac{r+n-1}{2}}) \\
&\quad \cdots\cdots\cdots \\
&- (y^{r+2} + \cdots + y^{n-3}) \\
&- (y^{r+1} + y^{r+3} + \cdots + y^{n-3} + y^{n-2})
\end{aligned}
\tag{32}
$$

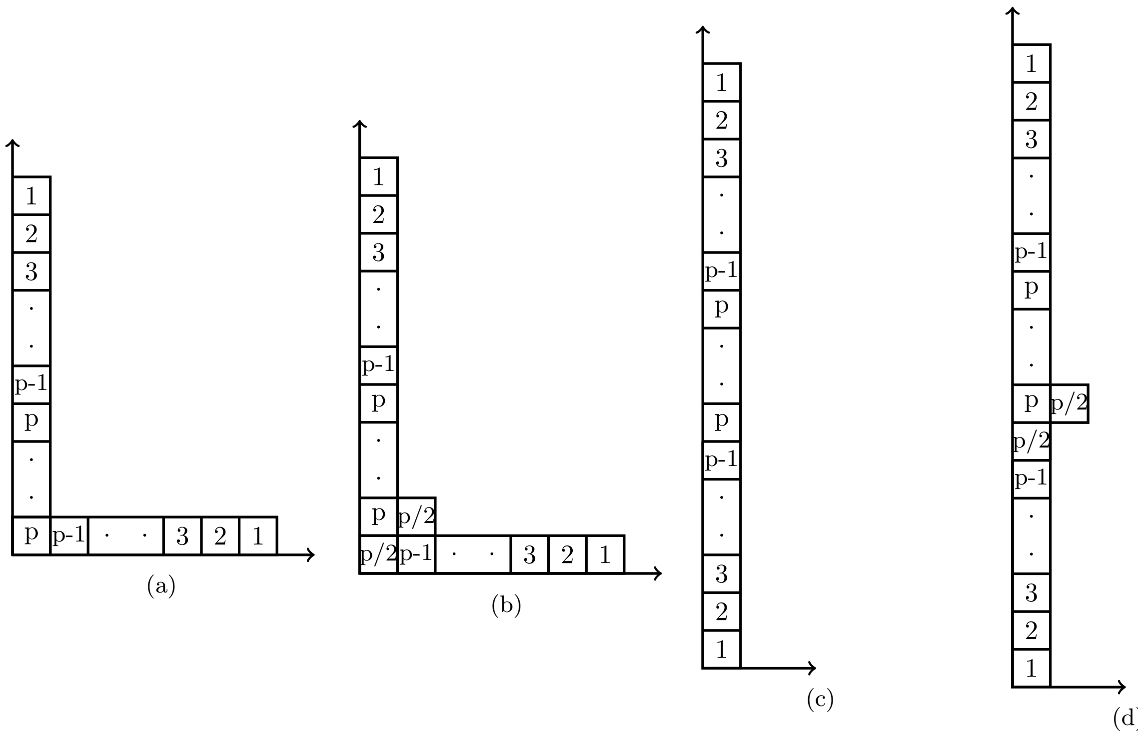


Figure 17: D-reductions of a rectangle.

If $p \geq n$, the sequence of remainders above is periodic with period n, given by the part of the sequence shown above, and the sum of any subsequence of n consecutive remainders is 0. So if p is divisible by n, $P_{p,q}(y)$ is divisible by $Q(y)$. If p is not divisible by n, then doing first the cancellation as above and then using the symmetry present in the sequence of remainders, the sum of the sequence of remainders equals 0 only if $r+1=t$, that is, only if q is divisible by n.

If r is even, one has the following sequence of remainders, each remainder written in a separate pair of parentheses:

$$\begin{aligned} R_{p,q}(y) =& (1+y+y^2+\cdots+y^{r-1}) \\ &+(y+y^2+\cdots+y^{r-2}) \\ &+(y^2+\cdots+y^{r-3}) \\ &\quad\cdots\cdots\cdots \\ &+(y^{\frac{r-2}{2}}+y^{\frac{r}{2}})+(0)-(y^{\frac{r-2}{2}}+y^{\frac{r}{2}}) \\ &\quad\cdots\cdots\cdots \\ &-(y+y^2+\cdots+y^{r-2}) \\ &-(1+y+y^2+\cdots+y^{r-1}) \\ &+(y^{r+1}+y^{r+3}+\cdots+y^{n-3}+y^{n-2}) \\ &+(y^{r+2}+\cdots+y^{n-3}) \\ &\quad\cdots\cdots\cdots \\ &+(y^{\frac{r+n-1}{2}}+y^{\frac{r+n+1}{2}})+(0)-(y^{\frac{r+n-1}{2}}+y^{\frac{r+n+1}{2}}) \\ &\quad\cdots\cdots\cdots \\ &-(y^{r+2}+\cdots+y^{n-3}) \\ &-(y^{r+1}+y^{r+3}+\cdots+y^{n-3}+y^{n-2}) \\ &\quad\cdots\cdots\cdots \end{aligned} \tag{33}$$

If $p \geq n$, the sequence of remainders above is periodic with period n, given by the part of the sequence shown above, and the sum of any subsequence of n consecutive remainders is 0. So if p is divisible by n, $P_{p,q}(y)$ is divisible by $Q(y)$. If p is not divisible by n, then doing first the cancellation as above and then using the symmetry present in the sequence of remainders, the sum of the sequence of remainders equals 0 only if $r+1=t$, that is, only if q is divisible by n.

2) We assume now that n divides p or q and count the extra tiles B that appears. They are counted by the coefficients of the quotient, call it $S(y)$, of the division of $P_{p,q}(y)$ by $Q(y)$. We need to compute the sum S_1 of the coefficients in $S(y)$ of the even powers of y and the sum S_2 of the coefficients in $S(y)$ of the odd powers of y. The difference $S_1 - S_2$ gives the number of extra tiles B that we need to consider.

We use the equation relating the derivatives:

$$P'_{p,q}(y) = Q'(y)S(y) + Q(y)S'(y). \tag{34}$$

Note that $Q(-1) = 0, Q'(-1) = n/2, S(-1) = S_1 - S_2$. Plugging in $x = -1$ gives:

$$S_1 - S_2 = S(-1) = \frac{2P'_{p,q}(-1)}{n}. \tag{35}$$

From (31) one has:

$$\begin{aligned} P'_{p,q}(y) &= 2\cdot 1 + 3\cdot 2y + 4\cdot 3y^2 + \cdots + (p-1)(p-2)y^{p-3} + p(p-1)y^{p-2} + \cdots + p(q-1)y^{q-2} \\ &\quad +(p-1)qy^{q-1} + (p-2)(q+1)y^q + \cdots + 2(p+q-3)y^{p+q-4} + (p+q-2)y^{p+q-3}. \end{aligned} \tag{36}$$

While computing $P_{p,q}(-1)$ we recall that n is even and distinguish the following cases:
Case A. p even, q odd.
Case B. p odd, q even.
Case C. p even, q even.
We need the following formulas:

$$\begin{aligned} 2\cdot 1 - 3\cdot 2 + 4\cdot 3 - \cdots - (p-1)(p-2) &= -\frac{p(p-2)}{2} \\ p(p-1) - p(p) + p(p+1) - \cdots + p(q-2) - p(q-1) &= -\frac{p(q-p+1)}{2} \\ (p-1)q - (p-2)(q+1) + \cdots + 3(p+q-4) - 2(p+q-3) + (p+q-2) &= \frac{pq}{2}. \end{aligned} \tag{37}$$

Case A. One has:

$$\begin{aligned} P'_{p,q}(-1) &= 2\cdot 1 - 3\cdot 2 + 4\cdot 3 - \cdots - (p-1)(p-2) + p(p-1) - \cdots - p(q-1) \\ &\quad +(p-1)q - (p-2)(q+1) + \cdots - 2(p+q-3) + (p+q-2) = \frac{p}{2}. \end{aligned} \tag{38}$$

The number of extra tiles B that we have in this case is

$$-\frac{p}{2} + \frac{p}{n} = \frac{p(1-k)}{n}. \tag{39}$$

In order to have a complete reduction, the number of B tiles has to be a multiple of $k-2$. As $k-1$ and $k-2$ are relatively prime, we have the condition that p is a multiple of $n(k-2)$.

Case B. One has:

$$\begin{aligned} P'_{p,q}(-1) &= 2\cdot 1 - 3\cdot 2 + 4\cdot 3 - \cdots + (p-1)(p-2) - p(p-1) + \cdots + p(q-1) \\ &\quad -(p-1)q + (p-2)(q+1) + \cdots - 2(p+q-3) + (p+q-2) = \frac{q}{2}. \end{aligned} \tag{40}$$

The number of extra tiles B that we have in this case is $\frac{q}{n}$. We have the condition that q is a multiple of $n(k-2)$.

Case C. One has:

$$P'_{p,q}(-1) = 2\cdot 1 - 3\cdot 2 + 4\cdot 3 - \cdots - (p-1)(p-2) + p(p-1) - \cdots + p(q-1) \\ -(p-1)q + (p-2)(q+1) + \cdots + 2(p+q-3) - (p+q-2) = 0. \tag{41}$$

The number of extra tiles B that we have in this case is

$$-\frac{p}{2} + \frac{p}{2} = 0. \tag{42}$$

So in this case a signed tiling is always possible.

5. Proof of Proposition 4

If k is even, finding a signed tiling for a k-inflated copy of the L n-omino can be reduced, via reductions by $C_3(k), C_4(k)$ tiles, to finding a signed tiling for a $nk \times k$ rectangle. From Theorem 3 follows that such a tiling always exists.

If k is odd, a reduction to a $kn \times k$ rectangle can be done only modulo a B tile, which does not belong to the ideal generated by $\mathcal{T}_n$.

Appendix

Here we show a signed tiling of a 16×1 rectangle by $\mathcal{T}_8$. To simplify the presentation, we use some tiles from the Gröbner basis, which we already know that can be signed tiled by $\mathcal{T}_8$. One uses the following formula:

$$xy^9H_1(8) + H_2(8) + xy^7C_4(8) + y^7C_3(8) - y^6C_5(8) = x\cdot\frac{y^{16}-1}{y-1}. \tag{43}$$

Acknowledgement

The author was partially supported by Simons Foundation Grant 208729.

References

References

[1] T. Becker, V. Weispfenning (in cooperation with Heinz Krendel), Gröbner Bases, Springer-Verlag, 1993.

[2] O. Bodini, B. Nouvel. Z-Tilings of polyominoes and standard basis, in Combinatorial Image Analysis, Springer (2004), 137–150.

[3] M. Chao, D. Levenstein, V. Nitica, R. Sharp, A coloring invariant for ribbon L-tetrominoes, Discrete Mathematics, *313* (2013) 611–621.

[4] J. H. Conway, J. C. Lagarias, Tilings with polyominoes and combinatorial group theory, Journal Combinatorial Theory, Series A *53* (1990), 183–208.

[5] M. M. Dizdarević, M. Timotijević, R. T. Živaljević, Signed polyomino tilings by n-in-line polyominoes and Gröbner bases, arxiv 1409.2745v1, 2014.

[6] S. W. Golomb, Checker boards and polyominoes, American Mathematical Monthly, *61* (1954) 675–682.

[7] S. W. Golomb, Replicating figures in the plane, Mathematical Gazette, *48* (1964) 403–412.

[8] S. W. Golomb, Polyominoes, Puzzles, Patterns, Problems, and Packings (2$^{\text{nd}}$ ed.), Princeton University Press, NJ, 1994.

[9] V. Nitica, Rep-tiles revisited, in the volume MASS Selecta: Teaching and Learning Advanced Undergraduate Mathematics, American Mathematical Society, 2003.

[10] V. Nitica, Signed tilings by ribbon L n-ominoes, n odd, via Gröbner bases, preprint 2015

[11] V. Nitica, Every tiling of the first quadrant by ribbon L n-ominoes follows the rectangular pattern. Open Journal of Discrete Mathematics, *5*, (2015) 11–25.

[12] I. Pak, Ribbon tile invariants, Transactions of the American Mathematical Society, *352* (2000) 5525–5561.