Renormalization Group Summation with Heavy Fields

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Abstract

The summation of logarithmic contributions to perturbative radiative corrections in physical processes through use of the renormalization group equation has proved to be a useful way of enhancing the information one can obtain from explicit calculation. However, it has proved difficult to perform this summation when massive fields are present. In this note we point out that if the masses involved are quite large, the decoupling theorem of Symanzik and of Appelquist and Carazzone can be used to make the summation of logarithms possible.

1 Introduction

Higher loop calculations in perturbative quantum chromodynamics (QCD) lead to results that depend on the logarithm of the unphysical renormalization mass scale μ^2, typically of the form $\ln\left(\frac{s}{\mu^2}\right)$ where s is the centre of mass energy in the process being considered. As μ^2 is unphysical, one has the renormalization group (RG) equation [1-5] which in many instances makes it possible to sum logarithmic corrections [6,7]. However, when massive fields are present, the form of the logarithms that arise is often so complicated that such summation is not feasible.

In the instance that the masses M^2 that arise are much greater than the energy scale of the process, a decoupling theorem due to Symanzik [8] and Appelquist and Carazzone [9] states that up to order $1/M^2$, these masses serve only to renormalize the parameters that characterize the full theory, leading to an effective low energy theory in which these massive fields are not present (ie, they "decouple"). As both the full theory and the effective low energy theory satisfy the RG equation, it proves possible to relate the running parameters of the full theory with those of the effective theory when one employs a mass independent renormalization scheme [10,11].

This result has the consequence of making it possible to perform RG summation of logarithmic contributions to radiative effects in the massless effective theory, and then to incorporate the contribution of heavy fields by invoking the Appelquist-Carazzone-Symanzik (ACS) theorem. This involves summing the logarithms appearing in the relationship between the parameters charactering the full theory and those characterizing the effective theory.

We focus on the process $R(e^+e^- \to \text{hadrons})$.

2 Summation of Logarithms and the ACS theorem

Let us denote the amplitude for the process $e^+e^- \to$ hadrons in the full theory by $\Gamma(s, a, M, \mu)$ and in the effective low energy theory (in which $s \ll M^2$) by $\Gamma^*(s, a^*, \mu)$. The ACS theorem implies that [10]

$$\Gamma^*(s, a^*, \mu) = Z\Gamma(s, a, M, \mu) + O\left(\frac{1}{M^2}\right) \tag{1}$$

where a and a^* are the couplings in the two theories, and

$$Z = Z(a, M/\mu) \tag{2a}$$

$$a^* = a^*(a, M/\mu). \tag{2b}$$

Since μ is an unphysical parameter, we have the RG equations

$$\mu\frac{d\Gamma}{d\mu} = \left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a} + M\delta(a)\frac{\partial}{\partial M}\right)\Gamma \tag{3a}$$

$$\begin{aligned} &= 0 \\ &= \frac{d\Gamma^*}{d\mu} = \left(\mu\frac{\partial}{\partial\mu} + \beta^*(a^*)\frac{\partial}{\partial a^*}\right)\Gamma^* \ , \end{aligned} \tag{3b}$$

where

$$\beta(a) = \mu\frac{da}{d\mu} = -ba^2(1 + ca + c_2a^2 + \ldots) \tag{4a}$$

$$\beta^*(a^*) = \mu\frac{da^*}{d\mu} = -b^*a^{*^2}(1 + c^*a^* + c_2^*a^{*^2} + \ldots) \tag{4b}$$

and

$$M\delta(a) = \mu\frac{dM}{d\mu} = Mfa(1 + g_1a + g_2a^2 + \ldots). \tag{5}$$

We are using the mass independent renormalization scheme [4,5] with minimal subtraction [4].

Together, eqs. (1-3) lead to [10]

$$\beta^*(a^*) = \left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a} + M\delta(a)\frac{\partial}{\partial M}\right)a^*(a, M/\mu); \tag{6}$$

this equation relates the couplings a and a^* in the low energy region. (A formal solution of eq.(6) is discussed in the appendix.)

If now the expansion

$$\Gamma^* = \sum_{n=0}^{\infty}\sum_{m=0}^{n} T^*_{nm} L^m a^{*n+1} \tag{7}$$

($L \equiv \ln(\mu/\sqrt{s})$) is substituted into eq. (3b), then

$$A_n(a^*) = -\frac{\beta^*(a^*)}{n}\frac{d}{da^*}A_{n-1}(a^*) \tag{8}$$

where

$$A_n(a^*) = \sum_{m=0}^{\infty} T^*_{n+m,n} a^{*n+m+1}. \tag{9}$$

If now we define

$$\eta = \int_{a_0^*}^{a^*(\eta)} \frac{dx}{\beta^*(x)} \qquad (a_0^* = a^*(0)) \tag{10}$$

then by eq. (8)

$$\begin{aligned} A_n(a^*(\eta)) &= -\frac{1}{n}\frac{d}{d\eta}A_{n-1}(a^*(\eta)) \\ &= \frac{(-1)^n}{n!}\frac{d^n}{d\eta^n}A_0(a^*(\eta)) \end{aligned} \tag{11}$$

and so by eqs. (7-11),

$$\begin{aligned} \Gamma^*(s, a^*, \mu) &= \sum_{n=0}^{\infty}\frac{(-L)^n}{n!}\frac{d^n}{d\eta^n}A_0(a^*(\eta)) \\ &= A_0(a^*(\eta - L)). \end{aligned} \tag{12}$$

As noted in ref. [12], $\eta - L = -\ln\left(\frac{e^{-\eta}\mu}{\sqrt{s}}\right)$ and so η can be absorbed into μ; we now denote $a^*(-L)$ by $\alpha^*(-L)$ to distinguish $a^*(\eta)$ in eq. (10) from the running coupling a^* of eq. (4b). In fact, by setting $L = 0$ in eqs. (1,12) we see that the boundary value $a_0^* = \alpha^*(0)$ is just the running coupling a^*, so that

$$\ln\frac{\sqrt{s}}{\mu} = \int_{a^*}^{\alpha^*\left(\ln\frac{\sqrt{s}}{\mu}\right)} \frac{dx}{\beta^*(x)}. \tag{13}$$

Differentiating eq. (13) with respect to $\ln\mu$ results in

$$-1 = \frac{1}{\beta^*\left(\alpha^*\left(\ln\frac{\sqrt{s}}{\mu}\right)\right)}\mu\frac{d}{d\mu}\alpha^*\left(\ln\frac{\sqrt{s}}{\mu}\right) - \frac{1}{\beta^*(a^*)}\left(\mu\frac{da^*}{d\mu}\right) \tag{14}$$

and so by eq. (4b)

$$\mu\frac{d\alpha^*\left(\ln\frac{\sqrt{(s)}}{\mu}\right)}{d\mu} = 0. \tag{15}$$

Thus changes in μ are compensated by changes in a^* in eq. (13) leaving $\alpha^*\left(\ln\frac{\sqrt(s)}{\mu}\right)$ unchanged; it is independent of μ. One assigns, using experimental results, a value to a^* appropriate to whatever value of μ that is chosen. The centre of mass energy s is some multiple of this chosen value of μ^2. The net result, following from this observation, is that eq. (12) can now be written in the form

$$\Gamma^*(s, a^*(\mu), \mu) = A_0\left(\alpha^*\left(\ln\frac{\sqrt{s}}{\mu}, a^*(\mu)\right)\right) \tag{16}$$

with the explicit dependence of α^* on μ through $\ln\frac{\sqrt{s}}{\mu}$ being compensated by the implicit dependence through $a^*(\mu)$.

By eqs. (9,16) we find that

$$\Gamma^* = \sum_{n=0}^{\infty} T_n^*(\alpha^*)^{n+1} \qquad (T_n^* \equiv T_{n0}^*). \tag{17}$$

In ref. [13] it is shown that renormalization scheme ambiguities when using mass independent renormalization can be characterized by $c_2^*, c_3^* \ldots$; in ref. [12] it is shown how these ambiguities result in T_n^* being expressed as

$$T_0^* = 1 \tag{18a}$$

$$T_1^* = \tau_1^* \tag{18b}$$

$$T_2^* = -c_2^* + \tau_2^* \tag{18c}$$

$$T_3^* = -2c_2^*\tau_1^* - \frac{1}{2}c_3^* + \tau_3^* \tag{18d}$$

etc. where τ_i^* are all renormalization scheme invariants.

We now return to eq. (6) which relates the running coupling a^* in the effective low energy theory to the running coupling a in the full theory. The function $a^*(a, M/\mu)$ is evidently of the form

$$a^* = a + \sum_{n=1}^{\infty}\sum_{m=1}^{n} \lambda_{nm}\Lambda^m a^n \tag{19}$$

where $\Lambda = \ln\frac{M}{\mu}$. If now

$$B_n(a) = \sum_{m=0}^{\infty} \lambda_{n+m,n} a^m \qquad (\lambda_\infty = 1) \tag{20}$$

then by eq. (19), eq. (6) becomes

$$\beta^*\left(\sum_{n=0}^{\infty} B_n\Lambda^n\right) = \left[(-1+\delta(a))\frac{\partial}{\partial\Lambda} + \beta(a)\frac{\partial}{\partial a}\right]\sum_{n=0}^{\infty} B_n\Lambda^n$$

or

$$\sum_{k=0}^{\infty}\frac{1}{k!}\beta^{*(k)}(a)\left(\sum_{n=0}^{\infty} B_n(a)\Lambda^n\right)^k = \sum_{n=0}^{\infty}\left[(-1+\delta(a))(nB_n\Lambda^{n-1}) + \beta(a)B_n'\Lambda^n\right]. \tag{21}$$

The functions $B_n(a)$ $(n = 1, 2\ldots)$ can be determined b y the requirement that eq. (21) is satisfied at each order in Λ. We find that

$$B_0(a) = 1 \tag{22a}$$

$$B_1(a) = \frac{\beta^*(a) - \beta(a)}{-1 + \delta(a)} \tag{22b}$$

$$B_2(a) = \frac{\beta^{*'}(a)B_1(a) - \beta(a)B_1'(a)}{2(-1 + \delta(a))} \tag{22c}$$

etc.

We can now write a^* as

$$a^* = \sum_{n=0}^{\infty} B_n(a)\Lambda^n \tag{23}$$

with $B_n(a)$ given by eq. (22). Together, eqs. (13, 17,23) show that Γ^* can now be expressed in terms of a, s and M. Thus we have achieved our objective of having a resummation of logarithmic corrections to the process $e^+e^- \to$ hadrons which takes into account the contributions of a field with a large mass. The argument that leads to eq. (15) is still valid; Γ^* is independent of μ. However, both a and M are to be determined experimentally at the value of μ that is chosen.

The question remains of the renormalization schemes used to compute Γ^* and Γ respectively. (This has been considered in the context of supersymmetry in ref. [14].) In principle these two choices can be made independently; there is no relationship between a and a^* that fixes how the parameters c_i of eq. (4a) and g_i of eq. (5) are related to the parameters c_i^* of eq. (4b). The values of c_i^* affect T_n^* through eq. (18); however, altering the values of c_i and g_i only serves to alter the relationship of a and M with μ. But as Γ^* is independent of μ, it is possible to select whatever value of c_i and g_i that is most convenient. For example, if $c_i = g_i = 0$, then eq. (22) is greatly simplified. The choice $c_i^* = 0$ further simplifies eq. (22); it also reduces $\alpha^*\left(\ln\frac{\sqrt{s}}{\mu}\right)$ in eq. (13) to being a Lambert W function [12]. One could also choose c_i^* so that $T_n^* = 0 \quad (n \geq 2)$, reducing the sum in eq. (17) to being just two terms.

3 Conclusion

We have combined RG summation with the ACS theorem to show how the calculation of the amplitude for the process $e^+e^- \to$ hadrons can take into account the presence of heavy field. We have also demonstrated how the result is independent of the renormalization scale μ and how one can make convenient choices for the parameters c_i^*, c_i and g_i that parameterize the renormalization schemes chosen.

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Appendix

In this section we discuss the formal solution to eq. (6) using the method of characteristics.

We begin by writing eq. (6) as

$$\left(\beta(a)\frac{\partial}{\partial a} + (-1+\delta(a))\frac{\partial}{\partial \Lambda}\right) a^*(a,\Lambda) = \beta^*(a^*(a,\Lambda))$$

where $\Lambda = \ln M/\mu$. Next, the characteristic functions $a(r,s)$, $\Lambda(r,s)$, $a^*(r,s)$ are introduced, satisfying

$$\frac{da(r,s)}{ds} = \beta(a(r,s)), \qquad a(r,0) = a \tag{A.1a}$$

$$\frac{d\Lambda(r,s)}{ds} = -1 + \delta(a(r,s)), \quad \Lambda(r,0) = \Lambda \tag{A.1b}$$

$$\frac{da^*(r,s)}{ds} = \beta^*(a^*(r,s)), \quad a^*(r,0) = a^* \ . \tag{A.1c}$$

Eq. (6) is satisfied by $a(r,s)$, $\Lambda(r,s)$ and $a^*(r,s)$ for all r and s. From eqs. (A.1a) and (A.1c) we see that

$$s = \int_{\overline{a}}^{a(r,s)} \frac{dx}{\beta(x)} + c_2(r) \tag{A.2a}$$

$$s = \int_{\overline{a}^*}^{a^*(r,s)} \frac{dx}{\beta^*(x)} + c_3(r) \tag{A.2b}$$

in addition, since

$$\frac{d\Lambda}{da} = \frac{-1+\delta(a)}{\beta(a)} \ , \tag{A.3}$$

we see that

$$\Lambda(r,s) = \int_{\overline{a}}^{a(r,s)} \frac{dx(-1+\delta(x))}{\beta(x)} + c_1(r) \ . \tag{A.4}$$

The function $c_i(r)$ are boundary conditions on eq. (A.1).

Implicit in eq. (19) is the boundary condition $a^*(r, \Lambda = 0) = a(r, \Lambda = 0) = r$ on eq. (A.1). From this condition, eqs. (A.2a, A.2b, A.4) lead to

$$0 = \int_{\overline{a}}^{r} \frac{dx}{\beta(x)} + c_2(r) \tag{A.5a}$$

$$0 = \int_{\overline{a}^*}^{r} \frac{dx}{\beta^*(x)} + c_3(r) \tag{A.5b}$$

$$0 = \int_{\overline{a}}^{r} \frac{dx(-1+\delta(x))}{\beta(x)} + c_1(r) \ . \tag{A.5c}$$

Eq. (A.5) fixes $c_i(4)$, and so eqs. (A.2a), (A.2b), (A.4) become

$$s = \int_{r}^{a(r,s)} \frac{dx}{\beta(x)} \tag{A.6a}$$

$$s = \int_{r}^{a^*(r,s)} \frac{dx}{\beta^*(x)} \tag{A.6b}$$

$$\Lambda(r,s) = \int_{r}^{a(r,s)} dx \frac{(-1+\delta(x))}{\beta(x)} \ . \tag{A.6c}$$

In principle now, from eq. (A.6c) one can determine r; this can be inserted into eq. (A.6a) to obtain s. With these values of s and r, a^* in eq. (A.6b) is determined in terms of $a^*(r,s)$, $\Lambda(r,s)$ thus giving the solution to eq. (A.1). To do this in closed form is not feasible, even when using the 't Hooft renormalization scheme in which

$$\beta(a) = -ba^2(1+ca) \tag{A.7a}$$

$$\beta^*(a^*) = -b^* a^{*^2}(1+c^* a^*) \tag{A.7b}$$

$$\delta(a) = fa\ . \tag{A.7c}$$

It is best to employ the expansion of eq. (19). However, for purposes of illustration, consider

$$\beta(a) = b/a \tag{A.8a}$$

$$\beta^*(a^*) = b^*/a^* \tag{A.8b}$$

$$\delta(a) = 0 \tag{A.8c}$$

in which case we have from eq. (A.6)

$$s = \frac{1}{2b}(a^2 - r^2) \tag{A.9a}$$

$$s = \frac{1}{2b^*}(a^{*^2} - r^2) \tag{A.9b}$$

$$\Lambda = \frac{-a^2 + r^2}{2b}\ . \tag{A.9c}$$

By eq. (A.9c)

$$r^2 = a^2 + 2b\Lambda$$

and so by eq. (A.9a)

$$s = -\Lambda$$

and so eq. (A.9b) becomes

$$a^{*^2} = a^2 + 2(b^* - b)\Lambda \tag{A.10}$$

which satisfies

$$\left(\frac{b}{a}\frac{\partial}{\partial a} - \frac{\partial}{\partial \Lambda}\right) a^* = \frac{b^*}{a^*}$$

with the boundary condition $a^* = a$ when $\Lambda = 0$.